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Corrigendum and Addendum to "Polarized Parallel Transport and Uniruled Divisors on Generalized Kummer Varieties"

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We correct the statement of the main result of [9] and provide some further precisions.

The goal of this short note is to state correctly the main result of [9]. For the definitions, the notations and the motivations we refer the reader to [9]. The correct statement is the following:

Theorem 0.1. Let $n \geq 1$ be an integer. Let $\mathfrak{M} = \cup_{d>0} \mathfrak{M}_{2d}$ be the union of the moduli spaces \mathfrak{M}_{2d} of projective irreducible holomorphic symplectic varieties of $K_n(A)$ -type polarized by a line bundle of degree $2d$. For all $(X, H) \in \mathfrak{M}$, outside at most a finite number of connected components determined by the monodromy orbit of H , the linear system $|mH|$, for some m , contains a uniruled divisor covered by rational curves of primitive class.

Let q be the Beauville–Bogomolov quadratic form on $H^2(X, \mathbb{Z})$. This induces an embedding $H^2(X, \mathbb{Z}) \hookrightarrow H_2(X, \mathbb{Z}), H \mapsto H^\vee$. By abuse of notation we denote again by q the quadratic form on $H_2(X, \mathbb{Z})$.

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Remark 0.2. The statement above insures precisely existence of uniruled divisors covered by primitive rational curves if there exist integers p, g , and ϵ such that $p \geq g$ and $\epsilon = 0$ or 1 with

- (i) the class $\alpha := \frac{H^\vee}{\text{div}(H)} \in H_2(X, \mathbb{Z})$ can be written as $\gamma + (2g - \epsilon)\eta$ with η in the monodromy orbit of the class of the exceptional curve on a $K_n(A)$;
- (ii) $\gamma \in \eta^\perp$, $q(\gamma) = 2p - 2$ (hence, $q(\alpha) = 2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}$).

Remark 0.3.

- (i) It follows from Proposition 2.1 that if $q(\alpha) > n + 1$, then a multiple of H is uniruled by primitive rational curves of class α .
- (ii) If $\rho(X) \geq 2$ then X always contains an ample uniruled divisor covered by primitive rational curves (cf. Corollary 2.3).
- (iii) If $n \leq 5$ then the conclusion of the theorem holds for *all* the connected components of \mathfrak{M} (cf. Remark 2.3).
- (iv) If $n + 1$ is a power of a prime number, then by [7] and [8], the monodromy group is maximal. Therefore, it suffices to check that the square $q(\alpha)$ is of the form $2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}$, with $p \geq g$.

The original proof was based on three ingredients: the 1st was a deformation theoretic statement, saying that rational curves whose deformations cover a divisor in irreducible holomorphic symplectic manifolds are non-obstructed [1, Corollary 3.5]. The 2nd is the characterization of polarized parallel transport operators on polarized irreducible holomorphic symplectic varieties (X, H) of $K_n(A)$ -type [9, Theorem 1.1] that allows to obtain an explicit description of the polarized deformation equivalence [9, Theorem 4.2]. These two ingredients are true. The 3rd argument consists in the construction of explicit examples of uniruled divisors on the generalized Kummer variety associated with a polarized abelian surface (A, H_A) with $NS(A) = \mathbb{Z}H_A$ such that $p_a(H_A) \geq g \geq 2$. The construction is also correct, but the examples that we provided cannot yield all the possible primitive polarizations, as we tacitely and erroneously assumed in [9]. Even without taking the monodromy orbit into account, this is simply because it may happen that the number $2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}$, $\epsilon = 0, 1$ is positive even with $p < g$, which obviously renders our geometric argument empty. Indeed, the rational curves are constructed as g_n^1 on the normalization of a nodal curve of geometric genus g lying in the hyperplane linear system $|H_A|$, which is supposed to have $p_a(H_A) = p$. We also take the occasion of this note to provide the full proof (Proposition 1.1) of a technical point that we claimed in [9, Section 4.2] to follow from a dimension count as

in [12, Example 4.1, 3]). The statement is correct, but the argument cannot be the same as in [12, Example 4.1, 3]) because we deal here with a locally closed subset (the Severi variety) of a complete linear system and not with the complete linear system.

