Polarized Parallel Transport and Uniruled Divisors on Deformations of Generalized Kummer Varieties
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Corrigendum and Addendum to “Polarized Parallel Transport and Uniruled Divisors on Generalized Kummer Varieties”

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We correct the statement of the main result of [9] and provide some further precisions.

The goal of this short note is to state correctly the main result of [9]. For the definitions, the notations and the motivations we refer the reader to [9]. The correct statement is the following:

**Theorem 0.1.** Let \( n \geq 1 \) be an integer. Let \( \mathcal{M} = \bigcup_{d > 0} \mathcal{M}_{2d} \) be the union of the moduli spaces \( \mathcal{M}_{2d} \) of projective irreducible holomorphic symplectic varieties of \( K_n(A) \)-type polarized by a line bundle of degree \( 2d \). For all \( (X, H) \in \mathcal{M} \), outside at most a finite number of connected components determined by the monodromy orbit of \( H \), the linear system \(|mH|\), for some \( m \), contains a uniruled divisor covered by rational curves of primitive class.

Let \( q \) be the Beauville–Bogomolov quadratic form on \( H^2(X, \mathbb{Z}) \). This induces an embedding \( H^2(X, \mathbb{Z}) \hookrightarrow H_2(X, \mathbb{Z}), H \mapsto H^\vee \). By abuse of notation we denote again by \( q \) the quadratic form on \( H_2(X, \mathbb{Z}) \).
Remark 0.2. The statement above insures precisely existence of uniruled divisors covered by primitive rational curves if there exist integers $p, g$, and $\epsilon$ such that $p \geq g$ and $\epsilon = 0$ or $1$ with

(i) the class $\alpha := \frac{H^\vee}{\dim (H)} \in H_2(X, \mathbb{Z})$ can be written as $\gamma + (2g - \epsilon)\eta$ with $\eta$ in the monodromy orbit of the class of the exceptional curve on a $K_n(A)$;

(ii) $\gamma \in \eta^\perp$, $q(\gamma) = 2p - 2$ (hence, $q(\alpha) = 2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}$).

Remark 0.3.

(i) It follows from Proposition 2.1 that if $q(\alpha) > n + 1$, then a multiple of $H$ is uniruled by primitive rational curves of class $\alpha$.

(ii) If $\rho(X) \geq 2$ then $X$ always contains an ample uniruled divisor covered by primitive rational curves (cf. Corollary 2.3).

(iii) If $n \leq 5$ then the conclusion of the theorem holds for all the connected components of $\mathcal{M}$ (cf. Remark 2.3).

(iv) If $n + 1$ is a power of a prime number, then by [7] and [8], the monodromy group is maximal. Therefore, it suffices to check that the square $q(\alpha)$ is of the form $2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}$, with $p \geq g$.

The original proof was based on three ingredients: the 1st was a deformation theoretic statement, saying that rational curves whose deformations cover a divisor in irreducible holomorphic symplectic manifolds are non-obstructed [1, Corollary 3.5]. The 2nd is the characterization of polarized parallel transport operators on polarized irreducible holomorphic symplectic varieties $(X,H)$ of $K_n(A)$-type [9, Theorem 1.1] that allows to obtain an explicit description of the polarized deformation equivalence [9, Theorem 4.2]. These two ingredients are true. The 3rd argument consists in the construction of explicit examples of uniruled divisors on the generalized Kummer variety associated with a polarized abelian surface $(A,H_A)$ with $\text{NS}(A) = \mathbb{Z}H_A$ such that $p_a(H_A) \geq g \geq 2$. The construction is also correct, but the examples that we provided cannot yield all the possible primitive polarizations, as we tacitly and erroneously assumed in [9]. Even without taking the monodromy orbit into account, this is simply because it may happen that the number $2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}, \epsilon = 0, 1$ is positive even with $p < g$, which obviously renders our geometric argument empty. Indeed, the rational curves are constructed as $g_1^n$ on the normalization of a nodal curve of geometric genus $g$ lying in the hyperplane linear system $|H_A|$, which is supposed to have $p_a(H_A) = p$. We also take the occasion of this note to provide the full proof (Proposition 1.1) of a technical point that we claimed in [9, Section 4.2] to follow from a dimension count as
in [12, Example 4.1, 3]). The statement is correct, but the argument cannot be the same as in [12, Example 4.1, 3]] because we deal here with a locally closed subset (the Severi variety) of a complete linear system and not with the complete linear system.

