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Stability of Shuffled Switched Linear Systems: A Joint Spectral Radius Approach ⋆

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Abstract

A switching signal for a switched system is said to be shuffled if all modes of the system are activated infinitely often. In this paper, we develop tools to analyze stability properties of discrete-time switched linear systems driven by shuffled switching signals. We introduce the new notion of shuffled joint spectral radius (SJSR), which intuitively measures how much the state of the system contracts each time the signal shuffles (i.e. each time all modes have been activated). We show how this notion relates to stability properties of the associated switched systems. In particular, we show that some switched systems that are unstable for arbitrary switching signals can be stabilized by using switching signals that shuffle sufficiently fast and that the SJSR allows us to derive an expression of the minimal shuffling rate required to stabilize the system. We then present several approaches to compute lower and upper bounds of the SJSR using tools such as the classical joint spectral radius, Lyapunov functions and finite state automata. Several tightness results of the bounds are established. Finally, numerical experiments are presented to illustrate the main results of the paper.

Key words: Switched systems, Shuffled stability, Joint spectral radius, Lyapunov methods, Automata theoretic techniques.

1 Introduction

Switched systems are dynamical systems with several modes of operation, each mode being described by a differential (continuous-time) or difference (discrete-time) equation. At each instant, the active mode is determined by a so-called switching signal. Switched systems are very useful in applications to describe in a faithful manner the execution of control algorithms on distributed computing infrastructures and thus for taking into account the constraints related to the sharing of computing and communication resources [2,12,26].

The question of the stability of switched systems has been extensively studied for almost three decades. Early works have mostly focused on stability for switching signals that are arbitrary or that satisfy some minimum or average dwell-time condition (see e.g. [22,21,27,23] and the references therein). For the class of discrete-time switched linear systems with arbitrary switching signals, a powerful notion for analyzing stability is that of joint spectral radius (JSR) [16], which allows to characterize the worst-case asymptotic growth of infinite products of matrices. However, computing the JSR is, in general, a difficult task [29], and several methods have been proposed to compute tight lower and upper bounds [14,8,30]. In the past decade, a significant amount of research has been carried out to analyze the stability of switched systems subject to constrained switching. Constraints on the switching signal are usually described by finite-state automata and methods for handling such constraints through Lyapunov functions have been presented in several works [20,18,25]. An alternative approach to analyze stability with such constrained switching signals is by computing the so-called constrained joint spectral radius [11,18,33].

There are however classes of constrained switching signals that cannot be described using classical finite-state

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automata. An example of such a class is that of shuffled switching signals. A switching signal is said to be shuffled if each mode of the switched system is activated an infinite number of times. It is known from formal language theory [4] that a non-trivial set of sequences cannot satisfy both liveness and safety properties. Then, the set of shuffled switching signals, which satisfies the liveness property, cannot be characterized using classical finite state automata that can only describe safety properties. To the best of our knowledge, shuffled stability has first been studied in [17] where it is proved that the set of stable shuffled switched linear systems is not semi-algebraic. However, no further characterization of shuffled stability is provided in that paper. A necessary and sufficient condition can be found in [15] in terms of the maximal spectral radius of an infinite set of matrices. However, its computation is non-trivial and not discussed in that paper, which therefore does not provide any practical way to check shuffled stability. In [31], it is shown that a notion of robust shuffled stability is equivalent to stability for arbitrary switching signals. In [13], a Lyapunov characterization of shuffled stability and a converse Lyapunov result have been established. While the latter paper provides a method to check shuffled stability of switched linear systems, it does not provide tight estimates of the convergence rate in particular with respect to the characteristics of the shuffling phenomena. Providing such estimates is the main contribution of this paper.

Stability analysis of shuffled switched systems is also interesting from the point of view of applications. An example of such applications is multi-agent consensus with switching communication topologies. Consider a symmetric and connected communication graph where only one edge is active at each time instant and is selected by a switching signal. If one uses a shuffled switching signal, then each edge is activated infinitely often so that the union of future communication graphs is connected at all time, and it is well-known (see e.g. [24,7]) that the consensus is asymptotically reached. On the contrary, if some edges are never activated the union of future communication graphs may be disconnected and the consensus is not attainable. Another potential application is the design of observers for switched systems where the dynamics in each mode is unobservable. Clearly, the dynamics of the estimation error cannot be stable for arbitrary switching signals, since keeping the same mode activated all the time makes the system unobservable (see e.g. [27]). However, considering shuffled switching signals, it may be possible to design asymptotically convergent observers [28]. This type of observer design problems is natural when considering a system with distributed sensors communication over a shared network. More generally, shuffled switching signals are of interest to describe applications where multiple components rely on a shared resource and where the access to this resource must be granted to each of the component infinitely often. An example of such application is shown in an illustrative case study at the end of this paper. An additional motivation for considering shuffled switching signals is that they constitute a simple example of more general ω-regular languages [4] that can be described using either non-deterministic Büchi automata, deterministic Rabin automata, deterministic Muller automata (which are finite state automata with various acceptance conditions), or using Linear Temporal Logic formulas. Moreover, it has been shown in [31] that the stability of a switched linear system whose switching signal belongs to an arbitrary ω-regular language is actually equivalent to the shuffled stability of a lifted switched linear system. Being able to analyze the stability of such systems gives the possibility to deal with complex applications involving control and scheduling co-design [32,3].

In this paper, we provide a new tool to analyze shuffled stability of discrete-time switched linear systems. The main contributions of this paper are as follows. We introduce a notion of shuffled joint spectral radius (SJSR), which intuitively measures how much the state of the system contracts each time the signal shuffles (i.e. each time all modes have been activated). We establish several properties of the SJSR and show how it relates to stability properties of switched systems driven by shuffled switching signals. In particular, we show that some switched systems that are unstable for arbitrary switching signals can be stabilized by using switching signals that shuffle sufficiently fast, the minimal shuffling rate being related to both the JSR and the SJSR. We also present two approaches to compute approximation of the SJSR. The first approach is based on the JSR of a finite set of matrices and allows us to compute asymptotically tight lower and upper bounds. The second approach is based on Lyapunov functions and automata theoretic techniques and allows us to compute upper bounds. The Lyapunov approach draws inspiration from [13] but we consider a different underlying automata which allow us to prove the tightness of our upper bound. Our Lyapunov conditions also resemble those of [19], however, in that work, the relation of such Lyapunov functions to shuffled switching signals was not established.

The organization of the paper is as follows. Section 2 presents the necessary background on switched systems and shuffled switching signals. In Section 3, the SJSR is introduced and several of its properties are established. Section 4 shows the relationship between the SJSR and stability properties of a switched system driven by shuffled switching signals. In Section 5, we present two approaches to compute approximations of the SJSR. Finally, two numerical examples are used in Section 6 to illustrate the main results of the paper.

Notations: $\mathbb{R}_0^+$ denotes the set of non-negative real numbers. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. We use $\|\cdot\|$ to denote an arbitrary norm on $\mathbb{R}^n$ and the

associated induced matrix norm defined for $M \in \mathbb{R}^{n\times n}$ by $\|M\| = \sup_{x\neq 0} \frac{\|Mx\|}{\|x\|}$. The spectral radius of a matrix M is denoted $\boldsymbol{\rho}(M)$, note that $\boldsymbol{\rho}(M) = \lim_{k\to+\infty} \|M^k\|^{1/k}$. Given a sequence of matrices $(M_k)_{k\in\mathbb{N}}$, with $M_k \in \mathbb{R}^{n\times n}$, we define for $K_1, K_2 \in \mathbb{N}$, with $K_1 \leq K_2$

$$\prod_{k=K_1}^{K_2} M_k = \begin{cases} M_{K_2} \cdots M_{K_1}, & \text{if } K_2 > K_1, \\ M_{K_1}, & \text{if } K_2 = K_1. \end{cases}$$

Given a set of matrices $\mathcal{M} \subseteq \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}$, we define $\alpha\mathcal{M} = \{\alpha M |\ M \in \mathcal{M}\}$. Given a set $\mathcal{I}$, we use $2^{\mathcal{I}}$ to denote its powerset, that is the set of subsets of $\mathcal{I}$, including the empty set $\emptyset$ and $\mathcal{I}$ itself.

2 Preliminaries

Let us consider a discrete-time switched system of the following form:

$$x(t+1) = A_{\theta(t)}x(t),\ t \in \mathbb{N} \tag{1}$$

where $x : \mathbb{N} \to \mathbb{R}^n$ is the *trajectory* and $\theta : \mathbb{N} \to \mathcal{I}$ is the *switching signal* belonging to the class $\mathcal{S}(\mathcal{I})$ of arbitrary switching signals. $\mathcal{I} = \{1, \ldots, m\}$, with $m \geq 2$, is the finite set of *modes* and $\mathcal{A} = \{A_i \in \mathbb{R}^{n\times n} | i \in \mathcal{I}\}$ is a collection of matrices indexed by the modes. For a switching signal θ, let $\mathbb{A}_{\theta,0} = I_n$, and

$$\mathbb{A}_{\theta,T} = \prod_{t=0}^{T-1} A_{\theta(t)},\ \forall T \geq 1.$$

Given an initial state $x_0 \in \mathbb{R}^n$, the trajectory defined by (1) with $x(0) = x_0$ is unique and is denoted $\mathbf{x}(., x_0, \theta)$, it satisfies for all $t \in \mathbb{N}$, $\mathbf{x}(t, x_0, \theta) = \mathbb{A}_{\theta,t}x_0$.

In this paper, we focus on a particular class of constrained switching signals called shuffled, i.e. signals for which each mode in $\mathcal{I}$ is activated infinitely often. A formal definition is given in [13] as:

Definition 1. *A switching signal $\theta : \mathbb{N} \to \mathcal{I}$ is shuffled if*

$$\forall i \in \mathcal{I}, \forall T \in \mathbb{N}, \exists t \geq T :\ \theta(t) = i.$$

The sequence of shuffling instants $(\tau_k^\theta)_{k\in\mathbb{N}}$ is defined by $\tau_0^\theta = 0$ and for all $k \in \mathbb{N}$,

$$\tau_{k+1}^\theta = \min\left\{ t > \tau_k^\theta \;\middle|\; \begin{array}{c} \forall i \in \mathcal{I}, \exists s \in \mathbb{N} : \\ \tau_k^\theta \leq s < t \text{ and } \theta(s) = i \end{array} \right\}.$$

The shuffling index $\kappa^\theta : \mathbb{N} \to \mathbb{N}$ is given by

$$\kappa^\theta(t) = \max\{k \in \mathbb{N} |\ \tau_k^\theta \leq t\}.$$

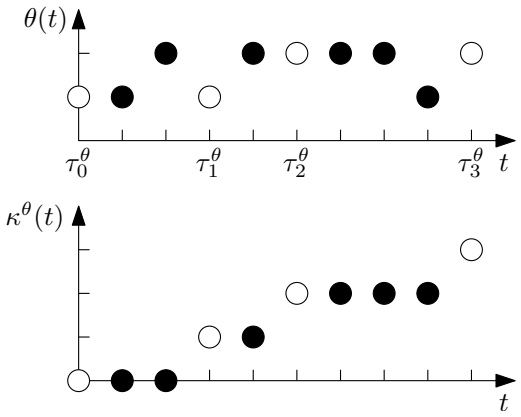


Fig. 1. A switching signal (with $m = 2$) and the associated shuffling instants and shuffling index [13].

The shuffling rate γ^θ is defined as

$$\gamma^\theta = \liminf_{t\to+\infty} \frac{\kappa^\theta(t)}{t}.$$

The set of shuffled switching signals is denoted $\mathcal{S}_s(\mathcal{I})$.

Intuitively, the period between two successive shuffling instants τ_k^θ and τ_{k+1}^θ is a period of minimal duration over which all modes are activated at least once. The shuffling index $\kappa^\theta(t)$ counts the number of times the signal has shuffled (i.e. the number of shuffling instants) up to time t. The shuffling rate γ^θ characterizes the frequency of shuffling instants over all time. Note that for all $k \in \mathbb{N}$, $\tau_k^\theta \geq km$ and therefore, for all $t \in \mathbb{N}$, we have $0 \leq \kappa^\theta(t) \leq \frac{t}{m}$, from which it follows that $\gamma^\theta \in [0, \frac{1}{m}]$. An example of shuffled switching signal is shown in Figure 1.

