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Active vibration control and stability analysis of a time-delay system subjected to friction-induced vibration

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ABSTRACT

The use of active vibration control may induce a delay leading to detrimental degradation of the performance of active vibration control. This is particularly true in the case of mechanical systems subjected to friction-induced vibration and noise for which such time-delays can lead to the appearance of undesirable instability. Furthermore, conducting a stability analysis of time-delay systems and estimation of the critical time delay are challenging, due to the infinite nature of the characteristic (quasi) polynomial of the associated closed-loop system, having an infinite number of roots.

The objective of this paper is to discuss a strategy for the estimation of the critical time delay for the problem of Friction-Induced Vibration and noise (FIVE). To achieve such an objective, the prediction of the stability analysis of time delay systems and the estimation of the associated critical time delay are first performed by applying the frequency sweep test and the eigenvalue problem approximation using the Taylor series expansion of the delayed term. In a second time, a mixed approach is proposed to predict effectively the real critical time delay of autonomous controlled systems subjected to friction-induced vibration. The efficiency of the proposed approach is illustrated by numerical examples for the prediction of self-sustaining vibrations of a phenomenological model with two degrees of freedom for which it is possible to provide a clear understanding and illustration of the phenomena involved and observed.

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1. Introduction

The problem of Friction-Induced Vibration and noise (FIVE) is crucial due to its huge impact for both the acoustic discomfort or the potential poor system performances [1–5]. Consequently the development of technological solutions making it possible to reduce or eliminate self-excited vibrations of a mechanical system subjected to friction-induced vibration is nowadays a crucial point. Many strategies can be used to control such a system. Thereby the use of passive or active vibration control is an attractive option to mitigate such vibrations. Various passive methods have been investigated in the literature such as adding excitation [6–9], nonlinear control by modulating the normal forces at the frictional interface [10], dynamic vibration absorbers [11,12] or Nonlinear Energy Sink [13]. Other approaches based on active methods for controlling friction-induced oscillations have also been put forward by researchers [14–24]. For a more general point of view concerning

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the control of machines with friction, the reader can refer to the survey paper of Armstrong-Helouvy [25] that discusses the problem of friction compensation in controlled mechanical systems survey with contributions from the tribology, physics and controls literature.

For several reasons, some time delays are often present in various mechanical systems. They may come from not only inherent time-delays in the components of the systems where the propagation of effects takes time but also the necessity of feedback controls to stabilize a system that can also introduce some inevitable delays. A special attention should be brought to the consideration of the concept of time delay to properly solve the problem of Friction-Induced Vibration and noise. Effectively, such time-delays can cause some significant problems such as the appearance of instability for a mechanical system subjected to friction-induced vibration [26]. Consequently the performance of active vibration control suffers severe degradation for the case in which the time delay has to be taken into account. This explains why time-delayed feedback control methods have received outstanding attention in the domain of mechanical systems subjected to friction-induced vibration in the past decades [15,17,20,21,26]. The interested reader can refer to the paper of Chatterjee [16] that gives an overview of efficiency of the time-delayed feedback control for the three well-known mechanisms of friction-induced instabilities (i.e. Stribeck instability governed by velocity-weakening characteristics of friction force, mode-coupling instability, and sprag-slip instability).

Moreover the presence of time delays leads to more complex analysis including infinite-dimensionality than in the case of system without delay that corresponds to a finite dimensional case. So it is not possible to analyze such time-delay systems with classical methods, especially, in checking stability. Such problems are often solved indirectly by using approximation such as the Taylor series expansion or Padé approximants as a substitute for the exponential time-delay terms in the characteristic equation. The main interest of Padé approximants and general rational fractional approximants [27,28] is that they make it possible to often provide better approximation of the function than truncating its Taylor series. However Padé approximants are rational functions and that artificial singular points may occur. Moreover Insperger [29] indicates that the approximation of delayed systems by Taylor series expansion must be used with care in engineering applications due to the fact that Taylor series approximation of the delayed term is not always an appropriate tool to analyze asymptotic behavior of time-delay systems. On the other hand methodology derived from the small gain theorem and based on the well-known receptance method can be applied in the case of linear systems with time-varying delay [30] and unstructured uncertainties [31]. This approach ensures the robust stability of the uncertain second-order system despite the time-varying delay. Other approaches can also be based on the frequency response methods combined to the receptance method as shown in [32] in the case of rank-one feedback controllers with time delay. The main advantage of the proposed method is to take into consideration the time delays without approximations and without *a posteriori* stability verification by using frequency response calculation.

Nowadays, one of the main scientific challenges is not only to investigate the effect of time delay for the problem of Friction-Induced Vibration and noise but also to predict the critical time delay for which instability appears. One of the main originality of the proposed study is to answer this question by developing a hybrid method which combines the efficiency of the frequency sweeping test [33] with an expansion based approach of the delayed term for approximating the eigenvalues problem. More specifically it will be shown in this paper that this hybrid method makes it possible to efficiently calculate the critical time delay in the context of mechanical systems subjected to friction-induced vibration, as opposed to using the frequency sweeping approach alone. Note that this work can be considered as an extension study of the original work of Singh and Ouyang [20] on the use of frequency sweeping test for estimating the critical time delay and a complementary study in the continuity of the previous works of Singh [34] and Singh et al. [35] that are outstanding essential steps referential in control of second-order mechanical systems with time delay. Until today only few works [22,36] have been published to discuss and contribute to this important subject in line with the work of Singh et al. in the field of mode-coupling instability and friction-induced vibration.

This paper is organized as follows. First of all, the mechanical system without control or regulated by state feedback control with time delay is briefly discussed. Then, some basic concepts based on the stability analysis of uncontrolled and controlled systems are presented. More specifically the frequency sweeping test and the approximation of the eigenvalues problem by using Taylor series expansion of the delayed term are discussed for predicting the stability analysis of time-delay systems and for estimating the associated critical time delay. Then numerical examples are undertaken to discuss the relevance of the two methodologies and a mixed strategy is developed to predict efficiently the critical time delay of the controlled system.

2. Preamble

2.1. Mechanical system under study

The mechanical system under study is shown in Fig. 1. This two-degrees-of-freedom model corresponds to a phenomenological model which makes it possible to analyze mono-instability and the generation of oscillations for mechanical systems subjected to friction-induced vibration. This phenomenological model has been previously proposed by Hulten [37,38] and extensively studied by Sinou and Jézéquel [39] and Sinou et al. [40] for predicting mode coupling instability in friction induced vibrations and its dependency on system parameters including damping.

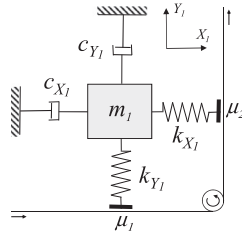


Fig. 1. Mechanical system under study.