The $K3^{[n]}$-type case, initially treated in [1], is subject to the same considerations and will be treated in [2].

We realized our mistake after the appearance of [10], which provides counterexamples in the $K3^{[n]}$-case that apply exactly in all cases not covered by the similar geometric constructions for the Hilbert scheme of points on a general projective $K3$. Contrary to the $K3^{[n]}$-type case as far as we know there are no known counterexamples to the existence of uniruled divisors ruled by a primitive curve class in the $K_n(A)$-type case. Nevertheless, we have no reasons to believe that the $K_n(A)$-type case could be exempt from this type of sporadic pathologies.

1 Existence of Uniruled Divisors on $K_n(A)$

In [9, Section 4.2 “Examples”) we claimed that “the natural map from $\tilde{\mathcal{C}}_{g+1} \to A^{[g+1]}$ is finite onto its image” invoking a dimension count made in [12, Example 4.1, 3]]. However, the same argument cannot work because we do not work with the full continuous system but with a locally closed subset (the Severi variety). Hence, we take the occasion to provide a full proof of that statement in the following.

Proposition 1.1. Let $g$ be an integer $\geq 2$ and $(A, H_A)$ be a general polarized abelian surface with $p_a(H_A) =: p \geq g$. Then $A^{[g+1]}$ contains a uniruled divisor covered by the $g_{g+1}$ on nodal genus $g$ curves in the continuous system $(H_A)$.

Proof. To prove the statement we can actually work over a very general polarized abelian surface, so let us suppose that $NS(A) =\mathbb{Z}H_A$. We will prove this statement by induction on $g$. It is sufficient to show it on the symmetric product of $A$.

Observe that, by [4, Theorem. 1.1], for all $2 \leq g \leq p_a(H_A)$, that the Severi variety parametrizing nodal genus $g$ curves inside $(H_A)$ is non-empty of the expected dimension $g$.

It is sufficient to show the claim on the symmetric product $A^{(g+1)}$ of $A$. More precisely, we will prove the following statement: there exists an irreducible component $V$ of the (Zariski closure of the) Severi variety parametrizing nodal genus $g$ curves inside $(H_A)$ such that, if $\mathcal{C}_V \to V$ denotes the universal curve and $\mathcal{C}_V^{(g+1)} \to V$ the relative symmetric product, the natural morphism

$$\mathcal{C}_V^{(g+1)} \to A^{(g+1)}$$
is generically finite onto its image. Note that this is equivalent to saying that \((g + 1)\) generic points on a generic curve of the family lie only on a finite number of curves of the family.

Indeed as

\[
\dim C^{(g+1)} = \text{relind}(C^{(g+1)}) + \dim V = (g + 1) + g = 2g + 1
\]

it follows that the image is a divisor inside \(A^{(g+1)}\). Since the \(k\)-th symmetric product of a curve is uniruled for \(k\) greater than the genus of the curve, as a by-product we have that such divisor is uniruled.

Note also that positive dimensional fibers of the morphism \(C^{(g+1)} \to A^{(g+1)}\) cannot lie in a fiber of \(C^{(g+1)} \to V\), as \(C^{(g+1)}_{t}\) injects into \(A^{(g+1)}\) for every \(t \in V\).

We start with the case \(g = 2\). Let \(C\) be one of the (finitely many) nodal curves of geometric genus 2 inside the linear system \(|H_{A}|\). In this case the points of the component \(V\) of the Severi variety containing \(C\) are given by all the translates of \(C\). The 3rd symmetric product \(C^{(3)}\) injects as a three-dimensional subvariety inside \(A^{(3)}\). The action of \(A\) on \(C^{(3)}\) by translation has no positive-dimensional stabilizer (as \(A\) is general, hence simple). Therefore the orbit of \(C^{(3)}\) under this action is a divisor. Using the notation above such divisor is the image of \(C^{(2+1)}_{V}\) in \(A^{(3)}\).