It is well known (see e.g. [16, Corollary 1.1]) that the system (1) is asymptotically stable for arbitrary switching signals if and only if $\boldsymbol{\rho}(\mathcal{A}) < 1$, where $\boldsymbol{\rho}(\mathcal{A})$ is the *joint spectral radius* (JSR) of $\mathcal{A}$, formally defined for a bounded set of matrices $\mathcal{M} \subseteq \mathbb{R}^{n\times n}$ by:

$$\boldsymbol{\rho}(\mathcal{M}) = \lim_{k\to+\infty} \left(\sup \left\{ \left\| \prod_{j=1}^{k} M_j \right\|^{1/k} \;\middle|\; \begin{array}{c} M_j \in \mathcal{M}, \\ j = 1, \ldots, k \end{array} \right\} \right).$$

However, the stability condition based on the JSR turns out to be conservative in the case of constrained switching sequences. Some works have adapted the notion of joint spectral radius to switching signals generated by deterministic [11,18,33] or stochastic [15,10] finite state automata. However, none of these constraints allow to deal with shuffled switching signals. Therefore, we introduce, in the next section, a new notion of joint spectral radius adapted for shuffled switching signals.

3 Shuffled Joint Spectral Radius

In this section, we introduce the shuffled joint spectral radius and establish several of its properties.

Definition 2. *Let $\rho > \boldsymbol{\rho}(\mathcal{A})$, the Shuffled Joint Spectral Radius relative to $(\mathcal{A}, \rho)$ (ρ-SJSR for short) is defined as*

$$\boldsymbol{\lambda}(\mathcal{A}, \rho) = \limsup_{k \to +\infty} \left(\sup_{\theta \in \mathcal{S}_s(\mathcal{I})} \left(\frac{\left\| \mathbb{A}_{\theta, \tau_k^\theta} \right\|}{\rho^{\tau_k^\theta}} \right)^{1/k} \right). \tag{2}$$

To better understand the characteristics of the ρ-SJSR, we are going to analyze some properties of the function $\boldsymbol{\lambda}(\mathcal{A}, \cdot)$. We first show that the ρ-SJSR of $\mathcal{A}$ can be seen as the JSR of an infinite but bounded set of matrices. For $\rho > \boldsymbol{\rho}(\mathcal{A})$, let us consider the following set of matrices:

$$\mathcal{M}_\rho = \left\{ \frac{1}{\rho^N} \prod_{k=1}^{N} A_{j_k}, \middle| \begin{array}{c} j_1, \dots, j_N \in \mathcal{I},\ N \in \mathbb{N}, \\ \forall i \in \mathcal{I}, \exists k, j_k = i, \\ \text{and } \forall k \neq N, j_k \neq j_N \end{array} \right\}.$$

Since N is unbounded, it is clear that $\mathcal{M}_\rho$ generally has an infinite number of elements. The following Lemma shows that this infinite set is bounded for $\rho > \boldsymbol{\rho}(\mathcal{A})$.

Lemma 1. *For all $\rho > \boldsymbol{\rho}(\mathcal{A})$, $\mathcal{M}_\rho$ is a bounded set of matrices.*

Proof. Let $\rho > \boldsymbol{\rho}(\mathcal{A})$, then $\boldsymbol{\rho}(\frac{1}{\rho}\mathcal{A}) = \frac{\boldsymbol{\rho}(\mathcal{A})}{\rho} < 1$. Therefore, from [6, Theorem 1 (b)], we get that $\frac{1}{\rho}\mathcal{A}$ is a left convergent product (following the terminology in [6]). Since the elements of $\mathcal{M}_\rho$ are products of matrices belonging to the set $\frac{1}{\rho}\mathcal{A}$, it follows from [6, Theorem 1 (a)] that $\mathcal{M}_\rho$ is a bounded set of matrices. □

Since $\mathcal{M}_\rho$ is a bounded set of matrices, its JSR is well defined. Moreover, in any matrix of $\mathcal{M}_\rho$, all modes in $\mathcal{I}$ appear at least once with the last mode j_N appearing only once. Hence, the set $\mathcal{M}_\rho$ coincides with the set of all possible matrices $\frac{1}{\rho^{\tau_1^\theta}} \mathbb{A}_{\theta, \tau_1^\theta}$ for $\theta \in \mathcal{S}_s(\mathcal{I})$.

Lemma 2. *For all $\rho > \boldsymbol{\rho}(\mathcal{A})$, $\boldsymbol{\lambda}(\mathcal{A}, \rho) = \boldsymbol{\rho}(\mathcal{M}_\rho)$.*

Proof. Let $\rho > \boldsymbol{\rho}(\mathcal{A})$ and $k \in \mathbb{N}$ with $k \geq 1$, observe that for every $\theta \in \mathcal{S}_s(\mathcal{I})$ the sequence $(\theta_j)_{j \in \mathbb{N}}$ of switching signals in $\mathcal{S}_s(\mathcal{I})$ defined by $\theta_j(t) = \theta(t + \tau_j^\theta)$ for $t, j \in \mathbb{N}$ is such that

$$\mathbb{A}_{\theta, \tau_k^\theta} = \prod_{j=0}^{k-1} \mathbb{A}_{\theta_j, \tau_1^{\theta_j}} \text{ and } \tau_k^\theta = \sum_{j=0}^{k-1} \tau_1^{\theta_j}. \tag{3}$$

On the other hand, given a sequence $(\theta_j)_{j \in \mathbb{N}}$ in $\mathcal{S}_s(\mathcal{I})$ one can construct a switching signal $\theta \in \mathcal{S}_s(\mathcal{I})$ such that $\tau_i^\theta = \sum_{j=0}^{i-1} \tau_1^{\theta_j}$ for $i \geq 1$ and $\theta(t) = \theta_i(t - \tau_i^\theta)$ for $\tau_i^\theta \leq t < \tau_{i+1}^\theta$, so that (3) holds true. Then, it follows that

$$\begin{aligned} &\sup_{\theta \in \mathcal{S}_s(\mathcal{I})} \left(\frac{\left\| \mathbb{A}_{\theta, \tau_k^\theta} \right\|}{\rho^{\tau_k^\theta}} \right)^{1/k} \\ &\qquad = \sup_{\theta_0, \dots, \theta_{k-1} \in \mathcal{S}_s(\mathcal{I})} \left(\frac{\left\| \prod_{j=0}^{k-1} \mathbb{A}_{\theta_j, \tau_1^{\theta_j}} \right\|}{\rho^{\tau_1^{\theta_0} + \cdots + \tau_1^{\theta_{k-1}}}} \right)^{1/k} \\ &\qquad = \sup_{M_1, \dots, M_k \in \mathcal{M}_\rho} \left\| \prod_{j=1}^{k} M_j \right\|^{1/k}. \end{aligned}$$

Then, taking the limit superior on both sides when k goes to infinity yields the expected result. □

Due to Lemma 2, the ρ-SJSR inherits several properties of the JSR.

Proposition 1. *For all $\rho > \boldsymbol{\rho}(\mathcal{A})$,*

$$\boldsymbol{\lambda}(\mathcal{A}, \rho) = \lim_{k \to +\infty} \left(\sup_{\theta \in \mathcal{S}_s(\mathcal{I})} \left(\frac{\left\| \mathbb{A}_{\theta, \tau_k^\theta} \right\|}{\rho^{\tau_k^\theta}} \right)^{1/k} \right). \tag{4}$$

Moreover, the value $\boldsymbol{\lambda}(\mathcal{A}, \rho)$ does not change if we replace the induced matrix norm in (2) or in (4) by any other matrix norm.

Proof. The result is a direct consequence of Lemma 2 and of the properties of the JSR stated in [16, Lemma 1.2] and [16, Page 10]. □

We provide further useful properties of $\boldsymbol{\lambda}(\mathcal{A}, \cdot)$ in the proposition below.

Proposition 2. *One of the following properties holds:*

(i) The function $\rho \mapsto \boldsymbol{\lambda}(\mathcal{A}, \rho)$ is decreasing, takes values in $(0, 1)$ and, for all $\boldsymbol{\rho}(\mathcal{A}) < \rho_1 \leq \rho_2$,

$$\boldsymbol{\lambda}(\mathcal{A}, \rho_2) \leq \left(\frac{\rho_1}{\rho_2} \right)^m \boldsymbol{\lambda}(\mathcal{A}, \rho_1). \tag{5}$$

(ii) For all $\rho > \boldsymbol{\rho}(\mathcal{A})$, $\boldsymbol{\lambda}(\mathcal{A}, \rho) = 0$ and for all $\theta \in \mathcal{S}_s(\mathcal{I})$ $\mathbb{A}_{\theta, t} = 0$, for all $t \geq \tau_n^\theta$.

Proof. Let $\rho > \boldsymbol{\rho}(\mathcal{A})$, similar to the proof of Lemma 1, it follows from [6, Theorem 1] applied to the set of matrices $\frac{1}{\rho}\mathcal{A}$ that there exists $C_1 \geq 1$ such that

$$0 \leq \frac{\left\| \mathbb{A}_{\theta, T} \right\|}{\rho^T} \leq C_1, \quad \forall \theta \in \mathcal{S}(\mathcal{I}), \forall T \in \mathbb{N}. \tag{6}$$

By considering $T = \tau_k^\theta$, raising to the power $1/k$ and taking the supremum over shuffled switching signals, we find

$$0 \leq \sup_{\theta \in \mathcal{S}_s(\mathcal{I})} \left(\frac{\left\| \mathbb{A}_{\theta,\tau_k^\theta} \right\|}{\rho^{\tau_k^\theta}} \right)^{1/k} \leq C_1^{1/k}.$$

Taking the limit of all terms yields $\boldsymbol{\lambda}(\mathcal{A}, \rho) \in [0, 1]$.

Let us assume that there exists $\rho' > \boldsymbol{\rho}(\mathcal{A})$ such that $\boldsymbol{\lambda}(\mathcal{A}, \rho') = 0$ and let us consider an arbitrary shuffled switching signal $\theta \in \mathcal{S}_s(\mathcal{I})$. Let us consider the sequence $(\theta_j)_{j\in\mathbb{N}}$ of switching signals in $\mathcal{S}_s(\mathcal{I})$, defined as in the proof of Lemma 2. From Lemma 2, $\boldsymbol{\rho}(\mathcal{M}_{\rho'}) = 0$ and therefore the joint spectral radius of any finite subfamily of $\mathcal{M}_{\rho'}$ is also zero. In particular, we get for the following subfamily of n elements:

$$\boldsymbol{\rho}\left(\left\{ \frac{1}{\rho'^{\tau_1^{\theta_0}}} \mathbb{A}_{\theta,\tau_1^{\theta_0}}, \ldots, \frac{1}{\rho'^{\tau_1^{\theta_{n-1}}}} \mathbb{A}_{\theta,\tau_1^{\theta_{n-1}}} \right\} \right) = 0 \, .$$

By applying [16, Proposition 2.1] and using (3) we get $\mathbb{A}_{\theta,\tau_n^\theta} = 0$. It follows that $\mathbb{A}_{\theta,t} = 0$, for all $t \geq \tau_n^\theta$, for all $\theta \in \mathcal{S}_s(\mathcal{I})$. Then, we get that $\boldsymbol{\lambda}(\mathcal{A}, \rho) = 0$ for all $\rho > \boldsymbol{\rho}(\mathcal{A})$.

