n-Cyclic Refined Neutrosophic Groups
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Abstract

This paper introduces for the first time the concept of n-cyclic refined neutrosophic group as a direct application of the concept of n-cyclic refined neutrosophic set. Also, it discusses some of its elementary properties such as AH-subgroups, kernels, and direct products.

Keywords: n-cyclic refined neutrosophic group, AH-subgroup, AH-homomorphism

1. Introduction

Neutrosophy is a philosophical concept founded by Smarandache to generalize the fuzzy logic [1,32]. Neutrosophic sets were very applicative in many areas of science such as topology [2,27,60,61,63], decision making [21,35], applied mathematics [3,4,20,25,28,57,64], and pure mathematics [9,10,19,39,43,44,47,50,51,52].

Neutrosophic sets were used in the study of algebraic structures such as modules [6], rings [12], groups [13], and matrices [6,53,62]. In the literature, we find three interesting generalizations of neutrosophic sets, where refined neutrosophic sets [13] n-refined neutrosophic sets [15,50], and n-cyclic refined neutrosophic sets [31]. These kinds lead us to many interesting generalizations of classical algebraic structures such as n-refined neutrosophic modules [5], n-refined spaces and matrices [8,46,54], refined neutrosophic ring [23,48,49], and n-cyclic refined neutrosophic modules [31].

In this work, we use the concept of n-cyclic refined neutrosophic set to defined n-cyclic refined neutrosophic groups. These groups will be studied carefully, and many elementary properties will be discussed through this paper, especially AH-subgroups, kernels, and homomorphisms.

2. Preliminaries

Definition 2.1: [31]

Let $(R,+,\times)$ be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define $R_n(I) = \{a_0 + a_1I + \cdots + a_nI_n; a_i \in R\}$ to be n-cyclic refined neutrosophic ring.

Operations on $R_n(I)$ are defined as:

$$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i, \sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_{i+j \mod n}$$

Where $\times$ is the multiplication on the ring $R$.

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It is obvious that $R_n(I)$ is a ring in the algebraic ordinary concept.

**Definition 2.2:** [31]

Let $R_n(I)$ be an n-cyclic refined neutrosophic ring, it is called commutative if $xy = yx$ for all $x, y \in R_n(I)$. If there is $1 \in R_n(I)$ such that $1 \cdot x = x \cdot 1 = x$, then it is called an n-cyclic refined neutrosophic ring with unity.

**Definition 2.3:** [31]

Let $R_n(I)$ be a commutative n-cyclic refined neutrosophic ring and $P: R_n(I) \rightarrow R_n(I)$ is a function defined as $P(x) =$ $\sum_{i=0}^{n} a_i x^i$ such that $a_i \in R_n(I)$, we call P an n-cyclic refined neutrosophic polynomial on $R_n(I)$.

We denote by $R_n(I)[x]$ to be the ring of n-cyclic refined neutrosophic polynomials over $R_n(I)$.

Since $R_n(I)$ is a classical ring, then $R_n(I)[x]$ is a classical ring.

**Definition 2.4:** [31]

Let $(M, +, \cdot)$ be a module over the ring $R$, we say that $M_n(I) = M + MI_1 + \cdots + MI_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n : x_i \in M\}$ is a weak n-cyclic refined neutrosophic module over the ring $R$. Elements of $M_n(I)$ are called n-cyclic refined neutrosophic vectors, elements of $R$ are called scalars.

If we take scalars from the n-cyclic refined neutrosophic ring $R_n(I)$, we say that $M_n(I)$ is a strong n-cyclic refined neutrosophic module over the n-cyclic refined neutrosophic ring $R_n(I)$. Elements of $M_n(I)$ are called n-refined neutrosophic scalars.

**Remark 2.5:** [31]

Addition on $M_n(I)$ is defined as:

\[ \sum_{i=0}^{n} a_i I_i + \sum_{i=0}^{n} b_i I_i = \sum_{i=0}^{n} (a_i + b_i) I_i. \]

Multiplication by a scalar $m \in R$ is defined as:

\[ m \cdot \sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} (m a_i) I_i. \]

Multiplication by an n-cyclic refined neutrosophic scalar $m = \sum_{i=0}^{n} m_i I_i \in R_n(I)$ is defined as:

\[ \sum_{i=0}^{n} m_i I_i \cdot \sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} (m_i a_i) I_i I_j. \]

Where $a_i \in M, m_i \in R, I_i I_j = I_{(i+j) \mod n}$.

