The calibration method for the Thermal Insulation functional

Camille Labourie, Emmanouil Milakis

The calibration method for the Thermal Insulation functional

C. Labourie*† E. Milakis*‡

Abstract

We provide minimality criteria for functionals arising in the theory of Thermal Insulation. We expand the principle of calibrations for minimizers in this type of problems.

AMS Subject Classifications: 49K10, 35R35.

Keywords: Thermal Insulation, Calibration Method, Free Boundary Problems.

Contents

1 Introduction **2**
- 1.1 Calibrations for free-discontinuity problems 2
- 1.2 The thermal insulation functional 3
- 1.3 Summary of results 4

2 A Dirichlet problem **5**
- 2.1 Statement of the problem and calibrations 5
- 2.2 The one dimensional case 8
  - 2.2.1 A short analysis 8
  - 2.2.2 The construction 10
- 2.3 Higher dimension 12

*University of Cyprus, Department of Mathematics & Statistics, P.O. Box 20537, Nicosia, CY- 1678 CYPRUS.

†labourie.camille@ucy.ac.cy

‡emilakis@ucy.ac.cy

§This work was co-funded by the European Regional Development Fund and the Republic of Cyprus through the Research and Innovation Foundation (Project: EXCELLENCE/1216/0025)

3 The Thermal Insulation Problem **15**
3.1 Definition of the problem and calibrations 15
3.1.1 Informal computations with $\Omega = B(0,1)$ 16
3.1.2 Three sufficient conditions of minimality 18
3.1.3 Miscellaneous remarks 20
3.2 Proof of Theorem 3.2 20
3.3 Proof of Theorem 3.4 21
3.4 Proof of Theorem 3.5 22

Appendices **31**

A Proof of Lemma 3.3 **31**

1 Introduction

1.1 Calibrations for free-discontinuity problems

Free-discontinuity problems consists in minimizing the energy $E(u, K)$ of a pair composed of a function $u \in C^1(\mathbf{R}^n \setminus K)$ and a set K of dimension $n-1$. The energy presents a competition between the Dirichlet energy of u in $\mathbf{R}^n \setminus K$ and the surface energy of K. The set K is interpreted as an hypersurface (with possibly singularities) where u jumps between different values. This notion of pair composed of a function and of its discontinuity set is alternatively formalized by the space SBV (special functions with bounded variations). The model case of this kind of problem is the Mumford-Shah functional coming from image segmentation.

These problems generally present two kind of Euler-Lagrange equations. Considering small perturbations of u (with the discontinuity set K being fixed), one obtains that u satisfy a PDE with a boundary condition on each connected component of $\mathbf{R}^n \setminus K$. As an example, minimizers of the Mumford-Shah functional are harmonic functions satisfying a Neumann boundary condition. Considering small perturbations of K under diffeormophisms, one obtains an equation that deals with the mean curvature of K. However, these equations do not entirely characterise the minimizers. We also point out that the minimizer may not be unique. We summarize these difficulties as a lack of convexity of the functional E.

In [1], Alberti, Bouchitté, Dal Maso have introduced a sufficient condition for minimality by adapting the calibration method that was known for minimal hypersurfaces. Here is a simplified summary. Let us consider a competitor (u, K). We define the complete graph Γ_u of u as the boundary of the subgraph of u. It is the reunion of the graph of u on $\mathbf{R}^n \setminus K$ and of vertical sides above K (the discontinuities of u). We denote by ν_{Γ_u} the normal vector to Γ_u pointing into the subgraph of u. A calibration for (u, K)

is a divergence-free vector field $\phi\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n \times \mathbf{R}$ such that

$$E(u,K) = \int_{\Gamma_u} \phi \cdot \nu_{\Gamma_u} \mathrm{d}\mathcal{H}^{n-1} \tag{1}$$

and for all competitor (v,L),

$$E(v,L) \geq \int_{\Gamma_v} \phi \cdot \nu_{\Gamma_v} \mathrm{d}\mathcal{H}^{n-1}. \tag{2}$$

One observes that the Gauss-Green theorem and the divergence-free property imply

$$\int_{\Gamma_u} \phi \cdot \nu_{\Gamma_u} \mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma_v} \phi \cdot \nu_{\Gamma_v} \mathrm{d}\mathcal{H}^{n-1} \tag{3}$$

so the existence of such a vector field proves that (u,K) is a minimizer. In [1, Lemma 3.7], the authors provide four axioms which ensure the properties (1), (2) above and it is convenient to take these axioms as a definition of calibrations.

In general, we don't know if calibrations exist for minimizers of this kind of problem. There is no general recipe to follow and the construction can be very difficult. Alberti, Bouchitté and Dal Maso give some example of applications for the Mumford-Shah functional in [1]. The crack-tip is famous example of minimizer of the Mumford-Shah functional for which a calibration has not been found.

1.2 The thermal insulation functional

A free boundary problem related to thermal insulation was recently studied by Caffarelli–Kriventsov ([3], [4]) and Bucur–Giacomini ([5]). Relaxing the problem in SBV, it consists in minimizing

$$E(u) = \int |\nabla u|^2 \,\mathrm{d}x + \beta \int_{J_u} (u^-)^2 + (u^+)^2 \,\mathrm{d}\mathcal{H}^{n-1} + \gamma^2 \mathcal{L}^n(\{\, u > 0 \,\}),$$

where $\beta, \gamma > 0$ and the competitors are functions $u \in SBV(\mathbf{R}^n)$ such that $u = 1$ on a given bounded open set $\Omega \subset \mathbf{R}^n$. Here J_u is the set of all jump points of u, that is the points x for which there exist two real numbers $u^- < u^+$ and a (unique) vector $\nu_u(x) \in \mathbf{S}^{n-1}$ such that

$$\lim_{r\to 0} \fint_{B_r \cap H^+} \left| u(y) - u^- \right| \mathrm{d}y = 0 \tag{4a}$$

$$\lim_{r\to 0} \fint_{B_r \cap H^-} \left| u(y) - u^+ \right| \mathrm{d}y = 0, \tag{4b}$$

where

$$H^+ = \{\, y \in \mathbf{R}^n \mid (y-x)\cdot \nu_u(x) > 0 \,\} \tag{5a}$$

$$H^- = \{\, y \in \mathbf{R}^n \mid (y-x)\cdot \nu_u(x) < 0 \,\}. \tag{5b}$$

The domain Ω can be interpreted as an initial data. The function u can be interpreted as the temperature (which is fixed to 1 on Ω) and J_u as an isolating layer which has no width and which has an infinite thermal conductivity. Note that without loss of generality, we can assume that $0 \leq u \leq 1$. Indeed, post-composing u with the orthogonal projection onto $[0,1]$ decreases the energy.

Caffarelli–Kriventsov and Bucur-Giacomini have shown the existence of minimizers in [3] and [5] respectively. Both authors prove a non-degeneracy property: there exists $0 < \delta < 1$ (depending only on n and Ω) such that such that $\operatorname{spt}(u) \subset B(0, \delta^{-1})$ and

$$u \in \{\, 0 \,\} \cup [\delta, 1] \quad \mathcal{L}^n\text{-a.e. on } \mathbf{R}^n. \tag{6}$$

They also prove that the jump set J_u is essentially closed, $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$, and that it satisfies uniform density estimates. In [4], it was proven that the jump set is locally the union of the graphs of two $C^{1,\alpha}$ functions provided that it is trapped between two planes which are sufficiently close. In [6], we have proved the higher integrability of the gradient for minimizers of the thermal insulation problem, an analogue of De Giorgi's conjecture for the Mumford-Shah functional, and deduced that the singular part of the free boundary has Hausdorff dimension strictly less than $n-1$.

The approaches in these three papers highlighted the similarities and the differences between the thermal insulation problem and the Mumford-Shah functional. In particular, for the thermal insulation problem, one has to deal with an harmonic function satisfying a Robin boundary condition at the boundary rather than a Neumann boundary condition.

1.3 Summary of results

The purpose of this article is to provide minimality criteria for the thermal insulation functional by construction of calibrations. The article is divided into two parts. In the first part, we fix an open set A of $\mathbf{R}^n$ and we consider the functional

$$E_0(u) = \int_A |\nabla u|^2 \,\mathrm{d}x + \beta \int_{J_u \cap A} (u^+)^2 + (u^-)^2 \,\mathrm{d}\mathcal{H}^{n-1}, \tag{7}$$

where $u \in \mathrm{SBV}(A)$. In Theorem 2.1, we present a sufficient condition so that a positive harmonic function $u \colon A \to \mathbf{R}$ is a minimizer of E_0 among all competitors $v \in \mathrm{SBV}(A)$ with $\{\, v \neq u \,\} \subset\subset A$. The condition is also necessary in dimension one but likely not in higher dimension. An anologous questions had been studied for the homogeneous Mumford-Shah functional

$$F_0(u, A) = \int_A |\nabla u|^2 \,\mathrm{d}x + \beta \mathcal{H}^{n-1}(J_u) \tag{8}$$

by Chambolle without calibrations ([8, Theorem 3.1(i)]) and Alberti, Bouchitté, Dal Maso with calibrations ([1, section 4.6]). In the second part, we come back to the full thermal insulation functional,

$$E(u) = \int |\nabla u|^2 \, \mathrm{d}x + \beta \int_{J_u} (u^+)^2 + (u^-)^2 \, \mathrm{d}\mathcal{H}^{n-1} + \gamma^2 \mathcal{L}^n(\{\, u > 0 \,\}) \qquad (9)$$

where $u \in \mathrm{SBV}(\mathbf{R}^n)$ is such that $u = 1$ on a given bounded open set Ω. The section starts with an informal discussion about the case $\Omega = B(0,1)$. We expect that the relevant competitors are either the indicator function $\mathbf{1}_\Omega$ or an harmonic function supported in a bigger ball. We can make explicit computations with these competitors to find minimality criterias. Then we prove and generalize these criterias to other initial data Ω using calibrations. In Theorem 3.2 and 3.4, we present two sufficient conditions so that $\mathbf{1}_\Omega$ is a minimizer. Coming back to the case $\Omega = B(0,1)$, we present a calibration to prove that an harmonic function supported in a bigger ball is a minimizer. However, we have not adapted this construction to other initial data Ω.

Acknowledgement: We would like to thank Antonin Chambolle for helpful discussions about extensions of normal vector fields.