The $K3^{[n]}$ -type case, initially treated in [1], is subject to the same considerations and will be treated in [2].

We realized our mistake after the appearance of [10], which provides counterexamples in the $K3^{[n]}$ -case that apply exactly in all cases not covered by the similar geometric constructions for the Hilbert scheme of points on a general projective $K3$. Contrary to the $K3^{[n]}$ -type case as far as we know there are no known counterexamples to the existence of uniruled divisors ruled by a primitive curve class in the $K_n(A)$ -type case. Nevertheless, we have no reasons to believe that the $K_n(A)$ -type case could be exempt from this type of sporadic pathologies.

1 Existence of Uniruled Divisors on $K_n(A)$

In [9, Section 4.2 “Examples”] we claimed that “the natural map from $\tilde{\mathcal{C}}_{g+1}^1 \rightarrow A^{[g+1]}$ is finite onto its image” invoking a dimension count made in [12, Example 4.1, 3]). However, the same argument cannot work because we do not work with the full continuous system but with a locally closed subset (the Severi variety). Hence, we take the occasion to provide a full proof of that statement in the following.

Proposition 1.1. Let g be an integer ≥ 2 and (A, H_A) be a general polarized abelian surface with $p_a(H_A) =: p \geq g$. Then $A^{[g+1]}$ contains a uniruled divisor covered by the \mathfrak{g}_{g+1}^1 on nodal genus g curves in the continuous system $\{H_A\}$.

Proof. To prove the statement we can actually work over a very general polarized abelian surface, so let us suppose that $NS(A) = \mathbb{Z}H_A$. We will prove this statement by induction on g . It is sufficient to show it on the symmetric product of A .

Observe that, by [4, Theorem. 1.1], for all $2 \leq g \leq p_a(H_A)$, that the Severi variety parametrizing nodal genus g curves inside $\{H_A\}$ is non-empty of the expected dimension g .

It is sufficient to show the claim on the symmetric product $A^{(g+1)}$ of A . More precisely, we will prove the following statement: there exists an irreducible component V of the (Zariski closure of the) Severi variety parametrizing nodal genus g curves inside $\{H_A\}$ such that, if $\mathcal{C}_V \rightarrow V$ denotes the universal curve and $\mathcal{C}_V^{(g+1)} \rightarrow V$ the relative symmetric product, the natural morphism

$$\mathcal{C}_V^{(g+1)} \rightarrow A^{(g+1)}$$

is generically finite onto its image. Note that this is equivalent to saying that $(g + 1)$ generic points on a generic curve of the family lie only on a finite number of curves of the family.

Indeed as

$$\dim \mathcal{C}_V^{(g+1)} = \text{reldim}(\mathcal{C}_V^{(g+1)}) + \dim V = (g + 1) + g = 2g + 1$$

it follows that the image is a divisor inside $A^{(g+1)}$. Since the k -th symmetric product of a curve is uniruled for k greater than the genus of the curve, as a by-product we have that such divisor is uniruled.

Note also that positive dimensional fibers of the morphism $\mathcal{C}_V^{(g+1)} \rightarrow A^{(g+1)}$ cannot lie in a fiber of $\mathcal{C}_V^{(g+1)} \rightarrow V$, as $\mathcal{C}_t^{(g+1)}$ injects into $A^{(g+1)}$ for every $t \in V$.

We start with the case $g = 2$. Let C be one of the (finitely many) nodal curves of geometric genus 2 inside the linear system $|H_A|$. In this case the points of the component V of the Severi variety containing C are given by all the translates of C . The 3rd symmetric product $C^{(3)}$ injects as a three-dimensional subvariety inside $A^{(3)}$. The action of A on $C^{(3)}$ by translation has no positive-dimensional stabilizer (as A is general, hence simple). Therefore the orbit of $C^{(3)}$ under this action is a divisor. Using the notation above such divisor is the image of $\mathcal{C}_V^{(2+1)}$ in $A^{(3)}$.