3. Main Concepts and discussion

**Definition 3.1:**

Let I be the neutrosophic element, which refers to indeterminacy, we define n-cyclic refining system of I by the set $L = \{ I^l = I, I^2, \ldots, I^{n-1} \}$, where $I^i \neq I^j$ for all $i \neq j$ and $i, j < n$.

We define a binary operation on L as follows:

\[ I^i, I^j = I^{i+j \mod n}. \] It is clear that L has a structure of the subset $(Z_n/\{0\}, +)$, where $Z_n$ is the additive group of integers modulo n.
Let \((G, \ast)\) be a group, and \(I\) is the neutrosophic element with property \(I^n = I; n \geq 2\), with
\[I^i \neq I^j \text{ for all } i \neq j \text{ and } i, j < n.\]
We call \(M(G) = G \cup GI \cup GI^2 \cup \ldots \cup GI^{n-1}\) an n-cyclic refined neutrosophic group.

It is easy to see that 2-cyclic refined neutrosophic group is the classical neutrosophic group.

**Remark 3.3:**
The sets \(GI^k\) are groups under the binary operation \((xl^k)(yl^k) = (xyl^k)\); \(k < n\) with identity \(I^k\) and each one of them must be isomorphic to \(G\).

**Definition 3.4:**
Let \(M(G)\) be an n-cyclic refined neutrosophic group, \(H\) be a subset of \(M(G)\). We call \(H\) an AH-subgroup if \(H = H_0 \cup H_1 I \cup H_2 I^2 \cup \ldots \cup H_{n-1} I^{n-1}\), where \(H_i\) is a subgroup of \(G\) for all \(i\).

We call \(H\) an AHS-subgroup if \(H_0 = H_1 = \ldots = H_{n-1}\).

We call \(H\) an AH-normal if \(H_i\) is normal subgroup of \(G\) for all \(i\).

We call \(H\) an AHS-normal if it is AHS-subgroup and AH-normal.

**Definition 3.5:**
Let \(M(G)\), \(M(H)\) be two n, m-cyclic refined neutrosophic groups respectively, \(f: M(G) \rightarrow M(H)\) be a map, we say that \(f\) is an AH-homomorphism if it is a homomorphism between \(G\), \(H\) i.e. \(f(xy) = f(x)f(y)\) for all \(x, y \in G\) and \(f(l^k) = (l^k)\) such \(l^k\) is the neutrosophic element of \(H\).

We define \(AH - Ker(f) = Kerf_0 \cup Kerf_1 I \cup \ldots \cup Kerf_{n-1} I^{n-1}\), we regard that \(AH - Ker(f)\) is an AHS-normal subgroup of \(M(G)\).

We say that \(f\) is an isomorphism if it is a correspondence one-to-one homomorphism.

If \(H = H_0 \cup H_1 I \cup \ldots \cup H_{n-1} I^{n-1}\) and \(K = K_0 \cup K_1 I \cup \ldots \cup K_{n-1} I^{n-1}\) are two AH-subgroups of \(M(G)\). We say that they are isomorphic if \(H_i \cong K_i\) for all \(i\).

**Definition 3.6:**
Let \(H, K\) be two AH-subgroups of \(M(G)\). We define
\[HK = H_0K_0 \cup H_1K_1 I \cup \ldots \cup H_{n-1}K_{n-1} I^{n-1}.\]

**Definition 3.7:**
Let \(M(G)\) be an n-cyclic refined neutrosophic group, \(H = H_0 \cup H_1 I \cup H_2 I^2 \cup \ldots \cup H_{n-1} I^{n-1}\) be an AH-normal subgroup of \(M(G)\). We define the corresponding AH-factor as \(M(G)/H = (G/H_0) \cup (G/H_1 I) \cup \ldots \cup (G/H_{n-1} I^{n-1}).\)

**Definition 3.8:**
Let \(M(G)\) be an n-cyclic refined neutrosophic group. We define the AH-center of \(M(G)\) by

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\[ Z(M(G)) = Z(G) \cup Z(G)I \cup \ldots \cup Z(G)I^{n-1}. \]

It is easy to see that \( Z(M(G)) \) is an AHS-normal subgroup of \( M(G) \).

**Definition 3.9:**

Let \( M(G) \) be an \( n \)-cyclic refined neutrosophic group. We say that \( M(G) \) is abelian if \( G \) is abelian, i.e. \( M(G) = Z(M(G)) \).

\( M(G) \) is said to be cyclic if \( G \) is cyclic.

**Theorem 3.10:**

Let \( M(G) \) be an \( n \)-cyclic refined neutrosophic group, then

(a) If \( H \) is an AH-normal subgroup and \( M(G) \) is abelian, then \( M(G)/H \) is abelian.

