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(Theta, triangle)-free and (even hole, K_4)-free graphs. Part 1: Layered wheels

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Abstract

We present a construction called layered wheel. Layered wheels are graphs of arbitrarily large treewidth and girth. They might be an outcome for a possible theorem characterizing graphs with large treewidth in terms of their induced subgraphs (while such a characterization is well-understood in terms of minors). They also provide examples of graphs of large treewidth and large rankwidth in well-studied classes, such as (theta, triangle)-free graphs and even-hole-free graphs with no K_4 (where a hole is a chordless cycle of length at least four, a theta is a graph made of three internally vertex disjoint paths of length at least two linking two vertices, and K_4 is the complete graph on four vertices).

1 Introduction

In this article, all graphs are finite, simple, and undirected. The vertex set of a graph G is denoted by $V(G)$ and the edge set by $E(G)$. A graph H is an *induced subgraph* of a graph G if some graph isomorphic to H can be obtained from G by deleting vertices. A graph H is a *minor* of a graph G if some graph isomorphic to H can be obtained from G by deleting vertices, deleting edges, and contracting edges.

When we say that G *contains* H without specifying *as a minor* or *as an induced subgraph*, we mean that H is an induced subgraph of G. A graph is *H-free* if it does not contain H (so, as an induced subgraph). For a family of graphs $\mathcal{H}$, G is *$\mathcal{H}$-free* if for every $H \in \mathcal{H}$, G is H-free. A class of graphs is *hereditary* if it is $\mathcal{H}$-free for some $\mathcal{H}$ or, equivalently, if it is closed under taking induced subgraphs. A *hole* in a graph is a chordless cycle of length at least four. It is *odd* or *even* according to its length (that is its number of edges). We denote by K_ℓ the complete graph on ℓ vertices.

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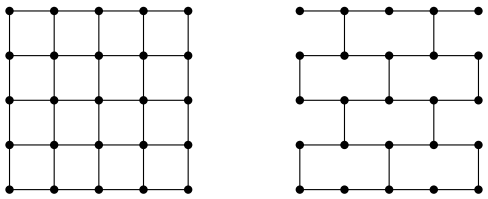


Figure 1: A grid and a wall

The present work is originally motivated by a question asked by Cameron et al. in [3]: is the treewidth (or cliquewidth) of an even-hole-free graph bounded by a function of its clique number? In this first part, we describe a construction called *layered wheel* showing that the answer is no. In the second part, we will show that under additional restrictions, the treewidth is bounded. We postpone the formal definition of a layered wheel to Section 3 although we use the term several times until then. There are three main motivations:

- When considering the induced subgraph relation (instead of the minor relation), is there a theorem similar to the celebrated grid-minor theorem of Robertson and Seymour?
- A better understanding of the classes defined by excluding the so-called Truemper configurations, that play an important role in hereditary classes of graphs.
- The structure of even-hole-free graphs.

We now give details on each of the three items.

The grid-minor theorem

The *treewidth* of a graph is an integer measuring how far is the graph from being a tree (far here means the difficulty of decomposing the graph in a kind of tree-structure). We give a formal definition of treewidth in Section 2.

The $(k \times k)$-*grid* is the graph on $\{(i,j) : 1 \leq i,j \leq k\}$ where two distinct ordered pairs (i,j) and (i',j') are adjacent whenever exactly one of the following holds: $|i-i'| = 1$ and $j = j'$, or $i = i'$ and $|j-j'| = 1$ (see Figure 1). Robertson and Seymour [21] proved that there exists a function f such that every graph with treewidth at least $f(k)$ contains a $(k \times k)$-grid as a minor (see [8] for the best function known so far). This is called the *grid-minor theorem*. The $(k \times k)$-*wall* is the graph obtained from the $(k \times k)$-grid by deleting all edges with form $(2i+1, 2j) - (2i+1, 2j+1)$ and $(2i, 2j+1) - (2i, 2j+2)$.

Subdividing k times an edge $e = uv$ of a graph, where $k \geq 1$, means deleting e and adding a path $uw_1 \dots w_k v$. The *k-subdivision* of a graph G is the graph obtained from G by subdividing k-times all its edges (simultaneously). Note that replacing "grid" by a more specific graph in the grid-minor theorem, such

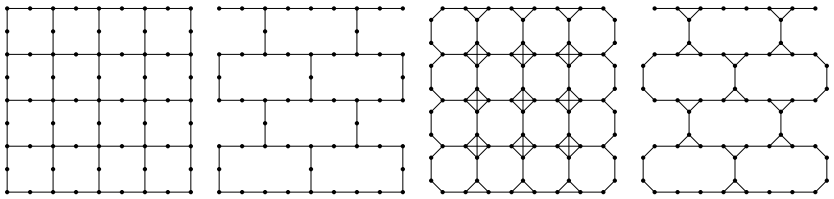


Figure 2: A subdivision of a grid, of a wall, and the line graphs of the former

as k-subdivision of a $(k \times k)$-grid, $(k \times k)$-wall, or k-subdivision of a $(k \times k)$-wall provides statements that are formally weaker (at the expense of a larger function), because a large grid contains a large subdivision of a grid, a large wall, and a large subdivision of a wall. However, these trivial corollaries are in some sense stronger, because walls, subdivisions of walls, and subdivision of grids are graphs of large treewidth that are more sparse than grids. So they somehow certify a large treewidth with less information. Since one can always subdivide more, there is no "ultimate" theorem in this direction.

It would be useful to have a similar theorem with "induced subgraph" instead of "minor". Simply replacing "minor" with "induced subgraph" in the statement is trivially false, and here is a list of known counter-examples: K_k, $K_{k,k}$, subdivisions of walls, line graphs of subdivisions of walls (see Figure 2), where K_k denotes the complete graph on k vertices, $K_{k,k}$ denotes the complete bipartite graph with each side of size k, and where the *line graph* of a graph R is the graph G on $E(R)$ where two vertices in G are adjacent whenever they are adjacent edges of R.

One of our results is that the simple list above is not complete. In section 3, we present a construction that we call *layered wheel*. Layered wheels have large treewidth and large girth (the *girth* of a graph is the length of its shortest cycle). Large girth implies that they contain no K_k, no $K_{k,k}$, and no line graphs of subdivisions of walls. Moreover, layered wheels contain no subdivisions of $(3, 5)$-grids (this is explained after Lemma 3.3).