Now let us assume that for all $\rho > \boldsymbol{\rho}(\mathcal{A})$, $\boldsymbol{\lambda}(\mathcal{A}, \rho) > 0$. Let $\boldsymbol{\rho}(\mathcal{A}) < \rho_1 \leq \rho_2$. By recalling that $\tau_k^\theta \geq km$, we have for all $k \in \mathbb{N}$ and for all $\theta \in \mathcal{S}_s(\mathcal{I})$

$$\frac{\left\| \mathbb{A}_{\theta,\tau_k^\theta} \right\|}{\rho_2^{\tau_k^\theta}} = \frac{\left\| \mathbb{A}_{\theta,\tau_k^\theta} \right\|}{\rho_1^{\tau_k^\theta}} \left(\frac{\rho_1}{\rho_2} \right)^{\tau_k^\theta} \leq \frac{\left\| \mathbb{A}_{\theta,\tau_k^\theta} \right\|}{\rho_1^{\tau_k^\theta}} \left(\frac{\rho_1}{\rho_2} \right)^{km}.$$

Raising to the power $1/k$ and taking the supremum over shuffled switching signals yields

$$\sup_{\theta \in \mathcal{S}_s(\mathcal{I})} \left(\frac{\left\| \mathbb{A}_{\theta,\tau_k^\theta} \right\|}{\rho_2^{\tau_k^\theta}} \right)^{1/k} \leq \sup_{\theta \in \mathcal{S}_s(\mathcal{I})} \left(\frac{\left\| \mathbb{A}_{\theta,\tau_k^\theta} \right\|}{\rho_1^{\tau_k^\theta}} \right)^{1/k} \left(\frac{\rho_1}{\rho_2} \right)^{m}.$$

Now we take the limit of both terms and we get (5), which implies that $\rho \mapsto \boldsymbol{\lambda}(\mathcal{A}, \rho)$ is decreasing. Then, let us assume that there exists $\rho_2 > \boldsymbol{\rho}(\mathcal{A})$ such that $\boldsymbol{\lambda}(\mathcal{A}, \rho_2) = 1$. It follows from (5) that for all $\rho_1 \in (\boldsymbol{\rho}(\mathcal{A}), \rho_2)$, $\boldsymbol{\lambda}(\mathcal{A}, \rho_2) < \boldsymbol{\lambda}(\mathcal{A}, \rho_1)$, which contradicts the fact that $\boldsymbol{\lambda}(\mathcal{A}, \rho_1) \in [0, 1]$. Hence, $\boldsymbol{\lambda}(\mathcal{A}, \rho) \in (0, 1)$ for all $\rho > \boldsymbol{\rho}(\mathcal{A})$. □

In order to get rid of the dependence of ρ in the ρ-SJSR, it is natural to introduce the following definition.

Definition 3. *The Shuffled Joint Spectral Radius (SJSR) of $\mathcal{A}$ is defined as*

$$\boldsymbol{\lambda}(\mathcal{A}) = \lim_{\rho \to \boldsymbol{\rho}(\mathcal{A})^+} \boldsymbol{\lambda}(\mathcal{A}, \rho). \tag{7}$$

Since by Proposition 2, $\boldsymbol{\lambda}(\mathcal{A}, \cdot)$ is bounded and non-increasing in ρ, the right limit at $\boldsymbol{\rho}(\mathcal{A})$ in (7) exists and the SJSR is well-defined. We can now show some properties of the SJSR.

Proposition 3. *The SJSR enjoys the following properties:*

(i) $\boldsymbol{\lambda}(\mathcal{A})$ belongs to $[0, 1]$ and is independent of the choice of the norm;
(ii) For all $K \in \mathbb{R}$, $K \neq 0$, we have $\boldsymbol{\lambda}(K\mathcal{A}) = \boldsymbol{\lambda}(\mathcal{A})$.

Proof. The first statement follows from (7) and by the properties of $\boldsymbol{\lambda}(\mathcal{A}, \cdot)$ proved in Propositions 1 and 2.

Concerning the second item, let $K \in \mathbb{R}$, $K \neq 0$, we have by (7)

$$\boldsymbol{\lambda}(K\mathcal{A}) = \lim_{\rho \to \boldsymbol{\rho}(K\mathcal{A})^+} \boldsymbol{\lambda}(K\mathcal{A}, \rho) = \lim_{\rho \to \boldsymbol{\rho}(\mathcal{A})^+} \boldsymbol{\lambda}(K\mathcal{A}, |K|\rho)$$

where the second equality comes from the property of the JSR, $\boldsymbol{\rho}(K\mathcal{A}) = |K|\boldsymbol{\rho}(\mathcal{A})$, see e.g. [16, Proposition 1.2]. Furthermore, from (2), one can deduce that for all $\rho > \boldsymbol{\rho}(\mathcal{A})$, $\boldsymbol{\lambda}(K\mathcal{A}, |K|\rho) = \boldsymbol{\lambda}(\mathcal{A}, \rho)$, therefore $\boldsymbol{\lambda}(K\mathcal{A}) = \boldsymbol{\lambda}(\mathcal{A})$. □

Let us remark that while the JSR $\boldsymbol{\rho}(\mathcal{A})$ belongs to $\mathbb{R}_0^+$, the SJSR $\boldsymbol{\lambda}(\mathcal{A})$ is always a value in $[0, 1]$. Intuitively, while the JSR provides an estimate of the contraction rate (when $\rho(\mathcal{A}) < 1$) or of the expansion rate (when $\rho(\mathcal{A}) > 1$) of the system state at each time step for arbitrary switching signals, the SJSR measures how much additional contraction of the state is obtained each time the signal shuffles. Theorem 1 in the next section provides theoretical ground to this interpretation.

4 Shuffled switched systems and ρ-SJSR

In this section, we show how the SJSR relates to stability properties of switched linear systems with shuffled switching signals. From Proposition 2, we already know that if $\boldsymbol{\lambda}(\mathcal{A}, \rho) = 0$ for some $\rho > \boldsymbol{\rho}(\mathcal{A})$ then all trajectories of (1) will stay at 0 after n shuffling instants. So, we focus in this section on the case when $\boldsymbol{\lambda}(\mathcal{A}, \rho) > 0$ for all $\rho > \boldsymbol{\rho}(\mathcal{A})$. In particular, we show how the ρ-SJSR allows us to compute bounds on the shuffling rate ensuring stability. The next result clarifies the relationship between the ρ-SJSR and the behavior of the trajectories of (1).

Theorem 1. *For all $\rho > \boldsymbol{\rho}(\mathcal{A})$, for all $\lambda \in (\boldsymbol{\lambda}(\mathcal{A}, \rho), 1]$, there exists $C \geq 1$ such that*

$$\begin{aligned} \|\mathbf{x}(t, x_0, \theta)\| \leq & C\rho^t \lambda^{\kappa^\theta(t)} \|x_0\|, \\ & \forall \theta \in \mathcal{S}_s(\mathcal{I}), \ \forall x_0 \in \mathbb{R}^n, \ \forall t \in \mathbb{N}. \end{aligned} \tag{8}$$

Conversely, if there exists $C \geq 1$, $\rho \geq 0$ and $\lambda \in [0, 1]$ such that (8) holds, then either $\rho > \boldsymbol{\rho}(\mathcal{A})$ and $\lambda \geq \boldsymbol{\lambda}(\mathcal{A}, \rho)$, or $\rho = \boldsymbol{\rho}(\mathcal{A})$ and $\lambda \geq \boldsymbol{\lambda}(\mathcal{A})$.

Proof. We start by proving the direct result. Let $\rho > \boldsymbol{\rho}(\mathcal{A})$ and $\lambda \in (\boldsymbol{\lambda}(\mathcal{A},\rho),1]$. By definition of $\boldsymbol{\lambda}(\mathcal{A},\rho)$, there exists $k_0 \geq 1$ such that

$$\sup_{\theta\in\mathcal{S}_s(\mathcal{I})}\left(\frac{\left\|\mathbb{A}_{\theta,\tau_k^\theta}\right\|}{\rho^{\tau_k^\theta}}\right)^{1/k} \leq \lambda, \qquad \forall k \geq k_0.$$

It follows that

$$\left\|\mathbb{A}_{\theta,\tau_k^\theta}\right\| \leq \rho^{\tau_k^\theta}\lambda^k, \quad \forall\theta\in\mathcal{S}_s(\mathcal{I}), \forall k \geq k_0. \tag{9}$$

Then, let $C_1 \geq 1$ be such that (6) holds. In particular, for $T = \tau_k^\theta$ and for shuffling switching signals, we obtain from (6) that

$$\left\|\mathbb{A}_{\theta,\tau_k^\theta}\right\| \leq C_1\rho^{\tau_k^\theta}, \quad \forall\theta\in\mathcal{S}_s(\mathcal{I}), \forall k \in \mathbb{N}. \tag{10}$$

Then, let $C_2 = C_1\lambda^{-k_0}$, it follows from (9) and (10) that

$$\left\|\mathbb{A}_{\theta,\tau_k^\theta}\right\| \leq C_2\rho^{\tau_k^\theta}\lambda^k, \quad \forall\theta\in\mathcal{S}_s(\mathcal{I}), \forall k \in \mathbb{N}. \tag{11}$$

Let $\theta \in \mathcal{S}_s(\mathcal{I})$, $t \in \mathbb{N}$, and $k = \kappa^\theta(t)$, we have $\mathbb{A}_{\theta,t} = \mathbb{A}_{\theta',t-\tau_k^\theta}\mathbb{A}_{\theta,\tau_k^\theta}$ where $\theta' \in \mathcal{S}_s(\mathcal{I})$ is given by $\theta'(s) = \theta(\tau_k^\theta + s)$, for all $s \in \mathbb{N}$. By (6), we get that

$$\left\|\mathbb{A}_{\theta',t-\tau_k^\theta}\right\| \leq C_1\rho^{t-\tau_k^\theta}. \tag{12}$$

Then, let $C = C_1C_2$, by (11) and (12), we get

$$\left\|\mathbb{A}_{\theta,t}\right\| \leq \left\|\mathbb{A}_{\theta',t-\tau_k^\theta}\right\|\left\|\mathbb{A}_{\theta,\tau_k^\theta}\right\| \leq C\lambda^{\kappa^\theta(t)}\rho^t.$$

Hence, (8) holds.

We now prove the converse result. By definition of induced matrix norm, (8) is equivalent to the following:

$$\|\mathbb{A}_{\theta,t}\| \leq C\rho^t\lambda^{\kappa^\theta(t)}, \quad \forall\theta\in\mathcal{S}_s(\mathcal{I}),\ \forall t\in\mathbb{N}. \tag{13}$$

Since $\lambda \in [0,1]$ and $\kappa^\theta(t) \in \mathbb{N}$ we also have

$$\|\mathbb{A}_{\theta,t}\| \leq C\rho^t, \quad \forall\theta\in\mathcal{S}_s(\mathcal{I}),\ \forall t\in\mathbb{N}.$$

Raising the previous terms to the power $1/t$ and taking the supremum over shuffling switching signals yields

$$\sup_{\theta\in\mathcal{S}_s(\mathcal{I})}\|\mathbb{A}_{\theta,t}\|^{1/t} \leq C^{1/t}\rho, \quad \forall t\in\mathbb{N}.$$

Using the fact that for every $\theta \in \mathcal{S}(\mathcal{I})$ and all $t \in \mathbb{N}$ there exists $\theta_t \in \mathcal{S}_s(\mathcal{I})$ that coincides with θ up to time t, the equation above can be re-written as

$$\sup_{\theta\in\mathcal{S}(\mathcal{I})}\|\mathbb{A}_{\theta,t}\|^{1/t} \leq C^{1/t}\rho, \quad \forall t\in\mathbb{N}.$$

Taking the limit as t goes to infinity, one obtains that $\boldsymbol{\rho}(\mathcal{A}) \leq \rho$. Recalling (13) and fixing $t = \tau_k^\theta$, we have

$$\|\mathbb{A}_{\theta,\tau_k^\theta}\| \leq C\rho^{\tau_k^\theta}\lambda^k, \quad \forall\theta\in\mathcal{S}_s(\mathcal{I}),\ \forall k\in\mathbb{N}$$

which is equivalent to

$$\left(\frac{\|\mathbb{A}_{\theta,\tau_k^\theta}\|}{\rho^{\tau_k^\theta}}\right)^{1/k} \leq C^{1/k}\lambda, \quad \forall\theta\in\mathcal{S}_s(\mathcal{I}),\ \forall k\in\mathbb{N}. \tag{14}$$

If $\rho > \boldsymbol{\rho}(\mathcal{A})$, taking the supremum over all shuffled switching signals and the limit as k goes to infinity yields $\boldsymbol{\lambda}(\mathcal{A},\rho) \leq \lambda$. If $\rho = \boldsymbol{\rho}(\mathcal{A})$, then for all $\rho' > \boldsymbol{\rho}(\mathcal{A})$, (14) gives

$$\left(\frac{\|\mathbb{A}_{\theta,\tau_k^\theta}\|}{\rho'^{\tau_k^\theta}}\right)^{1/k} \leq C^{1/k}\lambda, \quad \forall\theta\in\mathcal{S}_s(\mathcal{I}),\ \forall k\in\mathbb{N}.$$

Then, it follows that for all $\rho' > \boldsymbol{\rho}(\mathcal{A})$, $\boldsymbol{\lambda}(\mathcal{A},\rho') \leq \lambda$ and therefore $\boldsymbol{\lambda}(\mathcal{A}) \leq \lambda$. □

The previous theorem provides a bound on the growth of the state and can be used to derive conditions for stabilization of system (1) using shuffled switching signals with a minimal shuffling rate.

