HAL open science

Deformations of singular symplectic varieties and termination of the log minimal model program

Christian Lehn, Gianluca Pacienza

To cite this version:

Christian Lehn, Gianluca Pacienza. Deformations of singular symplectic varieties and termination of the log minimal model program. Algebraic Geometry, 2016, 3 (4), pp.392-406. 10.14231/AG-2016-018 . hal-03256222

HAL Id: hal-03256222

https://hal.science/hal-03256222v1

Submitted on 10 Jun 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Deformations of singular symplectic varieties and termination of the log minimal model program

Christian Lehn and Gianluca Pacienza

Abstract

We generalize Huybrechts' theorem on deformation equivalence of birational irreducible symplectic manifolds to the singular setting. More precisely, under suitable natural hypotheses, we show that two birational symplectic varieties are locally trivial deformations of one another. As an application we show the termination of any log-minimal model program for a pair (X, Δ) of a projective irreducible symplectic manifold X and an effective $\mathbb{R}$-divisor Δ. To prove this result we follow Shokurov's strategy and show that LSC and ACC for mlds hold for all the models appearing along any log-MMP of the initial pair.

1. Introduction

In the theory of irreducible symplectic manifolds an important result due to Huybrechts [Hu03, Theorem 2.5] insures that two birational[1] such manifolds X and X' are deformation equivalent. Even more is true, namely there exist smooth proper families $\pi : \mathcal{X} \longrightarrow S$ and $\pi' : \mathcal{X}' \longrightarrow S$ over a pointed disk $0 \in S$ and a birational map between $\mathcal{X}$ and $\mathcal{X}'$ which is an isomorphism over $S \setminus 0$ and coincides with the given birational map between $X = \pi^{-1}(0)$ and $X' = \pi'^{-1}(0)$ over 0. In particular, X and X' have isomorphic Hodge structures. Huybrechts' result yields a characterization of non-separated points in the moduli space of marked irreducible symplectic manifolds. Apart from its theoretical relevance, it has also been successfully applied to solve concrete problems (see e.g. [Be99, De99, AL14, Le15]).

2010 Mathematics Subject Classification 14E30, 14E05, 32S05, 32G05, 14B07

Keywords: minimal model program, deformation, local triviality, minimal log discrepancy, termination of flips, ACC and LSC conjectures, irreducible symplectic manifolds, symplectic variety

C.L. was supported by the DFG through the research grant Le 3093/1-1 while enjoying the hospitality of the IMJ, Paris, and during the revision of the article by the DFG research grant Le 3093/2-1. G.P. was partially supported by the University of Strasbourg Institute for Advanced Study (USIAS), as part of a USIAS Fellowship, and by the ANR project "CLASS" no. ANR-10-JCJC-0111.

[1] We will use the term birational instead of bimeromorphic also in the analytic context.

Recently, there has been renewed interest in the study of singular symplectic varieties. For instance, there is a very interesting line of research started by Greb, Kebekus and Peternell on varieties with numerically trivial canonical divisor and singularities that appear in the MMP, see [GKP]. On the other hand, singular symplectic varieties play an important role also in the study of smooth symplectic varieties, see e.g. [DV10, OG99, OG03]. Given the importance of Huybrechts' theorem to the theory of irreducible symplectic manifolds it is natural to ask whether a *singular* version of this result holds true.

Before stating our result recall that since the work of Beauville [Be00] there is a well-established notion of singular symplectic variety (see § 2). Let X and X' be singular symplectic varieties which are birational with one another. If they both admit crepant resolutions by irreducible symplectic manifolds, which is for example the case if X and X' show up in the log-MMP of a given irreducible symplectic manifold (cf. Lemma 4.1), then also their crepant resolutions are birational and hence deformation equivalent thanks to Huybrechts' theorem. By work of Namikawa this implies that also the two birational singular symplectic varieties will be deformation equivalent, but by construction this deformation does not preserve the singularity type. The question arises under which circumstances we may find a deformation that connects X and X' and preserves the singularities. Our main result gives precise conditions under which this is possible.

Theorem 1.1. *Let X and X' be $\mathbb{Q}$-factorial projective symplectic varieties having crepant resolutions by irreducible symplectic manifolds and suppose that $\phi : X \dashrightarrow X'$ is a birational map which is an isomorphism in codimension 1. Then there exist proper families $\mathcal{X} \longrightarrow S$ and $\mathcal{X}' \longrightarrow S$ of locally trivial deformations of X and X' over a pointed disk $0 \in S$ and a birational map between $\mathcal{X}$ and $\mathcal{X}'$ which is an isomorphism over $S \setminus 0$ and coincides with ϕ over 0. In particular, X and X' are homeomorphic and their local analytic isomorphism type is the same.*

The theorem says that just as in the smooth case X and X' are non-separated points in the space of marked symplectic varieties with fixed underlying topological space. Note that we can neither drop the $\mathbb{Q}$-factoriality hypothesis, as every small contraction on a smooth symplectic manifold would give a counter-example, nor the hypothesis that ϕ is an isomorphism in codimension 1, as every divisorial contraction from a smooth X to a singular X' would give a counter-example. Thus, our result is optimal. To the best of our knowledge, it is the first result giving information on singularity preserving deformations of birational singular symplectic varieties. Of course one can still ask whether the conclusion holds for arbitrary symplectic varieties, i.e., not necessarily possessing irreducible symplectic crepant resolutions. This is likely to be true but for the moment out of reach for technical reasons.

The proof of Theorem 1.1 heavily relies on Namikawa's foundational work on deformation theory of singular symplectic varieties and on his comparison results of these deformations with those of crepant resolutions, cf. [Na01, Na06, Na10]. Another important tool is Kaledin's local structure

theorem of symplectic singularities, see [Kal06, Theorem 2.3]. These are the two pillars for the main new technical contribution of this work, which is the proof of smoothness of certain deformation spaces (cf. Proposition 2.3). Together with Huybrechts' original strategy, whose simple geometric idea guides through the technicalities, this is an essential ingredient in our proof of our main result, Theorem 1.1. The unobstructedness result is obtained by invoking Ran's T^1-lifting principle [Ra92, Ka92, Ka97]. More precisely, given an irreducible symplectic crepant resolution $\pi : \tilde{X} \longrightarrow X$ of a symplectic variety X, thanks to the T^1-lifting principle we can relate the locally trivial deformations of X to those deformations of $\tilde{X}$ preserving the irreducible components of the exceptional locus of π (see Proposition 2.3 for the precise statement).

The second part of the paper is devoted to present our major application of Theorem 1.1; it is concerned with the minimal model program (MMP)[2]. Though there has been a lot of progress, its most important goals, finding good representatives, so-called *minimal models*, in every birational equivalence class of algebraic varieties, and connecting a given variety X to one of its minimal models by elementary birational transformations, have not yet been completely accomplished. The existence of minimal models as well as the termination of certain special MMPs have been established in many cases in the seminal paper [BCHM10], see also [CL10]. What is missing in general is termination of flips.

We show here that irreducible symplectic manifolds behave as good as possible with respect to the MMP. To put our result into perspective notice that termination of log-flips has been shown for irreducible symplectic manifolds by Matsushita-Zhang [MZ13] (see also [Ma12]) following a strategy due to Shokurov [ShoV] (see below). However, the termination of log-flips does not imply that every MMP on a symplectic manifold terminates, for smoothness plays a crucial role in Matsushita-Zhang's argument. For example, if the MMP produces not only flips but also divisorial contractions, the resulting variety will acquire singularities and then there could still be an infinite sequence of flips. As an application of Theorem 1.1, we show that this does not happen.

Theorem 1.2. *Let X be a projective irreducible symplectic manifold and let Δ be an effective $\mathbb{R}$-Cartier divisor on X, such that the pair (X, Δ) is log-canonical. Then every log-MMP for (X, Δ) terminates in a minimal model (X', Δ') where X' is a symplectic variety with canonical singularities and Δ' is an effective, nef $\mathbb{R}$-Cartier divisor.*

It is well-known that from the previous result one derives the following (see [Bi12] for the relevant definitions and further developments).