By inductive hypothesis, there exists an irreducible component W of the (Zariski closure of the) Severi variety parametrizing nodal genus $g - 1$ curves inside $\{H_A\}$ such that, if $\mathcal{C}_W \rightarrow W$ denotes the universal curve and $\mathcal{C}_W^{(g)} \rightarrow W$ the relative symmetric product, the natural morphism

$$\mathcal{C}_W^{(g)} \rightarrow A^{(g)}$$

is generically finite onto its image.

Now let V be (the Zariski closure of) an irreducible component of the Severi variety of nodal genus g curves in $\{H_A\}$ obtained by smoothing one node of the curves in W (which can be done by the regularity of the Severi variety, [3, Example 1.3]). By construction $W \subset V$. Let $\mathcal{C}_V \rightarrow V$ be the universal curve. Its restriction over W yields a map $\mathcal{C}_W \rightarrow W$. Let D be the image of the morphism

$$\mathcal{C}_V^{(g+1)} \rightarrow A^{(g+1)}.$$

Observe that D contains the image D_W of

$$\mathcal{C}_W^{(g+1)} \rightarrow A^{(g+1)}.$$

We claim that by the inductive hypothesis D_W has codimension 2, or, equivalently, that the morphism $\mathcal{C}_W^{(g+1)} \rightarrow A^{(g+1)}$ is generically finite onto its image. Indeed if $\xi = x_1 + \dots + x_{g+1}$ is a generic point of the image, then, say, $x_1 + \dots + x_g$ is a generic point of the image of the morphism $\mathcal{C}_W^{(g)} \rightarrow A^{(g)}$. By the inductive hypothesis the points x_1, \dots, x_g lie on finitely many curves of the family W , *a fortiori* that will be true for x_1, \dots, x_g, x_{g+1} and the claim follows.

We want to prove that D contains D_W strictly. If this were not the case, by irreducibility, we would have $D = D_W$. Let $U \subset D$ be an open subset over which the morphisms $\mathcal{C}_W^{(g+1)} \rightarrow A^{(g+1)}$ and $\mathcal{C}_V^{(g+1)} \rightarrow A^{(g+1)}$ are smooth, and let $p_1 + p_2 + \dots + p_{g+1}$ be a point in U . Let C be a nodal genus g curve in V containing these points. Let us fix the 1st g points p_1, \dots, p_g . By induction these points are contained inside a finite number of curves of genus $g - 1$ belonging to W . Let B_1, \dots, B_m be all such curves. Let $U_C \subset C$ be an open subset such that for all $q \in U_C$ we have $p_1 + \dots + p_g + q \in U$. As we have seen above p_1, \dots, p_g, q lie on finitely many curves of genus $g - 1$ belonging to W , and these curves must be B_1, \dots, B_m . Therefore, as q varies in U_C , we deduce that U_C is a subset of a finite union of genus $(g - 1)$ curves. As C is irreducible, there is an i such that $C = B_i$, which is clearly a contradiction. Therefore, D must strictly contain D_W and be a divisor, which is necessarily uniruled. ■

The rest of the proof remains the same and we refer the reader to [9] for the details.

2 Where It Does Not Work

In this section we prove that, for every dimension, there is at most a finite number of components of the moduli space of polarized manifolds (X, H) of $K_n(A)$ -type where the strategy of the previous section does not work. The uniruled divisors we constructed have a cohomology class that is a multiple of $H_A - (2g)\tau$ (or $H_A - (2g - 1)\tau$) where $2p - 2 = H_A^2$ and H_A is the primitive polarization on the abelian surface. We have the following:

Proposition 2.1. Let X be a projective irreducible holomorphic symplectic variety of $K_n(A)$ -type. Let $C \in H_2(X, \mathbb{Z}) \cap N_1(X)$ be a primitive class such that its square $q(C)$ with respect to the Beauville–Bogomolov form is $> n + 1$. Then, the class C is deformation equivalent to the class of one of the curves constructed in the previous section.