We leave an open question asked by Zdeněk Dvořák (personal communication): is it true that for some function f every graph with treewidth at least $f(k)$ contains either K_k, $K_{k,k}$, a subdivision of the $(k \times k)$-wall, the line graph of some subdivision of the $(k \times k)$-wall, or some variant of the layered wheel with at least k layers? In the next paragraphs, we give variants of Dvořák's question.

Truemper configurations

A *prism* is a graph made of three vertex-disjoint chordless paths $P_1 = a_1 \dots b_1$, $P_2 = a_2 \dots b_2$, $P_3 = a_3 \dots b_3$ of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles and no edges exist between the paths except those of the two triangles. Such a prism is also referred to as a $3PC(a_1a_2a_3, b_1b_2b_3)$ or a $3PC(\Delta, \Delta)$ (3PC stands for *3-path-configuration*).

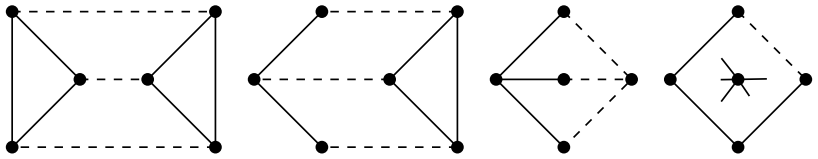


Figure 3: Prism, pyramid, theta, and wheel (dashed lines represent paths)

A *pyramid* is a graph made of three chordless paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$ of length at least one, two of which have length at least two, vertex-disjoint except at a, and such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident to a. Such a pyramid is also referred to as a $3PC(b_1b_2b_3, a)$ or a $3PC(\Delta, \cdot)$.

A *theta* is a graph made of three internally vertex-disjoint chordless paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of length at least two and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b. Such a theta is also referred to as a $3PC(a, b)$ or a $3PC(\cdot, \cdot)$.

Observe that the lengths of the paths in the three definitions above are designed so that the union of any two of the paths induces a hole. A *wheel* $W = (H, c)$ is a graph formed by a hole H (called the *rim*) together with a vertex c (called the *center*) that has at least three neighbors in the hole.

A *3-path-configuration* is a graph isomorphic to a prism, a pyramid, or a theta. A *Truemper configuration* is a graph isomorphic to a prism, a pyramid, a theta, or a wheel. They appear in a theorem of Truemper [23] that characterizes graphs whose edges can be labeled so that all chordless cycles have prescribed parities (3-path-configurations seem to have first appeared in a paper Watkins and Mesner [26]).

Truemper configurations play an important role in the analysis of several important hereditary graph classes, as explained in a survey of Vušković [25]. Let us simply mention here that many decomposition theorems for classes of graphs are proved by studying how some Truemper configurations contained in the graph attaches to the rest of the graph, and often, the study relies on the fact that some other Truemper configurations are excluded from the class. The most famous example is perhaps the class of *perfect graphs*. In these graphs, pyramids are excluded, and how a prism contained in a perfect graphs attaches to the rest of the graph is important in the decomposition theorem for perfect graphs, whose corollary is the celebrated *Strong Perfect Graph Theorem* due to Chudnovsky, Robertson, Seymour, and Thomas [5]. See also [22] for a survey on perfect graphs, where a section is specifically devoted to Truemper configurations. Many other examples exist, see [13] for a long list of them.

Some researchers started to study systematically classes defined by excluding some Truemper configurations [13]. We believe that among many classes that can be defined in that way, the class of theta-free graphs is one of the most interesting classes. This is because it generalizes claw-free graphs (since a theta

contains a claw), and so it is natural to ask whether it shares the most interesting features of claw-free graphs: a structural description (see [6]), a polynomial time algorithm for the maximum stable set (see [14]), an approximation algorithms for the chromatic number (see [18]), a polynomial time algorithm for the induced linkage problem (see [15]), and a polynomial χ-bounding function (see [17]).

In the attempt of finding a structural description of theta-free graphs, a seemingly easy case is when triangles are also excluded. Because then, every vertex of degree at least 3 is the center of a claw (therefore a possible start for a theta), so that excluding theta and triangle should enforce some structure. Supporting this idea, Radovanović and Vušković [20] proved that every (theta, triangle)-free graph is 3-colorable.

Hence, we believed when starting this work that (theta, triangle)-free graphs have bounded treewidth. But this turned out to be false: layered wheels are (theta, triangle)-free graphs of arbitrarily large treewidth.

However, on the positive side, we note that layered wheels need many vertices to increase the treewidth. More specifically, a layered wheel G is made of $l+1$ layers, where l is an integer. Each layer is a path and $|V(G)| \geq 2^l$ (see Lemma 3.2), $l \leq \text{tw}(G) \leq 2l$ (see Theorems 3.12 and 5.4). So, the treewidth of a layered wheel is "small" in the sense that it is logarithmic in the size of its vertex set. We wonder whether such a behavior is general in the sense of the following conjecture.

Conjecture 1.1. *For some constant c, if G is a (theta, triangle)-free graph, then the treewidth of G is at most $c \log |V(G)|$.*

This conjecture reflects our belief that constructions similar to the layered wheel must have an exponential number of vertices (exponential in the treewidth). It suggests the following variant of Dvořák's question: is it true that for some constant $c > 1$ and some function f, every graph with treewidth at least $f(k)$ contains either K_k, $K_{k,k}$, a subdivision of the $(k \times k)$-wall, the line graph of some subdivision of the $(k \times k)$-wall, or has at least $c^{f(k)}$ vertices?

Kristina Vušković observed that $K_{k,k}$ is a (prism, pyramid, wheel)-free graph, or equivalently an *only-theta graph* (because the theta is the only Truemper configuration contained in it). Moreover, walls are only-theta graphs, line graphs of subdivisions of walls are only-prism graphs, and triangle-free layered wheels are only-wheel graphs. Observe that complete graphs contain no Truemper configuration, so they are simultaneously only-prism, only-wheel, and only-theta. One may wonder whether a graph with large treewidth should contain an induced subgraph of large treewidth with a restricted list of induced subgraphs isomorphic to Truemper configurations.

Even-hole-free graphs

Our last motivation for this work is a better understanding of even-hole-free graphs. These are related to Truemper configurations because thetas and prisms obviously contain even holes (to see this, consider two paths of the same parity among the three paths that form the configuration). Also, call *even wheel* a

wheel $W = (H, c)$ where c has an even number of neighbors in H. It is easy to check that every even wheel contains an even hole.