2 A Dirichlet problem

2.1 Statement of the problem and calibrations

The ambient space is $\mathbf{R}^n$. We fix a parameter $\beta > 0$. Given a Borel set $A \subset \mathbf{R}^n$ and an SBV function u in a neighborhood of A, we define

$$E_0(u, A) = \int_A |\nabla u|^2 \, \mathrm{d}x + \beta \int_{J_u \cap A} (u^+)^2 + (u^-)^2 \, \mathrm{d}\mathcal{H}^{n-1}. \qquad (10)$$

We fix an open set A and an harmonic positive function $u \colon A \to \mathbf{R}$. We are interested in finding a sufficient condition so that the following minimality property holds true: for all $v \in \mathrm{SBV}(A)$ with $\{\, v \neq u \,\} \subset\subset A$, we have

$$E_0(u, A) \leq E_0(v, A). \qquad (11)$$

It is equivalent to require that for all open set $B \subset\subset A$, for all $v \in \mathrm{SBV}(A)$ with $v = u$ in $A \setminus \overline{B}$, we have

$$E_0(u, \overline{B}) \leq E_0(v, \overline{B}). \qquad (12)$$

Therefore, we can directly assume that u has an extension in a neighborhood V of $\overline{A}$ and find a sufficient condition so that for all $v \in \mathrm{SBV}(V)$ with $v = u$ in $V \setminus \overline{A}$, we have

$$E_0(u, \overline{A}) \leq E_0(v, \overline{A}). \qquad (13)$$

Theorem 2.1. *Let A be a bounded open set of $\mathbf{R}^n$ with Lipschitz boundary. Let $u\colon A \to \mathbf{R}$ be an harmonic function for which there exists $0 \leq m < M$ such that $m \leq u \leq M$ and for which $0 < \sup_A |\nabla u| < \infty$. We assume that*

$$\int_m^\delta 2(M-t)\,\mathrm{d}t \leq \beta_0(m^2+\delta^2), \tag{14}$$

where

$$\beta_0 = \beta\frac{(M-m)}{\sup_A |\nabla u|} \qquad \text{and} \qquad \delta = \frac{M}{1+\beta_0}. \tag{15}$$

We finally assume that u has a C^1 extension in open set V containing $\overline{A}$. Then u is a minimizer of $E_0(v,\overline{A})$ among all $v \in \mathrm{SBV}(V)$ such that $v = u$ in $V \setminus \overline{A}$.

Remark 2.2. The function

$$s \mapsto \beta_0(m^2+s^2) - \int_m^s 2(M-t)\,\mathrm{d}t \tag{16}$$

attains its minimum at $s = \frac{M}{1+\beta_0}$ so (14) is equivalent to say that for all $s \in [0, M]$,

$$\int_m^s 2(M-t)\,\mathrm{d}t \leq \beta_0(m^2+s^2). \tag{17}$$

Remark 2.3. Let us consider the case where $n = 1$, $A =]0, h[$ (where $h > 0$) and u is an affine function whose graph joins $(0, m)$ and (h, M). Then the theorem condition is necessary. In this case $\beta_0 = \beta h$, $\delta = \frac{M}{1+\beta h}$ and the condition (14) amounts to

$$h^{-1}(M-m)^2 \leq \beta(m^2+\delta^2) + h^{-1}(M-\delta)^2. \tag{18}$$

One recognizes that the right-hand side is the Dirichlet energy of u. The left-hand side is the energy of the jump function whose graph joins $(0, m)$, $(0, \delta)$ and (h, M).

Remark 2.4. The condition (14) is trivially satisfied if $m \geq \delta$, that is,

$$m \geq \beta^{-1} \sup_A |\nabla u|. \tag{19}$$

Remark 2.5. For the homogeneous Mumford-Shah functional,

$$F_0(u) = \int_A |\nabla u|^2 + \beta\mathcal{H}^{n-1}(J_u), \tag{20}$$

Chambolle found the condition $(M-m)\sup_A |\nabla u| \leq \beta$ (see [8, Theorem 3.1 (i)]). This is somewhat analogous to our condition because this can be rewritten

$$\int_m^M 2(M-t)\,\mathrm{d}t \leq \beta_0 \tag{21}$$

where $\beta_0 = \beta\frac{(M-m)}{\sup_A |\nabla u|}$.

We are going to state the notion of calibrations associated to our problem. This notion is justified by [1, Section 2 and 3]. Let A be a bounded open set of $\mathbf{R}^n$ with Lipschitz boundary, let V be an open set containing $\overline{A}$ and let $u \in SBV(V)$ be such that $0 \leq u \leq M$ on A (for some $M > 0$). A calibration for u in $\overline{A} \times [0, M]$ is a Borel map

$$\phi = (\phi^x, \phi^t) \colon \overline{A} \times [0, M] \to \mathbf{R}^n \times \mathbf{R} \tag{22}$$

which is bounded and approximately-regular in $\overline{A} \times [0, M]$, divergence-free in $A \times]0, M[$ and such that

(a) $\phi^t(x,t) \geq \frac{1}{4}|\phi^x(x,t)|^2$ for $\mathcal{L}^n$-a.e. $x \in \overline{A}$ and every $t \in [0, M]$;

(b) $\left|\int_r^s \phi^x(x,t)\,\mathrm{d}t\right| \leq \beta(r^2+s^2)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \overline{A}$ and every $r, s \in [0, M]$;

(a') $\phi^x(x,u) = 2\nabla u$ and $\phi^t(x,u) = \frac{1}{4}|\nabla u|^2$ for $\mathcal{L}^n$-a.e. $x \in \overline{A}$;

(b') $\int_{u^-}^{u^+} \phi^x(x,t)\,\mathrm{d}t = \beta\left[(u^-)^2 + (u^+)^2\right]\nu_u$ for $\mathcal{H}^{n-1}$-a.e. $x \in J_u \cap \overline{A}$.

The existence of such a vector field ϕ implies that u is a minimizer of $E_0(v, \overline{A})$ among $v \in SBV(V)$ such that $v = u$ in $V \setminus \overline{A}$.

Remark 2.6. With regard to the function u of Theorem 2.1, we have $J_u = \emptyset$ so we don't need to check (b').

Remark 2.7. Our Dirichlet problem is different than in [1, Section 3] because our boundary condition is one-sided (the competitor v might jump on ∂A). This is why we define ϕ up to $\partial A \times [0, M]$. However, the calibration method also applies to our setting. It requires a minor modification of [1, Lemma 2.10] to deal with the case where ϕ is defined only on $\Gamma_u \cap (\overline{A} \times [0, M])$.

Remark 2.8 (Scaling). We are going to detail the scaling properties of E_0. We write $E_0(u, \beta, A)$ to explicit the parameters β in the definition of E_0. For all $M \in \mathbf{R}$, we have

$$E_0(Mu, \beta, A) = M^2 E_0(u, \beta, A), \tag{23}$$

and for all $h > 0$,

$$E_0(u_h, \beta h, h^{-1}A) = h^{2-n} E_0(u, \beta, A). \tag{24}$$

where $u_h(x) = u(xh)$. Thus, for all $h > 0$ and $M \geq 0$,

$$E_0(M^{-1}u_h, \beta h, h^{-1}A) = M^{-2}h^{2-n} E_0(u, \beta, A). \tag{25}$$

Remark 2.9 (Slope along a jump). This Remark is an example of application of [1, Lemma 2.5] that will fluidify the next constructions. We consider three C^1 functions $\sigma, u, v \colon \mathbf{R}^n \to \mathbf{R}$. We work in $\mathbf{R}^n \times \mathbf{R}$ and we introduce the hypersurface

$$H = \{\, (x,t) \in \mathbf{R}^n \times \mathbf{R} \mid t = \sigma(x) \,\} \tag{26}$$

and the vector field

$$\phi = \begin{cases} (2\nabla u, |\nabla u|^2) & \text{for } t > \sigma(x) \\ (2\nabla v, |\nabla v|^2) & \text{for } t < \sigma(x). \end{cases} \tag{27}$$

A normal vector field to H is $(-\nabla\sigma, 1)$ and we have along H,

$$\phi(x, \sigma(x)^+) \cdot \begin{pmatrix} -\sigma \\ 1 \end{pmatrix} = \phi(x, \sigma(x)^-) \cdot \begin{pmatrix} -\sigma \\ 1 \end{pmatrix}. \tag{28}$$

if and only if $\nabla\sigma = \frac{1}{2}(\nabla u + \nabla v)$. Therefore, ϕ is divergence-free in the sense of distributions provided that u, v are harmonic and $\sigma = \frac{1}{2}(u + v)$ (modulo an additive constant).

2.2 The one dimensional case

The ambient space is $\mathbf{R}$. We fix constants $a < b$ and $0 \leq m < M$. We minimize

$$E(u) = \int |\nabla u|^2 \, \mathrm{d}x + \beta \int_{J_u} (u^+)^2 + (u^-)^2 \, \mathrm{d}\mathcal{H}^{n-1} \tag{29}$$

over the function $u \in \mathrm{SBV}(\mathbf{R}^n)$ such that $u = m$ on $]-\infty, a[$, $u = M$ on $]b, \infty]$ and $0 \leq u \leq M$. By scaling, it suffices to study the case $[a, b] = [0, 1]$.

2.2.1 A short analysis

We compare affine and jump fonctions. Here is a summary (without proof) of what can be observed through computations. First, we try to find the jump function which has the minimal energy. It is never convenient to do more than one jump. The optimal value of the function at each side of the jump corresponds to the Robin condition $\partial_\nu u = \beta u$ (where ν is the inward unit normal vector). The optimal location of the jump may be $x_0 = 0$ or some $x_0 \in]0, 1[$. It cannot be $x_0 = 1$ though and this comes from the fact that $m < M$. If $m < \frac{M}{1+\beta}$, the best location is $x_0 = 0$ and the jump is going from m to $\frac{M}{1+\beta}$. If $m > \frac{M}{1+\beta}$, the best location is a certain $x_0 \in]0, 1[$ but then the affine function is necessarily a better competitor. We conclude that the affine function is a minimizer if and only if it is better than the jump at $x = 0$, that is

$$(M - m)^2 \leq \beta m^2 + \left(M - \frac{M}{1+\beta}\right)^2. \tag{30}$$

We define $\delta = \frac{M}{1+\beta}$ and we observe that (30) is equivalent to

$$\int_m^\delta 2(M - t) \, \mathrm{d}t \leq \beta(m^2 + \delta^2). \tag{31}$$

We try to guess what could be a calibration in the limit case

$$\int_m^\delta 2(M-t)\,\mathrm{d}t = \beta(m^2+\delta^2). \tag{32}$$

In this case, we have necessarily $m < \delta$. There are two minimizers; the first one is the affine function u which joins the points $(0,m)$, $(1,M)$ and the second one is the jump function v which joins the points $(0,m)$, $(0,\delta)$, $(1,M)$. A nice property of calibrations is that they calibrate all minimizers simultaneously. We should have $\phi = (2\nabla u, |\nabla u|^2)$ on the graph of u, that is $\phi = (2(M-m),(M-m)^2)$ on $t = m + (1-m)x$. And we should have as well $\phi = (2\nabla v, |\nabla v|^2)$ on the graph of v, that is $\phi = (2(M-\delta),(M-\delta)^2)$ on $t = m + (M-\delta)x$. Finally, we should have

$$\int_m^\delta \phi^x(0,t)\,\mathrm{d}t = m^2+\delta^2. \tag{33}$$

A simple solution is to set

$$\phi = \left(\frac{2(M-t)}{1-x}, \left[\frac{M-t}{1-x}\right]^2\right) \tag{34}$$

for $t \geq m + (1-m)x$. Next, we try to determine $\phi^x(0,t)$ for $t \in [0,m]$. We have necessarily

$$\begin{aligned}\int_0^m \phi^x(0,t)\,\mathrm{d}t &= \int_0^\delta \phi^x(0,t)\,\mathrm{d}t - \int_m^\delta \phi^x(0,t)\,\mathrm{d}t &(35)\\ &\leq \beta\delta^2 - \beta(m^2+\delta^2) &(36)\\ &\leq -\beta m^2. &(37)\end{aligned}$$

We suggest to set $\phi^x(0,t) = -2\beta t$ on $[0,m]$. Then we try to extend ϕ in a simple way while respecting the axioms of calibrations and we arrive at

$$\phi = \begin{cases} \left(\frac{2(M-t)}{1-x}, \left[\frac{M-t}{1-x}\right]^2\right) & \text{if } m+\tau x \leq t \leq M \\ \left(2(M-m), [M-m]^2\right) & \text{if } m+\sigma x \leq t \leq m+\tau x \\ \left(-2\beta m, \beta^2 m^2\right) & \text{if } m \leq t \leq m+\sigma x \\ \left(-2\beta t, \beta^2 m^2\right) & \text{if } 0 \leq t \leq m. \end{cases} \tag{38}$$

where $\tau = M-m$ and $\sigma = \frac{1}{2}((M-m)-\beta m)$. The slope σ has been chosen as in Remark 2.9. In the next section, we generalize this construction when there is only an inequality in (32).