(b) If \( M(G) \) is finite and \( H \) is an AHS-subgroup, then \( O(H) \) divides \( O(M(G)) = nO(G) \).

(c) If \( H \) is an AH-normal subgroup and \( M(G) \) is cyclic then \( M(G)/H \) is cyclic.

Proof:

(a) Since \( M(G)/H = (G/H_0) \cup (G/H_1)I \cup \ldots \cup (G/H_{n-1})I^{n-1} \) and \( G/H_i \) is abelian for all \( i \), then \( M(G)/H \) is abelian.

(b) We have that \( O(H) = nO(H_0) \) and \( O(H_0) \) divides the order of \( G \) then \( O(H) \) divides \( O(M(G)) = nO(G) \).

(c) Since \( G/H_i \) is cyclic for all \( i \) then \( M(G)/H \) is cyclic.

**Theorem 3.11:**

Let \( M(G) \) be an \( n \)-cyclic refined neutrosophic group and \( H, K \) be two AH-subgroups, then

(a) \( H \cap K \) is an AH-subgroup.

(b) If \( H, K \) are AHS-subgroups, then \( H \cap K \) is an AHS-subgroup.

(c) If \( H, K \) are AH-normal subgroups, then \( H \cap K \) and \( HK \) are AH-normal subgroups.

(d) If \( H, K \) are AHS-normal subgroups, then \( H \cap K \) and \( HK \) are AHS-normal subgroups.

Proof: Suppose that \( H = H_0 \cup H_1I \cup \ldots \cup H_{n-1}I^{n-1} \) and \( K = K_0 \cup K_1I \cup \ldots \cup K_{n-1}I^{n-1} \) then \( H \cap K = (H_0 \cap K_0) \cup (H_1 \cap K_1)I \cup \ldots \cup (H_{n-1} \cap K_{n-1})I^{n-1} \) by this argument we can easily find that the proof holds.

**Theorem 3.12:**

Let \( M(G), M(H) \) be two \( n, m \)-cyclic refined neutrosophic groups respectively, and \( f: M(G) \rightarrow M(H) \) be a homomorphism, then

(a) \( n \geq m \).

(b) If \( K \) is an AH-subgroup of \( M(G) \) then \( f(K) \) is an AH-subgroup of \( M(H) \).

(c) If \( K \) is an AHS-subgroup of \( M(G) \) then \( f(K) \) is an AHS-subgroup of \( M(H) \).
(d) If $K$ is an AH-normal subgroup of $M(G)$ then $f(K)$ is an AH-normal subgroup of $f(M(G))$.

(e) If $K$ is an AHS-normal subgroup of $M(G)$ then $f(K)$ is an AHS-normal subgroup of $f(M(G))$.

(f) $M(G)/\text{Ker}f \cong f(M(G))$.

Proof:

(a) Suppose that $n < m$ then $I' = f(I) = f(I^n) = (I')^n$ and this is a contradiction thus $n \geq m$.

(b) Suppose that $K = K_0 \cup K_1 I \cup ... \cup K_{n-1} I^{n-1}$, then $f(K) = f(K_0) \cup f(K_1) I \cup ... \cup f(K_{n-1}) I^{n-1}$ with subgroups $f(K_i)$ for all $i$ of $M(H)$, so that $f(K)$ is an AH-subgroup of $M(H)$.

(c) It is obvious that if $K_i = K_j, then f(K_i) \cong f(K_j)$, thus $f(K)$ is an AHS-subgroup of $M(H)$.

(d), (e) hold directly from (b) and (c) and from the fact that if $K_i$ is normal, then $f(K_i)$ is normal.

(f) From the definition, we find $M(G)/\text{Ker}f = (G/\text{Ker}f_0) \cup (G/\text{Ker}f_1) I \cup ... \cup (G/\text{Ker}f_{n-1}) I^{n-1}$; but $G/\text{Ker}f_G \cong f(G)$, thus $M(G)/\text{Ker}f \cong f(G)$.

Theorem 3.13:

Let $M(G)$ be an n-cyclic refined neutrosophic group and $H, K$ be two AH-normal subgroups with $K \leq H$, then $M(G)/K \cong M(G)/H$.

Proof:

Suppose that $H_0 \leq H_0 \cup H_1 I \cup ... \cup H_{n-1} I^{n-1}$ and $K = K_0 \cup K_1 I \cup ... \cup K_{n-1} I^{n-1}$ with $K \leq H$, then $M(G)/K \cong M(G)/H \cong M(G)/H_0 \cup (G/H_0) I \cup ... \cup (G/H_{n-1}) I^{n-1}$.