Even-hole-free graphs were originally studied to experiment techniques that would help to settle problems on perfect graphs. This has succeeded, in the sense that the decomposition theorem for even-hole-free graphs (see [24]) is in some respect similar to the one that was later on discovered for perfect graphs (see [5]). However, classical problems such as graph coloring or maximum stable set, are polynomial time solvable for perfect graphs, while they are still open for even-hole-free graphs. This is a bit strange because the decomposition theorem for even-hole-free graphs is in many respect simpler than the one for perfect graphs. Moreover, it is easy to provide perfect graphs of arbitrarily large treewidth (or even rankwidth), such as bipartite graphs, or their line graphs. On the other hand, for even-hole-free graphs, apart from complete graphs, it is not so easy. Some constructions are known, see [1].

But so far, every construction of even-hole-free graphs of arbitrarily large treewidth (or rankwidth) contains large cliques. Moreover, it is proved in [4] that (even hole, triangle)-free graphs have bounded treewidth. This is based on a structural description of the class from [9]. Hence, Cameron et al. [3] asked whether (even hole, K_4)-free graphs have bounded treewidth. We prove in this article that it is not the case, by a variant of the layered wheel construction (see Theorem 3.10). As for (theta, triangle)-free, we need a large number of vertices to grow the treewidth, so we propose the following conjecture.

Conjecture 1.2. *There exists a constant c such that for any (even hole, K_4)-free graph G, the treewidth of G is at most $c \log |V(G)|$.*

Our construction of even-hole-free layered wheels contains diamonds (a *diamond* is a graph obtained for K_4 by removing an edge). We therefore propose the following conjecture.

Conjecture 1.3. *Even-hole-free graphs with no K_4 and no diamonds have bounded treewidth.*

(Even hole, pyramid)-free graphs attracted some attention (see [7]). It is therefore worth noting that even-hole-free layered wheels are pyramid-free (see Theorem 3.11). We remark that it is also possible to obtain a variant of even-hole-free layered wheel that does contain pyramids. We omit giving all details about this construction that is still of interest because it might give indications about how an even-hole-free graph can be decomposed (or not) around a pyramid.

We note that for the classes where we prove unbounded treewidth, the cliquewidth (and therefore the rankwidth), to be defined later, is also large (see Theorems 3.15 and 4.16).

Outline of the article

In Section 2, we introduce the terminology used in our proofs.

In Section 3, we describe the construction of layered wheels for two classes of graphs: (theta, triangle)-free graphs and (even hole, K_4)-free graphs (in fact, we prove it for a more restricted class namely (even hole, K_4, pyramid)-free graphs). We prove that the constructions actually yield graphs in the corresponding classes (this is non-trivial, see Theorems 3.5, 3.10, and 3.11). We then prove that layered wheels have unbounded treewidth (see Theorem 3.12) and cliquewidth (see Theorem 3.15).

In Section 4, we recall the definition of rankwidth. We exhibit (theta, triangle)-free graphs and (even hole, K_4)-free graphs with large rankwidth. This is a trivial corollary of Theorem 3.15, but the computation is more accurate (see Theorem 4.16).

In Section 5, we give an upper bound on the treewidth of layered wheels. We prove a stronger result: the so-called *pathwidth* of layered wheels is bounded by some linear function of the number of its layers (see Theorem 5.4).

2 Summary of the main results and terminology

The treewidth, cliquewidth, rankwidth, and pathwidth of a graph G are denoted by $\mathrm{tw}(G)$, $\mathrm{cw}(G)$, $\mathrm{rw}(G)$, and $\mathrm{pw}(G)$ respectively. The following lemma is well-known.

Lemma 2.1 (See [11] and [19])**.** *For every graph G, the followings hold:*

- $\mathrm{rw}(G) \leq \mathrm{cw}(G) \leq 2^{\mathrm{rw}(G)+1}$*;*
- $\mathrm{cw}(G) \leq 3 \cdot 2^{\mathrm{tw}(G)} - 1$*;*
- $\mathrm{tw}(G) \leq \mathrm{pw}(G)$*.*

The first item of the lemma is proved in [11], and the second item is proved in [19]. The third item follows because pathwidth is a special case of treewidth (see Section 5). All results presented in this article can be summarized in the next two theorems.

Theorem 2.2. *For every integers $l \geq 1$ and $k \geq 4$, there exists a graph $G_{l,k}$ such that the followings hold:*

- *$G_{l,k}$ is theta-free and has girth at least k (in particular, $G_{l,k}$ is triangle-free);*
- $l \leq \mathrm{tw}(G_{l,k}) \leq \mathrm{pw}(G_{l,k}) \leq 2l$*;*
- $l \leq \mathrm{rw}(G_{l,k}) \leq \mathrm{cw}(G_{l,k}) \leq 3 \cdot 2^{\mathrm{tw}(G)} - 1 \leq 3 \cdot 2^{2l} - 1 \leq |V(G_{l,k})|$*.*

Theorem 2.3. *For every integers $l \geq 1$ and $k \geq 4$, there exists a graph $G_{l,k}$ such that the followings hold:*

- *$G_{l,k}$ is (even hole, K_4, pyramid)-free and every hole in $G_{l,k}$ has length at least k;*

- $l \leq \mathrm{tw}(G_{l,k}) \leq \mathrm{pw}(G_{l,k}) \leq 2l$;
- $l \leq \mathrm{rw}(G_{l,k}) \leq \mathrm{cw}(G_{l,k}) \leq 3 \cdot 2^{\mathrm{tw}(G)} - 1 \leq 3 \cdot 2^{2l} - 1 \leq |V(G_{l,k})|$.

A graph H is a *subgraph* of a graph G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a graph G and a subset $X \subseteq V(G)$, we let $G[X]$ denote the subgraph of G *induced* by X, i.e. $G[X]$ has vertex set X, and $E(G[X])$ consists of the edges of G that have both ends in X.

For simplicity, sometimes we do not distinguish between a vertex set and the graph induced by the vertex set. So we write $G \setminus H$ instead of $G[V(G) \setminus V(H)]$. Also for a vertex $v \in V(G)$, we write $G \setminus v$ (instead of $G[V(G) \setminus \{v\}]$) and similarly, we write $G \setminus S$ for some $S \subseteq V(G)$. For $v \in V(G)$, we denote by $N_H(v)$, the set of neighbors of v in H that is called the *neighborhood* of v, and $N_G(v)$ is also denoted by $N(v)$.