As a reminder, this model has one mass m_1 held against moving bands by two plates. Each contact between mass and plates is modeled by linear springs and dampings: (k_{X_1}, c_{X_1}) and (k_{Y_1}, c_{Y_1}) . It is assumed that the moving band and the mass are always in contact: the friction forces are determined with a classical Coulomb's law and the friction coefficient at the different contacts between the masses and the band is supposed to be constant and equal to μ_j (for $j = 1, 2$, see Fig. 1 for more details). The tangential friction forces F_t are related to the normal forces F_n by the following relation: $F_t = \mu_j F_n$.

The second-order differential equations describing the dynamic behaviour of the mechanical system under study can be written in the following matrix form:

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + (\mathbf{K} + \mathbf{K}_\mu)\mathbf{X}(t) = \mathbf{0} \quad (1)$$

where $\mathbf{X}(t)$ defines the vector of the two degrees-of-freedom of the mechanical system (i.e. $\mathbf{X} = [X_1(t) Y_1(t)]^T$). The mass, damping and stiffness matrices as well as the contributions \mathbf{K}_μ due to the friction forces are defined as follows

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} c_{X_1} & 0 \\ 0 & c_{Y_1} \end{bmatrix}; \quad \mathbf{K} = \begin{bmatrix} k_{X_1} & 0 \\ 0 & k_{Y_1} \end{bmatrix}; \quad \mathbf{K}_\mu = \begin{bmatrix} 0 & \mu_2 k_{Y_1} \\ -\mu_1 k_{X_1} & 0 \end{bmatrix} \quad (2)$$

The main properties of the mechanical system under study are : $m_1 = 1$ kg; $c_{X_1} = c_{Y_1} = 1$ Ns/m; $k_{X_1} = 6000$ N/m; $k_{Y_1} = 3000$ N/m. For the sake of simplicity it is also assumed that the two coefficients of friction at each contact between the plates of the moving band are equal (i.e. $\mu_1 = \mu_2$).

The stability of the system can be determined by using the well-known Complex Eigenvalue Analysis (CEA). If λ defines the complex eigenvalues of the system such as

$$\det(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K} + \mathbf{K}_\mu) = 0 \quad (3)$$

stability is determined by considering the real part of λ . If all eigenvalues have negative real parts, the system is stable. If at least one eigenvalue has a positive real part, the system is unstable. The imaginary part of the associated positive eigenvalue defines the angular frequency of the unstable mode.

For the case under study, the Hopf bifurcation point is detected at $\mu_{Hopf} = 0.35$. The system is stable for the friction coefficient $\mu \in [0; \mu_{Hopf}[$ and become unstable for $\mu > \mu_{Hopf}$. The angular frequency of the unstable mode is equal to 67.1 rad.s⁻¹ at the Hopf bifurcation point.

2.2. System regulated by state feedback control with time delay

The previous system regulated by state feedback control with time delay can be represented as

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + (\mathbf{K} + \mathbf{K}_\mu)\mathbf{X}(t) = \mathbf{b}\mathbf{u}(t - \tau) \quad (4)$$

with

$$\mathbf{u}(t - \tau) = -\mathbf{f}^T \dot{\mathbf{X}}(t - \tau) - \mathbf{g}^T \mathbf{X}(t - \tau) \quad (5)$$

where τ corresponds to the constant time delay between the measurements of the state and the actuation of the control. To be noted that the time delays for the velocity and displacement state feedback are assumed to be equal. \mathbf{f} and \mathbf{g} are the vectors of proportional velocity and displacement control gains. Substituting Eq. (5) in Eq. (4), the closed-loop system with time-delay can be defined as

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{b}\mathbf{f}^T \dot{\mathbf{X}}(t - \tau) + (\mathbf{K} + \mathbf{K}_\mu)\mathbf{X}(t) + \mathbf{b}\mathbf{g}^T \mathbf{X}(t - \tau) = \mathbf{0} \quad (6)$$

As discussed in [41,42], by considering $\mathbf{X}(t) = \mathbf{Z}e^{st}$ where \mathbf{Z} is a constant vector, the eigenvalue problem of the closed-loop system defined in Eq. (6) is given by

$$(s^2 \mathbf{M} + s(\mathbf{C} + \mathbf{b}\mathbf{f}^T e^{-s\tau}) + \mathbf{K} + \mathbf{K}_\mu + \mathbf{b}\mathbf{g}^T e^{-s\tau})\mathbf{Z} = \mathbf{0} \quad (7)$$

To be noted that for the closed-loop system without time delay, the problem is reduced to

$$(s^2 \mathbf{M} + s(\mathbf{C} + \mathbf{b}\mathbf{f}^T) + \mathbf{K} + \mathbf{K}_\mu + \mathbf{b}\mathbf{g}^T)\mathbf{Z} = \mathbf{0} \quad (8)$$

The non-trivial solutions ($s_i, \mathbf{Z}_i \neq \mathbf{0}$) for the closed-loop system with time delay (without time delay, respectively) are eigenpairs of the system (7) (system defined in Eq. (8), respectively). As a reminder Eq. (3) defines the eigenvalue problem of the corresponding open-loop system.

3. Stability analysis of time-delay systems and estimation of the critical time delay

The aim of this section is to briefly described two approaches for predicting the stability of time-delay systems and more particularly for estimating the critical time delay. The first approach is called the frequency sweeping tests [33] and the second method is based on the approximation of the eigenvalues problem by using Taylor series expansion of the delayed term.

3.1. Stability based on frequency sweeping tests

The stability analysis of the closed-loop system with time-delay defined in Eq. (6) can be performed by considering the well-known frequency-sweeping approach [33].

First of all, Eq. (6) can be rewritten in a state space form

$$\dot{\mathbf{Y}}(t) = \mathbf{A}_0 \mathbf{Y}(t) + \mathbf{A}_1 \mathbf{Y}(t - \tau) \tag{9}$$

where the state vector \mathbf{Y} is defined by $\mathbf{Y} = [\mathbf{X} \dot{\mathbf{X}}]^T$. The matrices \mathbf{A}_0 and \mathbf{A}_1 of size $(2p, 2p)$ (where p defined the number of degrees of freedom of the system) are given by

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{K} + \mathbf{K}_\mu) & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}; \mathbf{A}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{b}\mathbf{g}^T & \mathbf{M}^{-1}\mathbf{b}\mathbf{f}^T \end{bmatrix} \tag{10}$$

The previous eigenvalue problem defined in Eq. (7) can be rewritten as the pseudo-polynomial

$$\det(s\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_1 e^{-\tau s}) = 0 \tag{11}$$

The system for a given value of time delay τ is asymptotically stable if and only if all roots of this transcendental function lie in the open left half of the complex plane. Moreover, a system is said to be delay-independent stable if its stability does not depend on the delay value (for $\tau \in [0; +\infty[$). Otherwise, the system is said to be delay-dependent stable. A necessary and sufficient condition for stability independent of delay [33] is given by:

- \mathbf{A}_0 and $\mathbf{A}_0 + \mathbf{A}_1$ are Hurwitz matrices (or stable matrices, i.e. every eigenvalue of \mathbf{A}_0 and $\mathbf{A}_0 + \mathbf{A}_1$ has strictly negative real part);
- $\rho((j\omega\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_1) < 1 \forall \omega > 0$ where $\rho(\mathbf{A})$ corresponds to the spectral radius of the matrix \mathbf{A} .