By inductive hypothesis, there exists an irreducible component \(W\) of the (Zariski closure of the) Severi variety parametrizing nodal genus \(g - 1\) curves inside \(\{H_{A}\}\) such that, if \(C_{W} \to W\) denotes the universal curve and \(C^{(g)}_{W} \to W\) the relative symmetric product, the natural morphism

\[
C^{(g)}_{W} \to A^{(g)}
\]

is generically finite onto its image.

Now let \(V\) be (the Zariski closure of) an irreducible component of the Severi variety of nodal genus \(g\) curves in \(\{H_{A}\}\) obtained by smoothing one node of the curves in \(W\) (which can be done by the regularity of the Severi variety, [3, Example 1.3]). By construction \(W \subset V\). Let \(C_{V} \to V\) be the universal curve. Its restriction over \(W\) yields a map \(C_{W} \to W\). Let \(D\) be the image of the morphism

\[
C^{(g+1)}_{V} \to A^{(g+1)}.
\]

Observe that \(D\) contains the image \(D_{W}\) of

\[
C^{(g+1)}_{W} \to A^{(g+1)}.
\]
We claim that by the inductive hypothesis $D_W$ has codimension 2, or, equivalently, that the morphism $\varphi^{(g+1)}_W \to A^{(g+1)}$ is generically finite onto its image. Indeed if $\xi = x_1 + \ldots + x_{g+1}$ is a generic point of the image, then, say, $x_1 + \ldots + x_g$ is a generic point of the image of the morphism $\varphi^{(g)}_W \to A^{(g)}$. By the inductive hypothesis the points $x_1, \ldots, x_g$ lie on finitely many curves of the family $W$, a fortiori that will be true for $x_1, \ldots, x_g, x_{g+1}$ and the claim follows.

We want to prove that $D$ contains $D_W$ strictly. If this were not the case, by irreducibility, we would have $D = D_W$. Let $U \subset D$ be an open subset over which the morphisms $\varphi^{(g+1)}_W \to A^{(g+1)}$ and $\varphi^{(g+1)}_V \to A^{(g+1)}$ are smooth, and let $p_1 + p_2 + \cdots + p_{g+1}$ be a point in $U$. Let $C$ be a nodal genus $g$ curve in $V$ containing these points. Let us fix the 1st $g$ points $p_1, \ldots, p_g$. By induction these points are contained inside a finite number of curves of genus $g - 1$ belonging to $W$. Let $B_1, \ldots, B_m$ be all such curves. Let $U_C \subset C$ be an open subset such that for all $q \in U_C$ we have $p_1 + \ldots + p_g + q \in U$. As we have seen above $p_1, \ldots, p_g, q$ lie on finitely many curves of genus $g - 1$ belonging to $W$, and these curves must be $B_1, \ldots, B_m$. Therefore, as $q$ varies in $U_C$, we deduce that $U_C$ is a subset of a finite union of genus $(g - 1)$ curves. As $C$ is irreducible, there is an $i$ such that $C = B_i$, which is clearly a contradiction. Therefore, $D$ must strictly contain $D_W$ and be a divisor, which is necessarily uniruled. ■

The rest of the proof remains the same and we refer the reader to [9] for the details.

2 Where It Does Not Work

In this section we prove that, for every dimension, there is at most a finite number of components of the moduli space of polarized manifolds $(X, H)$ of $K_n(A)$-type where the strategy of the previous section does not work. The uniruled divisors we constructed have a cohomology class that is a multiple of $H_A - (2g)\tau$ (or $H_A - (2g - 1)\tau$) where $2p - 2 = H^2_A$ and $H_A$ is the primitive polarization on the abelian surface. We have the following:

Proposition 2.1. Let $X$ be a projective irreducible holomorphic symplectic variety of $K_n(A)$-type. Let $C \in H_2(X, \mathbb{Z}) \cap N_1(X)$ be a primitive class such that its square $q(C)$ with respect to the Beauville–Bogomolov form is $> n + 1$. Then, the class $C$ is deformation equivalent to the class of one of the curves constructed in the previous section.