A *path* in G is a sequence P of distinct vertices $p_1 \dots p_n$, where for $i, j \in \{1, \dots, n\}$, $p_i p_j \in E(G)$ if and only if $|i - j| = 1$. For two vertices $p_i, p_j \in V(P)$ with $j > i$, the path $p_i p_{i+1} \dots p_j$ is a *subpath of* P that is denoted by $p_i P p_j$. The subpath $p_2 \dots p_{n-1}$ is called the *interior* of P. The vertices p_1, p_n are the *ends* of the path, and the vertices in the interior of P are called the *internal* vertices of P.

A *cycle* is defined similarly, with the additional properties that $n \geq 4$ and $p_1 = p_n$. The *length* of a path P is the number of edges of P. The length of cycle is defined similarly.

We now give a formal definition of treewidth. A *tree decomposition* of a graph G is a pair $(T, \{X_t\}_{t \in V(T)})$, where T is a tree whose every node t is assigned a vertex subset $X_t \subseteq V(G)$, called a *bag*, such that the following three conditions hold:

(T1) $\bigcup_{t \in V(T)} X_t = V(G)$, i.e., every vertex of G is in at least one bag.

(T2) For every $uv \in E(G)$, there exists a node t of T such that bag X_t contains both u and v.

(T3) For every $u \in V(G)$, the set $T_u = \{t \in V(T) : u \in X_t\}$, i.e., the set of nodes whose corresponding bags contain u, induces a connected subtree of T.

The *width* of tree decomposition $(T, \{X_t\}_{t \in V(T)})$ equals $\max_{t \in V(T)} |X_t| - 1$, that is, the maximum size of its bag minus 1. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum possible width of a tree decomposition of G.

3 Construction and treewidth

In this section, we describe the construction of layered wheels for two classes of graphs, namely the class of (theta, triangle)-free graphs and the class of (even hole, K_4)-free graphs. We also give a lower bound on their treewidth.

(Theta, triangle)-free layered wheels

We now present *ttf-layered-wheels* which are theta-free graphs of girth at least k, containing K_{l+1} as a minor, for all integers $l \geq 1, k \geq 4$ (see Figure 4).

Construction 3.1. *Let $l \geq 0$ and $k \geq 4$ be integers. An (l,k)-ttf-layered-wheel, denoted by $G_{l,k}$, is a graph consisting of $l+1$ layers, which are paths $P_0, P_1, \ldots, P_l$. The graph is constructed as follows.*

(A1) *$V(G_{l,k})$ is partitioned into $l+1$ vertex-disjoint paths $P_0, P_1, \ldots, P_l$. So, $V(G_{l,k}) = V(P_0) \cup \cdots \cup V(P_l)$. The paths are constructed in an inductive way.*

(A2) *The path P_0 consists of a single vertex.*

(A3) *For every $0 \leq i \leq l$ and every vertex u in P_i, we call* ancestor of u *any neighbor of u in $V(P_0) \cup \cdots \cup V(P_{i-1})$. The* type of u *is the number of its ancestors (as we will see, the construction implies that every vertex has type 0 or 1). Observe that the unique vertex of P_0 has type 0. We will see that the construction implies that for every $1 \leq i \leq l$, the ends of P_i are vertices of type 1.*

(A4) *Suppose inductively that $l \geq 1$ and layers $P_0, P_1, \ldots, P_{l-1}$ are constructed. The l^{th}-layer P_l is built as follows.*

For any $u \in P_{l-1}$ we define a path Box_u (that will be a subpath of P_l), in the following way:

- *if u is of type 0, Box_u contains three neighbors of u, namely u_1, u_2, u_3, in such way that $\mathrm{Box}_u = u_1 \ldots u_2 \ldots u_3$.*
- *if u is of type 1, let v be its unique ancestor. Box_u contains six neighbors of u, namely $u_1, \ldots, u_6$, and three neighbors of v, namely v_1, v_2, v_3, in such a way that*

$$\mathrm{Box}_u = u_1 \ldots u_2 \ldots u_3 \ldots v_1 \ldots v_2 \ldots v_3 \ldots u_4 \ldots u_5 \ldots u_6.$$

The neighbors of u and the neighbors of v in Box_u are of type 1, the other vertices of Box_u are of type 0. We now specify the lengths of the boxes and how they are connected to form P_l.

(A5) *The path P_l goes through the boxes of P_l in the same order as vertices in P_{l-1}. For instance, if uvw is a subpath of P_{l-1}, then P_l goes through Box_u, Box_v, and Box_w, in this order along P_l. Note that the vertices of P_l that are in none of the boxes are of type 0. Note that for $u \neq v$, we have $\mathrm{Box}_u \cap \mathrm{Box}_v = \emptyset$.*

(A6) *Let w, w' be vertices of type 1 in P_l (so vertices from the boxes), and consecutive in the sense that the interior of wP_lw' contains no vertex of type 1. Then wP_lw' is a path of length at least $k-2$.*

(A7) Observe that every vertex in P_l has type 0 or 1.

(A8) There are no other vertices or edges apart from the ones specified above.

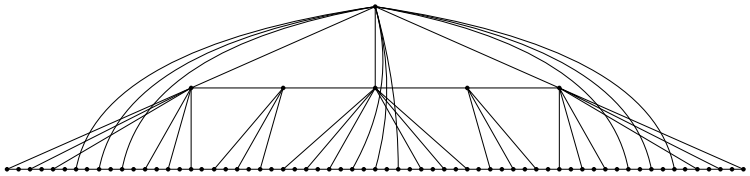


Figure 4: A ttf-layered-wheel $G_{2,4}$

Observe that the construction is not fully deterministic because in (A6), we just indicate a lower bound on the length of wP_lw', so there may exist different ttf-layered-wheels $G_{l,k}$. This flexibility will be convenient below to exhibit ttf-layered-wheels of arbitrarily large rankwidth.

Lemma 3.2. *For $0 \leq i \leq l-1$ and $i+1 \leq j \leq l$, every vertex $u \in V(P_i)$ has at least 3^{j-i} neighbors in P_j.*

Proof. We prove the lemma by induction on j. If $j = i+1$, then (A4) implies that for every $0 \leq i \leq l-1$ and every vertex u in P_i, u has three or six neighbors in P_{i+1}. If $j > i+1$, then by the induction hypothesis, every vertex $u \in V(P_i)$ has at least 3^{j-1-i} neighbors in P_{j-1}. Hence by (A4), it has at least $3 \cdot 3^{j-1-i} = 3^{j-i}$ neighbors in P_j. □

Lemma 3.2 implies in particular that every vertex of layer i has neighbors in all layers $i+1, \ldots, l$. Construction 3.1 is in fact the description of an inductive algorithm that constructs $G_{l,k}$. So, the next lemma is clear.