If the system defined in Eq. (9) is stable at $\tau = 0$ and $\text{rank}(\mathbf{A}_1) = q$, the closed-loop system with time-delay is stable for the interval delay value $\tau \in [0; \tau_{lim}]$ and becomes unstable at $\tau = \tau_{lim}$ with

$$\tau_{lim} = \min_{1 \leq i \leq q} \tau_{lim}^i \tag{12}$$

and

$$\tau_{lim}^i = \begin{cases} \min_{1 \leq k \leq n} \frac{\theta_k^i}{\omega_k^i} & \text{if } \lambda_i((j\omega_k^i - \mathbf{A}_0), \mathbf{A}_1) = e^{-j\theta_k^i} \text{ for some } \omega_k^i \in (0, \infty), \theta_k^i \in [0, 2\pi] \\ \infty & \text{if } \rho((j\omega\mathbf{I} - \mathbf{A}_0), \mathbf{A}_1) > 1 \forall \omega > 0 \end{cases} \tag{13}$$

3.2. Taylor series expansion

By considering a Taylor series expansion of the delayed term $\mathbf{u}(t - \tau)$ up to the n th order such as

$$\mathbf{u}(t - \tau) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \tau^n \mathbf{u}^{(n)}(t) \tag{14}$$

where $\mathbf{u}^{(n)}$ denotes the n th derivative with respect to time, the closed-loop system (4) can be approximated by

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + (\mathbf{K} + \mathbf{K}_\mu)\mathbf{X}(t) &= \mathbf{B}\mathbf{f}^T \left(-\dot{\mathbf{X}}(t) + \tau\ddot{\mathbf{X}}(t) + \dots + \frac{(-1)^n}{n!} \tau^n \mathbf{X}^{(n+1)}(t) \right) \\ &+ \mathbf{B}\mathbf{g}^T \left(-\mathbf{X}(t) + \tau\dot{\mathbf{X}}(t) + \dots + \frac{(-1)^n}{n!} \tau^n \mathbf{X}^{(n)}(t) \right) \end{aligned} \tag{15}$$

Using the previous expression $\mathbf{X}(t) = \mathbf{Z}e^{st}$ with \mathbf{Z} a constant vector and the first-order approximation of Eq. (14), solving eigenvalues of the problem corresponds to the quadratic eigenvalues problem

$$(s^2(\mathbf{M} - \tau\mathbf{B}\mathbf{f}^T) + s(\mathbf{C} + \mathbf{B}\mathbf{f}^T - \tau\mathbf{B}\mathbf{g}^T) + \mathbf{K} + \mathbf{K}_\mu + \mathbf{B}\mathbf{g}^T)\mathbf{Z} = \mathbf{0} \tag{16}$$

So the associated first-order realization of the generalized eigenvalue problem is provided by

$$(\mathbf{P}_1 - s\mathbf{Q}_1)\mathbf{V}_1 = \mathbf{0} \tag{17}$$

with

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{K} + \mathbf{K}_\mu + \mathbf{B}\mathbf{g}^T & \mathbf{C} + \mathbf{B}\mathbf{f}^T - \tau\mathbf{B}\mathbf{g}^T \end{bmatrix}; \mathbf{Q}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} + \tau\mathbf{B}\mathbf{f}^T \end{bmatrix}; \mathbf{V}_1 = \begin{bmatrix} \mathbf{Z} \\ s\mathbf{Z} \end{bmatrix} \tag{18}$$

Similarly the second-order approximations of Eq. (14) yields the cubic eigenvalue problems defined by

$$\left(\frac{\tau^2}{2}\mathbf{B}\mathbf{f}^T s^3 + s^2 \left(\mathbf{M} - \tau\mathbf{B}\mathbf{f}^T + \frac{\tau^2}{2}\mathbf{B}\mathbf{g}^T \right) + s(\mathbf{C} + \mathbf{B}\mathbf{f}^T - \tau\mathbf{B}\mathbf{g}^T) + \mathbf{K} + \mathbf{K}_\mu + \mathbf{B}\mathbf{g}^T \right) \mathbf{Z} = \mathbf{0} \tag{19}$$

and the associated first-order realization of the generalized eigenvalue problem is provided by

$$(\mathbf{P}_2 - s\mathbf{Q}_2)\mathbf{V}_2 = \mathbf{0} \tag{20}$$

with

$$\mathbf{P}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{K} + \mathbf{K}_\mu + \mathbf{B}\mathbf{g}^T & \mathbf{C} + \mathbf{B}\mathbf{f}^T - \tau\mathbf{B}\mathbf{g}^T & \mathbf{M} - \tau\mathbf{B}\mathbf{f}^T + \frac{\tau^2}{2}\mathbf{B}\mathbf{g}^T \end{bmatrix}; \mathbf{Q}_2 = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{\tau^2}{2}\mathbf{B}\mathbf{f}^T \end{bmatrix}; \mathbf{V}_2 = \begin{bmatrix} \mathbf{Z} \\ s\mathbf{Z} \\ s^2\mathbf{Z} \end{bmatrix} \tag{21}$$

More generally, the first-order realization of the generalized eigenvalue problem for a Taylor series expansion of the delayed term $\mathbf{u}(t - \tau)$ up to the n th order is given by

$$(\mathbf{P}_n - s\mathbf{Q}_n)\mathbf{V}_n = \mathbf{0} \tag{22}$$

with the matrices \mathbf{P}_n and \mathbf{Q}_n of size $(n + 1, n + 1)$ and the vector \mathbf{V}_n of size $(n+1)$ such as

$$\mathbf{P}_n = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{I} \\ \mathbf{K} + \mathbf{K}_\mu + \mathbf{B}\mathbf{g}^T & \mathbf{C} + \mathbf{B}\mathbf{f}^T - \tau\mathbf{B}\mathbf{g}^T & \mathbf{M} - \tau\mathbf{B}\mathbf{f}^T + \frac{\tau^2}{2}\mathbf{B}\mathbf{g}^T & \dots & (-1)^{(n-1)} \frac{\tau^{n-1}}{(n-1)!} \mathbf{B}\mathbf{f}^T + (-1)^n \frac{\tau^n}{n!} \mathbf{B}\mathbf{g}^T \end{bmatrix} \tag{23}$$

$$\mathbf{Q}_n = \text{diag} \left(\mathbf{I}, \dots, \mathbf{I}, (-1)^n \frac{\tau^n}{n!} \mathbf{B}\mathbf{f}^T \right); \mathbf{V}_n = [\mathbf{Z} \quad s\mathbf{Z} \quad \dots \quad s^n\mathbf{Z}] \tag{24}$$