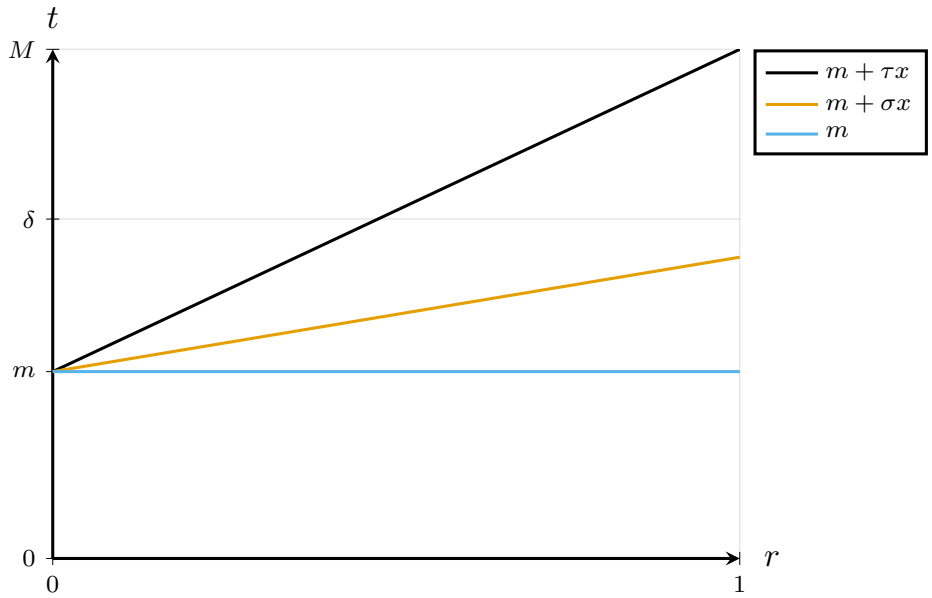

Figure 1: $\beta = \frac{1}{2}$

2.2.2 The construction

We recall that $\delta = \frac{M}{1+\beta}$ and we assume

$$\int_m^\delta 2(M-t)\,\mathrm{d}t \leq \beta(m^2+\delta^2). \tag{39}$$

We consider a real number $0 \leq \lambda \leq \beta m$ such that

$$\int_m^\delta 2(M-t)\,\mathrm{d}t \leq \lambda m + \beta\delta^2. \tag{40}$$

We will also need the conditions $\lambda \leq M-m$ and $\lambda \leq \frac{\beta M}{1+\beta}$. If $m \leq \delta$, these two last conditions are automatically satisfied. Otherwise, it suffices to take $\lambda = 0$ for example. Finally, we define

$$\phi = \begin{cases} \left(\frac{2(M-t)}{1-x}, \left[\frac{M-t}{1-x}\right]^2\right) & \text{si } m+\tau x \leq t \leq M \\ \left(2(M-m), [M-m]^2\right) & \text{si } m+\sigma x \leq t \leq m+\tau x \\ \left(-2\lambda, \lambda^2\right) & \text{si } m \leq t \leq m+\sigma x \\ \left(-2\lambda m^{-1} t, \lambda^2\right) & \text{si } 0 \leq t \leq m. \end{cases} \tag{41}$$

where $\tau = M-m$ and $\sigma = \frac{1}{2}((M-m)-\lambda)$. Note that we have $0 \leq \sigma \leq \tau$ because $0 \leq \lambda \leq M-m$. The slope σ has been chosen as in Remark 2.9.

We prove that for all x and all $r \leq s$,

$$\left| \int_r^s \phi^x(x,t)\,\mathrm{d}t \right| \leq \beta(r^2+s^2). \tag{42}$$

We have globally

$$\phi^x \geq -2\lambda\tfrac{t}{m} \geq -2\beta t \tag{43}$$

so it suffices to control $\int_r^s \phi^x\,\mathrm{d}t$ from above. Let us fix t. We are going to see that the function

$$r \mapsto \beta(r^2+s^2) - \int_r^s \phi^x(x,t)\,\mathrm{d}t \tag{44}$$

is non-decreasing on $[0,M]$. If $0 \leq r \leq m$,

$$\beta(r^2+s^2) - \int_r^s \phi^x(x,t)\,\mathrm{d}t = \beta r^2 - \tfrac{\lambda}{m} r^2 + (...) \tag{45}$$

where $(...)$ does not depend on r. The right-hand side is non-decreasing on $[0,m]$ because $\lambda \leq \beta m$. If $m \leq r \leq m + \sigma x$,

$$\beta(r^2+s^2) - \int_r^s \phi^x(x,t)\,\mathrm{d}t = \beta r^2 - 2\lambda r + (...) \tag{46}$$

where $(...)$ does not depend on r. The right-hand side is non-decreasing on $[m, m+\sigma x]$ becaus $\lambda \leq \beta m$. Finally, the function in non-decreasing on $[m+\sigma x, M]$ because ϕ^x is non-negative on this interval. We conclude that it suffice to show that for all x and t,

$$\int_0^s \phi^x(x,t)\,\mathrm{d}t \leq \beta s^2. \tag{47}$$

This is trivial if $s \leq m$ so we assume $s > m$. There exists x_0 such that $s = m + (M-m)x_0$. The function

$$x \mapsto \int_r^s \phi^x(x,t)\,\mathrm{d}t \tag{48}$$

is non-decreasing on $[x_0, 1]$ because the part where $\phi^x \leq 0$ gets bigger at the expense of the part where $\phi^x \geq 0$. Now, it suffices to show that for all x and $m + (M-m)x \leq s \leq M$,

$$\int_0^s \phi^x(x,t)\,\mathrm{d}t \leq \beta s^2. \tag{49}$$

We apply the divergence theorem in the polygone delimited by the cycle $(x,0),(x,s),(1,M)(0,1),(0,0)$. We get

$$\int_0^s \phi^x(x,t)\,\mathrm{d}t = -\frac{(M-s)^2}{1-x} + (M-m)^2 - \lambda m + \lambda^2 x \tag{50}$$

so

$$\beta s^2 - \int_0^s \phi^x(x,t)\,\mathrm{d}t = \beta s^2 + \frac{(M-s)^2}{1-x} - \lambda^2 x + \lambda m - (M-m)^2. \tag{51}$$

The function $s \mapsto \beta s^2 + \frac{(M-s)^2}{1-x}$ is minimal for $s = \frac{M}{1+\beta(1-x)}$ and its the minimum value is $\frac{\beta M^2}{1+\beta(1-x)}$. We have therefore

$$\beta s^2 - \int_0^s \phi^x(x,t)\,\mathrm{d}t \geq \frac{\beta M^2}{1+\beta(1-x)} - \lambda^2 x + \lambda m - (M-m)^2. \tag{52}$$

The function $x \mapsto \frac{\beta M}{1+\beta(1-x)} - \lambda^2 x$ is non-decreasing on $[0,1]$ because $\lambda \leq \frac{\beta M}{1+\beta}$. We are thus led to

$$\beta s^2 - \int_0^s \phi^x(x,t)\,\mathrm{d}t \geq \frac{\beta M^2}{1+\beta} + \lambda m - (M-m)^2 \tag{53}$$

As $\frac{\beta M^2}{1+\beta} = \beta\delta^2 + (M-\delta)^2$, the previous quantity can be rewritten

$$\lambda m + \beta\delta^2 - \int_m^\delta 2(M-t)\,\mathrm{d}t \tag{54}$$

and this is non-negative by assumption.

2.3 Higher dimension

The ambient space is $\mathbf{R}^n$. We recall that A is a bounded open set of $\mathbf{R}^n$ with Lipschitz boundary, we recall that

$$\beta_0 = \beta\frac{(M-m)}{\sup_A |\nabla u|} \qquad \text{and} \qquad \delta = \frac{M}{1+\beta_0} \tag{55}$$

and we assume

$$\int_m^\delta 2(M-t)\,\mathrm{d}t \leq \beta_0(m^2+\delta^2). \tag{56}$$

We proceed to build the calibration. We consider a real number $0 \leq \lambda \leq \beta_0 m$ such that

$$\int_m^\delta 2(M-t)\,\mathrm{d}t \leq \lambda m + \beta_0\delta^2. \tag{57}$$

We will also need the conditions $\lambda \leq M-m$ and $\lambda \leq \frac{\beta_0 M}{1+\beta_0}$. If $m \leq \delta$, this two last conditions are automatically satisfied. Otherwise, it suffices to take $\lambda = 0$ for example. Finally, we define

$$\phi = \left(\varphi^x \cdot \frac{\nabla u(x)}{M-m}, \varphi^t \cdot \left|\frac{\nabla u(x)}{M-m}\right|^2\right) \tag{58}$$

where

$$\varphi = \begin{cases} \left(2(M-m)\frac{M-t}{M-u}, \left[(M-m)^2\frac{M-t}{M-u}\right]^2\right) & \text{if } u(x) \le t \le M \\ \left(2(M-m), [M-m]^2\right) & \text{if } \sigma(x) \le t \le u(x) \\ \left(-2\lambda, \lambda^2\right) & \text{if } m \le t \le \sigma(x) \\ \left(-2\lambda\frac{t}{m}, \lambda^2\right) & \text{if } 0 \le t \le m. \end{cases} \tag{59}$$

and

$$\sigma(x) = m + \frac{1}{2}(u-m)(1 - \tfrac{\lambda}{M-m}) \tag{60}$$

$$= m + \frac{1}{2}\frac{u-m}{M-m}(M-m-\lambda). \tag{61}$$

As $0 \le \lambda \le M-m$, we see that $m \le \sigma(x) \le u(x)$. We prove that for all x and $r \le s$,

$$\left|\int_r^s \varphi^x(x,t)\,\mathrm{d}t\right| \le \beta_0(r^2+s^2). \tag{62}$$