Proof. We know by [9, Theorem 4.2] that $C$ is deformation equivalent to either $H_A - 2g\tau$ or $H_A - (2g - 1)\tau$, with $g \leq n + 1$. If $q(C) > n + 1$, the square of $H_A - (2n + 2)\tau$ is positive,
that is, $H_A^2 = 2p - 2$ with $p > n + 1$. Thus, $p > n + 1 \geq g$, which means that $H_A - 2g\tau$ can be represented by the class of a $q^1_{g+1}$ on a nodal curve in $[H_A]$. $lacksquare$

**Corollary 2.2.** Let $\mathcal{M}_n$ be the moduli space of all polarized manifolds of $K_n(A)$-type with $n$ fixed. Then, the number of components of $\mathcal{M}_n$ whose general points $(X, H)$ do not have a uniruled divisor ruled by a rational curve of primitive class is at most finite.

**Proof.** The components of $\mathcal{M}_n$ are in bijective correspondence with the monodromy orbits of a given class of positive square in $L_n := U^3 \oplus (-2n - 2) \cong H^2(X, \mathbb{Z})$; see [11, Theorem 2.8].

For a fixed square of $H$, there is a finite number of orbits (computed again in [11, Theorem 2.8]), so it follows that if $X$ has a uniruled divisor when $q(H)$ is big enough, our claim will hold. The dual curve to $H$ is given by $H/\text{div}(H)$, where $\text{div}(H)$ is the divisibility of $H$, which is the positive generator of the ideal $q(H, H^2(X, \mathbb{Z}))$. The divisibility is at most $2n + 2$; therefore, if $q(H) \geq (2n + 2)^2(n + 1)$ the dual curve has square at least $n + 1$, so that Proposition 2.1 applies and our claim follows. $lacksquare$

**Corollary 2.3.** Let $X$ be a projective manifold of $K_n(A)$-type with Picard rank at least two. Then $X$ has an ample divisor ruled by primitive rational curves.

**Proof.** Since $X$ is projective and has Picard rank at least two, its Picard lattice is indefinite and contains primitive elements of positive arbitrary Beauville–Bogomolov square and so does the ample cone. Let $H$ be an ample divisor such that $q(H) \geq (2n + 2)^2(n + 1)$. Let $C$ be its dual curve in $H_2(X, \mathbb{Z})$. As the divisibility of $H$ is at most $2n + 2$ it follows that $q(C) \geq n + 1$, and Proposition 2.1 yields our claim. $lacksquare$

**Remark 2.4** The estimate of Proposition 2.1 is definitely not sharp; indeed, all primitive curves of positive square on manifolds of $K_n(A)$-type with $n \leq 5$ are deformation equivalent to the curves we construct in Proposition 1.1. Indeed, by [9, Theorem 4.2] we can suppose that our pair $(X, C)$ with $q(C) > 0$ is $(K_n(A), H_A - \mu\tau)$ with $0 \leq \mu \leq n + 1$ and $A$ is an abelian surface of genus $p$. The class $H_A - \mu\tau$ is given by the class of the rational curves constructed in Proposition 1.1, which have class $H_A - 2g\tau$, with the eventual addition of a tail of class $\tau$, so that $2g \leq n + 2$. By contradiction let us suppose that $g > p$ and $n \leq 5$. We have $q(H_A - 2g\tau) = 2p - 2 - 2\frac{q^2}{n+1} \leq 2p - 2 - 2\frac{(p+1)^2}{n+1} \leq 2p - 2 - 2\frac{(p+1)^2}{6}$. However, the last value is never positive; hence, $q(H_A - 2g\tau)$ cannot be positive and we reach a contradiction. Analogously, for $C = H_A - (2g - 1)\tau$, we have $q(C) \leq \frac{20p - 25 - 4p^2}{12}$ with $g \geq p + 1$ and $2g \leq n + 2$, which is again not positive.
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