As discussed in [42], the characteristic equation of a controlled system with time delay has infinite number of roots over the complex field. For a p -dimensional system without time delay, the location of the $2p$ roots in the complex plan defines the stability of the controlled system. As previously discussed in Section 2, the system is unstable if at least one eigenvalue has a positive real part. In the case of a controlled system with time delay, the characteristic equation has infinite number of roots over the complex field [42]. Due to the fact that the approximated generalized eigenvalue problem up to the n th order give only $2 \times (p + n - 1)$ roots, unstable closed-loop poles that are roots of the eigenvalue problem (7) may be missing. So additional stability analysis needs to be performed to ensure that all the poles of the controlled system with time delay are stable.

For the interested reader, it should be noted that Araujo [36] proposed to resume the work of Singh [34] and Ram et al. [42] by using the classical Padé approximation for the time delay exponential instead the truncation of the Taylor series. To give a fair comparison, a detailed analysis is conducted for the k th order Padé and $(k + 1)$ th order Taylor series that give the same final order of approximated system to evaluate the eigenvalue problem. The results indicate that the well-known Padé approximants for time delay in frequency domain is shown to be as accurate as that the truncation of the Taylor exponential expansion proposed in [42].

4. Application and numerical examples

In this section the relevance and effectiveness of the two previous methodologies (i.e. the frequency sweeping approach and the Taylor series expansion) are discussed for the prediction of the stability analysis of time-delay systems and estimation of the critical time delay. Finally a mixed strategy is proposed to predict efficiency the value of the critical time delay. Four reference cases will be considered for all the rest of the study. In the case of no active control with time delay, these four reference cases can be defined as follows: the first one for $\mu = 0.34$ corresponds to a stable system with a friction coefficient just before the Hopf bifurcation. The second case for $\mu = 0.36$ corresponds to the unstable system with a friction coefficient just after the Hopf bifurcation. The last two cases (for $\mu = 0.4$ and $\mu = 0.6$) are two configurations far from

Table 1
Poles chart for the uncontrolled system and the controlled system without time delay.

Case	μ	Mode	Uncontrolled system		Controlled system	
			ω (rad.s ⁻¹)	Real part	ω (rad.s ⁻¹)	Real part
1	0.34	1	64	-0.5	64	-0.5
		2	70	-0.5	70	-0.5
2	0.36	1	67	1.6	67	-1.6
		2	67	-2.6	67	-2.6
3	0.4	1	67	5.4	67	-5.4
		2	67	-6.4	67	-6.4
4	0.6	1	69	14.5	69	-14.5
		2	69	-15.5	69	-15.5

Table 2
Estimation of τ_{lim} based on the frequency sweeping approach.

μ	τ_{lim}
0.34	0.0005
0.36	0.00482
0.4	0.0123
0.6	0.0097

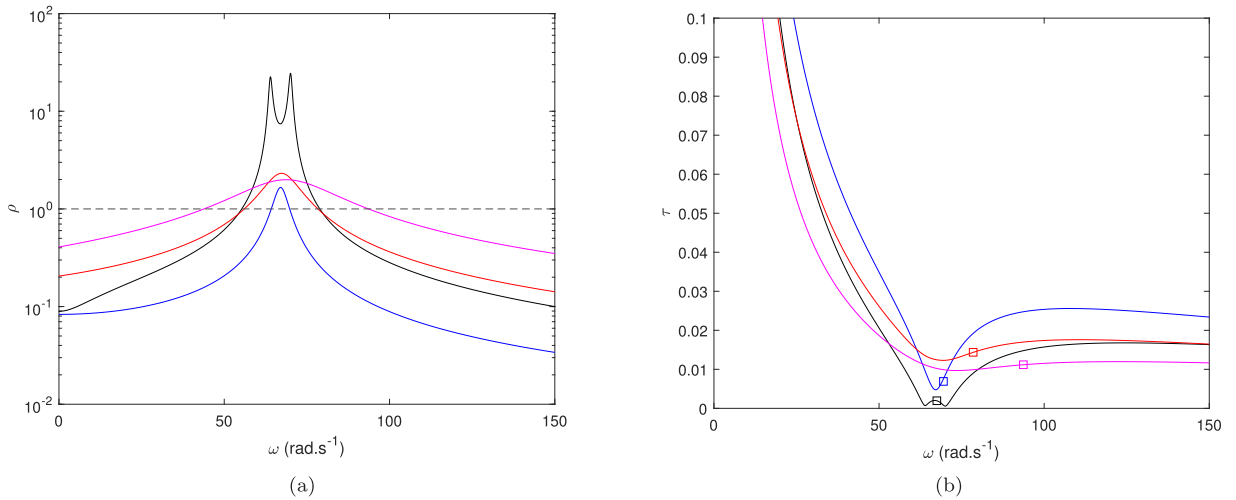


Fig. 2. Evolution of (a) the spectrum radius ρ and (b) the critical time delay τ versus the angular frequency ω based on the frequency sweeping approach (— $\mu = 0.34$; — $\mu = 0.36$; — $\mu = 0.4$; — $\mu = 0.6$); \square : prediction of (ω_c, τ_c) based on direct numerical simulations.

the Hopf bifurcation point. **Table 1** summarizes the stability analysis for each case (see the columns corresponding to the uncontrolled system).

First of all, we consider the stability analysis based on frequency sweeping tests. In this case, the matrices \mathbf{A}_0 and $\mathbf{A}_0 + \mathbf{A}_1$ are Hurwitz matrices for the case of the system regulated by state feedback control with time delay. As a reminder, the uncontrolled system is stable for $\mu \in [0; \mu_{Hopf}[$ with $\mu_{Hopf} = 0.34$ which is consistent with the fact that the matrix \mathbf{A}_0 is Hurwitz stable. For the other three case studies (i.e. $\mu = 0.36$, $\mu = 0.4$ and $\mu = 0.6$), only the matrix $\mathbf{A}_0 + \mathbf{A}_1$ is a stable matrix (i.e. every eigenvalue of $\mathbf{A}_0 + \mathbf{A}_1$ has strictly negative real part). Then **Fig. 2(a)** gives the spectrum radius of the matrix $(j\omega\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_1$ for the four cases under study. Due to the fact that this quantity can be superior to one for the frequency range of interest, we conclude that the system is delay-dependent stable for each case (to be noted that the black dashed line in **Fig. 2(a)** corresponds to the spectral radius equal to one). In addition **Fig. 2(b)** illustrates the curves (ω, τ) that defines the system stability border predicted by the frequency sweeping approach. The critical time delay value τ_{lim} for the frequency range of interested corresponds to the minimum value of τ , as also indicated in **Eq. (12)**. Values of τ_{lim} for the four cases under study are given in **Table 2**.