Corollary 1.3. Let X be a projective irreducible symplectic manifold and let Δ be an effective $\mathbb{R}$-Cartier divisor on X. Then birationally Δ has a Zariski decomposition in the sense of Fujita

[2]Strictly speaking, rather its logarithmic version (log-MMP). We will however mostly use the term MMP for simplicity.

and in the sense of Cutkosky-Kawamata-Moriwaki.

The proof of Theorem 1.2 follows Shokurov's strategy. Let us go a little more into detail. To show the termination of flips, Shokurov introduced the so-called *minimal log discrepancy* (mld for short), which is a local invariant associated to (X, Δ) and which increases under flips. It is nowadays interpreted as an invariant of the singularity of (X, Δ) at a given point. Ambro and Shokurov have made two strong conjectures about the behaviour of mlds. These are the lower semi-continuity conjecture (LSC) and the ascending chain condition conjecture (ACC), see paragraph 3.4. Shokurov proved that these two conjectures imply termination of flips [ShoV]. For smooth varieties, LSC holds by the fascinating paper [EMY03] and if all varieties in a sequence of flips are smooth, ACC holds for trivial reasons. However, even if we start with a smooth variety X, the MMP easily carries us out of the class of smooth varieties. Matsushita-Zhang's key point is that a flip of a smooth symplectic variety remains smooth by deep results of Namikawa [Na06], see section 4 for more details.

The ACC and LSC conjectures seem to be out of reach for arbitrary varieties. Starting with some variety X and running an MMP might a priori produce a huge variety of different singularities. Nevertheless, if we can bound the class of singularities of varieties that show up in intermediate steps of the MMP, then there is hope that Shokurov's strategy can be used. In our case, as recalled before, this class of varieties will be the class of proper varieties with symplectic singularities which have a crepant resolution by an irreducible symplectic manifold. We first prove the following result.

Theorem 3.8. *Let Y be a normal projective $\mathbb{Q}$-Gorenstein variety and let Δ be an effective $\mathbb{R}$-Cartier divisor on Y such that (Y, Δ) is log-canonical. If $\pi : X \longrightarrow Y$ is a crepant morphism and LSC holds on X, then LSC holds for (Y, Δ).*

Then we use in a crucial way Theorem 1.1 to show that in a sequence of flips of singular symplectic varieties the singularities, more precisely, their local analytic isomorphism type *does not change*. This allows us to invoke a result of Kawakita [Kaw12] to deduce that ACC holds along any log MMP of an irreducible symplectic manifold and conclude.

The paper is organised as follows: in § 2 we prove Theorem 1.1. Then we turn to the application and prove Theorem 3.8 in § 3, after having recalled the basic definitions and results on minimal log-discrepancies. Finally, we put all the ingredients together and show how to deduce termination in § 4.

Acknowledgements

It is a pleasure to thank Sébastien Boucksom, Stéphane Druel, Masayuki Kawakita, James McKernan and Yusuke Nakamura for helpful discussions. We thank Florin Ambro and Caucher

Birkar for answering questions by e-mail. Furthermore, we would like to thank Vladimir Lazić for giving us the opportunity to present the material at the winter school *Birational methods in hyperkähler geometry* in December 2014 in Bonn.

2. Deformations

We work over the field of complex numbers. A deformation of a variety Y is a flat morphism $\mathscr{Y} \longrightarrow S$ of complex spaces to a pointed space $(S, 0)$ such that the fiber $\mathscr{Y}_0$ over $0 \in S$ is (isomorphic to) Y. We will mostly work with space germs, i.e., equivalence classes of deformations where two deformations $\mathscr{Y} \longrightarrow S$ and $\mathscr{Y}' \longrightarrow S'$ are equivalent if S and S' are isomorphic in some small neighborhoods of their distinguished points and moreover $\mathscr{Y}$ and $\mathscr{Y}'$ are isomorphic in neighborhoods of $\mathscr{Y}_0$ and $\mathscr{Y}_0'$ and in a way which is compatible with the maps to the base. We will often use representatives of these equivalence classes and shrink them if necessary without mention. A *symplectic variety* is a normal projective variety X admitting an everywhere non-degenerate closed 2-form ω on the regular locus X_{reg} of X such that, for any resolution $f : \tilde{X} \longrightarrow X$ with $f^{-1}(X_{reg}) \cong X_{reg}$ the 2-form ω extends to a regular 2-form on $\tilde{X}$. An *irreducible symplectic manifold* is a simply connected compact Kähler manifold X admitting an everywhere non-degenerate closed 2-form ω such that $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \omega$.

In this section we are going to prove Theorem 1.1, which should be interpreted as an analogue of the well-known result of Huybrechts [Hu03, Theorem 2.5]. This is the only section where we make use of the complex numbers, which however does not seem to be essential and results as well as proofs should carry over mutatis mutandis to any algebraically closed field of characteristic zero.

The proof relies on Ran's T^1-lifting principle [Ra92, Ka92, Ka97]. Essentially it says that a given deformation problem is unobstructed if the tangent space T_X^1 to the deformation space is "deformation invariant" in the sense that for every small deformation $\mathcal{X} \longrightarrow S$ of X its relative versions $T^1_{\mathcal{X}/S}$ are free $\mathcal{O}_S$-modules. We refer to [GHJ, § 14] for a concise account. Recall e.g. from [Se06, 1.2] that the tangent space to the deformation functor $\mathrm{Def}_X^{\mathrm{lt}}$ of locally trivial deformations of an algebraic variety X is $H^1(T_X)$, opposed to arbitrary deformations, where the tangent space is $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$. An obstruction space for $\mathrm{Def}_X^{\mathrm{lt}}$ is given by $H^2(T_X)$.

Let us recall the following well-known result on the local structure of singular symplectic varieties. For convenience we sketch the proof which is due to Kaledin and Namikawa, see [Na10]. Note that by convention we consider the singular locus as a subscheme (or complex subspace) with the induced reduced structure.

Proposition 2.1. Let X be a symplectic variety and let $\Sigma \subset X$ be the singular locus of X^{sing}. Then $\mathrm{codim}_X \Sigma \geqslant 4$ and every $x \in U := X \setminus \Sigma$ has a neighbourhood which is locally analytically isomorphic to $(\mathbb{C}^{2n-2}, 0) \times (S, p)$ where $2n = \dim X$ and (S, p) is the germ of a smooth point or a rational double point on a surface. This isomorphism can be chosen to preserve the symplectic

structure.

Proof. Kaledin's result [Kal06, Theorem 2.3] implies that Σ has codimension $\geqslant 4$ and that every point of U admits the sought for product decomposition in the formal category. By [Ar69, Corollary 2.6] the decomposition exists analytically. The last statement is [Na10, Lemma 1.3]. $\square$

Proposition 2.2. Let X be a $\mathbb{Q}$-factorial compact symplectic variety, let $\pi : \tilde{X} \longrightarrow X$ be a crepant resolution by a compact Kähler manifold $\tilde{X}$ and let $U \subset X$ be as in Proposition 2.1. Then the restriction $H^1(X, T_X) \longrightarrow H^1(U, T_U)$ is an isomorphism and $h^1(T_X) = h^1(T_{\tilde{X}}) - m$ where m is the number of irreducible components of the exceptional divisor of π.