Theorem 3.14:

Let $M(G)$ be an n-cyclic refined neutrosophic group, and $H$ is an AH-normal subgroup, then for each AH-subgroup $T$ of $M(G)/H$ there is an AH-subgroup of $M(G)$ contains $H$.

Proof: It can be proved as the classical case.

Definition 3.15:

Let $M(G), M(H)$ be two n-cyclic refined neutrosophic groups,

we define $M(G) \times M(H) = (G \times H) \cup (G \times H) I^1 \cup ... \cup (G \times H) I^n$ with $(I^k)^k = I^k (I')^k$ for all $k$, it is clear that $M(G) \times M(H)$ is an n-generalized neutrosophic group with neutrosophic element $I^k$.

Theorem 3.16:

Let $M(G), M(H)$ be two n-cyclic refined neutrosophic groups, then

(a) If $M(G), M(H)$ are abelian then $M(G) \times M(H)$ is abelian.

(b) If $T, S$ are two AH-subgroups of $M(G), M(H)$ respectively, then $T \times S$ is an AH-subgroup of $M(G) \times M(H)$.

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(c) If $T,S$ are two AH-normal subgroups of $M(G), M(H)$ respectively, then $T \times S$ is an AH-normal subgroup of $M(G) \times M(H)$.

(d) If $T,S$ are two AHS-subgroups of $M(G), M(H)$ respectively, then $T \times S$ is an AHS-subgroup of $M(G) \times M(H)$.

(e) If $T,S$ are two AHS-normal subgroups of $M(G), M(H)$ respectively then $T \times S$ is an AHS-normal subgroup of $M(G) \times M(H)$.

Proof:

(a) It is clear since $G \times H$ is abelian.

(b) Assume that $T = T_0 \cup \ldots \cup T_{n-1}$ and $S = S_0 \cup \ldots \cup S_{n-1}$, then

$M(G) \times M(H)/T \times S \cong M(G)/T \times M(H)/S$.

Proof:

Suppose that $T = T_0 \cup \ldots \cup T_{n-1}$ and $S = S_0 \cup \ldots \cup S_{n-1}$, then

$M(G)/T \times M(H)/S = (G/T_0 \times H/S_0) \cup (G/T_1 \times H/S_1) \cup \ldots \cup (G/T_{n-1} \times H/S_{n-1})$.

$\cong G \times H/T_0 \times S_0 \cup \ldots \times (G \times H/T_{n-1} \times S_{n-1}) = M(G) \times M(H)/T \times S$.

Example 3.18:

Consider the additive group $(R^+,\cdot)$ and the integer $n = 3$. The corresponding 3-cyclic refined neutrosophic group is $M(R) = \{a,bl,cl^2; a,b,c \in R^+\}$.

1) We know that $(Q^+,\cdot)$ is a subgroup of $(R^+,\cdot)$, hence $L=Q^+ \cup Q^+I \cup Q^+I^2 = \{x, yI, zI^2; x, y, z \in Q^+\}$ is a 3-cyclic refined AHS-subgroup.

2) The corresponding AH-factor is $M(R)/L = R^*/Q^+ \cup R^*/Q^+I \cup R^*/Q^+I^2$.

Example 3.19:
Consider the following two groups \( G = (\mathbb{Z}, +), H = (2\mathbb{Z}, +) \), and the integer \( n = 4 \), the corresponding 4-cyclic refined neutrosophic groups are 
\[
M(G) = \{a, bI, cI^2, dI^3; a, b, c, d \in \mathbb{Z}\}, \quad M(H) = \{2a, 2bI, 2cI^2, 2dI^3; a, b, c, d \in \mathbb{Z}\}.
\]

1) \( g : G \to H; g(x) = 4x \) is a group homomorphism. \( \text{Ker}(g) = \{0\} \).

2) The corresponding AH-homomorphism is \( f : M(G) \to M(H); f(x) = 4x, \text{ and } f(xl^l) = 4xl^l; 1 \leq l \leq 3 \).

3) The corresponding AH-kernel is \( AH - \text{Ker}(f) = \text{Ker}(g) \cup \text{Ker}(g)I \cup \text{Ker}(g)I^2 \cup \text{Ker}(g)I^3 = \{0, 0 + I, 0 + I^2, 0 + I^3\} \).

**Conclusion**

In this paper, we have defined for the first time the concept of n-cyclic refined neutrosophic group as a new application of n-cyclic refined neutrosophic sets. We have discussed some of their elementary properties such as AH-subgroups, AH-kernels and direct products.

As a future research directions, we aim to study the n-cyclic refined neutrosophic semi groups and loops.

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