Then, **Table 3** gives estimation of the critical time delay and the associated angular frequency of the system by performing direct numerical simulations for various values of τ . Theses values are respectively denoted τ_c and ω_c in the rest of the

Table 3
Estimation of τ_c and ω_c based on direct numerical simulations.

μ	τ_c	ω_c (rad.s ⁻¹)
0.34	0.00195	68
0.36	0.0069	69
0.4	0.01436	78
0.6	0.01118	94

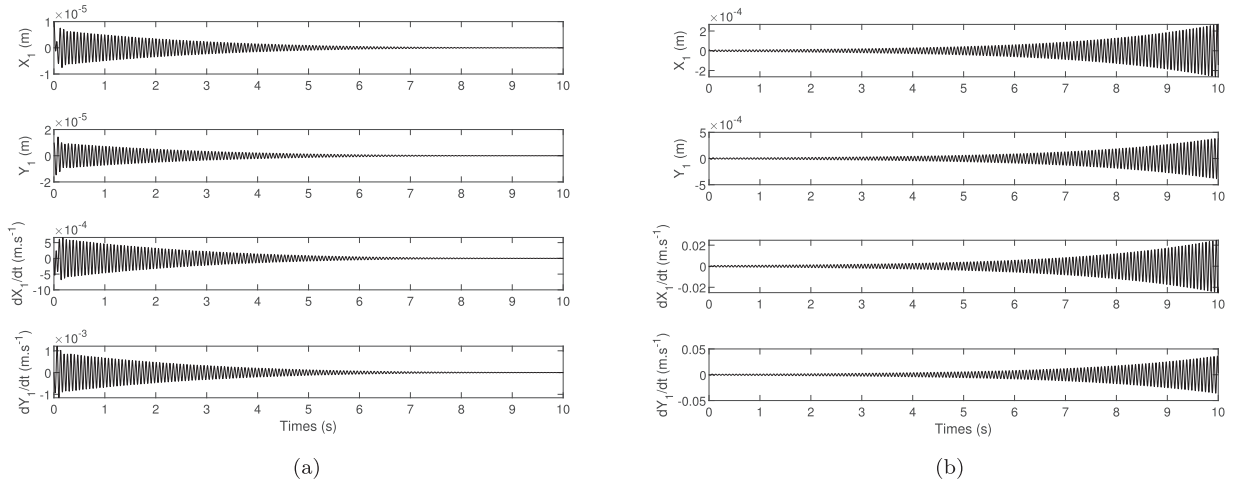


Fig. 3. Time responses of the controlled system for $\mu = 0.6$ and $\chi = 0.01$ (a) just before the critical time delay with $\tau_c(1 - \chi)$ and (b) just after the critical time delay with $\tau_c(1 + \chi)$.

study. In practice, numerical simulations have been carried out for different values of τ . For each simulation, a temporal integration of the system given in Eq. (6) has been performed and the stability of the system regulated by state feedback control with time delay has been carried out by considering the growing (unstable behaviour) or decreasing (stable behaviour) in vibration amplitudes. The critical time delay τ_c is then identified when the system goes from decreasing oscillations to increasing oscillations. Fig. 3 illustrates the time responses of the controlled system just before and just after the critical time delay τ_c for the case $\mu = 0.6$. The numerical simulations are performed with $\tau = \tau_c(1 \pm \chi)$ and $\chi = 0.01$. To be noted that a small disturbance on the displacement vector (i.e. initial conditions at time $t = 0$ given by $\mathbf{X} = [\epsilon \ \epsilon]^T$ and $\dot{\mathbf{X}} = \mathbf{0}$) has to be initially applied on the time-delay system. Several initial conditions for the temporal integration scheme have been tested and the results obtained for the estimation of the values (ω_c, τ_c) are always the same. It is clearly observed that the system is stable for $\tau = \tau_c(1 - \chi)$ with a decreasing of the initial oscillations (see Fig. 3(a)), whereas the system becomes unstable for $\tau = \tau_c(1 + \chi)$ with an increase of oscillations (see Fig. 3(b)). These results demonstrate that the critical value τ_{lim} given by the frequency sweeping test does not correspond to τ_c . More precisely it is observed that the controlled system is stable for a time delay greater than the critical value τ_{lim} (i.e. $\tau_c > \tau_{lim}$).

In addition, values (ω_c, τ_c) are plotted in Fig. 2(b) (see the square symbol \square for each case under study). It is clearly shown that the identified value of (ω_c, τ_c) corresponds to a specific point of the stability border predicted by the frequency sweeping approach.

The value τ_c for which the system regulated by state feedback control with time delay becomes unstable does not coincide with the critical time delay value τ_{lim} . Moreover it is observed that the range $[\tau_{lim}; \tau_c]$ is not negligible, which means that the system may remain stable even if the time delay is much higher than τ_{lim} . In fact τ_{lim} indicates only the minimum value of τ for which the system remains stable whatever the pulsation frequency of the system. Therefore, for a mechanical system subjected to friction-induced vibration (i.e. an autonomous system without external excitation), the interesting quantity to find for the critical time delay corresponds to the couple of values (ω, τ) where ω defines the angular frequency of the autonomous system (which is a priori an unknown quantity for a given τ).

So all these results demonstrate without any ambiguity the limitation of the use of the frequency sweep approach alone to efficiently estimate the real value of the critical time delay τ_c which does not generally correspond to the lower bound τ_{lim} of the system stability border (ω, τ). Now the question to answer is how to estimate this quantity (ω_c, τ_c) as accurately as possible. The novelty of the proposed study will be to develop a hybrid method based on the extension of the frequency sweeping test with an expansion based approach of the delayed term for approximating the eigenvalues problem. This new hybrid approach will make it possible to efficiently calculate the real critical time delay τ_c and the associated pulsation ω_c in the context of mechanical systems subjected to friction-induced vibration.