Corollary 1. *Assume $\boldsymbol{\lambda}(\mathcal{A},\rho) > 0$ for every $\rho > \boldsymbol{\rho}(\mathcal{A})$. Let $\theta \in \mathcal{S}_s(\mathcal{I})$, if there exists $\rho > \boldsymbol{\rho}(\mathcal{A})$ such that $\gamma^\theta > -\frac{\ln(\rho)}{\ln(\boldsymbol{\lambda}(\mathcal{A},\rho))}$, then*

$$\lim_{t\to+\infty}\|\mathbf{x}(t,x_0,\theta)\| = 0, \quad \forall x_0\in\mathbb{R}^n. \tag{15}$$

Proof. If $\boldsymbol{\lambda}(\mathcal{A},\rho) > 0$ and $\gamma^\theta > -\frac{\ln(\rho)}{\ln(\boldsymbol{\lambda}(\mathcal{A},\rho))}$, then there exists $\lambda \in (\boldsymbol{\lambda}(\mathcal{A},\rho),1)$ such that $\gamma^\theta > -\frac{\ln(\rho)}{\ln(\lambda)}$ and $\epsilon > 0$ such that $\epsilon < \gamma^\theta + \frac{\ln(\rho)}{\ln(\lambda)}$. Then, from the definition of shuffling rate, there exists $t_0 \in \mathbb{N}$ such that

$$\frac{\kappa^\theta(t)}{t} > \gamma^\theta - \frac{\epsilon}{2}, \qquad \forall t \geq t_0.$$

From Theorem 1, there exists $C \geq 1$ such that for all $x_0 \in \mathbb{R}^n$, for all $t \geq t_0$.

$$\begin{aligned}\|\mathbf{x}(t,x_0,\theta)\| &\leq C\rho^t\lambda^{\kappa^\theta(t)}\|x_0\| \\ &\leq C\rho^t\lambda^{(\gamma^\theta-\frac{\epsilon}{2})t}\|x_0\| \\ &\leq C\rho^t\lambda^{(-\frac{\ln(\rho)}{\ln(\lambda)}+\frac{\epsilon}{2})t}\|x_0\| \\ &= C\lambda^{\frac{\epsilon}{2}t}\|x_0\|,\end{aligned}$$

from which (15) follows. □

A practical consequence of the previous corollary is that even if the switched system is unstable for arbitrary switching (i.e. if $\boldsymbol{\rho}(\mathcal{A}) > 1$), it can be stabilized by shuffling sufficiently fast. Note that since for all $\theta \in \mathcal{S}_s(\mathcal{I})$, the shuffling rate $\gamma^\theta \leq \frac{1}{m}$, it follows that stabilization by shuffling can only be done if there exists $\rho > \boldsymbol{\rho}(\mathcal{A})$ such that $-\frac{\ln(\boldsymbol{\lambda}(\mathcal{A},\rho))}{\ln(\rho)} > m$.

Remark 1. *As a consequence of Corollary 1, we have that if $\boldsymbol{\lambda}(\mathcal{A}) \in (0,1)$, $\gamma^\theta > -\frac{\ln(\boldsymbol{\rho}(\mathcal{A}))}{\ln(\boldsymbol{\lambda}(\mathcal{A}))}$ implies (15). However, we are unable to show that, in general, the function $\rho \mapsto -\frac{\ln(\rho)}{\ln(\boldsymbol{\lambda}(\mathcal{A},\rho))}$ is non-decreasing, hence replacing the assumption of Corollary 1 by $\gamma^\theta > -\frac{\ln(\boldsymbol{\rho}(\mathcal{A}))}{\ln(\boldsymbol{\lambda}(\mathcal{A}))}$ may increase the conservativeness of the result.*

5 Approximation of the ρ-SJSR

Similar to the JSR, the exact computation of the ρ-SJSR appears in general out of reach, and an interesting problem is to look for theoretical and numerical methods allowing to estimate the ρ-SJSR. Such estimates would have an impact on the applicability of the stability results, Theorem 1 and Corollary 1. From Lemma 2, it follows that $\boldsymbol{\lambda}(\mathcal{A},\rho)$ could be approximated by computing bounds on $\boldsymbol{\rho}(\mathcal{M}_\rho)$. However, off-the-shelf algorithms for computing approximations of the JSR only apply to finite set of matrices [30]. A lower bound of $\boldsymbol{\lambda}(\mathcal{A},\rho)$ can still be computed by computing a lower bound of the JSR of a finite subfamily of $\mathcal{M}_\rho$. However, a similar approach does not allow to compute upper bounds. In the next subsection, we provide lower and upper bounds based on the computation of the joint spectral radius of a finite set of matrices. Methods to compute upper bounds for the ρ-SJSR based on Lyapunov techniques are then developed in Section 5.2.

5.1 Asymptotic estimate

In this section, we provide asymptotically tight bounds of $\boldsymbol{\lambda}(\mathcal{A},\rho)$ for a large enough ρ. We first recall the following classical result, which roughly speaking ensures, for any bounded set of matrices $\mathcal{M}$, the existence of a matrix norm such that $\boldsymbol{\rho}(\mathcal{M})$ is approximated by the maximum norm of the matrices in $\mathcal{M}$.

Proposition 4 ([6, Lemma 2])**.** *Let $\mathcal{M}$ be a bounded set of matrices. The following equality holds*

$$\boldsymbol{\rho}(\mathcal{M}) = \inf_{\|\cdot\|} \max_{M\in\mathcal{M}} \|M\|,$$

where the infimum is taken among all matrix norms induced from norms in $\mathbb{R}^n$.

When it exists, an induced norm $\|\cdot\|$ that satisfies $\boldsymbol{\rho}(\mathcal{M}) = \max_{M\in\mathcal{M}} \|M\|$ is called *extremal* for $\mathcal{M}$. We refer to [16, Theorem 2.2], and also to [5] and [17], for sufficient conditions and for necessary and sufficient conditions for the existence of extremal norms.

Let us consider the set $\mathcal{N}_\mathcal{I}$ of products of matrices where all modes in $\mathcal{I}$ appear exactly once:

$$\mathcal{N}_\mathcal{I} = \left\{ \prod_{k=1}^{m} A_{j_k} \;\middle|\; \begin{array}{c} j_1,\dots,j_m \in \mathcal{I}, \\ \forall i \in \mathcal{I}, \exists k \in \{1,\dots,m\}, j_k = i \end{array} \right\}.$$

The following proposition provides a lower bound for the ρ-SJSR.

Proposition 5. *For all $\rho > \boldsymbol{\rho}(\mathcal{A})$*

$$\boldsymbol{\lambda}(\mathcal{A},\rho) \geq \frac{\boldsymbol{\rho}(\mathcal{N}_\mathcal{I})}{\rho^m}.$$

Proof. Let us remark that we have $\frac{1}{\rho^m}\mathcal{N}_\mathcal{I} \subseteq \mathcal{M}_\rho$. Then, from the definition of the JSR, we get

$$\boldsymbol{\rho}(\mathcal{M}_\rho) \geq \boldsymbol{\rho}\Big(\frac{1}{\rho^m}\mathcal{N}_\mathcal{I}\Big) = \frac{\boldsymbol{\rho}(\mathcal{N}_\mathcal{I})}{\rho^m}.$$

Lemma 2 then allows us to conclude. □

It is natural to ask whether the lower bound provided in Proposition 5 can prove asymptotically tight. The following theorem provides an upper bound for the ρ-SJSR and answers to this question.

Theorem 2. *The following results hold true.*

(i) For all $K > \boldsymbol{\rho}(\mathcal{N}_\mathcal{I})$, there exists $R \geq \boldsymbol{\rho}(\mathcal{A})$ such that for all $\rho \geq R$,

$$\boldsymbol{\lambda}(\mathcal{A},\rho) \leq \frac{K}{\rho^m}. \tag{16}$$

(ii) We have the asymptotic estimate

$$\lim_{\rho\to+\infty} \rho^m \boldsymbol{\lambda}(\mathcal{A},\rho) = \boldsymbol{\rho}(\mathcal{N}_\mathcal{I}). \tag{17}$$

(iii) If there exists a norm $\|\cdot\|_*$ that is extremal for $\mathcal{N}_\mathcal{I}$, then there exists $R \geq \boldsymbol{\rho}(\mathcal{A})$ such that for all $\rho \geq R$,

$$\boldsymbol{\lambda}(\mathcal{A},\rho) = \frac{\boldsymbol{\rho}(\mathcal{N}_\mathcal{I})}{\rho^m}. \tag{18}$$

Proof. Let us fix $K > \boldsymbol{\rho}(\mathcal{N}_\mathcal{I})$. By Proposition 4 there exists an induced matrix norm $\|\cdot\|_*$ such that $\|M\|_* \leq K$ for every $M \in \mathcal{N}_\mathcal{I}$. Considering now the set $\hat{\mathcal{M}}_\rho = \rho^m \mathcal{M}_\rho$, we have $\mathcal{N}_\mathcal{I} \subseteq \hat{\mathcal{M}}_\rho$. We want to show that $\|M\|_* \leq K$ for every $M \in \hat{\mathcal{M}}_\rho \setminus \mathcal{N}_\mathcal{I}$, provided that ρ is large enough. For this purpose, we first take any

$\tilde{R} > \boldsymbol{\rho}(\mathcal{A})$. Then, again by Proposition 4, there exists an induced matrix norm $\|\cdot\|_{**}$ such that $\|A\|_{**} \leq \tilde{R}$ for every $A \in \mathcal{A}$. Taking $\tilde{K} \geq 1$ such that $\|A\|_* \leq \tilde{K}\|A\|_{**}$ for every matrix $A \in \mathbb{R}^{n\times n}$ we obtain

$$\|\mathbb{A}_{\theta,i}\|_* \leq \tilde{K}\|\mathbb{A}_{\theta,i}\|_{**} \leq \tilde{K}\tilde{R}^i, \qquad \forall \theta \in \mathcal{S}(\mathcal{I}),\ \forall i \in \mathbb{N}.$$

Observe now that every element of $\hat{\mathcal{M}}_\rho \setminus \mathcal{N}_\mathcal{I}$ takes the form $\frac{1}{\rho^{\tau_1^\theta - m}}\mathbb{A}_{\theta,\tau_1^\theta}$ for some $\theta \in \mathcal{S}_s(\mathcal{I})$ such that $\tau_1^\theta > m$. Setting $R = \max\left\{\tilde{R}, \frac{\tilde{K}}{K}\tilde{R}^{m+1}\right\}$ we have for every $\rho \geq R$

$$\left\|\frac{1}{\rho^{\tau_1^\theta - m}}\mathbb{A}_{\theta,\tau_1^\theta}\right\|_* \leq \frac{1}{\rho^{\tau_1^\theta - m}}\tilde{K}\tilde{R}^{\tau_1^\theta} = \tilde{K}\tilde{R}^m\left(\frac{\tilde{R}}{\rho}\right)^{\tau_1^\theta - m} \leq \tilde{K}\tilde{R}^m\left(\frac{\tilde{R}}{R}\right)^{\tau_1^\theta - m} \leq \frac{\tilde{K}\tilde{R}^{m+1}}{R} \leq K.$$

This proves that $\|M\|_* \leq K$ for every $M \in \hat{\mathcal{M}}_\rho \setminus \mathcal{N}_\mathcal{I}$, hence for every $M \in \hat{\mathcal{M}}_\rho$, if $\rho \geq R$. We deduce that

$$\boldsymbol{\lambda}(\mathcal{A},\rho) = \frac{1}{\rho^m}\boldsymbol{\rho}(\hat{\mathcal{M}}_\rho) \leq \frac{1}{\rho^m}\sup_{M\in\hat{\mathcal{M}}_\rho}\|M\|_* \leq \frac{K}{\rho^m},$$

for $\rho \geq R$, which proves (16).

Setting $K = \boldsymbol{\rho}(\mathcal{N}_\mathcal{I}) + \epsilon$ for an arbitrary $\epsilon > 0$ we then get the existence of $R \geq \boldsymbol{\rho}(\mathcal{A})$ such that, for all $\rho \geq R$,

$$0 \leq \rho^m\boldsymbol{\lambda}(\mathcal{A},\rho) - \boldsymbol{\rho}(\mathcal{N}_\mathcal{I}) \leq \epsilon,$$

where the inequality on the left follows from Propositions 5. Letting ϵ tend to zero, this implies (17).