Lemma 3.3. *For every integers $l \geq 0$ and $k \geq 4$, there exists an (l,k)-ttf-layered-wheel.*

We now prove that Construction 3.1 produces a theta-free graph with arbitrarily large girth and treewidth. Observe that any subdivision of the (3,5)-grid contains a theta. Thus, Theorem 3.5 implies that a ttf-layered-wheel does not contain any subdivision of (3,5)-grid as mentioned in the introduction.

The next lemma is useful to prove Theorem 3.5. For a theta consisting of three paths P_1, P_2, P_3, the common ends of those paths are called the *apexes* of the theta. Let G be graph containing a path P. The path P is *special* if

- there exists a vertex $v \in V(G \setminus P)$ such that $|N_P(v)| \geq 3$; and
- in $G \setminus v$, every vertex of P has degree at most 2.

Note that in the next lemma, we make no assumption on G, that in particular may contain triangles.

Lemma 3.4. *Let G be a graph containing a special path P. For any theta that is contained in G (if any), every apex of the theta is not in P.*

Proof. Let v be a vertex satisfying the properties as in the definition of special path. For a contradiction, suppose that P contains some vertex u which is an apex of some theta Θ in G. Note that u must have degree 3, and is therefore a neighbor of v. Consider two subpaths of P, u_1Pu_2 and u_2Pu_3 such that $u \in \{u_1, u_2, u_3\} \subseteq N(v)$ and both u_1Pu_2, u_2Pu_3 have no neighbors of v in their interior. This exists since $|N_P(v)| \geq 3$. Since u is an apex, either $H_1 = vu_1Pu_2v$ or $H_2 = vu_2Pu_3v$ is a hole of Θ. Without loss of generality suppose that $V(H_1) \subseteq V(\Theta)$. Hence the other apex of Θ must be also contained in H_1. Since $u_1v, u_2v \in E(G)$ and all vertices of $H_1 \setminus \{u_1, v, u_2\}$ have degree 2, u_1, u_2 must be the two apexes of Θ. Since $d(u_2) = 3$, $V(u_2Pu_3) \subseteq \Theta$. But then v has degree 3 in Θ while not being an apex, a contradiction. This completes the proof. □

Theorem 3.5. *For every integers $l \geq 0$ and $k \geq 4$, every (l, k)-ttf-layered-wheel $G_{l,k}$ is theta-free graph with girth at least k.*

Proof. We first show by induction on l that $G_{l,k}$ has girth at least k. This is clear for $l \leq 1$, so suppose that $l \geq 2$ and let H be a cycle in $G_{l,k}$ whose length is less than k. We may assume that layer P_l contains some vertex of H, for otherwise H is a cycle in $G_{l-1,k}$, so it has length at least k by the induction hypothesis. Let $P = u \dots v$ be a path such that $V(P) \subseteq V(H) \cap V(P_l)$ and with the maximum length among such possible paths. Note that P contains at least two vertices. Indeed, if P contains a single vertex, then such a vertex must have at least two ancestors, since it has degree 2 in H, which is impossible by the construction of $G_{l,k}$. So $u \neq v$. Moreover, note that as P is contained in a cycle, both u and v must have an ancestor. Let u' and v' be the ancestor of u and v respectively. By (A6) of Construction 3.1 P has length at least $k - 2$. Hence $u'uPvv'$ has length at least $k - 1$, so H has length at least k. This completes the proof.

Now we show that $G_{l,k}$ is theta-free. For a contradiction, suppose that it contains a theta. Let Θ be a theta with minimum number of vertices, and having u and v as apexes. As above, without loss of generality, we may assume that P_l contains some vertex of Θ. Note that every vertex of P_l is contained in a special path of $G_{l,k}$. Hence, by Lemma 3.4, $u, v \notin V(P_l)$. In particular, every vertex of $V(P_l) \cap V(\Theta)$ has degree 2 in Θ.

Let $P = x \dots y$ for some $x, y \in P_l$, be a path such that $V(P) \subseteq V(\Theta) \cap V(P_l)$ and it is inclusion-wise maximal with respect to this property. Since every vertex of P_l has at most one ancestor, $x \neq y$. Moreover, both x and y must have an ancestor, because every vertex of Θ has degree 2 or 3 in Θ. Let x' and y' be the ancestor of x and y respectively. By the maximality of P, both x' and y' are also in Θ. Note that no vertex in the interior of P is adjacent to x' or y', since otherwise such a vertex would have degree 3 in Θ, meaning that it is an apex, a contradiction.

Claim 1. *We have $x' \neq y'$, $x'y' \notin E(G_{l,k})$, and some internal vertex of P is of type 1.*

Proof of Claim 1. Otherwise, $x' = y'$ or $x'y' \in E(G_{l,k})$, or every internal vertex of P is of type 0. In the last case, we also have $x' = y' \in V(P_{l-1})$ or $x'y' \in E(G_{l,k})$ by the construction of $G_{l,k}$. Hence, in all cases, $V(P) \cup \{x', y'\}$ induces a hole in Θ, that must contain both u and v. Since $u, v \notin V(P_l)$, we have $u, v \in \{x', y'\}$. But this is not possible as $x' = y'$ or $x'y' \in E(G_{l,k})$. This proves Claim 1.

We now set $P' = x'xP_lyy'$ (that is a path by Claim 1).

Claim 2. *There exists no vertex of type 0 in P_{l-1} that has a neighbor in the interior of P.*

Proof of Claim 2. For a contradiction, let $t \in V(P_{l-1})$ be of type 0 that has neighbors in the interior of P. Note that $t \notin V(\Theta)$ because internal vertices of P have degree 2 in Θ. Let Q be the shortest path from x' to y' in $G_{l,k}[V(P') \cup \{t\}]$. Note that Q is shorter than P', because it does not go through one vertex of $N_P(t)$. So, P' can be substituted for Q in Θ, which provides a theta from u to v with less vertices, a contradiction to the minimality of Θ. This proves Claim 2.

Claim 3. *We may assume that:*

- *$x' \in V(P_{l-1})$ and x' has type 0.*
- *$y' \notin V(P_{l-1})$.*
- *y' has a neighbor w in P_{l-1} and $x'w \in E(G_{l,k})$.*
- *Every vertex in P has type 0, except x, y, and three neighbors of w. Observe that w has type 1 and has three more neighbors in P_l that are not in P.*

Proof of Claim 3. Suppose first that x', y' are both in P_{l-1}. Then by Claim 1, the path $x'P_{l-1}y'$ has length at least two. Moreover, by Claim 2, all its internal vertices are of type 1, because they all have neighbors in the interior of P. It follows that $x'P_{l-1}y'$ has length exactly two. We denote by z its unique internal vertex. Substituting $x'zy'$ for P', we obtain a theta that contradicts the minimality of Θ. Observe that the ancestor of z is not in $V(\Theta)$, because it has three neighbors in P. This proves that x', y' are not both in P_{l-1}.