Proof. The exceptional set of π is a divisor by the $\mathbb{Q}$-factoriality hypothesis and each of its irreducible components meets $\pi^{-1}(U)$ by the semi-smallness property of symplectic resolution, cf. [Kal06, Lemma 2.11]. Let us consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(T_X) & \longrightarrow & Ext^1(\Omega_X, \mathcal{O}_X) & \longrightarrow & H^0(T^1_X) \\ & & \downarrow & & \downarrow \phi & & \downarrow \\ 0 & \longrightarrow & H^1(T_U) & \longrightarrow & Ext^1(\Omega_U, \mathcal{O}_U) & \longrightarrow & H^0(T^1_U) \end{array}$$

with exact lines, where ϕ is an isomorphism by [Na01, Proposition 2.1]. As $T^1_X = \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$, the space $H^1(T_X)$ consists of extensions that are locally split. Analyzing the construction of the map ϕ described in [KM92, (12.5.6) Lemma] we see that an extension on X is locally split if and only if its restriction to U is and thus $H^1(T_X) \longrightarrow H^1(T_U)$ is an isomorphism.
Put $\tilde{U} := \pi^{-1}(U) \subset \tilde{X}$ and consider the following sequence

$$0 \longrightarrow H^1(\pi_* T_{\tilde{U}}) \longrightarrow H^1(T_{\tilde{U}}) \longrightarrow H^0(R^1\pi_* T_{\tilde{U}}) \longrightarrow 0 \tag{2.1}$$

It is exact and we have $h^1(T_U) = h^1(T_{\tilde{U}}) - m$ as was shown in (ii) of the proof of [Na01, Theorem 2.2]. As $h^1(T_{\tilde{U}}) = h^1(T_{\tilde{X}})$ by [Na01, Proposition 2.1], the claim follows. $\square$

In the situation of Theorem 1.1, let $\pi : \tilde{X} \longrightarrow X$ be a crepant resolution and let $D = \sum_{i=1}^m D_i$ be the exceptional divisor with its decomposition into irreducible components D_i. We put $L_i := \mathcal{O}_{\tilde{X}}(D_i)$ and denote by $\tilde{\mathscr{X}} \longrightarrow \mathrm{Def}(\tilde{X})$ the universal deformation of $\tilde{X}$. This is the germ of a smooth space of dimension $h^{1,1}(\tilde{X})$ by the Bogomolov-Tian-Todorov theorem. We consider the following subspaces of $\mathrm{Def}(\tilde{X})$:

- $\mathrm{Def}(\tilde{X}, \underline{L}) \subset \mathrm{Def}(\tilde{X})$ is the base of the universal deformation of $(\tilde{X}, L_1, \ldots, L_m)$, see [Hu99, (1.14)]. As the D_i define linearly independent classes in $H^2(\tilde{X}, \mathbb{C})$, $\mathrm{Def}(\tilde{X}, \underline{L})$ is smooth and of codimension m in $\mathrm{Def}(\tilde{X})$ by loc. cit.
- $\mathrm{Def}(\tilde{X}, \underline{D}) \subset \mathrm{Def}(\tilde{X})$ is the image of the components containing all D_i of the relative Douady space $\mathscr{D}(\tilde{\mathscr{X}} / \mathrm{Def}(\tilde{X})) \longrightarrow \mathrm{Def}(\tilde{X})$. This is the space where all components D_i deform along with $\tilde{X}$.

We clearly have $\mathrm{Def}(\tilde{X}, \underline{D}) \subset \mathrm{Def}(\tilde{X}, \underline{L})$ Consequently, $\dim \mathrm{Def}(\tilde{X}, \underline{D}) \leqslant h^{1,1}(\tilde{X}) - m$.

The key step will be to prove the smoothness of the space of locally trivial deformations of the singular variety X. Recall from [FK87] that the universal locally trivial deformation of X exists and that it is just the restriction of the universal deformation to the locally trivial locus $\mathrm{Def}^{\mathrm{lt}}(X) \subset \mathrm{Def}(X)$ in the Kuranishi space which is a closed subspace.

Proposition 2.3. Let $\pi : \tilde{X} \longrightarrow X$ be as above. Let $\tilde{\mathscr{X}} \longrightarrow \mathrm{Def}(\tilde{X}, \underline{D})$ and $\mathscr{X} \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X)$ be the universal deformations. Then there is a diagram

$$\begin{tikzcd} \tilde{\mathscr{X}} \arrow[r, "\Pi"] \arrow[d] & \mathscr{X} \arrow[d] \\ \mathrm{Def}(\tilde{X}, \underline{D}) \arrow[r, "\pi_*"] & \mathrm{Def}^{\mathrm{lt}}(X) \end{tikzcd} \tag{2.2}$$

with the following properties:

(i) $\mathrm{Def}^{\mathrm{lt}}(X)$ is smooth of dimension $h^{1,1}(\tilde{X}) - m$.

(ii) π_* is the restriction of the natural finite morphism $\mathrm{Def}(\tilde{X}) \longrightarrow \mathrm{Def}(X)$ and π_* is an isomorphism.

(iii) $\dim \mathrm{Def}(\tilde{X}, \underline{D}) = h^{1,1}(\tilde{X}) - m$, in particular $\mathrm{Def}(\tilde{X}, \underline{D}) = \mathrm{Def}(\tilde{X}, \underline{L})$.

Proof. We will first show that $\mathrm{Def}^{\mathrm{lt}}(X)$ is smooth. Let $U \subset X$ be as in Proposition 2.1. The restriction $H^1(T_X) \longrightarrow H^1(T_U)$ is an isomorphism by Proposition 2.2, in other words, deformations and their local triviality are determined on U. Let $j : X^{\mathrm{reg}} \longrightarrow U$ denote the inclusion. As T_U is reflexive, we have that $T_U \cong j_* T_{X^{\mathrm{reg}}}$. Hence, $H^1(T_X) = H^1(U, j_*\Omega_{X^{\mathrm{reg}}}) = \mathbb{H}^2(U, j_*\Omega^{\geqslant 1}_{X^{\mathrm{reg}}})$ which is deformation invariant, as we will show next. Consider the exact sequence of complexes

$$0 \longrightarrow j_*\Omega^{\geqslant 1}_{X^{\mathrm{reg}}} \longrightarrow j_*\Omega^{\bullet}_{X^{\mathrm{reg}}} \longrightarrow \mathcal{O}_U \longrightarrow 0. \tag{2.3}$$

By Grothendieck's theorem for V-manifolds $\mathbb{H}^k(j_*\Omega^{\bullet}_{X^{\mathrm{reg}}}) = H^k(U, \mathbb{C})$, see for example the footnote in the proof of [Na06, Proposition (1.11)]. Moreover, we have

$$H^1(\mathcal{O}_U) = H^1(\mathcal{O}_X) = H^1(\mathcal{O}_{\tilde{X}}) = 0,$$

where the first equality holds because X is Cohen-Macaulay and $\mathrm{codim}_X(X \setminus U) \geqslant 4$ and the second because X has rational singularities. In the same way, one finds

$$H^2(\mathcal{O}_U) = H^2(\mathcal{O}_X) = H^2(\mathcal{O}_{\tilde{X}}) \cong \mathbb{C}.$$

so that (2.3) gives an exact sequence

$$0 \longrightarrow H^1(j_*\Omega_{X^{\mathrm{reg}}}) \longrightarrow H^2(U, \mathbb{C}) \longrightarrow H^2(\mathcal{O}_U) \longrightarrow 0,$$

where the last map is surjective because the composition $H^2(\tilde{X}, \mathbb{C}) \longrightarrow H^2(\mathcal{O}_{\tilde{X}}) \xrightarrow{\cong} H^2(\mathcal{O}_U)$ is. The same line of arguments works identically in a relative situation and shows that $H^1(T_{\mathcal{X}/S}) = H^1(j_*\Omega_{(\mathcal{X}/S)^{\mathrm{reg}}})$ is a free $\mathcal{O}_S$-module for any small deformation $\mathcal{X} \longrightarrow S$ over a local artinian

scheme S. In other words, the tangent space to the deformation functor $H^1(T_X)$ is deformation invariant, hence by the T^1-lifting argument $\mathrm{Def}^{\mathrm{lt}}(X)$ is smooth. In particular, $\dim \mathrm{Def}^{\mathrm{lt}}(X) = \dim H^1(T_X)$ which is equal to $h^{1,1}(\tilde{X}) - m$ by Proposition 2.2.