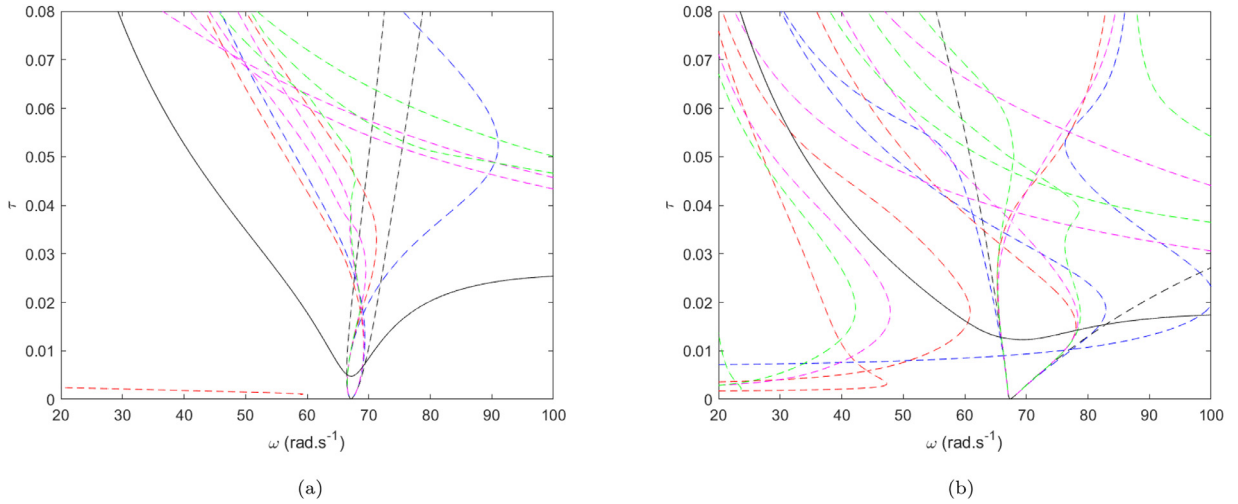


Fig. 4. Evolution of (ω, τ) based on the Taylor expansion of n th order for (a) $\mu = 0.36$ and (b) $\mu = 0.4$ (- - - 1st order; - - - 2nd order; - - - 3rd order; - - - 4th order; - - - 5th order); — $\tau(\omega)$ based on the frequency sweeping approach.

In an attempt to meet this objective, we first propose to investigate the stability analysis by considering the generalized eigenvalue problem for a Taylor series expansion of the delayed term $\mathbf{u}(t - \tau)$ up to the n th order as previously explained in Section 3.2. The eigenvalue problem of the closed-loop system defined in Eq. (7) has generally infinite number of roots over the complex field. Considering a Taylor series expansion of n th order leads to the calculation of a limited finite number of eigenvalues. Fig. 4(a) and (b) illustrate the imaginary parts (i.e. ω versus τ) of eigenvalues calculated with Taylor series expansions of n th order for $\mu = 0.36$ and $\mu = 0.4$, respectively. The primary and secondary eigenvalues [42] for $\tau \neq 0$ changed continuously (to be noted that the associated real parts are not shown for the sake of simplicity).

As suggested by Ram et al. [42], the system with no time delay (i.e. without τ) that has precisely $2p$ eigenvalues (with p the number of degrees of freedom) can be assimilated as the limiting situation of the time-delay system. This fact can be useful to distinguish between the primary and secondary eigenvalues for $\tau \neq 0$: by increasing continuously τ with the starting point $\tau = 0$, the location of each primary eigenvalue evolves continuously. Hence, assuming that the dynamic behaviour of the system evolves continuously from $\tau = 0$ to the critical time delay τ_c , it is possible to predict the evolution of the primary resonances and the associated stability behaviour by considering Taylor series for a chosen order. If one real part of eigenvalues which are linked with primary resonances is superior to zero then the system is unstable. By increasing continually the time delay τ , the first crossing from negative to positive value for real parts of the primary resonances give the critical time delay τ_{Tn} . The associated imaginary part ω_{Tn} corresponds to the angular pulsation of the self-excited vibration. The notation τ_{Tn} denotes the value of the quantity of interest by using a n th Taylor series expansion of the delayed term.

By applying this methodology, evolution of the two primary resonances based on the Taylor expansion of n th order (with $n=1, \dots, 5$) are given in Fig. 5 for the four cases under study ($\mu = 0.34$, $\mu = 0.36$, $\mu = 0.4$ and $\mu = 0.6$). For the interested reader, the system with no time delay (i.e. the starting point) is located by a circle \circ . Table 4 gives results for the detection of (ω_{Tn}, τ_{Tn}) at which the system becomes unstable (i.e. for $\tau > \tau_{Tn}$ at least one eigenvalue is outside the stability area). The couple of values (ω_{Tn}, τ_{Tn}) are also represented in Fig. 5 by a cross. Analyzing these results in detail, we can conclude that:

- a low order Taylor expansion does not allow to detect a critical time delay in some cases (see for example the 1st order for $\mu = 0.4$ and $\mu = 0.6$) while in other cases the estimated value (ω_{Tn}, τ_{Tn}) is sometimes far from the desired reference value (see for example the 1st order for $\mu = 0.34$ and $\mu = 0.36$ or the 2nd order for $\mu = 0.4$ and $\mu = 0.6$);
- upon reaching a certain order of the Taylor expansion, a perfect agreement between the estimate of the critical time delay value τ_c and the prediction of τ_{Tn} is observed for all the cases under study. Moreover, it is found that the value of ω_{Tn} (i.e. the imaginary part of the eigenvalue crossing the stability area at τ_{Tn}) fits with the value of the angular frequency ω_c of the self-excited vibrations;
- for an order of Taylor series expansion of sufficiently large, it is of course found that the couple (ω_{Tn}, τ_{Tn}) is very close to the frontier line defined by the frequency sweeping approach;
- the angular frequencies of the stable and unstable modes evolve in a non-negligible way as a function of τ . This is one of the explanations for the fact that the value predicted by τ_{lim} is not very close to the critical time delay τ_c .