So up to symmetry, we may assume that $y' \notin V(P_{l-1})$. Since y' has neighbor in P_l, it must be that y' has a neighbor $w \in V(P_{l-1})$, and that along P_l, one visits in order three neighbors of w, then y and two other neighbors of y', and then three other neighbors of w.

Let w' be the neighbor of w in P_{l-1}, chosen so that w' has neighbors in P. Since w' has type 0, by Claim 2, we have $w' = x'$. Hence, as claimed, $x' \in V(P_{l-1})$ and $x'w \in E(G)$. This proves Claim 3.

Let a, b, c, a', b', c' be the six neighbors of w in P_l appearing in this order along P_l, in such a way that $a, b, c \in V(P)$ and $a', b', c' \notin V(P)$. We have $\{a', b', c'\} \cap V(\Theta) \neq \emptyset$, since otherwise we obtain a shorter theta from u to v by replacing P' with $x'wy'$, a contradiction to the minimality of Θ. Let y'' be the neighbor of y' in yP_la' closest to a' along yP_la'. Since $w \notin V(\Theta)$, $V(y'y''P_lc') \subseteq V(\Theta)$.

If $y' \notin \{u, v\}$, then by replacing $x'P'y'y''P_lc'$ with $x'wc'$, we obtain a theta, a contradiction to the minimality of Θ. So, $y' \in \{u, v\}$. Without loss of generality, we may assume that $y' = v$.

If $u \neq x'$, then by replacing $V(x'P'y'y''P_lc')$ with $\{x', w, y', c'\}$ in Θ, we obtain a theta from w to u which contains less vertices than Θ, a contradiction to the minimality of Θ. So, $u = x'$.

Recall that x' has type 0. Let $z \neq w$ be the neighbor of x' in P_{l-1}. Moreover, let z' and z'' be the neighbor of z and x' in P_l respectively, such that all vertices in the interior of $z'P_lz''$ have degree 2. Since Θ goes through P, $w \notin V(\Theta)$. Therefore $z, z', z'' \in V(\Theta)$. This implies the hole $zx'z''P_lz'z$ is a hole of Θ, a contradiction because the other apex $v = y'$ is not in the hole. This completes the proof that $G_{l,k}$ is theta-free. □

Even-hole-free layered wheels

Recall that (even hole, triangle)-free graphs have treewidth at most 5 (see [4]), and as we will see, ttf-layered-wheels of arbitrarily large treewidth exist. Hence, some ttf-layered-wheels contain even holes (in fact, it can be checked that they contain even wheels). We now provide a construction of layered wheel that is (even hole, K_4)-free, but that contains triangles (see Figure 6). Its structure is similar to ttf-layered-wheel, but slightly more complicated.

The construction of ehf-layered-wheel that we are going to discuss emerges from the structure of wheels that may exist in a graph of the studied class (namely, even-hole-free graphs with no K_4). In the class of even-hole-free graphs, a wheel may have several centers while having the same rim. Those centers may be adjacent or not. In Figure 5, we give examples of wheels that may exist in an even-hole-free graph. Formally, we do not need to prove that these wheels are even-hole-free, and therefore we omit the (straightforward) proof.

Now we are ready to describe the construction of ehf-layered-wheel.

Construction 3.6. *Let $l \geq 1$ and $k \geq 4$ be integers. An (l, k)-ehf-layered-wheel, denoted by $G_{l,k}$, consists of $l+1$ layers, which are paths $P_0, P_1, \ldots, P_l$. We view these paths as oriented from left to right. The graph is constructed as follows.*

- *(B1) $V(G_{l,k})$ is partitioned into $l+1$ vertex-disjoint paths $P_0, P_1, \ldots, P_l$. So, $V(G_{l,k}) = V(P_0) \cup \cdots \cup V(P_l)$. The paths are constructed in an inductive way.*
- *(B2) The first layer P_0 consists of a single vertex r. The second layer P_1 is a path such that $P_1 = r_1P_1r_2P_1r_3$, where $\{r_1, r_2, r_3\} = N_{P_1}(r)$ and for $j = 1, 2$, $r_jP_1r_{j+1}$ is of odd length at least $k-2$.*

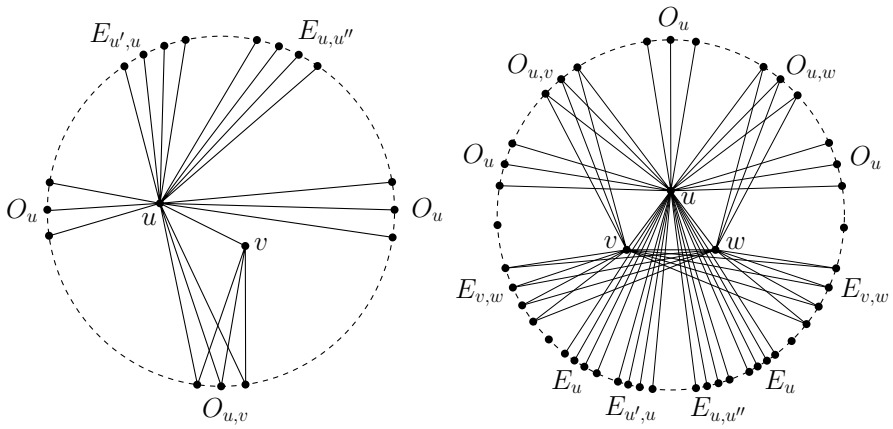


Figure 5: Wheels in an even-hole-free graph whose centers induce an edge or a triangle, with the corresponding zones as described in Construction 3.6 (dashed lines between two vertices represent paths of odd length)

(B3) *For every $0 \leq i \leq l$ and every vertex u in P_i, we call* ancestor of u *any neighbor of u in $G_{l,k}[P_0 \cup \cdots \cup P_{i-1}]$. The* type of u *is the number of its ancestors (as we will see, the construction implies that every vertex has type 0, 1, or 2). Observe that the unique vertex of P_0 has type 0, and P_1 consists only of vertices of type 0 or type 1. Moreover, we will see that if u is of type 2, then its ancestors are adjacent. Also, the construction implies that for every $1 \leq i \leq l$, the ends of P_i are vertices of type 1.*