As explained in [Na06, § 3] there is a diagram as (2.2) for arbitrary instead of locally trivial deformations. In particular, there is a finite map $\pi_* : \mathrm{Def}(\tilde{X}) \longrightarrow \mathrm{Def}(X)$ and for each $t \in \mathrm{Def}^{\mathrm{lt}}(X)$ and every $s \in \mathrm{Def}(\tilde{X})$ mapping to t the morphism $\tilde{\mathscr{X}_s} \longrightarrow \mathscr{X}_t$ is a crepant resolution. By abuse of notation we will denote by $\mathscr{X} \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X)$ also the restriction of the universal family to the locally trivial locus $\mathrm{Def}^{\mathrm{lt}}(X)$. Let $\mathscr{S} \subset \mathscr{X}$ be the singular locus of the singular locus of the morphism $\mathscr{X} \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X)$, i.e., the relative version of the subvariety $\Sigma \subset X$ introduced in Proposition 2.1, and denote $\mathscr{U}$ its complement in $\mathscr{X}$. By the local triviality property, $\mathscr{S}$ is a locally trivial hence flat deformation of Σ, in particular, it is fiberwise of codimension $\geqslant 4$ in $\mathscr{X}$. If we denote $\widetilde{\mathscr{U}} := \Pi^{-1}(\mathscr{U})$, then $\widetilde{\mathscr{U}_s} \longrightarrow \mathscr{U}_t$ is a crepant resolution of singularities for $t = \pi_*(s)$. Note that by choice of $\mathscr{U}$ its singular locus is a locally trivial deformation of an ADE-surface singularity, thus $\widetilde{\mathscr{U}} \longrightarrow \mathscr{U}$ is (fiberwise) the unique minimal relative resolution. In particular, $\widetilde{\mathscr{U}}$ is a locally trivial deformation of its central fiber and the exceptional divisors of $\widetilde{\mathscr{U}} \longrightarrow \mathscr{U}$ are locally trivial deformations of those of its central fiber $\widetilde{U} \longrightarrow U$. As $\mathscr{S}$ is fiberwise of codimension $\geqslant 4$, by taking closures in $\widetilde{\mathscr{X}}$ we obtain deformations of the D_i over $\pi_*^{-1}(\mathrm{Def}^{\mathrm{lt}}(X))$ with irreducible fibers. Therefore, the inclusion $\pi_*^{-1}(\mathrm{Def}^{\mathrm{lt}}(X)) \subset \mathrm{Def}(\tilde{X}, \underline{D})$ holds. As $\dim \mathrm{Def}(\tilde{X}, \underline{D}) \leqslant h^{1,1}(\tilde{X}) - m$, this inclusion is an equality and the restriction of π_* to $\mathrm{Def}(\tilde{X}, \underline{D})$ gives the desired morphism. Also (iii) follows from this.

To see that $\pi_* : \mathrm{Def}(\tilde{X}, \underline{D}) \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X)$ is an isomorphism, it suffices to show that its differential $T_{\mathrm{Def}(\tilde{X},\underline{D}),0} \longrightarrow T_{\mathrm{Def}^{\mathrm{lt}}(X),0} = H^1(T_X)$ is so. We have just seen that $\mathrm{Def}(\tilde{X}, \underline{D}) = \mathrm{Def}(\tilde{X}, \underline{L})$ and by invoking [Hu99, (1.14)] once more, we see that $T_{\mathrm{Def}(\tilde{X},\underline{D}),0} = \ker\left(\mathrm{c}_1(\underline{L}) : H^1(T_{\tilde{X}}) \xrightarrow{\phi} H^2(\mathcal{O}_{\tilde{X}})^m\right)$ where ϕ is given by cup product with $\mathrm{c}_1(L_i)$ and contraction in the i-th component. As explained before, for dimension reasons we may replace X by U and $\tilde{X}$ by $\tilde{U} := \pi^{-1}(U)$ in all cohomolgies involved. The differential of $\pi_* : \mathrm{Def}(\tilde{X}) \longrightarrow \mathrm{Def}(X)$ is thus a map $H^1(T_{\tilde{U}}) \longrightarrow \mathrm{Ext}^1(\Omega_U, \mathcal{O}_U)$ whose restriction to the subspace $H^1(\pi_* T_{\tilde{U}}) \subset H^1(T_{\tilde{U}})$ identifies the latter with $H^1(T_U) \subset \mathrm{Ext}^1(\Omega_U, \mathcal{O}_U)$. Thus, it remains to show that $H^1(\pi_* T_{\tilde{U}}) = \ker \phi|_{\tilde{U}}$. We have already seen that under the symplectic form $H^1(T_{\tilde{U}}) \cong H^1(j_* \Omega_{X^{\mathrm{reg}}}) \subset H^2(U, \mathbb{C})$ and on the other side $\ker \phi$ is identified with the subspace of those $\alpha \in H^{1,1}(\tilde{X})$ which satisfy $q_{\tilde{X}}(\alpha, \mathrm{c}_1(L_i)) = 0$ for $i = 1, \ldots, m$ where $q_{\tilde{X}}$ is the Bogomolov-Beauville-Fujiki form by [Hu99, (1.8)]. Certainly, classes which are pullback from X are among them, for dimension reasons we have equality, which completes the proof. □

The following lemma is probably well-known, we include its proof for convenience.

Lemma 2.4. Let $\tilde{X}$ be an irreducible symplectic manifold and let $\pi : \tilde{X} \longrightarrow X$ be a proper birational morphism to a Kähler complex space X. If X carries a line bundle L such that

$q_{\tilde{X}}(\pi^*L) > 0$, then X is projective.

Proof. It follows from Huybrechts' projectivity criterion [Hu97, Theorem 3.11] that in such a situation $\tilde{X}$ is projective. Then X is Kähler, Moishezon and has rational singularities, hence is projective by [Na02, Theorem 1.6]. □

Proposition 2.5. Let $X \dashrightarrow X'$ be as in Theorem 1.1 and let $\mathscr{X} \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X)$ and $\mathscr{X}' \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X')$ be the universal locally trivial deformations of X respectively X' and let $\pi : \tilde{X} \longrightarrow X$, $\pi' : \tilde{X}' \longrightarrow X'$ be crepant resolutions of singularities. Then there is an isomorphism $\gamma : \mathrm{Def}^{\mathrm{lt}}(X) \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X')$ fitting into a commutative diagram

$$\begin{tikzcd} \mathrm{Def}(\tilde{X}, \underline{D}) \arrow[r, "\tilde{\gamma}"] \arrow[d, "\pi_*"'] & \mathrm{Def}(\tilde{X}', \underline{D}') \arrow[d, "\pi'_*"'] \\ \mathrm{Def}^{\mathrm{lt}}(X) \arrow[r, "\gamma"] & \mathrm{Def}^{\mathrm{lt}}(X') \end{tikzcd}$$

of isomorphisms such that for each $t \in \mathrm{Def}^{\mathrm{lt}}(X)$ we have a birational map $\phi_t : \mathscr{X}_t \dashrightarrow \mathscr{X}'_{\gamma(t)}$. For very general t, the map ϕ_t is an isomorphism.