Finally, it may also be noted that considering high orders of the Taylor series expansion may lead to ill-conditioned matrix for $\tau \rightarrow 0$. The problem is that the matrix \mathbf{Q}_n can be very close to being singular for $\tau \rightarrow 0$, although it is not. Nothing can be done to solve it due to the fact the expression of \mathbf{Q}_n is the starting input of the generalized eigenvalue

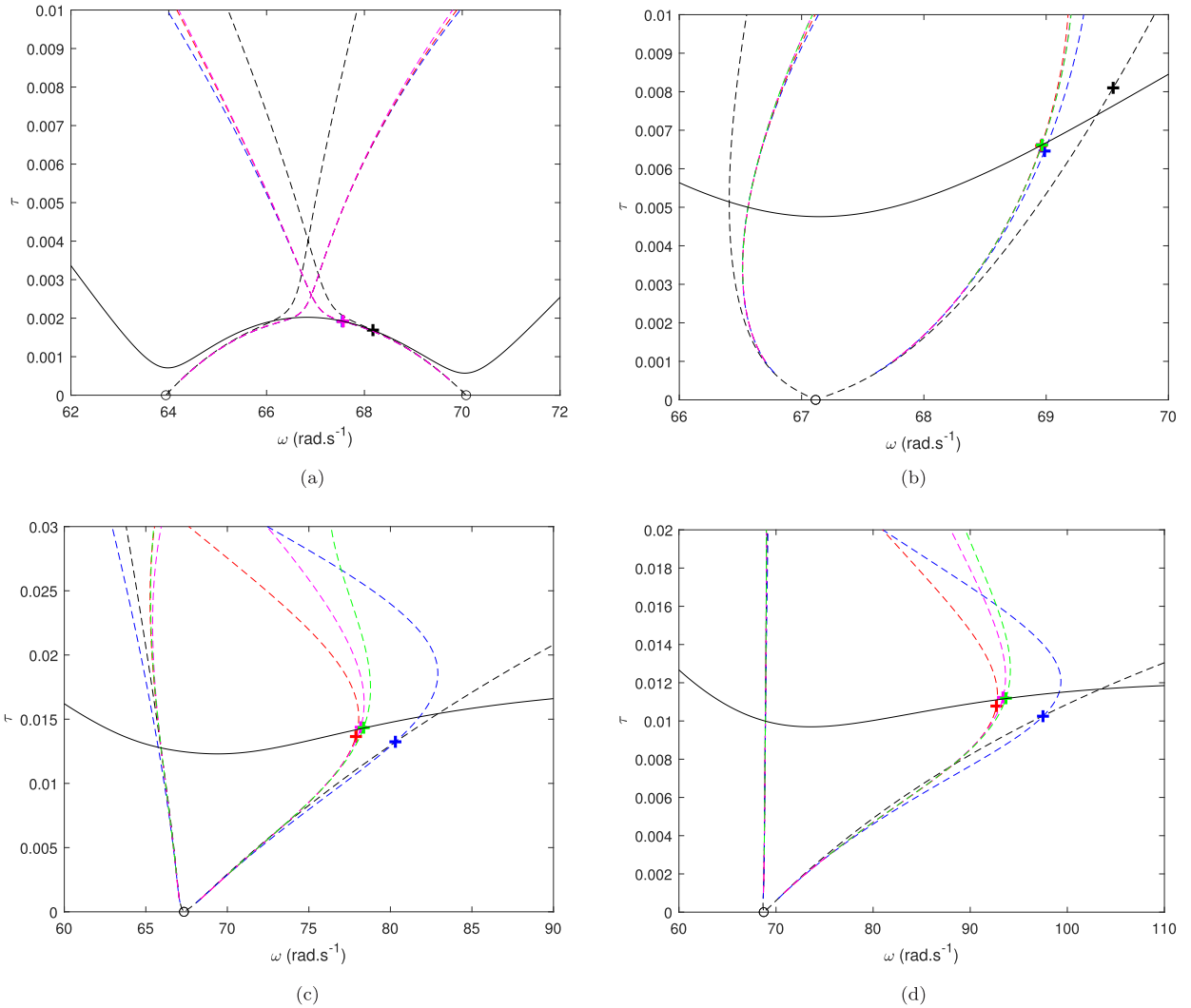


Fig. 5. Evolution of (ω, τ) based on the Taylor expansion of n th order for (a) $\mu = 0.34$ (b) $\mu = 0.36$ (c) $\mu = 0.4$ (d) $\mu = 0.6$ (--- 1st order; --- 2nd order; --- 3rd order; --- 4th order; --- 5th order); + prediction of (ω_{Tn}, τ_{Tn}) ; — $\tau(\omega)$ based on the frequency sweeping approach; \circ angular frequency for the system without time delay.

problem for a Taylor series expansion of the delayed term $\mathbf{u}(t - \tau)$ up to the n th order. This implies that the continuous tracking of primary resonances close to $\tau = 0$ can be difficult to achieve in practice for a high order in the Taylor series expansion. A basic solution is then to go back to a low order of the Taylor series (i.e. 1st or 2nd order for example) which is generally sufficient to initiate the process and then switch to a higher order to find a fine prediction of the critical time delay.

In this last part of this paper, we propose a mixed approach for the determination of the critical time delay. The proposed methodology is based on both the frontier stability curve estimated by the frequency sweeping approach and the generalized eigenvalue problem for a Taylor series expansion up to the n th order. The main idea is to find the specific point of the frontier stability curve that coincide with the unstable eigenvalue of the closed-loop system with time-delay. In order to achieve such an objective we propose to solve the generalized eigenvalue problem (22) for different values of τ by moving only along the frontier curve defined by the frequency sweeping and to focus on output eigenvalues of the problem that coincide with the input starting couple point (ω, τ) . Among the advantages of this approach, we can mention that no eigenvalue resolution for $\tau \rightarrow 0$ is requested, and only one eigenvalue is searched during the resolution process. The general process can be summarized as follow:

- step 1: definition of the frontier stability curve (ω, τ) based on the frequency sweeping tests for $\omega = [\omega_{min}; \omega_{max}]$ where ω_{min} and ω_{max} are the minimum and maximum limits for the angular frequency range of interest;
- step 2: i th calculation of the first approximated eigenvalue λ^i closest to the frontier curve for a specific value $\tau^i = \tau$ and by scanning the angular frequency interval on the frontier stability curve (ω_i, τ_i) with ω_i the i th iteration on the angular

Table 4

Estimation of τ_{Tn} and ω_{Tn} based on the Taylor expansion and the detection of positive real part of eigenvalues for primary resonances.

μ	Order n	τ_{Tn}	ω_{Tn} (rad.s ⁻¹)
0.34	1	0.00169	68
	2	0.00191	68
	3	0.00192	68
	4	0.00192	68
0.36	1	0.0081	70
	2	0.0065	69
	3	0.0067	69
	4	0.0067	69
	5	0.0067	69
0.4	1	-	-
	2	0.01324	80
	3	0.01366	78
	4	0.01438	78
	5	0.01435	78
0.6	1	-	-
	2	0.01025	97
	3	0.01078	93
	4	0.01122	93
	5	0.01119	94

Table 5

Estimation of τ_{Tn}^{mix} and ω_{Tn}^{mix} based on the mixed approach.