Proof. We know by [9, Theorem 4.2] that C is deformation equivalent to either $H_A - 2g\tau$ or $H_A - (2g - 1)\tau$, with $g \leq n + 1$. If $q(C) > n + 1$, the square of $H_A - (2n + 2)\tau$ is positive,

that is, $H_A^2 = 2p - 2$ with $p > n + 1$. Thus, $p > n + 1 \geq g$, which means that $H_A - 2g\tau$ can be represented by the class of a \mathfrak{g}_{g+1}^1 on a nodal curve in $\{H_A\}$. ■

Corollary 2.2. Let \mathcal{M}_n be the moduli space of all polarized manifolds of $K_n(A)$ -type with n fixed. Then, the number of components of \mathcal{M}_n whose general points (X, H) do not have a uniruled divisor ruled by a rational curve of primitive class is at most finite.

Proof. The components of \mathcal{M}_n are in bijective correspondence with the monodromy orbits of a given class of positive square in $L_n := U^3 \oplus (-2n - 2) \cong H^2(X, \mathbb{Z})$; see [11, Theorem 2.8].

For a fixed square of H , there is a finite number of orbits (computed again in [11, Theorem 2.8]), so it follows that if X has a uniruled divisor when $q(H)$ is big enough, our claim will hold. The dual curve to H is given by $H/\text{div}(H)$, where $\text{div}(H)$ is the divisibility of H , which is the positive generator of the ideal $q(H, H^2(X, \mathbb{Z}))$. The divisibility is at most $2n + 2$; therefore, if $q(H) \geq (2n + 2)^2(n + 1)$ the dual curve has square at least $n + 1$, so that Proposition 2.1 applies and our claim follows. ■

Corollary 2.3. Let X be a projective manifold of $K_n(A)$ -type with Picard rank at least two. Then X has an ample divisor ruled by primitive rational curves.

Proof. Since X is projective and has Picard rank at least two, its Picard lattice is indefinite and contains primitive elements of positive arbitrary Beauville–Bogomolov square and so does the ample cone. Let H be an ample divisor such that $q(H) \geq (2n + 2)^2(n + 1)$. Let C be its dual curve in $H_2(X, \mathbb{Z})$. As the divisibility of H is at most $2n + 2$ it follows that $q(C) \geq n + 1$, and Proposition 2.1 yields our claim. ■

Remark 2.4 The estimate of Proposition 2.1 is definitely not sharp; indeed, all primitive curves of positive square on manifolds of $K_n(A)$ -type with $n \leq 5$ are deformation equivalent to the curves we construct in Proposition 1.1. Indeed, by [9, Theorem 4.2] we can suppose that our pair (X, C) with $q(C) > 0$ is $(K_n(A), H_A - \mu\tau)$ with $0 \leq \mu \leq n + 1$ and A is an abelian surface of genus p . The class $H_A - \mu\tau$ is given by the class of the rational curves constructed in Proposition 1.1, which have class $H_A - 2g\tau$, with the eventual addition of a tail of class τ , so that $2g \leq n + 2$. By contradiction let us suppose that $g > p$ and $n \leq 5$. We have $q(H_A - 2g\tau) = 2p - 2 - 2\frac{g^2}{n+1} \leq 2p - 2 - 2\frac{(p+1)^2}{n+1} \leq 2p - 2 - 2\frac{(p+1)^2}{6}$. However, the last value is never positive; hence, $q(H_A - 2g\tau)$ cannot be positive and we reach a contradiction. Analogously, for $C = H_A - (2g - 1)\tau$, we have $q(C) \leq \frac{20p-25-4p^2}{12}$ with $g \geq p + 1$ and $2g \leq n + 2$, which is again not positive.

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