Finally, by taking $\|\cdot\|_*$ extremal for $\mathcal{N}_\mathcal{I}$ in the argument above, we obtain that (16) holds true with $K = \boldsymbol{\rho}(\mathcal{N}_\mathcal{I})$. Together with Proposition 5, this implies (18). □

In the case where all the matrices A_i are invertible, a practical criterion to estimate the lower bound R in Theorem 2 is given by the following lemma. Before stating the lemma, we define, for each $i \in \mathcal{I}$, the set $\mathcal{N}_i$ of products of N matrices in $\mathcal{A}$ with $N < m$, where each mode in $\mathcal{I}$ appears at most once and i appears exactly once:

$$\mathcal{N}_i = \left\{\prod_{k=1}^N A_{j_k} \;\middle|\; \begin{array}{c} j_1,\dots,j_N \in \mathcal{I},\ N < m, \\ \forall k \neq k', j_k \neq j_{k'},\ \text{and } \exists k, j_k = i \end{array}\right\}.$$

Lemma 3. *Assume all the matrices in $\mathcal{A}$ are invertible. Let $K \geq \boldsymbol{\rho}(\mathcal{N}_\mathcal{I})$ and let $\|\cdot\|_*$ be an induced matrix norm such that $\|M\|_* \leq K$ for every $M \in \mathcal{N}_\mathcal{I}$. Then, setting*

$$R = \max_{i\in\mathcal{I}, M\in\mathcal{N}_i}\|MA_iM^{-1}\|_*, \tag{19}$$

we have that $R \geq \boldsymbol{\rho}(A)$ and (16) (and (18), if $K = \boldsymbol{\rho}(\mathcal{N}_\mathcal{I})$) holds true for $\rho \geq R$.

Proof. Following the proof of Theorem 2, it is enough to show that $\|\mathbb{A}_{\theta,\tau_1^\theta}\|_* \leq KR^{\tau_1^\theta - m}$ for every $\theta \in \mathcal{S}_s(\mathcal{I})$ such that $\tau_1^\theta > m$. The product of matrices $\mathbb{A}_{\theta,\tau_1^\theta} = A_{\theta(\tau_1^\theta-1)}\cdots A_{\theta(0)}$ is shuffling once, therefore all the modes appear at least once (with $\theta(\tau_1^\theta - 1)$ appearing exactly once). The total number of repetitions in the product is given by $n = \tau_1^\theta - m > 0$. Suppose that the first repeated mode is $\theta(j_1)$. In order to remove $A_{\theta(j_1)}$, using the fact that all matrices A_i are invertible, we multiply $\mathbb{A}_{\theta,\tau_1^\theta}$ on the right by

$$Q_1 = (A_{\theta(j_1-1)}\cdots A_{\theta(0)})^{-1}A_{\theta(j_1)}^{-1}(A_{\theta(j_1-1)}\cdots A_{\theta(0)}).$$

Then, $\mathbb{A}_{\theta,\tau_1^\theta}Q_1 = \mathbb{A}_{\theta^{(1)},\tau_1^{\theta^{(1)}}}$ where the switching signal $\theta^{(1)}$ is given by $\theta^{(1)}(t) = \theta(t)$ until the instant $j_1 - 1$, and with $\theta^{(1)}(t) = \theta(t+1)$ afterwards. It is clear that $\tau_1^{\theta^{(1)}} = \tau_1^\theta - 1$. By repeating this process n times, we get a switching signal $\theta^{(n)}$ without repetitions before its first shuffling instant (i.e. $\tau_1^{\theta^{(n)}} = \tau_1^\theta - n = m$). Then, we finally have

$$\mathbb{A}_{\theta,\tau_1^\theta}\Big(\prod_{k=1}^n Q_{n+1-k}\Big) = \mathbb{A}_{\theta^{(n)},\tau_1^{\theta^{(n)}}},$$

with $\mathbb{A}_{\theta^{(n)},\tau_1^{\theta^{(n)}}} \in \mathcal{N}_\mathcal{I}$. It follows that

$$\mathbb{A}_{\theta,\tau_1^\theta} = \mathbb{A}_{\theta^{(n)},\tau_1^{\theta^{(n)}}}\Big(\prod_{k=1}^n Q_k^{-1}\Big),$$

where for all $k = 1,\dots,n$,

$$Q_k^{-1} \in \bigcup_{i\in\mathcal{I}}\{MA_iM^{-1} | M \in \mathcal{N}_i\},$$

and therefore $\|Q_k^{-1}\|_* \leq R$, where R is given by (19). Moreover, since $A_i \in \mathcal{N}_i$, we have that $R \geq \max_{i\in\mathcal{I}}\|A_i\|_*$, which implies by Proposition 4 that $R \geq \boldsymbol{\rho}(\mathcal{A})$. Finally, from submultiplicativity of the induced matrix norm, we get that for all $\theta \in \mathcal{S}_s(\mathcal{I})$

$$\|\mathbb{A}_{\theta,\tau_1^\theta}\|_* \leq \|\mathbb{A}_{\theta^{(n)},\tau_1^{\theta^{(n)}}}\|_*\Big(\prod_{k=1}^n\|Q_k^{-1}\|_*\Big) \leq KR^n\,,$$

which yields the expected result since $n = \tau_1^\theta - m$. □

We have seen with Theorem 2 and Lemma 3 that, under certain conditions, we can find tight bounds on the ρ-SJSR. In particular, if one can find a norm $\|\cdot\|_*$ that

is extremal for $\mathcal{N}_\mathcal{I}$, (19) provides an $R \geq \boldsymbol{\rho}(\mathcal{A})$ such that for all $\rho \geq R$ the exact value of ρ-SJSR is given by (18). For all $\rho \in [\boldsymbol{\rho}(\mathcal{A}), R)$ we can still use the lower bound provided by Proposition 5, but we need an effective way to compute an upper bound. To address this problem, in the next section we present two Lyapunov-based methods.

Before that, we end this section by exhibiting a class of switched systems for which the value of the ρ-SJSR can be explicitly determined for all values of $\rho > \rho(\mathcal{A})$. Let us consider a set of commuting matrices $\mathcal{A} = \{A_1, \cdots, A_m\}$, it is well known (see e.g. [9]) that in that case the JSR is equal to the maximal value of the spectral radii of the matrices of $\mathcal{A}$, that is $\boldsymbol{\rho}(\mathcal{A}) = \max(\boldsymbol{\rho}(A_1), \dots \boldsymbol{\rho}(A_m))$. Interestingly, in that case, we are also able to determine the expression of the ρ-SJSR:

Proposition 6. *For a finite set of commuting matrices $\mathcal{A} = \{A_1, \cdots, A_m\}$, we have for all $\rho > \boldsymbol{\rho}(\mathcal{A})$,*

$$\boldsymbol{\lambda}(\mathcal{A}, \rho) = \frac{\boldsymbol{\rho}(A_1 \cdots A_m)}{\rho^m}.$$

Proof. Since the matrices commute, all matrices $\mathcal{N}_\mathcal{I}$ are equal to $A_1 \cdots A_m$. Then, Proposition 5 provides a lower bound on $\boldsymbol{\lambda}(\mathcal{A}, \rho)$ with $\boldsymbol{\rho}(\mathcal{N}_\mathcal{I}) = \boldsymbol{\rho}(A_1 \cdots A_m)$. We need to show that it is also an upper bound.

From Proposition 4 we know that, for all $\rho > \boldsymbol{\rho}(\mathcal{A})$, there exists a (submultiplicative) matrix norm $\|\cdot\|_*$ such that $\rho > \max_{i \in \mathcal{I}}(\|A_i\|_*)$. In the case of commuting matrices, the product $\mathbb{A}_{\theta,\tau_k^\theta}$ can be rearranged as follows

$$\mathbb{A}_{\theta,\tau_k^\theta} = (A_1 \cdots A_m)^k A_1^{i_1} \cdots A_m^{i_m}$$

where $mk + i_1 + \cdots + i_m = \tau_k^\theta$. Whereby we have

$$\frac{\|\mathbb{A}_{\theta,\tau_k^\theta}\|_*}{\rho^{\tau_k^\theta}} \leq \frac{\|A_1 \cdots A_m\|_*^k \|A_1\|_*^{i_1} \cdots \|A_m\|_*^{i_m}}{\rho^{km}\rho^{i_1} \cdots \rho^{i_m}}.$$

Moreover, recalling that $\frac{\|A_j\|_*}{\rho} < 1$ for $j = 1, \dots, m$, we get

$$\frac{\|\mathbb{A}_{\theta,\tau_k^\theta}\|_*}{\rho^{\tau_k^\theta}} \leq \frac{\|A_1 \cdots A_m\|_*^k}{\rho^{km}}.$$

By raising the previous expression to the power $\frac{1}{k}$, by taking the supremum over $\theta \in \mathcal{S}_s(\mathcal{I})$ and the limit for k going to infinity, we get

$$\boldsymbol{\lambda}(\mathcal{A}, \rho) \leq \frac{\boldsymbol{\rho}(A_1 \cdots A_m)}{\rho^m}.$$

□

5.2 Computing bounds with Lyapunov techniques

This section details two methods for computing upper bounds on the JSR and the ρ-SJSR. Both approaches are based on Lyapunov functions and automata theoretic techniques. While the first approach is computationally more tractable, the second approach is shown to be tight.

5.2.1 Lyapunov function indexed by $\mathcal{I}$

Our first method is based on a Lyapunov function indexed by the set of modes $\mathcal{I}$. It can be seen as an extension of Theorem 1 in [13] in order to compute bounds on the JSR and the ρ-SJSR (in [13], only a stability certificate is provided).

Theorem 3. *If there exist $V : \mathcal{I} \times \mathbb{R}^n \to \mathbb{R}_0^+$, $\alpha_1, \alpha_2, \rho > 0$ and $\lambda \in [0, 1]$ such that the following inequalities hold true for every $x \in \mathbb{R}^n$*

$$\alpha_1\|x\|^2 \leq V(i, x) \leq \alpha_2\|x\|^2, \qquad i \in \mathcal{I} \quad (20)$$
$$V(i, A_{i'}x) \leq \rho^2 V(i, x), \qquad i, i' \in \mathcal{I}, i \neq i' \quad (21)$$
$$V(i + 1, A_i x) \leq \rho^2 V(i, x), \qquad i \in \mathcal{I} \setminus \{m\} \quad (22)$$
$$V(1, A_m x) \leq \rho^2 \lambda^{2m} V(m, x), \quad (23)$$

then the bound (8) *holds.*

Proof. The proof follows the main lines of that of Theorem 1 in [13]. Let us consider an initial condition $x_0 \in \mathbb{R}^n$ and a shuffled switching signal θ, let us denote $x(\cdot) = \mathbf{x}(\cdot, x_0, \theta)$. Let $\eta : \mathbb{N} \to \mathcal{I}$ be defined by $\eta(0) = 1$ and the following rules:

$$\begin{cases} \text{if } \theta(t) \neq \eta(t), & \text{then } \eta(t+1) = \eta(t); \\ \text{if } \theta(t) = \eta(t) \text{ and } \eta(t) \neq m, & \text{then } \eta(t+1) = \eta(t) + 1; \\ \text{if } \theta(t) = \eta(t) \text{ and } \eta(t) = m, & \text{then } \eta(t+1) = 1. \end{cases} \quad (24)$$

An equivalent description of the evolution of η can be given in terms of a finite state automaton, like the one shown in Figure 2 for $m = 3$. Inspecting the dynamics of η, one can easily see that η takes transitions from m to 1 an infinite number of times if and only if θ is a shuffled switching signal. Moreover, there is at least one such transition every m shuffling instants.

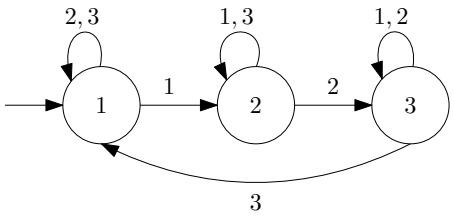


Fig. 2. Automaton describing the dynamics of η in (24) for $m = 3$. State labels correspond to the value of η, transition labels correspond to the value of θ.