This is a minor variant of the one-dimensional case. We have globally

$$\varphi^x \ge -2\lambda\tfrac{t}{m} \ge -2\beta_0 t \tag{63}$$

so it suffices to control $\int_r^s \varphi^x\,\mathrm{d}t$ from above. Let us fix t. We are going to see that the function

$$r \mapsto \beta_0(r^2+s^2) - \int_r^s \varphi^x(x,t)\,\mathrm{d}t \tag{64}$$

is non-decreasing on $[0,M]$. If $0 \le r \le m$,

$$\beta_0(r^2+s^2) - \int_r^s \varphi^x(x,t)\,\mathrm{d}t = \beta_0 r^2 - m^{-1}\lambda r^2 + (...) \tag{65}$$

where $(...)$ does not depend on r. The right-hand side is non-decreasing on $[0,m]$ because $\lambda \le \beta_0 m$. If $m \le r \le \sigma(x)$,

$$\beta_0(r^2+s^2) - \int_r^s \varphi^x(x,t)\,\mathrm{d}t = \beta_0 r^2 - 2\lambda r + (...) \tag{66}$$

where $(...)$ does not depend on r. The right-hand side is non-decreasing on $[m,\sigma(x)]$ because $\lambda \le \beta_0 m$. Finally, the function in non-decreasing on $[u(x),M]$ because φ^x is non-negative on this interval. We conclude that it suffice to show that for all x and s,

$$\int_0^s \varphi^x(x,t)\,\mathrm{d}t \le \beta_0 s^2. \tag{67}$$

This is trivial if $s \leq m$ so we assume $s > m$. There exists x_0 such that $s = u(x_0)$. Since $\sigma \leq u$, we have in particular $\sigma(x_0) \leq s$. For x such that $\sigma(x) \leq s \leq u(x)$, we compute

$$\int_0^s \varphi^x(x,t)\,\mathrm{d}t = -2\lambda m - 2\lambda(\sigma - m) + 2(M-m)(s-\sigma) \tag{68}$$

$$= -2\lambda m - 2\lambda(\sigma - m) + 2(M-m)(s-m) + 2(M-m)(m-\sigma) \tag{69}$$

$$= -2\lambda m - 2(\sigma - m)(M - m + \lambda) + 2(M-m)(s-m). \tag{70}$$

As $u(x_0) \leq u(x)$ and $0 \leq \lambda \leq M - m$, we have $\sigma(x_0) \leq \sigma(x)$ and it follows that

$$\int_0^s \varphi^x(x,u) \leq \int_0^s \varphi^x(x_0,u)\,\mathrm{d}. \tag{71}$$

Now, it suffices to show that for all x and $u(x) \leq s \leq m$,

$$\int_0^s \varphi^x(x,t)\,\mathrm{d}t \leq \beta_0 s^2. \tag{72}$$

We compute

$$\int_0^s \varphi^x(x,t)\,\mathrm{d}t = -\lambda m - 2\lambda(\sigma - m) + 2(M-m)(u-\sigma) + \frac{M-m}{M-u}\left((M-u)^2 - (M-s)^2\right). \tag{73}$$

By definition

$$\sigma - m = \tfrac{1}{2}\frac{u-m}{M-m}(M - m - \lambda) \tag{74}$$

$$u - \sigma = \tfrac{1}{2}\frac{u-m}{M-m}(M - m + \lambda) \tag{75}$$

so

$$-2\lambda(\sigma - m) + 2(M-m)(u-\sigma) = \frac{u-m}{M-m}((M-m)^2 + \lambda^2). \tag{76}$$

We also write

$$(M-m)(M-u) = (M-m)^2 + (M-m)(m-u) \tag{77}$$

$$= (M-m)^2 - (M-m)^2\frac{u-m}{M-m} \tag{78}$$

and we arrive at

$$\int_0^s \varphi^x(x,t)\,\mathrm{d}t = -\lambda m + \lambda^2\frac{u-m}{M-m} + (M-m)^2 - \frac{M-m}{M-u}(M-s)^2. \tag{79}$$

Therefore

$$\beta_0 s^2 - \int_0^s \varphi^x(x,t)\,\mathrm{d}t = \beta_0 s^2 + \frac{M-m}{M-u}(M-t)^2 - \lambda^2\frac{u-m}{M-m} + \lambda m - (M-m)^2. \quad (80)$$

For a fixed x, the function $s \mapsto \beta_0 s^2 + \frac{M-m}{M-u}(M-s)^2$ is minimal for

$$s = \frac{M}{1+\beta_0\frac{M-u}{M-m}} \quad (81)$$

and its minimum value is

$$\frac{\beta_0 M^2}{1+\beta_0\frac{M-u}{M-m}}. \quad (82)$$

We have therefore

$$\beta_0 s^2 - \int_0^s \varphi^x(x,t)\,\mathrm{d}t \geq \frac{\beta_0 M^2}{1+\beta_0\frac{M-u}{M-m}} - \lambda^2\frac{u-m}{M-m} + 2\lambda m - (M-m)^2. \quad (83)$$

The function

$$x \mapsto \frac{\beta_0 M^2}{1+\beta_0\frac{M-x}{M-m}} - \lambda^2\frac{x-m}{M-m} \quad (84)$$

is non decreasing on $[m, M]$ because $\lambda \leq \frac{\beta_0 M}{1+\beta_0}$. We are thus led to

$$\beta_0 s^2 - \int_0^s \varphi^x(x,t)\,\mathrm{d}t \geq \frac{\beta_0 M^2}{1+\beta_0} + \lambda m - (M-m)^2 \quad (85)$$

As $\frac{\beta_0 M^2}{1+\beta_0} = \beta_0\delta^2 + (M-\delta)^2$, the right-hand side can be rewritten

$$\lambda m + \beta_0\delta^2 - \int_m^\delta 2(M-t)\,\mathrm{d}t. \quad (86)$$

3 The Thermal Insulation Problem

3.1 Definition of the problem and calibrations

The ambient space is $\mathbf{R}^n$. We fix a bounded open set $\Omega \subset \mathbf{R}^n$ with Lipschitz boundary. It plays the role of an initial data. We fix parameters $\beta > 0$ and $\gamma > 0$. We minimize

$$E(u) = \int |\nabla u|^2\,\mathrm{d}x + \beta\int_{J_u} (u^+)^2 + (u^-)^2\,\mathrm{d}\mathcal{H}^{n-1} + \gamma^2\mathcal{L}^n(\{\,u > 0\,\}) \quad (87)$$

over $u \in \mathrm{SBV}(\mathbf{R}^n)$ such that $u = 1$ on Ω. In particular, we are interested in the case where Ω is a ball B_1 of center 0 and radius 1. We can assume without loss of generality that competitors satisfy $0 \leq u \leq 1$. It is shown in [3] that minimizers have necessarily a compact support. Therefore, we can also assume without loss of generality that the competitors have a compact support.

Now we state the notion of calibration associated to this problem. Let $u \in \mathrm{SBV}(\mathbf{R}^n)$ be such that $u = 1$ on Ω, $0 \leq u \leq 1$ and u has a compact support. A calibration for u is a Borel map

$$\phi = (\phi^x, \phi^t) \colon (\mathbf{R}^n \setminus \Omega) \times \mathbf{R} \to \mathbf{R}^n \times \mathbf{R} \tag{88}$$

which is bounded and approximately-regular in $(\mathbf{R}^n \setminus \Omega) \times [0,1]$, divergence-free in $(\mathbf{R}^n \setminus \overline{\Omega}) \times]0,1[$ and such that

(a) $\phi^t(x,t) \geq \frac{1}{4}|\phi^x(x,t)|^2 - \gamma^2 \mathbf{1}_{]0,1]}(t)$ for $\mathcal{L}^n$-a.e. $x \in \mathbf{R}^n \setminus \overline{\Omega}$ and every $t \in [0,1]$;

(b) $\left|\int_r^s \phi^x(x,t)\,\mathrm{d}t\right| \leq \beta(r^2 + s^2)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbf{R}^n \setminus \Omega$ and every $r < s$;

(a') $\phi^x(x,u) = 2\nabla u$ and $\phi^t(x,u) = |\nabla u|^2 - \gamma^2 \mathbf{1}_{]0,1]}(u)$ for $\mathcal{L}^n$-a.e. $x \in \mathbf{R}^n \setminus \overline{\Omega}$;

(b') $\int_{u^-}^{u^+} \phi^x(x,t)\,\mathrm{d}t = \beta\left[(u^-)^2 + (u^+)^2\right] \nu_u$ for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$.

We say that u is calibrated if such a vector field exists. This implies that u is a minimizer of $E(v)$ among $v \in SBV(\mathbf{R}^n)$ such that $v = 1$ on Ω.

3.1.1 Informal computations with $\Omega = B(0,1)$

We present informal computations in the case $\Omega = B_1$ (the open unit ball centred at the origin). We introduce for $r > 0$,

$$\Gamma(r) = \begin{cases} r - 1 & \text{si } n = 1 \\ \ln(r) & \text{si } n = 2 \\ \frac{1}{n-2}(1 - r^{2-n}) & \text{si } n \geq 3. \end{cases} \tag{89}$$

Given $x \in \mathbf{R}^n$, we write r for $|x|$. For example, $x \mapsto \Gamma(r)$ is an harmonic positive function on $\mathbf{R}^n \setminus \overline{B_1}$ which is 0 on ∂B_1.

We assume without proof that the only relevant competitors u are of the following form. Either u is the indicator function of B_1. Either there exists $R > 1$ and $0 < \delta < 1$ such that u is radial continuous in $\overline{B}_R$, $u = 1$ on $\overline{B_1}$, $u = \delta$ on ∂B_R and $u = 0$ on $\mathbf{R}^n \setminus \overline{B}_R$. For a fixed $R > 1$, the first Euler-Lagrange equation says that u should be harmonic in $B_R \setminus \overline{B_1}$ and that $\delta = \delta(R)$ should be determined by the Robin boundary condition

$$-\partial_r u = \beta u \text{ on } \partial B_R. \tag{90}$$

We find

$$\delta(R) = \frac{1}{1 + \beta R^{n-1}\Gamma(R)} \tag{91}$$

and

$$u(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 1 - \beta\delta(R)R^{n-1}\Gamma(x) & \text{for } 1 \leq |x| \leq R \\ 0 & \text{for } |x| > R. \end{cases} \tag{92}$$

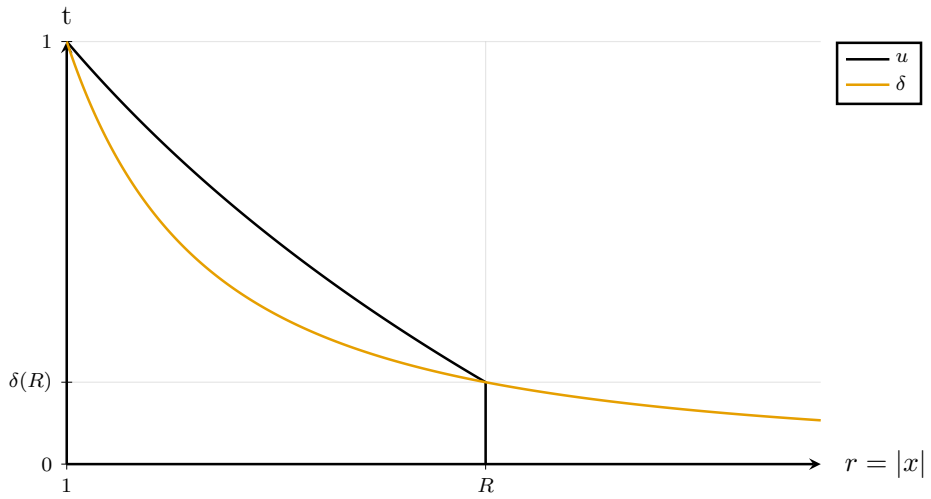

Figure 2: $n = 2$, $\beta = 3$ and $R = 2$

An integration by parts combined with the Dirichlet condition $u = 1$ on ∂B_1 and the Robin condition $-\partial_r u = \beta u$ on ∂B_R shows that

$$\int_{B_R \setminus B_1} |\nabla u|^2 \, \mathrm{d}x + \beta \int_{\partial B_R} u^2 \, \mathrm{d}\mathcal{H}^{n-1} = \beta \int_{\partial B_R} u \, \mathrm{d}\mathcal{H}^{n-1}. \tag{93}$$