(B4) *Suppose inductively that $l \geq 2$ and $P_0, P_1, \ldots, P_{l-1}$ are constructed. The l^{th}-layer P_l is built as follows.*

For all $0 \leq i \leq l-1$, any vertex $u \in V(P_i)$ has an odd number of neighbors in P_l, that are into subpaths of P_l that we call zones. *These zones are labeled by E_u or O_u according to their parity: a zone labeled E_u contains four neighbors of u, and a zone labeled O_u contains three neighbors of u. All these four or three neighbors are of type 1, and all the other vertices of the zone are of type 0.*

There are also zones that contain common neighbors of two vertices u, v. We label them $E_{u,v}$ (or $O_{u,v}$). A zone $E_{u,v}$ (resp. $O_{u,v}$) contains four (resp. three) common neighbors of u and v. All these four or three neighbors are of type 2, and all the other vertices of the zone are of type 0.

The ends of a zone E_u (resp. O_u) are neighbors of u. The ends of a zone $E_{u,v}$ (resp. $O_{u,v}$) are common neighbors of u and v. Distinct zones are disjoint.

(B5) *For any $u \in P_{l-1}$, we define the box* Box_u*, that is a subpath of P_l, as follows:*

- *If u is of type 0 (so it is an internal vertex of P_{l-1}), then let u' and u'' be the neighbors of u in P_{l-1}, so that $u'uu''$ is a subpath of P_{l-1}. In this case, Box_u goes through three zones $E_{u',u}$, O_u, $E_{u,u''}$ that appear in this order along P_l (see Figure 6).*
- *If u is of type 1, then let $v \in P_i$, $i < l-1$ be its ancestor.*
  If u is an internal vertex of P_{l-1}, then let u' and u'' be the neighbors of u in P_{l-1}, so that $u'uu''$ is a subpath of P_{l-1}. In this case, Box_u is made of five zones $E_{u',u}$, O_u, $O_{u,v}$, O_u, $E_{u,u''}$ (see Figure 6).
  If u is the left end of P_{l-1}, then let u'' be the neighbor of u in P_{l-1}. In this case, Box_u is made of four zones O_u, $O_{u,v}$, O_u, $E_{u,u''}$.
  If u is the right end of P_{l-1}, then let u' be the neighbor of u in P_{l-1}. In this case, Box_u is made of four zones $E_{u',u}$, O_u, $O_{u,v}$, O_u.
- *If u is of type 2 (so it is an internal vertex of P_{l-1}), then let $v \in P_i$ and $w \in P_j$, $j \leq i$ be its ancestors. If $i = j$, we suppose that v and w appear in this order along P_i (viewed from left to right). It turns out that either w is an ancestor of v, or v, w are consecutive along some path P_i (because as one can check, all vertices of type 2 that we create satisfy this statement). In this case, Box_u is made of 11 zones, namely $E_{u',u}$, E_u, $E_{v,w}$, O_u, $O_{u,v}$, O_u, $O_{u,w}$, O_u, $E_{v,w}$, E_u, and $E_{u,u''}$ (see Figure 6).*

Note that for any two adjacent vertices $u, v \in P_{l-1}$, Box_u and Box_v are not disjoint.

(B6) The path P_l visits all the boxes Box_- of P_l in the same order as vertices in P_{l-1}. For instance, if uvw is a subpath of P_{l-1}, then Box_u, Box_v, and Box_w appear in this order along P_l.

(B7) Let u and v be two vertices of P_l, both of type 1 or 2, and consecutive in the sense that every vertex in the interior of uP_lv is of type 0. If u and v have a common ancestor, then uP_lv has odd length, at least $k-2$. If u and v have no common ancestor, then uP_lv has even length, at least $k-2$.

(B8) Observe that every vertex in P_l has type 0, 1, or 2. Moreover, as announced, every vertex of type 2 has two adjacent ancestors.

(B9) There are no other vertices or edges apart from the ones specified above.

For the same reason as for ttf-layered-wheels, we allow flexibility in Construction 3.6, by just giving lower bounds for the lengths of paths described in (B7). So there may exist different ehf-layered-wheels $G_{l,k}$ for the same value of l and k.

Lemma 3.7. *For $0 \leq i \leq l-1$ and $i+1 \leq j \leq l$, every vertex $u \in V(P_i)$ has at least 3^{j-i} neighbors in P_j.*

Proof. We omit the proof since it is similar to the proof of Lemma 3.2. □

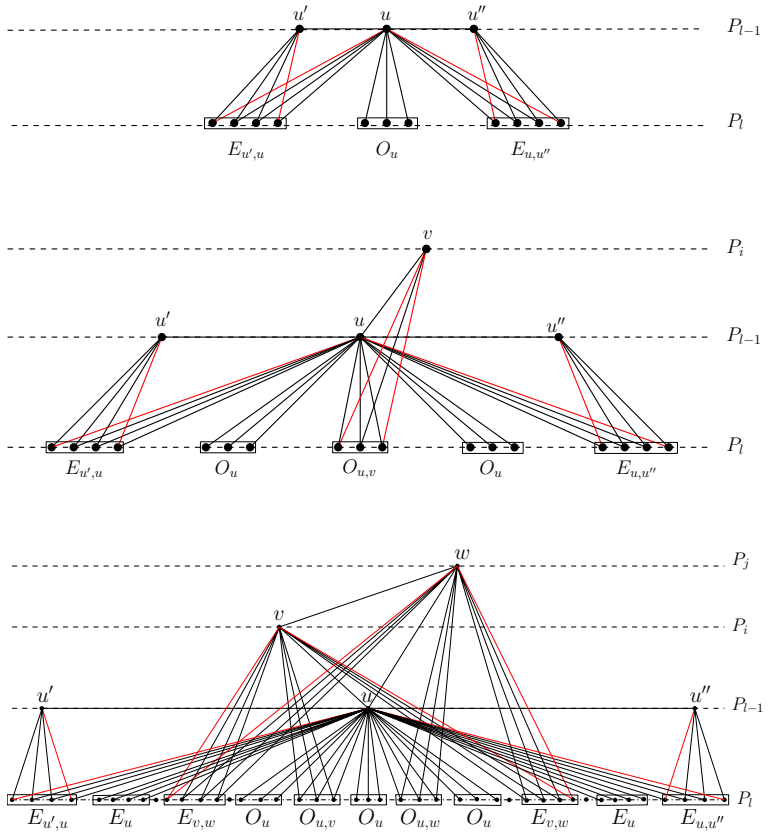


Figure 6: The neighborhood of a type 0, type 1, or type 2 vertex $u \in V(P_{l-1})$ in P_l (dashed lines between two vertices in P_l represent paths of odd length, and red edges represent non-internal edges as in the proof of Theorem 3.10)

Lemma 3.7 implies that every vertex of layer i has neighbors in all layers $i+1, \ldots, l$. The next lemma is clear.