Proof. As $\tilde{X} \dashrightarrow \tilde{X}'$ is an isomorphism in codimension 1, the local Torelli theorem gives an isomorphism $\tilde{\gamma} : \mathrm{Def}(\tilde{X}, \underline{D}) \longrightarrow \mathrm{Def}(\tilde{X}', \underline{D})$. The isomorphism γ is obtained from composition with the isomorphisms $\mathrm{Def}(\tilde{X}, \underline{D}) \xrightarrow{\pi_*} \mathrm{Def}^{\mathrm{lt}}(X)$ and $\mathrm{Def}(\tilde{X}', \underline{D}) \xrightarrow{\pi'_*} \mathrm{Def}^{\mathrm{lt}}(X')$ from Proposition 2.3 as $\gamma = \pi'_* \circ \tilde{\gamma} \circ (\pi_*)^{-1}$. As $\tilde{X}$ and $\tilde{X}'$ are birational by assumption, they are deformation equivalent by Huybrechts' result [Hu03, Theorem 2.5]. So for $s \in \mathrm{Def}(\tilde{X}, \underline{D})$ the fibers $\tilde{\mathscr{X}_s}$ and $\tilde{\mathscr{X}'_{s'}}$ with $s' = \tilde{\gamma}(s)$ are deformation equivalent and have the same periods, hence they are birational by Verbitsky's global Torelli theorem [Ve13, Theorem 1.17]. If we denote $t = \pi_*(s) \in \mathrm{Def}^{\mathrm{lt}}(X)$, $t' = \pi'_*(s') = \gamma(t) \in \mathrm{Def}^{\mathrm{lt}}(X')$, the morphisms $\pi_s : \tilde{\mathscr{X}_s} \longrightarrow \mathscr{X}_t$ and $\pi'_{s'} : \tilde{\mathscr{X}'_{s'}} \longrightarrow \mathscr{X}'_{t'}$ contract the same divisors and we obtain a birational map $\mathscr{X}_t \dashrightarrow \mathscr{X}'_{t'}$, which is isomorphic in codimension one. If t is close enough to $0 \in \mathrm{Def}^{\mathrm{lt}}(X)$, then $\mathscr{X}_t$ and $\mathscr{X}'_t$ are Kähler by [Na02, Proposition 5]. The last statement follows as the general projective deformation of X has Picard number one and birational maps between $\mathbb{Q}$-factorial K-trivial varieties of Picard number 1 are isomorphisms. Note that the subspace of $H^2(\tilde{X}, \mathbb{R})$ spanned by the classes $[D_1], \ldots, [D_m]$ is negative definite with respect to the Bogomolov-Beauville form q_X. This follows for example from [Bo04, Thm 4.5], see also [Dr11, Thm 1.3]. We can thus indeed always deform to projective varieties by [GHJ, Proposition 26.6] and Lemma 2.4. □

Proof of Theorem 1.1. The proof of [Hu99, Theorem 4.6] works with minor modifications to give a proof of Theorem 1.1. For convenience we sketch Huybrechts' argument with emphasis on where we have to argue differently. Let L' be an ample line bundle on X' and denote by L the $\mathbb{Q}$-line bundle on X obtained from L' by taking pullback to a resolution of indeterminacies, pushforward to X and double dual. Here we use $\mathbb{Q}$-factoriality. Replacing L' and L by multiples,

we may assume that L is a line bundle. Let us denote by $\pi : \tilde{X} \longrightarrow X$ a resolution of singularities where $\tilde{X}$ is an irreducible symplectic manifold and by D as before the exceptional divisor of π. Recall from Proposition 2.3 that $\mathrm{Def}^{\mathrm{lt}}(X) \cong \mathrm{Def}(\tilde{X}, \underline{D})$. Then points in the Kuranishi space $\mathrm{Def}^{\mathrm{lt}}(X, L) \cong \mathrm{Def}(\tilde{X}, \pi^*L, \underline{D})$ of the pair (X, L) parametrize deformations of X together with a line bundle whose pullback to a resolution of singularities by an irreducible symplectic manifold has positive Beauville-Bogomolov square; in particular, these are projective deformations by Lemma 2.4. We take a one-dimensional disk $S \subset \mathrm{Def}^{\mathrm{lt}}(X, L)$ which passes through the origin and which is very general in the sense that the fibers of the (restriction to S of the) universal family $\psi : \mathscr{X} \longrightarrow S$ for a very general $s \in S$ have Picard number one. Denote by $\mathscr{L}$ the universal line bundle restricted to $\mathscr{X}$, which is a deformation of L. Then as in [Hu99, Theorem 4.6] one shows that $\mathscr{L}_s$ is ample for very general $s \in S$ and that $h^0(\mathscr{X}_t, \mathscr{L}_t^m)$ is independent of t in a neighborhood of $0 \in S$ where we have to replace the Kodaira vanishing theorem by the one of Kawamata-Viehweg. Note that $H^i(X, L) = H^i(\tilde{X}, \pi^*L)$ for all i as X has rational singularities. We may now apply [Hu99, Proposition 4.5] which produces a deformation $\mathscr{X}' \longrightarrow S$ (maybe after shrinking S) together with a line bundle $\mathscr{L}'$ on $\mathscr{X}'$ on such that $(\mathscr{X}'_0, \mathscr{L}'_0) = (X', L')$ and an S-birational map $\mathscr{X} \dashrightarrow \mathscr{X}'$ which is an isomorphism outside the central fiber and the birational map $\phi : X \dashrightarrow X'$ we started with. Let us admit this result for a moment and let us see how to complete the proof. It remains to show that the deformation $\mathscr{X}' \longrightarrow S$ is locally trivial. For this we recall that the universal locally trivial deformation of X' is nothing else than the restriction of the universal deformation of X' to the locally trivial locus $\mathrm{Def}^{\mathrm{lt}}(X') \subset \mathrm{Def}(X')$ in the Kuranishi space. But the locally trivial locus is a Hodge locus by Proposition 2.3 and so we may deduce local triviality from comparison of the periods of $\mathscr{X}$ and $\mathscr{X}'$.

Let us now comment on the proof of [Hu99, Proposition 4.5]. It is contained in [Hu97, Proposition 4.2] and works roughly like this. By the hypothesis that $h^0(\mathscr{X}_t, \mathscr{L}_t)$ be independent of t we obtain that the coherent sheaf $\psi_*\mathscr{L}$ is locally free and its fiber at $t \in S$ is exactly $H^0(\mathscr{L}_t)$, at least after shrinking S. One considers the rational map $\varphi_{\mathscr{L}} : \mathscr{X} \dashrightarrow \mathbb{P}_S((\pi_*\mathscr{L})^*)$ and defines $\psi' : \mathscr{X}' \longrightarrow S$ to be its image. One easily shows that the general fiber of $\mathscr{X}' \longrightarrow S$ is irreducible and that $X' \subset \mathscr{X}'_0$. This inclusion is then shown to be an equality by comparing the number of sections of the tautological bundle (which is just L' when restricted to X'). Birationality is shown by a similar argument and all these steps carry over without changes for singular varieties. $\square$

3. Minimal log discrepancies

A *log pair* (X, Δ) consists of a normal variety X and a $\mathbb{R}$-Weil divisor $\Delta \geqslant 0$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A *log resolution* of a log pair (X, Δ) is a projective birational morphism $\pi : \tilde{X} \longrightarrow X$ such that $\tilde{X}$ is smooth and $\pi^*\Delta + Exc(\pi)$ has simple normal crossing support. A birational morphism $f : \tilde{X} \longrightarrow X$ between varieties for which K_X and $K_{\tilde{X}}$ are well-defined is called *crepant* if $\pi^*K_X = K_{\tilde{X}}$. A *crepant resolution* is a resolution of singularities which is also a crepant

morphism.

3.1 Elementary properties of mlds

If (X, Δ) is a log pair and $\pi : \tilde{X} \longrightarrow X$ is a log-resolution of (X, Δ), then we define the log discrepancy $a(E, X, \Delta)$ for a divisor E over X by the formula

$$K_{\tilde{X}} + \tilde{\Delta} = \pi^*(K_X + \Delta) + \sum_{E \subset \tilde{X}} (a(E, X, \Delta) - 1)E,$$

where $\tilde{\Delta}$ is the strict transform of Δ.