We conclude that the energy of u is

$$E = n\omega_n \beta R^{n-1}\delta(R) + \omega_n \gamma^2 R^n. \tag{94}$$

Now, we consider E as functions of $R \in [1, +\infty[$ that we try to optimize. We observe that $n\omega_n R^{n-1}\beta\delta(R)$ is the flux of the vector field $x \mapsto \beta\delta(r)e_r$ through ∂B_R. We compute

$$\operatorname{div}(\delta(r)e_r) = \delta'(r) + \frac{(n-1)}{r}\delta(r) \tag{95}$$

$$= -\left(\beta - \frac{(n-1)}{r}\right)\delta(r)^2 \tag{96}$$

so

$$E(R) = E(1) + \int_{B_R \setminus B_1} \left[\gamma^2 - \left(\beta^2 - \frac{(n-1)\beta}{r} \right) \delta(r)^2 \right] \mathrm{d}x, \tag{97}$$

where r means $|x|$. This shows in particular that

$$E'(R) = n\omega_n R^{n-1} \left[\gamma^2 - \left(\beta^2 - \frac{(n-1)\beta}{R} \right) \delta(R)^2 \right]. \tag{98}$$

The critical radii $R > 1$ are characterised by the equations

$$\left(\beta^2 - \frac{\beta(n-1)}{R} \right) \delta(R)^2 = \gamma^2. \tag{99}$$

Remark 3.1. As expected, this coincides with the second Euler-Lagrange equation

$$|\nabla u|^2 + \gamma^2 + \beta u^2 (H - 2\beta) = 0 \tag{100}$$

where $H = (n-1)R^{-1}$ is the mean scalar curvature of ∂B_R with respect to $-e_r$ (see [11, Definition 7.32], not to be confused with the arithmetic mean of the principal curvatures which is equal to R^{-1}). A proof of this formula for $C^{1,\alpha}$ surfaces is presented in [4, Theorem 15.1].

The condition $E'(R) = 0$ does not suffice to characterize minimizers. Depending on the parameters β, γ, the function E may not be convex (we will see in section 3.1.3 that it is convex at least when $n = 1$ or $\beta \geq n$). It can be for example increasing-decreasing-increasing. In this case, it attains a local minimum at some $R > 1$ which might be greater, equal or less than $E(1)$. In the equality case, there are two minimizers.

3.1.2 Three sufficient conditions of minimality

An obvious condition for $\mathbf{1}_{B(0,1)}$ to be a minimizer is that for all $r \geq 1$,

$$\left(\beta^2 - \frac{(n-1)\beta}{r} \right) \delta(r)^2 \leq \gamma^2. \tag{101}$$

Indeed, the function $r \mapsto E(r)$ is then non-decreasing on $[1, +\infty[$. Note that it suffices that $\beta \leq \gamma$. We generalize this observation to other initial data Ω in the two next theorems.

Theorem 3.2. *Let Ω be a C^1 bounded open set of $\mathbf{R}^n$ and assume that its outward unit normal vector field has a continuous extension on $\mathbf{R}^n \setminus \Omega$ such that $|\nu| \leq 1$ and which is divergence-free on $\mathbf{R}^n \setminus \overline{\Omega}$ in the sense of distributions. If $\beta \leq \gamma$, then $\mathbf{1}_\Omega$ is calibrated.*

According [9, Proposition 15], the assumption holds true if Ω is a C^2 bounded open convex of $\mathbf{R}^2$.

Before stating the second theorem, we want to generalize the previous function Γ to other initial data Ω. The proof of the next Lemma is postponed in Appendix because this is not a new result.

Lemma 3.3. *Let Ω be a non-empty C^2 star-shaped bounded open set of $\mathbf{R}^n$. There exists a continuous function $\Gamma\colon \mathbf{R}^n \setminus \Omega \to \mathbf{R}$ such that*

(i) Γ *is harmonic positive on* $\mathbf{R}^n \setminus \overline{\Omega}$*;*

(ii) $\Gamma = 0$ *on* $\partial\Omega$*,*

(iii) For all $x \in \mathbf{R}^n \setminus \overline{\Omega}$*,* $\nabla\Gamma(x) \neq 0$

(iv) Γ *has a* C^1 *extension up to the boundary and for all* $x \in \partial\Omega$*, there exists* $t > 0$ *such that* $\nabla\Gamma(x) = t\nu(x)$*, where* $\nu(x)$ *is the outward unit normal vector of* Ω *at* x*.*

Theorem 3.4. *Let Ω be a non-empty C^2 star-shaped bounded open set of $\mathbf{R}^n$ and let Γ be a function as in Lemma 3.3 (modulo a positive multiplicative constant). We define for $x \in \mathbf{R}^n \setminus \overline{\Omega}$,*

$$\nu(x) = \frac{\nabla\Gamma}{|\nabla\Gamma|} \qquad \text{and} \qquad \delta(x) = \frac{1}{1 + \beta|\nabla\Gamma|^{-1}\Gamma} \tag{102}$$

and we assume that $\mathrm{div}(\nu)$ *is bounded. If for all* $x \in \mathbf{R}^n \setminus \overline{\Omega}$*,*

$$\left(\beta^2 - \beta\mathrm{div}(\nu)\right)\delta^2 \leq \gamma^2, \tag{103}$$

then $\mathbf{1}_\Omega$ *is calibrated.*

Finally, we come back to $\Omega = B(0,1)$ and we give a sufficient condition so that an harmonic function in a bigger ball. We refer to (89) and (91) for the definition of Γ and δ.

Theorem 3.5. *Let Ω be the open ball $B(0,1)$. We assume that $\beta \geq n-1$ and that there exists $R > 1$ such that for all $r \geq R$,*

$$\left(\beta^2 - \frac{\beta(n-1)}{r}\right)\delta(r)^2 \leq \gamma^2, \tag{104}$$

with equality at $r = R$. Then the function

$$u(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 1 - \beta\delta(R)R^{n-1}\Gamma(x) & \text{for } 1 \leq |x| \leq R \\ 0 & \text{for } |x| > R, \end{cases} \tag{105}$$

is calibrated.

3.1.3 Miscellaneous remarks

We add here miscellaneous remarks that play no parts in the rest of the article.

Remark 3.6. We assume $\beta \geq n-1$ and we show that $r \mapsto r^{n-1}\delta(r)$ is decreasing on $[1,+\infty[$. We observe that

$$\operatorname{div}(\delta(r)e_r) = \frac{(r^{n-1}\delta)'}{r^{n-1}} \tag{106}$$

so by (96),

$$(r^{n-1}\delta)' = -\left(\beta^2 - \frac{\beta(n-1)}{r}\right) r^{n-1}\delta(r)^2 \tag{107}$$

and this is negative when $r > 1$.

Remark 3.7. We assume $n = 1$ or $\beta \geq n$ and we show that E is convex on $[1,+\infty[$. It suffices to show that $r \mapsto r^{n-1}\delta(r)$ is convex on $[1,+\infty[$. Using the previous remark, we write first

$$(r^{n-1}\delta)' = -\left(\beta^2 - \frac{\beta(n-1)}{r}\right) \frac{(r^{n-1}\delta)^2}{r^{n-1}}. \tag{108}$$

We compute

$$\left[\left(\beta^2 - \frac{\beta(n-1)}{r}\right) \frac{1}{r^{n-1}}\right]' = -\frac{(n-1)}{r^n}\left(\beta^2 - \frac{\beta n}{r}\right) \tag{109}$$

so $r \mapsto \left(\beta^2 - \frac{\beta(n-1)}{r}\right)\frac{1}{r^{n-1}}$ is decreasing on $[1,+\infty[$. As $r \mapsto r^{n-1}\delta$ is also decreasing, we deduce that $r \mapsto (r^{n-1}\delta)'$ is increasing.

3.2 Proof of Theorem 3.2

Let Ω be a C^1 bounded open set of $\mathbf{R}^n$ (we will add more assumptions as we advance in the construction). We want to build a calibration for $\mathbf{1}_\Omega$. First, we search for a continuous function $\phi^x \colon (\mathbf{R}^n \setminus \Omega) \times [0,1] \to \mathbf{R}^n$ such that

(i) for all $x \in \mathbf{R}^n \setminus \overline{\Omega}$, $\phi^x(x,0) = 0$;

(ii) for all $x \in \partial\Omega$,

$$\int_0^1 \phi^x(x,t)\,\mathrm{d}t = -\beta\nu(x), \tag{110}$$

where ν is the outward unit normal vector field of Ω;

(iii) for all $x \in \mathbf{R}^n \setminus \Omega$ and for all $0 \leq r \leq s \leq 1$,

$$\left|\int_r^s \phi^x(x,u)\,\mathrm{d}u\right| \leq \beta(r^2+s^2). \tag{111}$$

A simple starting point is to define for $x \in \partial\Omega$ and $0 \leq t \leq 1$, $\phi^x = -2\beta t\nu(x)$. Let us assume that ν has a continuous extension on $\mathbf{R}^n \setminus \Omega$ such that $|\nu| \leq 1$. Then we can extend ϕ^x on $(\mathbf{R}^n \setminus \Omega) \times [0,1]$ with the formula

$$\phi^x(x,t) = -2\beta t\nu(x). \tag{112}$$

It is clear that that for all $x \in \mathbf{R}^n \setminus \Omega$ and for all $r < s$, we have $\left|\int_r^s \phi^x\right| \leq \beta(r^2 + s^2)$ because $|\phi^x(x,t)| \leq 2\beta t$. Let us assume furthermore that ν is divergence-free on $\mathbf{R}^n \setminus \overline{\Omega}$ in the sense of distributions. The function ϕ^t is derived by the conditions $\mathrm{div}(\phi) = 0$ and $\phi^t(x,0) = 0$ for $x \in \mathbf{R}^n \setminus \overline{\Omega}$. This yields simply

$$\phi^t(x,t) = 0. \tag{113}$$

Finally, the condition $\phi^t \geq \frac{1}{4}|\phi^x|^2 - \gamma^2 \mathbf{1}_{]0,1]}(t)$ amounts to

$$\beta|\nu| \leq \gamma \tag{114}$$

and this requires $\beta \leq \gamma$.

3.3 Proof of Theorem 3.4

Let Ω be a C^1 bounded open set of $\mathbf{R}^n$ (we will add more assumptions as we advance in the construction). We are more careful than in the previous section and we define ϕ in two pieces. However, we still arrange ϕ so that it is globally continuous.

The principle is the same as before. We denote by ν the outward unit normal vector field of Ω. The starting point is to define for $x \in \partial\Omega$ and $0 \leq t \leq 1$, $\phi^x(x,t) = -2\beta t\nu(x)$. Then we extend ϕ^x and we derive ϕ^t by the conditions $\partial_t \phi^t = -\mathrm{div}_x(\phi^x)$ and $\phi^t(x,0) = 0$ for $x \in \mathbf{R}^n \setminus \overline{\Omega}$.