Let us consider $W : \mathbb{N} \to \mathbb{R}_0^+$, defined by $W(t) = \frac{V(\eta(t),x(t))}{\rho^{2t}}$, for all $t \in \mathbb{N}$. It follows from (21), (22) and (23) that $W(t+1) \leq W(t)$, thus $W(t)$ is non-increasing. Proceeding as in the proof of Theorem 1 in [13], we can show that

$$W(\tau_{jm}^\theta) \leq \lambda^{2jm} W(0), \quad \forall j \in \mathbb{N}.$$

Let $t \in [\tau_{jm}^\theta, \tau_{(j+1)m}^\theta)$, then $\kappa^\theta(t) \leq (j+1)m$. From the monotonicity of $W(t)$ we have

$$W(t) \leq W(\tau_{jm}^\theta) \leq \lambda^{2\kappa^\theta(t)-2m} W(0).$$

From (20) we have $W(0) \leq \alpha_2 \|x_0\|^2$ and we get for all $t \in \mathbb{N}$

$$V(\eta(t), x(t)) \leq \alpha_2 \rho^{2t} \lambda^{2\kappa^\theta(t)-2m} \|x_0\|^2.$$

From (20) we also have, for all $t \in \mathbb{N}$, $\|x(t)\|^2 \leq \frac{V(\eta(t),x(t))}{\alpha_1}$. Whence, by taking $C = \lambda^{-m}\sqrt{\frac{\alpha_2}{\alpha_1}}$, we obtain the bound (8). □

The previous result allows us to compute upper-bounds on the JSR and the ρ-SJSR. However, there is some conservativeness that is mostly due to the fact that the Lyapunov function is guaranteed to contract only every m shuffling instants. In the following subsection, we show an approach that makes it possible to compute a Lyapunov function that contracts at each shuffling instant. This also allows us to compute tight upper bounds on the JSR and the ρ-SJSR.

5.2.2 Lyapunov function indexed by $2^{\mathcal{I}}$

Our second approach is also based on the use of a Lyapunov function and of an automaton. The main difference is that we use an automaton that visits a given state at each shuffling instant (instead of at most each m shuffling instants as in the previous section). In order to do that, we need to use a Lyapunov function indexed on the powerset of $\mathcal{I}$. While this significantly increases the complexity of the approach, $2^{\mathcal{I}}$ having exponentially more elements than $\mathcal{I}$, this allows us to propose a method that provides tight bounds on the JSR and the ρ-SJSR.

Theorem 4. *If there exist $V : (2^{\mathcal{I}} \setminus \{\mathcal{I}\}) \times \mathbb{R}^n \to \mathbb{R}_0^+$, $\alpha_1, \alpha_2, \rho > 0$ and $\lambda \in [0,1]$ such that the following inequalities hold true for every $x \in \mathbb{R}^n$*

$$\alpha_1 \|x\|^2 \leq V(J,x) \leq \alpha_2 \|x\|^2, \qquad \forall J \subsetneq \mathcal{I} \quad (25)$$
$$V(J \cup \{i\}, A_i x) \leq \rho^2 V(J,x), \quad \text{if } J \cup \{i\} \neq \mathcal{I} \quad (26)$$
$$V(\emptyset, A_i x) \leq \rho^2 \lambda^2 V(J,x), \quad \text{if } J \cup \{i\} = \mathcal{I} \quad (27)$$

then the bound (8) holds. Conversely, if the matrices A_i are invertible, for all $i \in \mathcal{I}$ and the bound (8) holds for some $\rho > 0$, $\lambda \in [0,1]$ and $C \geq 1$, then there exists a function $V : (2^{\mathcal{I}} \setminus \{\mathcal{I}\}) \times \mathbb{R}^n \to \mathbb{R}_0^+$ such that the inequalities (25), (26) and (27) are satisfied.

Proof. We first prove the direct part. Let us consider an initial condition $x_0 \in \mathbb{R}^n$ and a shuffled switching signal θ, let us denote $x(\cdot) = \mathbf{x}(\cdot, x_0, \theta)$. Let $\eta : \mathbb{N} \to 2^{\mathcal{I}} \setminus \{\mathcal{I}\}$ be defined by $\eta(0) = \emptyset$ and the following rules:

$$\begin{cases} \text{if } \eta(t) \cup \{\theta(t)\} \neq \mathcal{I}, & \text{then } \eta(t+1) = \eta(t) \cup \{\theta(t)\}; \\ \text{if } \eta(t) \cup \{\theta(t)\} = \mathcal{I}, & \text{then } \eta(t+1) = \emptyset. \end{cases} \quad (28)$$

An equivalent description of the evolution of η can be given in terms of a finite state automaton, like the one shown in Figure 3 for $m = 3$. It is straightforward to see from the dynamics of η that $\eta(t) = \emptyset$ if and only if $t = \tau_k^\theta$ for some $k \in \mathbb{N}$. Hence, the automaton state $\emptyset$ is visited at each shuffling instant.

Now, let us consider $W : \mathbb{N} \to \mathbb{R}_0^+$, defined by $W(t) = \frac{V(\eta(t),x(t))}{\rho^{2t}}$, for all $t \in \mathbb{N}$. It follows from (26) and (27) that for all $t \in \mathbb{N}$, $W(t+1) \leq W(t)$, thus $W(t)$ is non-increasing. Also since for all $k \in \mathbb{N}$, we have $\eta(\tau_k^\theta) = \emptyset$, (27) gives $W(\tau_k^\theta) \leq \lambda^2 W(\tau_k^\theta - 1)$ for every $k \in \mathbb{N}$, $k \geq 1$. The latter relation and the monotonicity of $W(t)$ implies that $W(\tau_k^\theta) \leq \lambda^2 W(\tau_{k-1}^\theta)$. Hence, by induction we conclude that

$$W(\tau_k^\theta) \leq \lambda^{2k} W(0), \quad \forall k \in \mathbb{N}.$$

Let $t \in [\tau_k^\theta, \tau_{(k+1)}^\theta)$, then $\kappa^\theta(t) = k$, from the monotonicity of $W(t)$ we have:

$$W(t) \leq W(\tau_k^\theta) \leq \lambda^{2\kappa^\theta(t)} W(0).$$

From this point on the proof is similar to the final part of that of Theorem 3. Therefore we finally obtain the bound (8).

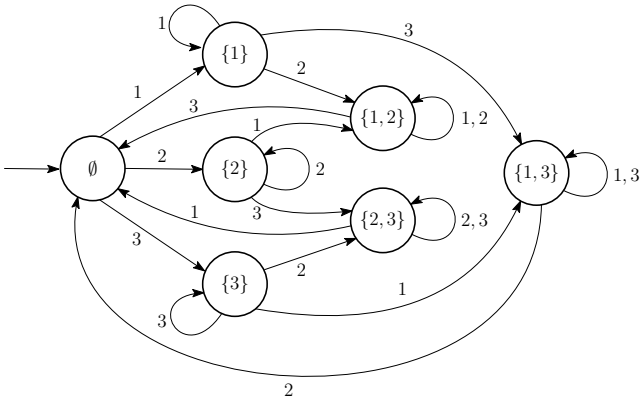


Fig. 3. Automaton describing the dynamics of η in (28) for $m = 3$. State labels correspond to the value of η, transition labels correspond to the value of θ.

We prove now the converse result. We first consider the case $\rho = 1$. For $J \subseteq \mathcal{I}$ with $J \neq \emptyset$, let $\mathcal{M}_J$ consist of all finite products of N matrices in $\mathcal{A}$, with $N \in \mathbb{N}$, where each mode in J appears at least once, and the last mode j_N belongs to J and appears exactly once:

$$\mathcal{M}_J = \left\{ \prod_{k=1}^{N} A_{j_k}, \left| \begin{array}{c} j_1, \ldots, j_N \in \mathcal{I},\ N \in \mathbb{N}, \\ \forall i \in J, \exists k, j_k = i, \\ j_N \in J \text{ and } \forall k \neq N, j_k \neq j_N \end{array} \right. \right\}.$$

In particular, $\mathcal{M}_{\mathcal{I}}$ is the set of all possible matrices $\mathbb{A}_{\theta,\tau_\theta^1}$ for $\theta \in \mathcal{S}_s(\mathcal{I})$. Since $\rho = 1$, it follows from (8) that for all $k \in \mathbb{N}$

$$\|\mathbb{A}_{\theta,\tau_\theta^k}\| \leq C\lambda^k, \forall \theta \in \mathcal{S}_s(\mathcal{I}) \ ,$$

which is equivalent to

$$\Big\| \prod_{j=1}^{k} M_j \Big\| \leq C\lambda^k, \ \forall M_1, \ldots, M_k \in \mathcal{M}_{\mathcal{I}}.$$

From [6, Lemma 2] applied to the family of products of matrices in $\{\frac{1}{\lambda}M |\ M \in \mathcal{M}_{\mathcal{I}}\}$, it follows that there exists a norm $\|\cdot\|_*$ in $\mathbb{R}^n$ such that the corresponding induced matrix norm, also denoted by $\|\cdot\|_*$, satisfies $\sup_{M\in\mathcal{M}_{\mathcal{I}}} \|M\|_* \leq \lambda$.

Then, for all subsets $J \subsetneq \mathcal{I}$ and $x \in \mathbb{R}^n$ we define

$$V(J,x) = \sup_{M\in\mathcal{M}_{\mathcal{I}\setminus J}} \|Mx\|_*^2.$$

Let us first prove (25). For $J \subsetneq \mathcal{I}$, let M_J be an arbitrary element of $\mathcal{M}_{\mathcal{I}\setminus J}$, then $V(J,x) \geq \|M_J x\|_*^2$. All matrices M_J are invertible, $\|\cdot\|_*$ is equivalent to $\|\cdot\|$, and $2^{\mathcal{I}} \setminus \{\mathcal{I}\}$ is a finite set, therefore there exists $\alpha_1 > 0$ such that $\|M_J x\|_*^2 \geq \alpha_1 \|x\|^2$, for all $J \subsetneq \mathcal{I}, x \in \mathbb{R}^n$. Also, it follows from (8) with $\rho = 1$ and $\lambda \in [0,1]$ that for all $J \subsetneq \mathcal{I}$, for all $M \in \mathcal{M}_{\mathcal{I}\setminus J}$, $\|M\| \leq C$. Then, $\|\cdot\|_*$ being equivalent to $\|\cdot\|$, it follows that there exists $\alpha_2 > 0$ such that $V(J,x) \leq \alpha_2 \|x\|^2$, for all $J \subsetneq \mathcal{I}$, $x \in \mathbb{R}^n$.

To prove (26), we first notice that every product MA_i with $M \in \mathcal{M}_{\mathcal{I}\setminus(J\cup\{i\})}$ belongs to $\mathcal{M}_{\mathcal{I}\setminus J}$. Hence,

$$\begin{aligned} V(J\cup\{i\}, A_i x) &= \sup_{M\in\mathcal{M}_{\mathcal{I}\setminus(J\cup\{i\})}} \|MA_i x\|_*^2 \\ &\leq \sup_{M'\in\mathcal{M}_{\mathcal{I}\setminus J}} \|M'x\|_*^2 = V(J,x). \end{aligned}$$

In order to prove (27), we use the fact that if $J \subsetneq \mathcal{I}$ is such that $J \cup \{i\} = \mathcal{I}$ then $A_i \in \mathcal{M}_{\mathcal{I}\setminus J} = \mathcal{M}_{\{i\}}$. Consequently,

$$\begin{aligned} V(\emptyset, A_i x) &= \sup_{M\in\mathcal{M}_{\mathcal{I}}} \|MA_i x\|_*^2 \\ &\leq \lambda^2 \|A_i x\|_*^2 \\ &\leq \lambda^2 \sup_{M\in\mathcal{M}_{\mathcal{I}\setminus J}} \|Mx\|_*^2 = V(J,x). \end{aligned}$$

This concludes the proof in the case $\rho = 1$.

Let us consider now the general case $\rho \neq 1$. For the set of matrices $\frac{1}{\rho}\mathcal{A}$, the corresponding dynamics satisfies (8) with $\rho = 1$ and we can consider the function V defined above. It directly follows from the properties shown above and the fact that V is homogeneous of degree two that (25)-(26)-(27) hold true. The proof is complete. □

As a direct consequence of the previous results and Theorem 1 we have the following corollary.

Corollary 2. *Let $\rho > 0$ and $\lambda \in [0,1]$ such that there exists a function V satisfying the conditions of Theorem 3 or Theorem 4, then either $\rho > \boldsymbol{\rho}(\mathcal{A})$ and $\lambda \geq \boldsymbol{\lambda}(\mathcal{A},\rho)$, or $\rho = \boldsymbol{\rho}(\mathcal{A})$ and $\lambda \geq \boldsymbol{\lambda}(\mathcal{A})$. Conversely, if the matrices A_i are invertible, for all $i \in \mathcal{I}$, then for all $\rho > \boldsymbol{\rho}(\mathcal{A})$, for all $\lambda \in (\boldsymbol{\lambda}(\mathcal{A},\rho), 1]$, there exists a function V satisfying the conditions of Theorem 4.*

Corollary 2 shows that upper-bounds of the JSR and of the ρ-SJSR can be found by computing Lyapunov functions satisfying the conditions in Theorems 3 and 4. Limiting the search to quadratic Lyapunov function, the conditions (20)-(21)-(22)-(23) or (25)-(26)-(27) can straightforwardly be translated to linear matrix inequalities (LMIs) for which efficient solvers exist. However, the tightness of the conditions in Theorem 4 is lost when constraining the Lyapunov functions to be quadratic.