Let $c_X(E) \in X$ be the center of a divisor over X. This is a not necessarily closed point of X. The *minimal log discrepancy* at $x \in X$ is

$$\operatorname{mld}(x, X, \Delta) := \inf_{c_X(E)=x} a(E, X, \Delta)$$

and the minimal log discrepancy along a subvariety $Z \subset X$ is

$$\operatorname{mld}(Z, X, \Delta) := \inf_{x \in Z} \operatorname{mld}(x, X, \Delta).$$

Notice that from the definition we have that

$$Z \subset Z' \Rightarrow \operatorname{mld}(Z, X, \Delta) \geqslant \operatorname{mld}(Z', X, \Delta). \tag{3.1}$$

Frequently we will write $\operatorname{mld}(x)$ and $\operatorname{mld}(Z)$ if there is no danger of confusion. We refer to [Am99, § 1] for more details.

We collect some basic facts about mlds.

Lemma 3.2. Let $f : X \longrightarrow Y$ be a proper birational morphism with X normal and $\mathbb{Q}$-Gorenstein. Then

$$\operatorname{mld}(W, Y, D) = \operatorname{mld}(\pi^{-1}(W), X, \pi^*D - K_{X/Y}).$$

Proof. This is [EMY03, Proposition 1.3 (iv)]. □

For $k \in \mathbb{N}$ let us denote by $X^{(k)} \subset X$ the subset of points of dimension k endowed with the subspace topology. The dimension of a point $x \in X$ is defined to be the dimension of the Zariski closure of x.

Lemma 3.3. The function $\operatorname{mld} := \operatorname{mld}_{(X,\Delta)} : X^{(k)} \longrightarrow \mathbb{R} \cup \{-\infty\}$ takes only finitely many values.

Proof. This is [Am99, Theorem 2.3]. □

3.4 Conjectures about mlds

Ambro and Shokurov have made the two following conjectures about mlds in [Am99, ShoV]. The importance of these conjectures is that if they are fulfilled, then log-flips terminate by the main theorem of [ShoV].

Conjecture 3.5. **(ACC)** Let $\Gamma \subset [0,1]$ be a DCC-set, *i.e.,* all decreasing sequences in Γ become eventually constant. For a fixed integer k the set

$$\Omega_k := \left\{ \operatorname{mld}(Z, X, \Delta) \;\middle|\; \begin{array}{l} \dim X = k, \\ (X, \Delta) \text{ log pair} \\ Z \subset X \text{ closed subvariety} \\ \operatorname{coeff}(\Delta) \in \Gamma \end{array} \right\}$$

is an ACC-set, that is, every increasing sequence $\alpha_1 \leqslant \alpha_2 \leqslant \dots$ in Ω_k eventually becomes stationary.

Conjecture 3.6. **(LSC)** Let X be a normal $\mathbb{Q}$-Gorenstein variety and let Δ be an $\mathbb{R}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then for each d the function $\operatorname{mld}_{(X,\Delta)} : X^{(d)} \longrightarrow \mathbb{R} \cup \{-\infty\}$ is lower semi-continuous.

Remark 3.7. If LSC holds on X, then for each $a \in \mathbb{R}$ and $d \in \mathbb{N}$ the set

$$X^{(d)}_{\leqslant a} := \{x \in X^{(d)} \mid \operatorname{mld}(x) \leqslant a\} \tag{3.2}$$

is closed, that is, there is, $X^{(d)}_{\leqslant a} = X_{\leqslant a} \cap X^{(d)}$ where $X_{\leqslant a}$ is the closure of $X^{(d)}_{\leqslant a}$ in X. Moreover, $X^{(d)}_a := \{x \in X^{(d)} \mid \operatorname{mld}(x) = a\}$ is open in $X^{(d)}_{\leqslant a}$. All this follows directly from Lemma 3.3 which together with the lower semi-continuity implies that for $x \in X^{(d)}$ there is an open neighbourhood $U \subset X^{(d)}$ of x such that

$$\forall x' \in U : \ \operatorname{mld}(x) \leqslant \operatorname{mld}(x').$$

It is well-known and an easy consequence of [Am99, Prop. 2.5] that lower semi-continuity is equivalent to $\operatorname{mld} : X^{(0)} \longrightarrow \mathbb{R} \cup \{-\infty\}$ being lower semi-continuous. Moreover, by loc. cit. one also sees that ACC holds, as soon as

$$\Omega^{(0)}_k := \left\{ \operatorname{mld}(x, X, \Delta) \;\middle|\; \begin{array}{l} \dim X = k, \\ K_X + \Delta \ \mathbb{Q}\text{-Cartier}, \\ x \in X \text{ closed point} \\ \operatorname{coeff}(\Delta) \in \Gamma \end{array} \right\}$$

is an ACC set.

Next we show that LSC descends along crepant morphisms.

Theorem 3.8. *Let Y be a normal projective $\mathbb{Q}$-Gorenstein variety and let Δ be an effective $\mathbb{R}$-Cartier divisor on Y such that (Y, Δ) is log-canonical. If $\pi : X \longrightarrow Y$ is a proper, crepant morphism and LSC holds on X, then*

$$\operatorname{mld} : Y^{(0)} \longrightarrow \mathbb{R} \cup \{\infty\}$$

is lower semi-continuous.

Proof. Let us fix a closed point $y \in Y$ and denote $W := \pi^{-1}(y)$. By Lemma 3.2 we have

$$\operatorname{mld}(y, Y, D) = \operatorname{mld}(\pi^{-1}(y), X, \pi^* D). \tag{3.3}$$

We have to show that there is an open neighbourhood $U \subset Y$ of y such that

$$\mathrm{mld}(y') \geqslant \mathrm{mld}(y) \quad \forall y' \in U.$$

To this end we spot the "bad" subsets of Y. By Lemma 3.3 the function mld takes only finitely many values. If $a := \mathrm{mld}(y)$ is the smallest mld on $Y^{(0)}$, then there is nothing to prove. Otherwise, let us denote by b the maximal mld on Y with $b < a$. In view of (3.3), the search for mlds smaller than a can be carried out on X, but at the price of having to take into account not only closed points. Consider for each $0 \leqslant d \leqslant n := \dim X$ the set

$$C_d := \{x \in X^{(d)} \mid \dim \pi(x) = 0, \mathrm{mld}(x) \leqslant b\}.$$

Let $\overline{C}_d$ denote the Zariski closure of C_d in X. By assumption, (LSC) holds on X and hence all $x \in \overline{C}_d$ with $\dim x = d$ satisfy $\mathrm{mld}(x) \leqslant b$. Now we set

$$U := Y \setminus \bigcup_{d=0}^{n} \pi(\overline{C}_d),$$

where $n = \dim(X)$. As π is proper, U is open. We will consecutively prove the following claims.

(i) Every irreducible component of $\overline{C}_d$ has relative dimension at least d over its image.

(ii) $y \in U$.

(iii) $\mathrm{mld}(y') \geqslant \mathrm{mld}(y)$ for all $y' \in U$.

Let Σ denote an irreducible component of $\overline{C}_d$ for some d. As $\overline{C}_d$ is the closure of C_d, the set $C_d \cap \Sigma$ is not empty. Thus, the set $\Sigma_{\geqslant d} := \{x \in \Sigma \mid \dim \pi^{-1}\pi(x) \geqslant d\}$ is not empty. By the upper semi-continuity of the fiber dimension [Gr66, Corollaire 13.1.5], $\Sigma_{\geqslant d}$ is closed and by definition we have $\Sigma_{\geqslant d} \supset \Sigma \cap C_d$. Therefore, as Σ is a component of the closure of C_d we have $\Sigma_{\geqslant d} = \Sigma$ and the first claim follows.