We assume that ν has a continuous extension on $\mathbf{R}^n \setminus \Omega$ such that $|\nu| \leq 1$ and ν is C^1 on $\mathbf{R}^n \setminus \overline{\Omega}$. We also consider a continuous function $\delta \colon \mathbf{R}^n \setminus \Omega \to [0,1]$ such that $\delta = 1$ on $\partial\Omega$, $0 < \delta < 1$ on $\mathbf{R}^n \setminus \overline{\Omega}$ and δ is C^1 on $\mathbf{R}^n \setminus \overline{\Omega}$. We define for $x \in \mathbf{R}^n \setminus \overline{\Omega}$ and $0 \leq t \leq \delta(x)$,

$$\phi^x = -2\beta t\nu \tag{115}$$

$$\phi^t = \beta t^2 \mathrm{div}(\nu) \tag{116}$$

and for $\delta(x) < t \leq 1$,

$$\phi^x = -2(1-t)\frac{\beta\delta\nu}{1-\delta} \tag{117}$$

$$\phi^t = -(1-t)^2 \mathrm{div}\left(\frac{\beta\delta\nu}{1-\delta}\right) - C(x) \tag{118}$$

where $C(x)$ will be chosen so that ϕ is continuous on $(\mathbf{R}^n \setminus \overline{\Omega}) \times [0,1]$. Note that this is already the case for ϕ^x. We find

$$C(x) = -(1-\delta)^2 \mathrm{div}\left(\frac{\beta\delta\nu}{1-\delta}\right) - \beta\delta^2 \mathrm{div}(\nu). \tag{119}$$

However we compute

$$\operatorname{div}\left(\frac{\delta\nu}{1-\delta}\right) = \frac{\operatorname{div}(\delta\nu) - \delta^2\operatorname{div}(\nu)}{(1-\delta)^2} \tag{120}$$

so this simplifies to $C(x) = -\beta\operatorname{div}(\delta\nu)$. With regard to approximate regularity, the definition of ϕ^t on $\partial\Omega \times [0,1]$ does not matter. Indeed, let M be an hypersurface of $\mathbf{R}^n \times \mathbf{R}$. Then for $\mathcal{H}^n$-a.e. $(x_0, t_0) \in M \cap (\partial\Omega \times [0,1])$, the vector $n_0 = (\nu(x_0), 0)$ is a normal vector to M at x and $\phi \cdot n_0 = \phi^x$. In order for ϕ to be bounded, it suffices that $\operatorname{div}(\nu)$ and $\operatorname{div}(\delta\nu)$ are bounded. Finally, the condition $\phi^t \geq \frac{1}{4}|\phi^x|^2 - \gamma^2 \mathbf{1}_{]0,1]}(t)$ amounts to

$$\left(\beta^2|\nu|^2 - \beta\operatorname{div}(\nu)\right)\delta^2 \leq \gamma^2 \tag{121a}$$

$$-\beta\operatorname{div}(\delta\nu) \leq \gamma^2. \tag{121b}$$

It is tempting to choose δ and ν in such a way that

$$\left(\beta^2|\nu|^2 - \beta\operatorname{div}(\nu)\right)\delta^2 = -\beta\operatorname{div}(\delta\nu). \tag{122}$$

In that case, $\operatorname{div}(\delta\nu)$ is bounded provided that $\operatorname{div}(\nu)$ is bounded. According to (120), the equality (122) is equivalent to

$$\left(\frac{\beta|\delta\nu|}{1-\delta}\right)^2 = -\operatorname{div}\left(\frac{\beta\delta\nu}{1-\delta}\right) \tag{123}$$

A natural solution is to assume that Ω is C^2 star-shaped, to consider the function Γ of Lemma 3.3 and to choose δ, ν in such a way that

$$\frac{\beta\delta\nu}{1-\delta} = \frac{\nabla\Gamma}{\Gamma}. \tag{124}$$

We suggest to define

$$\nu(x) = \frac{\nabla\Gamma}{|\nabla\Gamma|} \tag{125}$$

and

$$\delta(x) = \frac{1}{1 + \beta|\nabla\Gamma|^{-1}\Gamma}. \tag{126}$$

In conclusion, the conditions (121) simplify to

$$\left(\beta^2 - \beta\operatorname{div}(\nu)\right)\delta^2 \leq \gamma^2. \tag{127}$$

3.4 Proof of Theorem 3.5

Let Ω be the open ball $B(0,1)$. We assume that $\beta \geq n-1$ and that there exists $R > 1$ such that for all $r \geq R$,

$$\left(\beta^2 - \frac{\beta(n-1)}{r}\right)\delta(r)^2 \leq \gamma^2, \tag{128}$$

with equality at $r = R$. We recall some notations. Given $x \in \mathbf{R}^n$, we write r for $|x|$. We define for $r \geq 1$,

$$\Gamma(r) = \begin{cases} r - 1 & \text{si } n = 1 \\ \ln(r) & \text{si } n = 2 \\ \frac{1}{n-2}(1 - r^{2-n}) & \text{si } n \geq 3. \end{cases} \tag{129}$$

and

$$\delta(r) = \frac{1}{1 + \beta r^{n-1}\Gamma(r)}. \tag{130}$$

We define for $x \in \mathbf{R}^n$,

$$u(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 1 - \beta\delta(R)R^{n-1}\Gamma(r) & \text{for } 1 \leq |x| \leq R \\ 0 & \text{for } |x| > R. \end{cases} \tag{131}$$

For $r \geq 0$, we write $u(r)$ for the value of u on ∂B_r. Thus, we consider u as a function of the real variable $r \in [0, +\infty[$. Now, we list a few useful formulas. For $1 < |x| < R$, we have

$$\nabla u = -\beta\delta(R) \left(\tfrac{R}{r}\right)^{n-1} e_r \tag{132}$$

and

$$\frac{\nabla u}{1-u} = -\frac{\beta\delta}{1-\delta} e_r = -\frac{1}{r^{n-1}\Gamma(r)} e_r. \tag{133}$$

With a slight abuse of notations, we consider that ∇u is defined on $1 \leq |x| \leq R$ by (132). We observe that

$$\operatorname{div}\left(\frac{\nabla u}{1-u}\right) = \left(\frac{|\nabla u|}{1-u}\right)^2. \tag{134}$$

or equivalently

$$-\operatorname{div}\left(\frac{\beta\delta e_r}{1-\delta}\right) = \left(\frac{\beta\delta}{1-\delta}\right)^2. \tag{135}$$

According to (120), the line (135) is also equivalent to

$$-\beta\operatorname{div}(\delta e_r) = \left(\beta^2 - \frac{\beta(n-1)}{r}\right)\delta(r)^2. \tag{136}$$

We are going to define the calibration. Although we define ϕ^x and ϕ^t in parallel, the relevant part is really ϕ^x. The function ϕ^t is derived by the axioms of calibrations. We consider a continuous function $\rho\colon [1, R] \to \mathbf{R}$ such that $\delta(R) \leq \rho \leq u$ and which is C^1 on $[1, R[$. We fix x such that

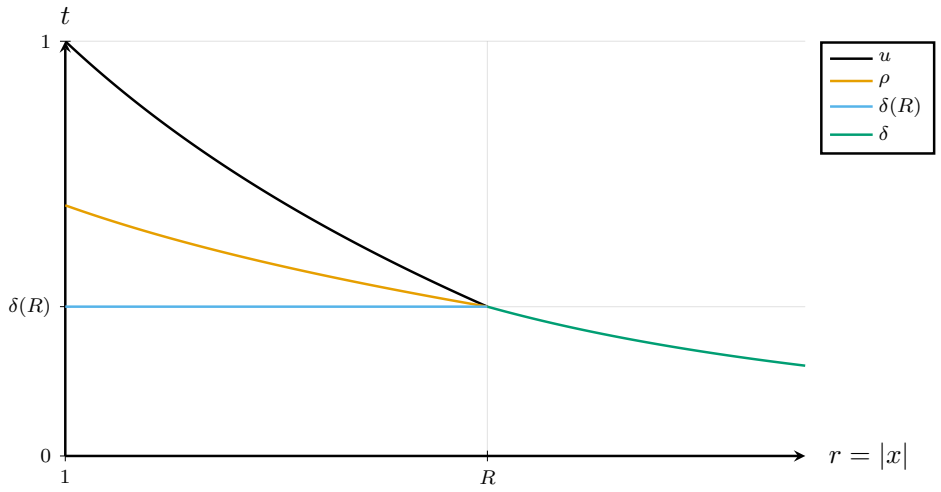

Figure 3: $n = 2$, $\beta = 3$ and $\gamma = \frac{1}{2}$

$1 \leq |x| \leq R$. We define for $0 \leq t \leq \delta(R)$,

$$\phi^x = -2\beta t e_r \tag{137}$$

$$\phi^t = \frac{(n-1)\beta t^2}{r}, \tag{138}$$

for $\delta(R) \leq t < \rho(r)$

$$\phi^x = -2\beta\delta(R) e_r \tag{139}$$

$$\phi^t = \frac{2(n-1)\beta\delta(R)t}{r} - \frac{(n-1)\beta\delta(R)^2}{r} \tag{140}$$

for $\rho(r) < t \leq u(r)$

$$\phi^x(x,t) = 2\nabla u \tag{141}$$

$$= -2\beta\delta(R)\left(\tfrac{R}{r}\right)^{n-1} e_r \tag{142}$$

$$\phi^t(x,t) = |\nabla u|^2 - \gamma^2 \tag{143}$$

$$= \beta^2\delta(R)^2\left(\tfrac{R}{r}\right)^{2n-2} - \gamma^2 \tag{144}$$

and for $u(r) \le t \le 1$,

$$\phi^x(x,t) = 2(1-t)\frac{\nabla u}{1-u} \tag{145}$$

$$= -2(1-t)\frac{\beta\delta e_r}{1-\delta} \tag{146}$$

$$\phi^t(x,t) = (1-t)^2\left(\frac{|\nabla u|}{1-u}\right)^2 - \gamma^2 \tag{147}$$

$$= (1-t)^2\left(\frac{\beta\delta}{1-\delta}\right)^2 - \gamma^2 \tag{148}$$

Next, we fix x such that $|x| \ge R$ and we use the same formula as in Section 3.3. We define for $t \le \delta(r)$,

$$\phi^x = -2\beta t e_r \tag{149}$$

$$\phi^t = \frac{(n-1)\beta t^2}{r} \tag{150}$$

and for $t \ge \delta(r)$,

$$\phi^x(x,t) = -2(1-t)\frac{\beta\delta e_r}{1-\delta} \tag{151}$$

$$\phi^t(x,t) = (1-t)^2\left(\frac{\beta\delta}{1-\delta}\right)^2 - \left(\beta^2 - \frac{\beta(n-1)}{r}\right)\delta^2 \tag{152}$$

Finally, the function ρ is given by the following complicated formula that the reader can ignore for the moment,

$$\rho(r) = \frac{\delta(R)}{2} + \frac{\beta\delta(R)r}{2}\left(\frac{\left(\frac{R}{r}\right)^{2n-2}\Gamma\left(\frac{R}{r}\right)}{\left(\frac{R}{r}\right)^{n-1}-1}\right) - \frac{\delta(R)r}{2n}\left(\beta - \frac{n-1}{R}\right)\left(\frac{\left(\frac{R}{r}\right)^n - 1}{\left(\frac{R}{r}\right)^{n-1}-1}\right). \tag{153}$$

Regardless of ρ, many properties can be checked. The vector field ϕ is bounded. It is continuous outside the graph

$$\{\,(x,t) \mid t = \rho(r),\ 1 \le r < R\,\}. \tag{154}$$

We point out that it is continuous through $|x| = R$ because

$$\gamma^2 = \left(\beta^2 - \frac{\beta(n-1)}{R}\right)\delta(R)^2 \tag{155}$$

and because for $|x| = R$, $\nabla u(x) = -\beta\delta(R)e_r$. The function ϕ is divergence free in the interior of each part. In order for ϕ to be divergence-free in $\mathbf{R}^n \setminus \overline{\Omega}$