6 Numerical examples

In this section, we illustrate the main results of the paper with two numerical examples [1] .

The first example shows an application of Proposition 5 to compute a lower bound of the ρ-SJSR, and of Theorem 2, Theorems 3 and 4 to compute upper bounds of the ρ-SJSR. The second example illustrates the result of Corollary 1 where a switched system that is unstable for arbitrary switching is stabilized by shuffling sufficiently fast. The switched systems considered in both examples are adapted from [13].

[1] The Matlab scripts of the two numerical examples are available at the following repository: `https://github.com/georgesaazan/Shuffled-systems`

6.1 Computing bounds of the ρ-SJSR

Let us consider a switched system in $\mathbb{R}^3$ with 2 modes, where $\mathcal{A} = \{A_1, A_2\}$ with:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1\cos(\phi_1) & -\mu_1\sin(\phi_1) \\ 0 & \mu_1\sin(\phi_1) & \mu_1\cos(\phi_1) \end{pmatrix}, \\ A_2 = \begin{pmatrix} \mu_2\cos(\phi_2) & -\mu_2\sin(\phi_2) & 0 \\ \mu_2\sin(\phi_2) & \mu_2\cos(\phi_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{29}$$

where the numerical values of the parameters are $\mu_1 = 0.9, \mu_2 = 0.2, \phi_1 = \frac{\pi}{6}, \phi_2 = \frac{\pi}{3}$. Let us notice that $\boldsymbol{\rho}(\mathcal{A}) = 1$ with the Euclidean norm $\|\cdot\|$ being extremal for $\mathcal{A}$.

We first compute a lower bound for the ρ-SJSR using Proposition 5. Let us remark that $\mathcal{N}_{\mathcal{I}} = \{A_1A_2, A_2A_1\}$. We use the JSR toolbox [30] to compute tight bounds on $\boldsymbol{\rho}(\mathcal{N}_{\mathcal{I}})$ and we obtain $\boldsymbol{\rho}(\mathcal{N}_{\mathcal{I}}) \in [\underline{\rho}, \overline{\rho}]$ where $\underline{\rho} = 0.7640321$ and $\overline{\rho} = 0.7640322$. It follows from Proposition 5, that

$$\forall \rho > 1,\ \boldsymbol{\lambda}(\mathcal{A}, \rho) \geq \frac{\underline{\rho}}{\rho^m}.$$

We now apply Theorem 2 and Lemma 3 to compute an upper bound on the ρ-SJSR. Considering $K = \overline{\rho}$, we search for an induced matrix norm $\|\cdot\|_*$ such that $\|A_1A_2\|_* \leq \overline{\rho}$ and $\|A_2A_1\|_* \leq \overline{\rho}$. Limiting the search to a quadratic norm of the form $\|x\|_* = \sqrt{x^\top Q x}, Q = Q^\top \geq 0$ with the associated induced matrix norm $\|M\|_* = \|Q^{1/2}MQ^{-1/2}\|$, the conditions above are equivalent to the following LMIs:

$$(A_1A_2)^\top Q A_1A_2 \leq \overline{\rho}^2 Q, \\ (A_2A_1)^\top Q A_2A_1 \leq \overline{\rho}^2 Q.$$

Solving these LMIs yields

$$Q = \begin{pmatrix} 0.2755 & 0.0688 & 0.0257 \\ 0.0688 & 0.3660 & 0.2248 \\ 0.0257 & 0.2248 & 0.3585 \end{pmatrix}.$$

Then, we compute R according to (19) and we get

$$R = \max(\|A_1\|_*, \|A_2\|_*) = 1.3279.$$

Then, from Lemma 3, we get that

$$\forall \rho \geq R,\ \boldsymbol{\lambda}(\mathcal{A}, \rho) \leq \frac{\overline{\rho}}{\rho^m}.$$

Now, we use the Lyapunov techniques presented in Theorems 3 and 4 to compute upper-bounds of the ρ-SJSR for $\rho \in (1, R)$. Searching for Lyapunov functions of the form $V(i, x) = x^\top Q_i x$, $i \in \mathcal{I}$ the conditions of Theorem 3 translate to the following LMIs:

$$\begin{aligned} I_n \leq & Q_i, && i \in \mathcal{I} \\ A_{i'}^\top Q_i A_{i'} \leq & \rho^2 Q_i, && i, i' \in \mathcal{I}, i \neq i' \\ A_i^\top Q_{i+1} A_i \leq & \rho^2 Q_i, && i \in \mathcal{I} \setminus \{m\} \\ A_m^\top Q_1 A_m \leq & \rho^2 \lambda^{2m} Q_m. \end{aligned}$$

Similarly, searching for Lyapunov functions of the form $V(J, x) = x^\top Q_J x$, $J \subsetneq \mathcal{I}$, the conditions of Theorem 4 translate to the following LMIs:

$$\begin{aligned} I_n \leq & Q_J, && J \subsetneq \mathcal{I} \\ A_i^\top Q_{J\cup\{i\}} A_i \leq & \rho^2 Q_J, && \text{if } J \cup \{i\} \neq \mathcal{I} \\ A_i^\top Q_1 A_i \leq & \rho^2 \lambda^2 Q_J, && \text{if } J \cup \{i\} = \mathcal{I}. \end{aligned}$$

Then, for each $\rho \in \Omega = \{1, 1.05, \cdots, 1.35\}$, we can find an upper bound $\overline{\boldsymbol{\lambda}}(\rho)$ on $\boldsymbol{\lambda}(\mathcal{A}, \rho)$ by searching (using a line search) for the smallest value of λ for which the LMIs above have a solution. Then, we can obtain an upper bound for the ρ-SJSR for all $\rho \in (1, R)$ using (5):

$$\forall \rho \in (1, R),\ \boldsymbol{\lambda}(\mathcal{A}, \rho) \leq \min_{\rho' \in \Omega, \rho' \leq \rho} \left(\frac{\rho'}{\rho}\right)^m \overline{\boldsymbol{\lambda}}(\rho').$$

The resulting upper-bounds as well as the lower bound are shown in Figure 4. One can check that the upper-bound computed using Theorem 4 is always tighter than that provided by Theorem 3. However, let us remark that the former approach involves solving a set of $m(m+1)$ LMIs while the latter requires solving a set of $(2^m - 1)(m+1)$ LMIs. For large values of m the latter approach is likely to be intractable.

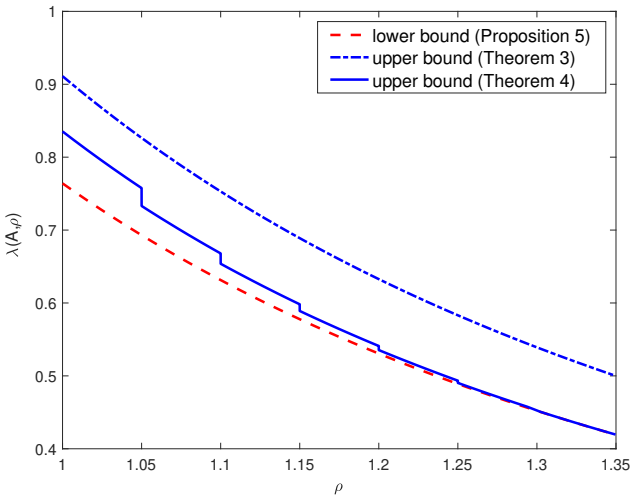


Fig. 4. Lower and upper bounds on the ρ-SJSR for (29).

6.2 Case study: synchronization over a network

We provide a short case-study regarding an application of our tools to the design of networked controllers for syn-

chronization of distributed oscillators. Let us consider $m+1$ identical components consisting of discrete-time oscillators whose dynamics is given by:

$$z_i(t+1) = Rz_i(t) + u_i(t),\ i = 1, \dots, m+1 \qquad (30)$$

where $z_i(t) \in \mathbb{R}^2, u_i(t) \in \mathbb{R}^2$ and $R = \begin{pmatrix} \mu\cos(\phi) & -\mu\sin(\phi) \\ \mu\sin(\phi) & \mu\cos(\phi) \end{pmatrix}$ with $\mu = 1.02$ and $\phi = \frac{\pi}{6}$. The input $u_i(t)$ is used for synchronization purpose and is based on the available information at time t. The information is exchanged over a single hop communication network, which consists of m communication channels between components i and $i+1$, $i = 1, \dots, m$. We assume that the network capacity is limited, in such a way that only one channel can be activated at a given time instant. Formally, the active channel is given by a switching signal $\theta : \mathbb{N} \to \mathcal{I} = \{1, \dots, m\}$. Then, the input value implementing the synchronization protocol is given as follows:

$$\begin{aligned}
u_1(t) &= \begin{cases} k(z_2(t) - z_1(t)), & \text{if } \theta(t) = 1 \\ 0, & \text{otherwise} \end{cases} \\
u_i(t) &= \begin{cases} k(z_{i-1}(t) - z_i(t)), & \text{if } \theta(t) = i-1 \\ k(z_{i+1}(t) - z_i(t)), & \text{if } \theta(t) = i \\ 0 & \text{otherwise} \end{cases} \\
&\text{for } i = 2, \dots, m, \\
u_{m+1}(t) &= \begin{cases} k(z_m(t) - z_{m+1}(t)), & \text{if } \theta(t) = m \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \qquad (31)$$

where $k \in (0,1)$ is a control gain. Denoting the vector of synchronization errors as $x(t) = (x_1(t)^\top, \dots, x_m(t)^\top)^\top$ with $x_i(t) = z_{i+1}(t) - z_i(t)$, the error dynamics is described by a $2m$-dimensional switched linear system of the form (1) with m modes. The expression of the matrices $A_i \in \mathbb{R}^{2m \times 2m}$, $i = 1, \dots, m$ can be easily derived from (30)-(31).

For $m \geq 2$, it is clear that the synchronization cannot be achieved using arbitrary switching signals. Indeed, considering for instance the constant switching signal $\theta(t) = 1$, for all $t \in \mathbb{N}$, components 1 and 2 will synchronize but other agents that do not get the opportunity to exchange information cannot synchronize. Hence, we use shuffled switching signals. In our case study, we aim at providing answers to the following questions:

(1) Co-design a control gain k and a minimal shuffling rate γ^* so that all oscillators synchronize;
(2) Determine the maximal number of oscillators that can be synchronized given the proposed network architecture and capacity.

We compute $\boldsymbol{\rho}(\mathcal{A})$ using the JSR toolbox. We then use Theorem 4 to compute an upper bound $\overline{\boldsymbol{\lambda}}(\rho)$ on the ρ-SJSR by solving the associated LMIs presented in the previous subsection. Note that

$$\gamma^* := \inf_{\rho > \boldsymbol{\rho}(\mathcal{A})} -\frac{\ln(\rho)}{\ln(\overline{\boldsymbol{\lambda}}(\rho))} \geq \inf_{\rho > \boldsymbol{\rho}(\mathcal{A})} -\frac{\ln(\rho)}{\ln(\boldsymbol{\lambda}(\mathcal{A}, \rho))}.$$

In this example, we observe numerically that the infimum is reached for $\rho = \boldsymbol{\rho}(\mathcal{A})$.

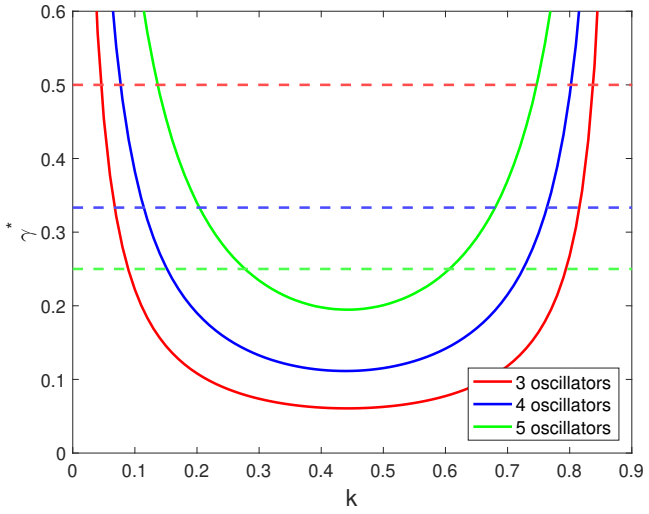


Fig. 5. Minimal shuffling rate γ^* as a function of the control gain k and of the number of oscillators $m+1$.