μ	Order n	τ_{Tn}^{mix}	ω_{Tn}^{mix} (rad.s ⁻¹)
0.34	1	0.00166	68
	2	0.00193	68
	3	0.00192	68
	4	0.00192	68
0.36	1	0.0074	69
	2	0.0067	69
	3	0.0066	69
	4	0.0066	69
	5	0.0066	69
0.4	1	-	-
	2	0.01518	82
	3	0.01419	78
	4	0.01423	78
	5	0.01431	78
0.6	1	-	-
	2	0.01149	97
	3	0.01111	93
	4	0.01115	93
	5	0.01117	94

frequency range of interest. The estimation of eigenvalues is performed via the generalized eigenvalue problem (22) for a Taylor series expansion up to the n th order. The notion of "closest to the frontier curve" can be easily performed by searching the eigenvalue with the imaginary part close to the angular frequency ω .

- step 3: inquiry on the compatibility between λ_i and (ω_i, τ_i) . If the imaginary part of the first approximated eigenvalue λ_i is equal to ω_i it means that the calculated eigenvalue is at the stability border. In this case, the real part of λ_i is examined: if the real part is negative, this eigenvalue λ^i corresponds to the stable primary resonance of the system (given by ω_i), i.e. the system remains stable. On the contrary, if the real part of λ_i is greater than zero, it means that the couple (ω_i, τ_i) of the frontier stability curve corresponds to the detection of the unstable mode and thus to the detection of the critical time delay and the associated angular frequency (ω_c, τ_c) .

One of the main advantage of the proposed mixed approach is that the problem of ill-conditioned matrix for \mathbf{Q}_n tends to disappear due to the fact that the calculation of eigenvalues is no longer carried out for $\tau \rightarrow 0$ but only for the specific values of τ that describe the frontier stability curve (ω, τ) based on the frequency sweeping approach.

Table 5 gives the critical time delay τ_{Tn}^{mix} and the associated angular frequency ω_{Tn}^{mix} . Results are in perfect agreement with the critical time delay previously calculated by the direct numerical simulations (to be compared with Table 3). Moreover

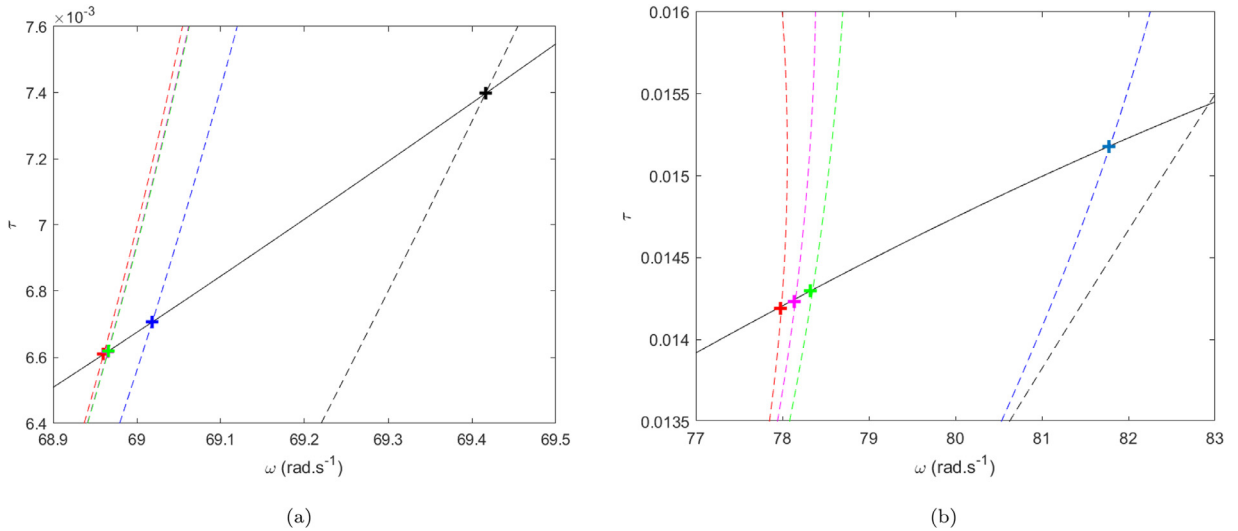


Fig. 6. Detection of the critical time delay based on the mixed approach for (a) $\mu = 0.36$ and (b) $\mu = 0.4$ - Predictions: + (ω_{T1}, τ_{T1}) , + (ω_{T2}, τ_{T2}) , + (ω_{T3}, τ_{T3}) , + (ω_{T4}, τ_{T4}) and + (ω_{T5}, τ_{T5}) ; — $\tau(\omega)$ based on the frequency sweeping approach and Taylor expansion for the - - - 1st order, - - - 2nd order, - - - 3rd order, - - - 4th order and - - - 5th order.

Fig. 6 illustrates the position of the critical time delay $(\omega_{Tn}^{mix}, \tau_{Tn}^{mix})$ based on the mixed approach for various n th order of the Taylor series expansion and for $\mu = 0.36$ and $\mu = 0.4$. To be noted that the evolutions of (τ, ω) based on the Taylor expansion of n th order are also visible (i.e. the dashed lines) even if these curves are not used in the mixed approach. They are only plotted in Fig. 6 to illustrate the fact that the mixed approach can also be understood as a way of finding the intersection between the evolution of the primary resonance of the system that becomes unstable and the stability frontier curve.

5. Conclusion

The stability analysis of a mechanical system regulated by state feedback control with time delay and the determination of the critical time delay τ_c are discussed. Based on comparison with direct time responses of the controlled system, it is demonstrated that the real critical time delay τ_c for the mechanical system subjected to friction-induced vibration does not coincide with the critical time delay τ_{lim} predicted by the frequency sweeping approach; it corresponds to a specific point of the stability border predicted by the frequency sweeping approach. More precisely, it is shown that the interesting quantity to find corresponds to the couple (ω_c, τ_c) where ω_c defines the angular frequency of the autonomous system (which is a priori an unknown quantity for a given time delay).

A first approach is to use the generalized eigenvalue problem approximation using the Taylor series expansion of the delayed term and to follow the evolution of the primary eigenvalues with the starting point corresponds to the system with no time delay (i.e. $\tau = 0$). This approach has the drawback of potentially causing ill-conditioned matrix if $\tau \rightarrow 0$ for high order of the Taylor series expansion and therefore makes it difficult to follow the evolution of the eigenvalues for low values of τ . Thereby we proposed a mixed approach for the determination of the critical time delay τ_c . This methodology is based on solving and analyzing the generalized eigenvalue problem approximation using the Taylor series expansion of the delayed term by focusing only on the frontier stability curve estimated by the frequency sweeping approach.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRedit authorship contribution statement

J.-J. Sinou: Conceptualization, Methodology, Software, Validation, Investigation, Resources, Writing - original draft, Writing - review & editing, Visualization, Supervision. **B. Chomette:** Conceptualization, Methodology, Software, Validation, Investigation, Resources, Writing - review & editing.

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