(in the sense of distributions), we have to choose ρ in such a way that for all $1 < |x| < R$

$$\phi(x, \rho(r)^-) \cdot \begin{pmatrix} -\rho'(r)e_r \\ 1 \end{pmatrix} = \phi(x, \rho(r)^+) \cdot \begin{pmatrix} -\rho'(r)e_r \\ 1 \end{pmatrix}. \tag{156}$$

This would imply that ϕ is approximately regular on $(\mathbf{R}^n \setminus \overline{\Omega}) \times [0,1]$ by [1, Remark 2.6]. We deal with the the approximately regularity of ϕ on on $\partial\Omega \times [0,1]$ as in the previous section. Let M be an hypersurface of $\mathbf{R}^n \times \mathbf{R}$. For $\mathcal{H}^n$-a.e. $(x_0, t_0) \in M \cap (\partial\Omega \times [0,1])$, we have $t_0 \neq \rho(x_0)$ and the vector $n_0 = (\nu(x_0), 0)$ is a normal vector to M at x. Then $\phi \cdot n_0 = \phi^x$ and we conclude by continuity of ϕ^x in an neighborhood of (x_0, t_0). It is clear from the construction that for $1 < |x| < R$,

$$\phi^x(x,u) = 2\nabla u \tag{157}$$

$$\phi^t(x,u) = |\nabla u|^2 - \gamma^2, \tag{158}$$

that for $|x| > R$,

$$\phi^x(x,u) = 0 = 2\nabla u \tag{159}$$

$$\phi^t(x,u) = 0 = |\nabla u|^2 \tag{160}$$

and that for $|x| = R$,

$$\int_0^{\delta(R)} \phi^x(x,t)\,\mathrm{d}t = \beta\delta(R)^2. \tag{161}$$

The condition $\phi^t \geq \frac{1}{4}|\phi^x|^2 - \gamma^2 \mathbf{1}_{]0,1]}(t)$ holds true because for every $r \geq R$,

$$\gamma^2 \geq \left(\beta^2 - \frac{\beta(n-1)}{r}\right)\delta(r)^2. \tag{162}$$

It is left to check that $\left|\int_s^t \phi^x\right| \leq \beta(s^2 + t^2)$ and to compute ρ. We state a preliminary Lemma which will simplify the discussion.

Lemma 3.8. *Let n be an integer ≥ 2 and let Γ be defined as in (129).*

(i) For all $s, t \geq 1$, $\Gamma(t) - \Gamma(s) = s^{2-n}\Gamma\left(\frac{t}{s}\right)$.

(ii) The function

$$t \mapsto \frac{t^{n-1}\Gamma(t)}{t^{n-1} - 1} \tag{163}$$

is increasing on $]1, +\infty[$.

(iii) The function

$$t \mapsto \frac{t^{n-1}\Gamma(t)}{t^n - 1} \tag{164}$$

is decreasing on $]1, +\infty[$.

(iv) For all $r \geq 1$,

$$\frac{r^{n-1}-1}{n-1} \leq r^{n-1}\Gamma(r) \leq \frac{r^n-1}{n} \tag{165}$$

Proof. There is no difficulty with the first point. For the second point, we are going to see that the function $r \mapsto r\Gamma\left(r^{\frac{1}{n-1}}\right)$ is convex on $[1,+\infty[$. This is clear from the fact that for $r \geq 1$,

$$r\Gamma(r^{\frac{1}{n-1}}) = \begin{cases} r\ln(r) & \text{si } n=2 \\ \frac{1}{n-2}(r-r^{\frac{1}{n-1}}) & \text{si } n \geq 3 \end{cases} \tag{166}$$

Therefore the function

$$r \mapsto \frac{r\Gamma(r^{\frac{1}{n-1}})}{r-1} \tag{167}$$

is increasing on $]1,+\infty[$ and by a change of variable, this also holds for

$$r \mapsto \frac{r^{n-1}\Gamma(r)}{r^{n-1}-1}. \tag{168}$$

Note that its limit when $r \mapsto 1$ is $(n-1)^{-1}$. Next, we are going to see that the function $r \mapsto r^{\frac{n-1}{n}}\Gamma\left(r^{\frac{1}{n}}\right)$ is concave on $[1,+\infty[$. We have for $r \geq 1$,

$$r^{\frac{n-1}{n}}\Gamma(r^{\frac{1}{n}}) = \begin{cases} \frac{1}{2}r^{\frac{1}{2}}\ln(r) & \text{si } n=2 \\ \frac{1}{n-2}\left(r^{\frac{n-1}{n}}-r^{\frac{1}{n}}\right) & \text{si } n \geq 3 \end{cases}, \tag{169}$$

we compute

$$\frac{d^2}{dr^2}\left[r^{\frac{1}{2}}\ln(r)\right] = -\frac{r^{-\frac{3}{2}}\ln(r)}{4} \leq 0 \tag{170}$$

and

$$\frac{d^2}{dr^2}\left[r^{\frac{n-1}{n}}-r^{\frac{1}{n}}\right] = -\frac{(n-1)r^{-\frac{1}{n}-1}(1-r^{\frac{2-n}{n}})}{n^2} \leq 0. \tag{171}$$

Therefore, the function

$$r \mapsto \frac{r^{n-1}\Gamma(r)}{r^n-1} \tag{172}$$

is decreasing on $]1,+\infty[$. Its limit when $r \mapsto 1$ is n^{-1}. □

We come back to the condition $\left|\int_r^s \phi^x\right| \leq \beta(r^2+s^2)$. It suffices to check that for all $x \in \mathbf{R}^n \setminus \Omega$ and for almost all $t \in [0,1]$, we have $|\phi^x(x,t)| \leq 2\beta t$. This is trivial in the area

$$\{\,(x,t) \mid r \geq R,\ 0 \leq t \leq 1\,\} \tag{173}$$

and also in

$$\{\,(x,t) \mid 1 \leq r \leq R,\ 0 \leq t < \rho(r)\,\}. \tag{174}$$

What about the area

$$\{\,(x,t)\mid 1\le r\le R,\ \rho(r)<t\le 1\,\}, \tag{175}$$

it amounts to check that for $1\le|x|\le R$,

$$|\nabla u(x)|\le\beta u(x). \tag{176}$$

We have

$$u(x)=\delta(R)(1+\beta R^{n-1}(\Gamma(R)-\Gamma(r)) \tag{177}$$

$$\nabla u(x)=-\beta\delta(R)\left(\tfrac{R}{r}\right)^{n-1}e_r \tag{178}$$

so we are led to show that for all $1\le r\le R$,

$$\left(\tfrac{R}{r}\right)^{n-1}\le 1+\beta R^{n-1}(\Gamma(R)-\Gamma(r) \tag{179}$$

For $n=1$, this is trivial. For $n\ge 2$, we apply Lemma 3.8 and we see that

$$R^{n-1}(\Gamma(R)-\Gamma(r))=r\left(\tfrac{R}{r}\right)^{n-1}\Gamma\left(\tfrac{R}{r}\right) \tag{180}$$

$$\ge\frac{r}{n-1}\left(\left(\tfrac{R}{r}\right)^{n-1}-1\right) \tag{181}$$

$$\ge\frac{1}{n-1}\left(\left(\tfrac{R}{r}\right)^{n-1}-1\right) \tag{182}$$

We conclude that (179) because $\beta\ge n-1$.

Remark 3.9. This is the only place where we use the assumption $\beta\ge n-1$.

We come finally to the computation of ρ. We recall that we are looking for a continuous function $\rho\colon[1,R]\to[0,1]$ such that $\delta(R)\le\rho\le u$, which is C^1 in $[1,R[$ and such that for $1<|x|<R$,

$$\phi(x,\rho(r)^-)\cdot\begin{pmatrix}-\rho'(r)e_r\\1\end{pmatrix}=\phi(x,\rho(r)^+)\cdot\begin{pmatrix}-\rho'(r)e_r\\1\end{pmatrix} \tag{183}$$

For $n=1$, we can take ρ equals to the constant $\delta(R)$ and then see that ϕ is globally continuous so there is nothing to check. We detail the computation of (183) for $n\ge 2$. It will be convenient to express the result in divergence form. We compute

$$\phi(x,\rho^-)\cdot\begin{pmatrix}-\rho' e_r\\1\end{pmatrix}=2\beta\delta(R)\rho'+\frac{2(n-1)\beta\delta(R)\rho}{r}-\frac{(n-1)\beta\delta(R)^2}{r} \tag{184}$$

$$=2\beta\delta(R)\mathrm{div}(\rho e_r)-\beta\delta(R)^2\mathrm{div}(e_r) \tag{185}$$

Next, we compute

$$\phi(x,\rho^+)\cdot\begin{pmatrix}-\rho' e_r\\1\end{pmatrix}=2\beta\delta(R)\left(\tfrac{R}{r}\right)^{n-1}\rho'+\beta^2\delta(R)^2\left(\tfrac{R}{r}\right)^{2n-2}-\gamma^2 \tag{186}$$

$$\begin{aligned}&=2\beta\delta(R)R^{n-1}\mathrm{div}\left(r^{1-n}\rho e_r\right)\\&\quad+\beta^2\delta(R)^2R^{2n-2}\mathrm{div}\left(r^{1-n}\Gamma(r)e_r\right)-\gamma^2\end{aligned} \tag{187}$$

and we observe that we can write

$$\gamma^2 = \mathrm{div}\left(\frac{\gamma^2 r}{n} e_r + \frac{c}{r^{n-1}} e_r\right) \tag{188}$$

where c is any real constant. A natural solution is to choose ρ such that

$$\begin{aligned} &2\beta\delta(R)\rho - \beta\delta(R)^2 \\ &\quad = 2\beta\delta(R)\left(\tfrac{R}{r}\right)^{n-1}\rho + \beta^2\delta(R)^2 R^{n-1}\left(\tfrac{R}{r}\right)^{n-1}\Gamma(r) - \frac{\gamma^2 r}{n} + \frac{c}{r^{n-1}}. \end{aligned} \tag{189}$$

We arrive at

$$\begin{aligned} 2\beta\delta(R)\left(\left(\tfrac{R}{r}\right)^{n-1} - 1\right)\rho = &-\beta\delta(R)^2 - \beta^2\delta(R)^2 R^{n-1}\left(\tfrac{R}{r}\right)^{n-1}\Gamma(r) \\ &+ \frac{\gamma^2 r}{n} - \frac{c}{r^{n-1}} \end{aligned} \tag{190}$$

The constant c is fixed by the fact that the left-hand side cancels at $r = R$, thus

$$\begin{aligned} 2\beta\delta(R)\left(\left(\tfrac{R}{r}\right)^{n-1} - 1\right)\rho = &\beta\delta(R)^2\left(\left(\tfrac{R}{r}\right)^{n-1} - 1\right) \\ &+ \beta^2\delta(R)^2 R^{n-1}\left(\tfrac{R}{r}\right)^{n-1}(\Gamma(R) - \Gamma(r)) - \frac{\gamma^2 r}{n}\left(\left(\tfrac{R}{r}\right)^n - 1\right) \end{aligned} \tag{191}$$

Remember that $\Gamma(R) - \Gamma(r) = r^{2-n}\Gamma\left(\frac{R}{r}\right)$ so

$$\begin{aligned} 2\beta\delta(R)\left(\left(\tfrac{R}{r}\right)^{n-1} - 1\right)\rho = &\beta\delta(R)^2\left(\left(\tfrac{R}{r}\right)^{n-1} - 1\right) \\ &+ \beta^2\delta(R)^2 r\left(\tfrac{R}{r}\right)^{2n-2}\Gamma\left(\tfrac{R}{r}\right) - \frac{\gamma^2 r}{n}\left(\left(\tfrac{R}{r}\right)^n - 1\right) \end{aligned} \tag{192}$$