We computed γ^* for several values of $k \in (0,1)$ and $m+1 \in \{3,4,5\}$ and we show the result in Figure 5. In this figure, the dashed lines correspond to the maximal achievable shuffling rate $\frac{1}{m}$. From Corollary 1, we know that the switched system can be stabilized if we use shuffled switching signals $\theta \in \mathcal{S}_s(\mathcal{I})$ whose shuffling rate $\gamma^\theta \in (\gamma^*, \frac{1}{m}]$. Hence, in order to stabilize the system, we should carefully select the control gain k such that the corresponding value of $\gamma^* < \frac{1}{m}$. This answers the first point of our case study. For instance, we can see that for a system composed of $m+1 = 3$ oscillators, for a control gain $k = 0.6$ we get $\gamma^* = 0.08$ and we can stabilize the system by imposing a shuffling rate $\gamma^\theta \in (0.08, 0.5]$. If we choose $k = 0.88$, then $\gamma^* > \frac{1}{m}$ and we are not able to stabilize the system by shuffling.

We also verified that for $m+1 = 6$ oscillators, $\gamma^* > \frac{1}{m}$ for all control gain $k \in (0,1)$. Therefore, for 6 oscillators, there is no suitable choice of control gain k such that the system can be stabilized by shuffling. This answers the second point of our case study.

We now proceed with some illustrative numerical simulations. Let us consider a system composed of $m+1 = 3$ oscillators with control gain $k = 0.4$, then from Figure 5, the corresponding lower bound on the shuffling rate is $\gamma^* = 0.06$. Let us consider the initial synchronization error $x(0) = (-1.5\ -0.5\ 2\ -1)^\top$. We consider random

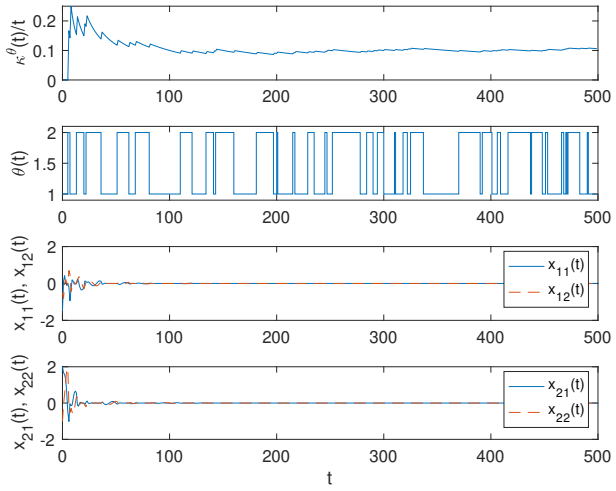


Fig. 6. Time evolution of the synchronization error $x(t)$ and the switching signal $\theta(t)$ for a shuffling rate higher than $\gamma^* = 0.06$.

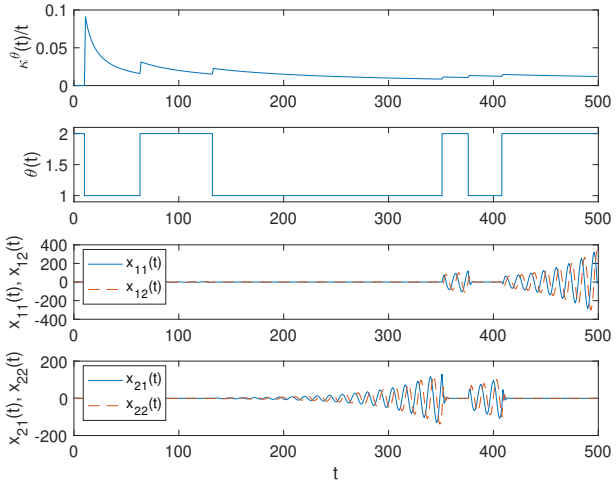


Fig. 7. Time evolution of the synchronization error $x(t)$ and the switching signal $\theta(t)$ for a shuffling rate lower than $\gamma^* = 0.06$.

switching signals generated by a discrete-time Markov chain with 2 states and the following transition matrix: $\left(\begin{smallmatrix} 1-p & p \\ p & 1-p \end{smallmatrix}\right)$ where $p \in (0,1)$ is the probability to switch at a given time instant. It is easy to see that such switching signals are shuffled almost surely. Moreover, the larger p the higher the shuffling rate of the switching signals. We first consider $p = 1/10$, Figure 6 shows the evolution of $\frac{\kappa^{\theta}(t)}{t}$, the switching signal $\theta(t)$ and the synchronization errors $x_1(t), x_2(t)$. It is interesting to note that $\frac{\kappa^{\theta}(t)}{t} > 0.06$ and that the system stabilizes as expected. Next, we consider $p = 1/70$, the corresponding simulation results are shown in Figure 7. We can check on the figure that in this case $\frac{\kappa^{\theta}(t)}{t} < 0.06$ and that the system does not stabilize, the shuffling being too slow.

7 Conclusion

In this paper, we introduced the ρ-SJSR and the SJSR, a new notion of joint spectral radius for discrete-time switched linear systems driven by shuffled switching signals. We have established some of their properties and highlighted their relation to stability properties of switched systems. We presented a method based on the JSR of a finite set of matrices to compute asymptotically tight lower and upper bounds of the ρ-SJSR. We also developed two approaches based on Lyapunov functions and automata theoretic techniques to compute upper-bounds of the ρ-SJSR. While the first approach is computationally more tractable, the second approach was shown to be tight. Numerical examples show the effectiveness of the proposed numerical approximation methods. We also showed an application of the SJSR to the synchronization of unstable oscillators.

The current work opens several research directions for the future. Firstly, even though the proposed Lyapunov conditions make it possible to compute tight upper bounds of the ρ-SJSR, their translation to linear matrix inequalities may introduce some conservatism. It is then interesting to investigate how techniques similar to the path-complete graph Lyapunov functions [1] can be used in order to derive linear matrix inequalities whose solution provide a tight upper bound of the ρ-SJSR. Secondly, it has been shown in [31] that stability of a switched system with switching signals belonging to an ω-regular language is equivalent to the shuffled stability of a lifted switched system, so our current approach can readily be used to analyze stability properties of such systems. However, since our approach is based on automata theoretic techniques, it is natural to think that one can derive stability conditions by working directly on the Büchi, Rabin or Muller automaton specifying the ω-regular language.

References

[1] A. A. Ahmadi, R. M. Jungers, P. A. Parrilo, and M. Roozbehani. Joint spectral radius and path-complete graph lyapunov functions. *SIAM Journal on Control and Optimization*, 52(1):687–717, 2014.

[2] R. Alur, A. d'Innocenzo, K. H. Johansson, G. J. Pappas, and G. Weiss. Compositional modeling and analysis of multi-hop control networks. *IEEE Transactions on Automatic control*, 56(10):2345–2357, 2011.

[3] R. Alur and G. Weiss. Regular specifications of resource requirements for embedded control software. In *EEE Real-Time and Embedded Technology and Applications Symposium*, pages 159–168, 2008.

[4] C. Baier and J.-P. Katoen. *Principles of model checking*. MIT press, 2008.

[5] N. E. Barabanov. Lyapunov indicator of discrete inclusions. 1. 2. 3. *Automation and Remote Control*, 49(2,3,5):152–157, 283–287, 558–565, 1988.

[6] M. A. Berger and Y. Wang. Bounded semigroups of matrices. *Linear Algebra and its Applications*, 166:21–27, 1992.

[7] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis. Convergence in multiagent coordination, consensus, and flocking. In *IEEE Conference on Decision and Control*, pages 2996–3000, 2005.

[8] V. D. Blondel and Y. Nesterov. Computationally efficient approximations of the joint spectral radius. *SIAM Journal on Matrix Analysis and Applications*, 27(1):256–272, 2005.

[9] V. D. Blondel, Y. Nesterov, and J. Theys. On the accuracy of the ellipsoid norm approximation of the joint spectral radius. *Linear Algebra and its Applications*, 394:91–107, 2005.

[10] Y. Chitour, G. Mazanti, and M. Sigalotti. On the gap between deterministic and probabilistic joint spectral radii for discrete-time linear systems. *Linear Algebra and its Applications*, 613:24–45, 2021.

[11] X. Dai. A Gel'fand-type spectral radius formula and stability of linear constrained switching systems. *Linear Algebra and its Applications*, 436(5):1099–1113, 2012.

[12] M. Donkers, W. Heemels, N. Van de Wouw, and L. Hetel. Stability analysis of networked control systems using a switched linear systems approach. *IEEE Transactions on Automatic control*, 56(9):2101–2115, 2011.

[13] A. Girard and P. Mason. Lyapunov functions for shuffle asymptotic stability of discrete-time switched systems. *IEEE Control Systems Letters*, 3(3):499–504, 2019.

[14] G. Gripenberg. Computing the joint spectral radius. *Linear Algebra and its Applications*, 234:43–60, 1996.

[15] L. Gurvits. Stability of discrete linear inclusion. *Linear algebra and its applications*, 231:47–85, 1995.

[16] R. Jungers. *The joint spectral radius: theory and applications*, volume 385. Springer Science & Business Media, 2009.

[17] V. Kozyakin. Algebraic unsolvability of problem of absolute stability of desynchronized systems. *Automation and Remote Control*, 51(6):754–759, 1990.

[18] V. Kozyakin. The Berger-Wang formula for the Markovian joint spectral radius. *Linear Algebra and its Applications*, 448:315–328, 2014.

[19] A. Kundu and D. Chatterjee. On stability of discrete-time switched systems. *Nonlinear Analysis: Hybrid Systems*, 23:191–210, 2017.

[20] J.-W. Lee and G. E. Dullerud. Uniformly stabilizing sets of switching sequences for switched linear systems. *IEEE Transactions on Automatic Control*, 52(5):868–874, 2007.

[21] D. Liberzon. *Switching in systems and control*. Springer Science & Business Media, 2003.

[22] D. Liberzon and A. S. Morse. Basic problems in stability and design of switched systems. *IEEE control systems magazine*, 19(5):59–70, 1999.

[23] H. Lin and P. J. Antsaklis. Stability and stabilizability of switched linear systems: a survey of recent results. *IEEE Transactions on Automatic control*, 54(2):308–322, 2009.

[24] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on automatic control*, 50(2):169–182, 2005.

[25] M. Philippe, R. Essick, G. E. Dullerud, and R. M. Jungers. Stability of discrete-time switching systems with constrained switching sequences. *Automatica*, 72:242–250, 2016.

[26] Y. Su and J. Huang. Stability of a class of linear switching systems with applications to two consensus problems. *IEEE Transactions on Automatic Control*, 57(6):1420–1430, 2011.

[27] Z. Sun and S. S. Ge. *Switched linear systems: control and design*. Springer Science & Business Media, 2006.

[28] A. Tanwani, H. Shim, and D. Liberzon. Observability for switched linear systems: characterization and observer design. *IEEE Transactions on Automatic Control*, 58(4):891–904, 2012.

[29] J. N. Tsitsiklis and V. D. Blondel. The Lyapunov exponent and joint spectral radius of pairs of matrices are hard - when not impossible - to compute and to approximate. *Mathematics of Control, Signals and Systems*, 10(1):31–40, 1997.

[30] G. Vankeerberghen, J. Hendrickx, and R. M. Jungers. JSR: A toolbox to compute the joint spectral radius. In *International conference on Hybrid Systems: Computation and Control*, pages 151–156, 2014.

[31] Y. Wang, N. Roohi, G. E. Dullerud, and M. Viswanathan. Stability analysis of switched linear systems defined by regular languages. *IEEE Transactions on Automatic Control*, 62(5):2568–2575, 2017.

[32] G. Weiss and R. Alur. Automata based interfaces for control and scheduling. In *International conference on Hybrid systems: computation and control*, pages 601–613, 2007.

[33] X. Xu and B. Acikmese. Approximation of the constrained joint spectral radius via algebraic lifting. *IEEE Transactions on Automatic Control*, 2020.