Suppose that $y \notin U$. Then we would have a point $x \in \overline{C}_d$ for some d with $\pi(x) = y$. By the previous statement, we have $\dim(W \cap \overline{C}_d) \geqslant d$, where, we recall, $W = \pi^{-1}(y)$. This implies that there is $x' \in W \cap \overline{C}_d$ with $\dim x' = d$ and hence $\mathrm{mld}(x') \leqslant b$ by the definition of B_d. But then, by (3.1)

$$a = \mathrm{mld}(y) \leqslant \mathrm{mld}(x') \leqslant b$$

contradicting the choice of $b < a$. Thus, $y \in U$.

Now if there were some $y' \in U$ with $\mathrm{mld}(y') < \mathrm{mld}(y) = a$ it would also be $\leqslant b$ by the maximality of b. Let $x \in \pi^{-1}(y')$ be a point with $\mathrm{mld}(x) = \mathrm{mld}(y')$ and denote $d := \dim(x)$. This would imply $x \in C_d \subset \overline{C}_d$ contrarily to the assumption $y' \in U$. This concludes the proof of the theorem. □

By [EMY03, Thm. 0.3], LSC holds on smooth varieties. This immediately yields the

Corollary 3.9. Let Y be a normal projective $\mathbb{Q}$-Gorenstein variety possessing a crepant resolution of singularities. Let Δ be an $\mathbb{R}$-Weil divisor on Y such that $K_Y + \Delta$ is $\mathbb{R}$-Cartier. Then the function $\mathrm{mld}_{(Y,\Delta)}$ is lower semi-continuous. □

4. Termination

In this section we prove our main application, namely Theorem 1.2. Notice that in its statement we could also drop the lc-assumption on (X, Δ), as thanks to $K_X = 0$ we can rescale Δ at any time. The proof of the theorem will occupy the rest of the section. Let (X, Δ) be a log pair on a projective irreducible symplectic manifold. By [BCHM10, Corollary 1.4.1], $(K_X + \Delta)$-flips exist and so we may run a $(K_X + \Delta)$-MMP. This produces a sequence

$$X = X_0 \overset{\phi_0}{\dashrightarrow} X_1 \overset{\phi_1}{\dashrightarrow} X_2 \dashrightarrow \dots \tag{4.1}$$

where the ϕ_i are either divisorial contractions or flips. Let us denote $\Delta_0 := \Delta$ and $\Delta_i = (\phi_i)_* \Delta_{i-1}$. Note that at each step K_{X_i} will be trivial and therefore we can rescale Δ_i such that (X_i, Δ_i) will be klt, hence the above result applies. We want to show that (4.1) terminates after a finite number of steps. First we notice:

Lemma 4.1. Each X_i is a symplectic variety and admits a crepant resolution.

Proof. By induction we may assume that X_{i-1} is a symplectic variety and has a crepant resolution $\tilde{\pi} : \tilde{X}_{i-1} \longrightarrow X_{i-1}$. Symplecticity of X_i is clear, as the exceptional locus of $X_{i-1} \dashrightarrow X_i$ on X_i has codimension $\geqslant 2$ and thus the symplectic form from X_{i-1} extends. By [BCHM10, Corollary 1.4.3] there exists a proper birational morphism $\pi : \tilde{X}_i \longrightarrow X_i$ such that $\tilde{X}_i$ has only $\mathbb{Q}$-factorial terminal singularities and π is crepant. Let $X_{i-1} \longrightarrow Z \longleftarrow X_i$ be the flipping contraction. Then the compositions $\tilde{X}_i \longrightarrow X_i \longrightarrow Z$ and $\tilde{X}_{i-1} \longrightarrow X_{i-1} \longrightarrow Z$ are crepant morphisms and $\tilde{X}_{i-1}$ is smooth, hence by [Na06, Corollary 1, p. 98] also $\tilde{X}_i$ is smooth. □

Proof of Theorem 1.2. In the course of the MMP, only a finite number of divisorial contractions can occur so that by the preceding lemma we can reduce to the following situation: $X = X_0$ is a symplectic variety having a crepant resolution, $\Delta = \Delta_0$ is an effective $\mathbb{R}$-divisor on X and we are given a sequence

$$X = X_0 \overset{\phi_0}{\dashrightarrow} X_1 \overset{\phi_1}{\dashrightarrow} X_2 \dashrightarrow \dots \tag{4.2}$$

where ϕ_i is a log-flip for the pair (X_i, Δ_i). We will show that such a sequence is finite by using Shokurov's criterion, see [ShoV, Theorem]. In fact, by [ShoV, Addendum 2], we only need LSC for the pairs (X_i, Δ_i) and ACC for the set

$$\Omega(X) := \{\mathrm{mld}(E_i, X_i, \Delta_i) | i \in \mathbb{N}\}$$

where $E_i \subset X_i$ denotes the exceptional locus of $\phi_i : X_i \longrightarrow X_{i+1}$, see also [HM10, § 3]. Recalling that LSC holds by Corollary 3.9 and Lemma 4.1 above, we are left with ACC. By a theorem of Kawakita [Kaw12], the set of all mlds for a fixed finite set of coefficients on a fixed projective variety is finite. More precisely, let Z be a projective variety, let $\Gamma \subset [0, 1]$ be a finite set and consider

$$M_\Gamma(Z) := \{\mathrm{mld}_{(Z,\Delta)}(x) \mid \mathrm{coeff}(\Delta) \in \Gamma, (Z, \Delta) \text{ lc at } x \in Z^{(0)}\}$$

Then by [Kaw12, Thm. 1.2], the set $M_\Gamma(Z)$ is finite. By loc. cit. also the bigger set

$$M_\Gamma^{\mathrm{loc}}(Z) := \{\mathrm{mld}_{(U,\Delta)}(x) \mid \mathrm{coeff}(\Delta) \in \Gamma, (U,\Delta) \text{ lc at } x, U \subset Z \text{ open}\}$$

is finite. Note that $U \subset Z$ is supposed to be open in the Euclidean topology and that Δ is not supposed to be the restriction of a divisor on Z. By Theorem 1.1 all X_i in the sequence (4.2) are locally trivial deformations of one another (notice that the $\mathbb{Q}$-factoriality is insured by [KM, Propositions 3.36 and 3.37]). Hence, $M_\Gamma^{\mathrm{loc}}(X_i)$ is independent of i, as mlds are local invariants. Consequently, $\Omega(X_0) \subset \bigcup_{i=0}^{\infty} M_\Gamma(X_i) \subset M_\Gamma^{\mathrm{loc}}(X_0)$. In particular, $\Omega(X_0)$ is finite and thus an ACC set and we may conclude the proof. □

Remark 4.2. The observation that equivalence by locally trivial deformations implies ACC has already been made by Nakamura in [Nak13, Corollary 1.4], where he considered the case of terminal quotient singularities.

Remark 4.3. It is tempting to try to deduce the termination of the MMP, proved here, from the termination of flips for irreducible symplectic manifolds. The proof of the latter by [MZ13] is much quicker and, notably, does not require any control at all on the local structure of symplectic singularities that appear along a log-MMP, as in the framework of [MZ13] every variety is smooth. Using the notation above, if we start from a $\mathbb{Q}$-factorial symplectic variety $X := X_0$ and a boundary divisor $\Delta := \Delta_0$ and if X admits a crepant resolution $f : Y \longrightarrow X$ together with the natural boundary divisor $\Gamma := f^*\Delta$, then the idea would be the following: given a log-flip $X_0 \dashrightarrow X_1$ one could try to run a MMP for (Y, Γ) in such away that flips and divisorial contractions are interchanged, that is, the MMP is supposed to produce a sequence of flips (Y_i, Γ_i) for $i = 0, \ldots, N$ and a divisorial contraction $Y_N \longrightarrow X_1$. Then by [MZ13] the sequence of flips for (Y, Γ) would be necessarily finite, however it seems very plausible that the MMP on (Y, Γ) might produce divisorial contractions right away which would disallow the use of [MZ13] and make this strategy useless.