We arrive at

$$\rho(r) = \frac{\delta(R)}{2} + \frac{\beta\delta(R)r}{2}\left(\frac{\left(\frac{R}{r}\right)^{2n-2}\Gamma\left(\frac{R}{r}\right)}{\left(\frac{R}{r}\right)^{n-1} - 1}\right) - \frac{\gamma^2 r}{2n\beta\delta(R)}\left(\frac{\left(\frac{R}{r}\right)^n - 1}{\left(\frac{R}{r}\right)^{n-1} - 1}\right) \tag{193}$$

As $\gamma^2 = \left(\beta^2 - \frac{\beta(n-1)}{R}\right)\delta(R)^2$, an alternative expression is

$$\begin{aligned} \rho(r) = &\frac{\delta(R)}{2} + \frac{\beta\delta(R)r}{2}\left(\frac{\left(\frac{R}{r}\right)^{2n-2}\Gamma\left(\frac{R}{r}\right)}{\left(\frac{R}{r}\right)^{n-1} - 1}\right) \\ &- \frac{\delta(R)r}{2n}\left(\beta - \frac{n-1}{R}\right)\left(\frac{\left(\frac{R}{r}\right)^n - 1}{\left(\frac{R}{r}\right)^{n-1} - 1}\right). \end{aligned} \tag{194}$$

It is clear that ρ is a continuous function of $r \in]1, R[$ and we also have $\lim_{r\to R} \rho(r) = \delta(R)$ because

$$\lim_{t\to 1^+} \frac{\Gamma(t)}{t^{n-1}-1} = \frac{1}{n-1} \tag{195}$$

and

$$\lim_{t\to 1^+} \frac{t^n-1}{t^{n-1}-1} = \frac{n}{n-1}. \tag{196}$$

We show that that $\rho \geq \delta(R)$. It suffices to justify that ρ is decreasing on $[1, R[$. For $0 \leq r < R$, we write $t = \frac{R}{r}$ so that

$$\rho(r) = \frac{\delta(R)}{2} + \frac{\beta\delta(R)Rt^{n-2}}{2}\left(\frac{t^{n-1}\Gamma(t)}{t^{n-1}-1}\right) - \frac{\delta(R)}{2n}\left(\beta - \frac{n-1}{R}\right) t^{-1}\left(\frac{t^n-1}{t^{n-1}-1}\right). \tag{197}$$

According to Lemma 3.8, the function $t \mapsto \left(\frac{t^{n-1}\Gamma(t)}{t^{n-1}-1}\right)$ is increasing on $]1, +\infty[$. It is also easy to see that the function $t^{-1}\left(\frac{t^n-1}{t^{n-1}-1}\right)$ is decreasing on $]1, +\infty[$ because

$$t^{-1}\left(\frac{t^n-1}{t^{n-1}-1}\right) = 1 + t^{-1}\frac{t-1}{t^{n-1}-1} \tag{198}$$

and $t \mapsto t^{n-1}$ is convex. We deduce that $r \mapsto \rho(r)$ is decreasing on $[1, R[$. Finally, we prove that for all $1 \leq r < R$, we have $\rho(r) \leq u(r)$. Remember that

$$u = \delta(R)\left[1 + \beta r\left(\frac{R}{r}\right)^{n-1}\Gamma\left(\frac{R}{r}\right)\right] \tag{199}$$

so we have to check that for all $1 \leq r < R$,

$$\frac{1}{2} + \frac{\beta r}{2}\left(\frac{\left(\frac{R}{r}\right)^{2n-2}\Gamma\left(\frac{R}{r}\right)}{\left(\frac{R}{r}\right)^{n-1}-1}\right) - \frac{r}{2n}\left(\beta - \frac{n-1}{R}\right)\left(\frac{\left(\frac{R}{r}\right)^n-1}{\left(\frac{R}{r}\right)^{n-1}-1}\right) \leq 1 + \beta r\left(\frac{R}{r}\right)^{n-1}\Gamma\left(\frac{R}{r}\right) \tag{200}$$

We rewrite this: for all $1 \leq r < R$,

$$\beta r\left(\frac{\left(\frac{R}{r}\right)^{n-1}\Gamma\left(\frac{R}{r}\right)}{\left(\frac{R}{r}\right)^{n-1}-1}\right)\left(\left(2-\frac{R}{r}\right)^{n-1}\right) - \frac{\beta r}{n}\left(\frac{\left(\frac{R}{r}\right)^n-1}{\left(\frac{R}{r}\right)^{n-1}-1}\right) \leq 1 - \frac{n-1}{n}\frac{r}{R}\left(\frac{\left(\frac{R}{r}\right)^n-1}{\left(\frac{R}{r}\right)^{n-1}-1}\right). \tag{201}$$

It suffices that for all $t > 1$,

$$\beta \left(\frac{t^{n-1}\Gamma(t)}{t^{n-1}-1} \right) \left(2 - t^{n-1}\right) - \frac{\beta}{n} \left(\frac{t^n - 1}{t^{n-1}-1} \right) \leq 0 \tag{202}$$

$$1 - \frac{n-1}{n} t^{-1} \left(\frac{t^n - 1}{t^{n-1}-1} \right) \geq 0. \tag{203}$$

The first point can be simplified as

$$\left(\frac{t^{n-1}\Gamma(t)}{t^n - 1} \right) \left(2 - t^{n-1}\right) \leq \frac{1}{n} \tag{204}$$

and this follows from Lemma 3.8. To prove the second point, we observe that $t \mapsto t^{-1} \left(\frac{t^n-1}{t^{n-1}-1} \right)$ is decreasing and that its limit when $t \to 1$ is $\frac{n}{n-1}$.

Appendices

A Proof of Lemma 3.3

We recall the statement.

Lemma. *Let Ω be a non-empty C^2 star-shaped bounded open set of $\mathbf{R}^n$. There exists a continuous function $\Gamma \colon \mathbf{R}^n \setminus \Omega \to \mathbf{R}$ such that*

(i) Γ is harmonic positive on $\mathbf{R}^n \setminus \overline{\Omega}$;

(ii) $\Gamma = 0$ on $\partial\Omega$,

(iii) For all $x \in \mathbf{R}^n \setminus \overline{\Omega}$, $\nabla\Gamma(x) \neq 0$

(iv) Γ has a C^1 extension up to the boundary and for all $x \in \partial\Omega$, there exists $t > 0$ such that $\nabla\Gamma(x) = t\nu(x)$, where $\nu(x)$ is the outward unit normal vector of Ω at x.

Proof. Without loss of generality, we assume that $0 \in \Omega$ and that Ω is star-shaped with respect to 0. We detail the case $n \geq 3$. According to [10, Section 3A, Theorem 3.40], there exists a unique function $p \in C(\mathbf{R}^n \setminus \Omega)$ such that

(i) p is harmonic on $\mathbf{R}^n \setminus \overline{\Omega}$,

(ii) p is harmonic at infinity,

(iii) $p = 1$ on $\partial\Omega$.

We refer to [10, Proposition 2.74] for the characterisations of functions which are harmonic at infinity. Now we define $\Gamma(x) = 1 - p(x)$ and we review the properties of the Lemma. It is clear that Γ is harmonic on $\mathbf{R}^n \setminus \overline{\Omega}$ and $\Gamma = 0$

on $\partial\Omega$. As p is harmonic at infinity, we have $\lim_{x\to\infty} p(x) = 0$ and thus $\lim_{x\to+\infty} \Gamma(x) = 1$. We can then apply the maximum principle to see that $\Gamma > 0$ on $\mathbf{R}^n \setminus \overline{\Omega}$. The function Γ has a C^1 extension up to the boundary thanks to the usual regularity results for Dirichlet problems. As Γ is constant on $\partial\Omega$, its tangential derivative is 0 along $\partial\Omega$. And according to the Hopf Lemma, the normal derivative (with respect to the outward normal vector) is > 0 along $\partial\Omega$. This proves that for all $x \in \partial\Omega$, there exists $t > 0$ such that $\nabla\Gamma(x) = t\nu(x)$. The fact that $\nabla\Gamma$ never vanishes comes from the fact that Ω is star-shaped. Indeed, the function $w\colon x \mapsto x \cdot \nabla\Gamma(x)$ is harmonic on $\mathbf{R}^n \setminus \overline{\Omega}$ and ≥ 0 on $\partial\Omega$. In addition, we see that $\lim_{x\to+\infty} w(x) = 0$ by applying [10, Proposition 2.75] to the function p. We can use the maximum principle to conclude that $w > 0$ on $\mathbf{R}^n \setminus \Omega$

In the case $n = 2$, we define p as the unique function $p \in C(\mathbf{R}^n \setminus \Omega)$ such that

(i) p is harmonic on $\mathbf{R}^n \setminus \overline{\Omega}$,

(ii) p is harmonic at infinity,

(iii) $p(x) = \ln(|x|)$ on $\partial\Omega$.

Then we define $\Gamma = \ln(|x|) - p$. As p is harmonic at infinity, it is bounded at infinity and thus $\lim_{x\to+\infty} \Gamma(x) = +\infty$. The rest of the proof is the same except that $\lim_{x\to+\infty} x \cdot \nabla\Gamma(x) = 1$. The case $n = 1$ is trivial. □

References

[1] G. Alberti; G. Bouchitte; G. Dal Maso The calibration method for the Mumford-Shah functional and free-discontinuity problems. Calc. Var. Partial Differential Equations 16 (2003), no. 3, 299-333.

[2] H. W. Alt; L. A. Caffarelli Existence and regularity for a minimum problem with free boundary. J. Reine Angew. Math., 325:105–144, 1981.

[3] L. A. Caffarelli and D. Kriventsov A free boundary problem related to thermal insulation. Comm. Partial Differential Equations 41 (2016), no. 7, 1149-1182.

[4] D. Kriventsov A free boundary problem related to thermal insulation: flat implies smooth. Calc. Var. Partial Differential Equations 58 (2019), no. 2, Paper No. 78, 83 pp.

[5] D. Bucur and A. Giacomini Shape optimization problems with Robin conditions on the free boundary. Ann. Inst. H. Poincaré Anal. Non Linéaire 33 (2016), no. 6, 1539–1568.

[6] C. Labourie and E. Milakis Higher integrability of the gradient for the Thermal Insulation problem. Submitted, arXiv:2101.09692

[7] B. Bogosel and M. Foare Numerical implementation in 1D and 2D of a shape optimization problem with Robin boundary conditions. Preprint available at http://www.cmap.polytechnique.fr/ beniamin.bogosel/pdfs/Robin.pdf

[8] N. Fusco An overview of the Mumford-Shah problem. Milan J. Math. 71 (2003), 95–119.

[9] A. Chambolle, V. Duval, G. Peyré, C. Poon Geometric properties of solutions to the total variation denoising problem. Inverse Problems 33 (2017), no. 1, 015002, 44 pp.

[10] G. B. Folland Introduction to partial differential equations. Second edition. Princeton University Press, Princeton, NJ, 1995. xii+324 pp. ISBN: 0-691-04361-2

[11] L. Ambrosio; N. Fusco; D. Pallara Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. xviii+434 pp. ISBN: 0-19-850245-1