References

Am99 F. Ambro. On minimal log discrepancies. *Math. Res. Lett.*, 6(5-6):573–580, 1999. – cited on p. 11, 12

AL14 N. Addington and M. Lehn. On the symplectic eightfold associated to a Pfaffian cubic fourfold. To appear in *J. Reine Angew. Math.*, `arXiv:1404.5657`, 2014. – cited on p. 1

Ar69 M. Artin. Algebraic approximation of structures over complete local rings. *Publ.Math. I.H.E.S.*, vol 36, 23-58, 1969. – cited on p. 6

Be99 A. Beauville. Counting rational curves on K3 surfaces. *Duke Math.*, 97, 99-108, 1999. – cited on p. 1

Be00 A. Beauville. Symplectic singularities. *Invent. math.*, 139, 541-549, 2000. – cited on p. 2

BCHM10 C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010. – cited on p. 3, 14

Bi12 C. Birkar. On existence of log minimal models and weak Zariski decompositions. *Math. Annalen*, 354, Number 2, 787–799, 2012. – cited on p. 3

Bo04 S. Boucksom. Divisorial Zariski decompositions on compact complex manifolds. *Ann. Sci. ENS* (4) 37, no. 1, 45-76, 2004. – cited on p. 9

CL10 P. Cascini and V. Lazić. New outlook on Mori theory I. *Duke Math. J.* 161, no. 12, 2415â2467, 2012. – cited on p. 3

De99 O. Debarre. On the Euler characteristic of generalized Kummer varieties. *Amer. J. Math.*, 121, 577-586, 1999. – cited on p. 1

Dr11 S. Druel. Quelques remarques sur la décomposition de Zariski divisorielle sur les variétés dont la première classe de Chern est nulle. *Math. Z.*, 267, 1-2, p. 413-423, 2011. – cited on p. 9

DV10 O. Debarre and C. Voisin. Hyper-Kähler fourfolds and Grassmann geometry. *J. reine angew. Math.*, 649, 63–87, 2010. – cited on p. 2

EMY03 M. Mustaţă, L. Ein, and T. Yasuda. Jet schemes, log discrepancies and Inversion of Adjunction. *Invent. Math.*, 153, 119-135, 2003. – cited on p. 4, 11, 13

FK87 H. Flenner and S. Kosarew. On locally trivial deformations. *Publ. Res. Inst. Math. Sci.*, 23(4):627–665, 1987. – cited on p. 7

GKP D. Greb, S. Kebekus, and T. Peternell. Singular spaces with trivial canonical class. Preprint: `arXiv:1110.5250`, To appear in Advanced Studies in Pure Mathematics (Mori 60 volume), Mathematical Society of Japan, Tokyo. – cited on p. 2

GHJ M. Gross, D. Huybrechts, and D. Joyce. *Calabi-Yau manifolds and related geometries*, Universitext. Springer-Verlag, Berlin, 2003. – cited on p. 5, 9

Gr66 A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966. – cited on p. 13

HM10 C. Hacon and J. McKernan. Flips and flops. *Proceedings of the International Congress of Mathematicians.*, Volume II, 513-539, Hindustan Book Agency, New Delhi, 2010. – cited on p. 14

Hu97 D. Huybrechts. Birational symplectic manifolds and their deformations. *J. Differential Geom.* 45, no. 3, 488-513, 1997. – cited on p. 9, 10

Hu99 D. Huybrechts. Compact hyperkähler manifolds: Basic results. *Invent. Math.*, 135, 63-113,1999. – cited on p. 6, 8, 9, 10

Hu03 D. Huybrechts. The Kähler cone of a compact hyperkähler manifold, *Math. Ann.*, 326, 499-513, 2003. – cited on p. 1, 5, 9

Kal06 D. Kaledin. Symplectic singularities from the Poisson point of view. *J. Reine Angew. Math.*, 600, 135-156, 2006. – cited on p. 3, 6

Kaw12 M. Kawakita. Discreteness of log discrepancies over log canonical triples on a fixed pair, Preprint `arXiv:1204.5248` , *J. Algebraic Geom.*, Published electronically: Feb 25, 2014. – cited on p. 4, 14, 15

Ka92 Y. Kawamata. Unobstructed deformations. A remark on a paper of Z. Ran: *Deformations of manifolds with torsion or negative canonical bundle*, *J. Algebraic Geom.*, 1(2), 183-190, 1992. – cited on p. 3, 5

Ka97 Y. Kawamata. Erratum on: *Unobstructed deformations. A remark on a paper of Z. Ran: "Deformations of manifolds with torsion or negative canonical bundle*, *J. Algebraic Geom.*, 6(4), 803-804, 1997. – cited on p. 3, 5

KM J. Kollár, S. Mori, *Birational geometry of algebraic varieties,* Cambridge Tracts in Mathematics, **134**, Cambridge University Press, Cambridge, 1998. – cited on p. 15

KM92 J. Kollár and S. Mori. Classification of three-dimensional flips. *J. Amer. Math. Soc.*, 5 (1992), 533-703. – cited on p. 6

Le15 C. Lehn. Twisted cubics on singular cubic fourfolds - On Starr's fibration, Preprint `arXiv:1504.06406`. – cited on p. 1

Ma12 D. Matsushita. On almost holomorphic lagrangian fibrations. Preprint `arXiv:1209.1194`. – cited on p. 3

MZ13 D. Matsushita and D.-Q. Zhang. Zariski F-decomposition and Lagrangian fibration on hyper-Kähler manifolds, Preprint `arXiv:09075311v5`, October 2013. To appear in Math. Research Letters. – cited on p. 3, 15

Nak13 Y. Nakamura. On semi-continuity problems for minimal log discrepancies. Preprint `arXiv:1305.1410`. – cited on p. 15

Na01 Y. Namikawa. Deformation theory of singular symplectic n-folds. *Math. Ann.*, 319, no. 3, 597–623, 2001. – cited on p. 2, 6

Na02 Y. Namikawa. Projectivity criterion of Moishezon spaces and density of projective symplectic varieties. *Internat. J. Math.* 13, no. 2, 125-135, 2002. – cited on p. 9

Na06 Y. Namikawa. On deformations of Q-factorial symplectic varieties. *J. Reine Angew. Math.*, 599, 97–110, 2006. – cited on p. 2, 4, 7, 8, 14

Na10 Y. Namikawa. Poisson deformations of affine symplectic varieties, *Duke Math. J.*, Volume 156, Number 1, 51–85, 2011. – cited on p. 2, 5, 6

OG99 K. O'Grady. Desingularized moduli spaces of sheaves on a K3, *J. Reine Angew. Math.* 512, 49-117, 1999. – cited on p. 2

OG03 K. O'Grady. A new six-dimensional irreducible symplectic variety, *J. Algebraic Geom.* 12, no. 3, 435-505, 2003. – cited on p. 2

Ra92 Z. Ran. Deformations of manifolds with torsion or negative canonical bundle, *J. Algebraic Geom.*, 1(2), 279-291, 1992. – cited on p. 3, 5

Se06 E. Sernesi. Deformations of algebraic schemes. Grundlehren der Mathematischen Wissenschaften, 334. Springer-Verlag, Berlin, 2006. – cited on p. 5

ShoV V.V. Shokurov. Letters of a bi-rationalist V - mld's and the termination of log flips. *Tr. Mat. Inst. Steklova* 246; translation in *Proc. Steklov Inst. Math.* 2004, no. 3 (246), 315–336, 2004. – cited on p. 3, 4, 11, 14

Ve13 M. Verbitsky. Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Math. J.*, Volume 162, Number 15, 2929-2986, 2013. – cited on p. 9

Christian Lehn lehn@math.uni-hannover.de

Christian Lehn, Institut für Algebraische Geometrie, Gottfried Wilhelm Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

Gianluca Pacienza pacienza@math.unistra.fr

Gianluca Pacienza, Institut de Recherche Mathématique Avancée, Université de Strasbourg et CNRS, 7 rue René Descartes, 67084 Strasbourg Cedex, France