Lemma 3.8. *For every integers $l \geq 1$ and $k \geq 4$, there exists an (l,k)-ehf-layered-wheel.*

We need some properties of lengths of some paths in ehf-layered-wheel. It is convenient to name specific subpaths of boxes first (see Figure 6).

- Suppose that u is a vertex in P_{l-1} (of any type).

  If u is not an end of P_{l-1}, then a subpath of Box_u is a *shared part of* Box_u if it is either the zone $E_{u',u}$ or the zone $E_{u,u''}$. The *private part of* Box_u is the path from the rightmost vertex of $E_{u',u}$ to the leftmost vertex of $E_{u,u''}$.

Otherwise, if u is the left end of P_{l-1} (and therefore of type 1), then u has only one shared part, that is the zone $E_{u,u''}$, where $u'' \in N_{P_{l-1}}(u)$. The private part of u is the path from the leftmost vertex of the leftmost zone O_u to the leftmost vertex of $E_{u,u''}$.

Similarly, if u is the right end of P_{l-1}, then u has only one shared part, that is the zone $E_{u',u}$, where $u' \in N_{P_{l-1}}(u)$. The private part of u is the path from the rightmost vertex of $E_{u',u}$ to the rightmost vertex of the rightmost zone O_u.

Observe that $\textsc{Box}_u$ is edgewise partitioned into a private part and some shared parts (namely zero if $l = 1$ and u is the unique vertex of layer P_0, one if $l > 1$ and u is an end of P_{l-1}, two otherwise).

- Suppose that u is of type 1 and v is its ancestor.

  If u is not the left end of P_{l-1}, then the *left escape of v in* $\textsc{Box}_u$ is the subpath of $\textsc{Box}_u$ from the rightmost vertex of $E_{u',u}$ to the leftmost vertex of $O_{u,v}$.

  If u is not the right end of P_{l-1}, then the *right escape of v in* $\textsc{Box}_u$ is the subpath of $\textsc{Box}_u$ from the rightmost vertex of $O_{u,v}$ to the leftmost vertex of $E_{u,u''}$.

- Suppose that u is of type 2 and v, w are its ancestors as in Construction 3.6. Note that u is not an end of P_{l-1}.

  The *left escape of v (resp. of w) in* $\textsc{Box}_u$ is the subpath of $\textsc{Box}_u$ from the rightmost vertex of $E_{u',u}$ to the leftmost vertex of the zone $E_{v,w}$ that is the closest to $E_{u',u}$.

  The *right escape of v (resp. of w) in* $\textsc{Box}_u$ is the subpath of $\textsc{Box}_u$ from the rightmost vertex of the zone $E_{v,w}$ that is the closest to $E_{u,u''}$, to the leftmost vertex of $E_{u,u''}$.

Lemma 3.9. *Let $G_{l,k}$ be an ehf-layered-wheel with $l \geq 1$ and u be a vertex in the layer P_{l-1}. Then the following hold:*

- *Shared parts of* $\textsc{Box}_u$ *are paths of odd length.*
- *The private part of* $\textsc{Box}_u$ *is a path of even length if u is not an end of P_{l-1}; and it is of odd length otherwise.*
- *If u has type 1 or 2, then all the left and right escapes of its ancestors in* $\textsc{Box}_u$ *are paths of even length.*

Proof. To check the lemma, it is convenient to follow the path $\textsc{Box}_u$ on Figure 6 from left to right. Along this proof, we refer to Construction 3.6, and we follow the notation given in Figure 6.

By (B7), shared parts of $\textsc{Box}_u$ have obviously odd length.

If u has type 0, then along the private part of $\textsc{Box}_u$, one meets 1 common neighbor of u and u', then 3 private neighbors of u, and then 1 common neighbor

of u and u''. In total, from the leftmost neighbor of u to its rightmost neighbor, one goes through 4 subpaths of Box_u, each of odd length by (B7) (2 of the paths are in zones, while 2 of them are between zones). The private part of Box_u has therefore even length.

If u has type 1, then the proof is similar. If it is not an end of P_{l-1}, then along the private part of Box_u, one visits 10 subpaths (6 in zones, 4 between zones), each of odd length by (B7). Otherwise, one visits 9 subpaths (6 in zones, 3 between zones), each of odd length by (B7).

If u has type 2 then u is not an end of P_{l-1}. Now there are more details to check. Along the private part of Box_u, one visits 32 subpaths. Among them, 22 are in zones and have odd length by (B7), and 10 are between zones. But 4 of the subpaths between zones have even length by (B7), namely, the paths linking E_u to $E_{v,w}$ (because $\{u\} \cap \{v, w\} = \emptyset$), $E_{v,w}$ to O_u, O_u to $E_{v,w}$, and $E_{v,w}$ to E_u. The 6 remaining subpaths between zones have odd length by (B7). In total, the private part of Box_u has even length as claimed.

For the left and right escapes, the proof is similar. If u is of type 1, then the escape is made of 4 paths each of odd length. If u is of type 2, then the escape is made of the path between zones $E_{v,w}$ and E_u that is of even length, three paths in zone E_u each of an odd length, and the path between zone E_u and $E_{u',u}$ or $E_{u,u''}$ that is of odd length. So, every left and every right escape is of even length. □

Theorem 3.10. *For every integers $l \geq 1$ and $k \geq 4$, every (l, k)-ehf-layered-wheel $G_{l,k}$ is (even hole, K_4)-free and every hole in $G_{l,k}$ has length at least k.*

Proof. It is clear from the construction that $G_{l,k}$ does not contain K_4. Moreover, it follows from (B7) that apart from triangles, any chordless cycle in $G_{l,k}$ is of length at least k (we omit the formal proof that is similar to the proof that ttf-layered-wheels have girth at least k).

For a contradiction, consider an ehf-layered-wheel $G_{l,k}$ that contains an even hole H. Suppose that l is minimal, and under this assumption that H has minimum length. Hence, layer P_l contains some vertex of H, for otherwise $G_{l,k}[P_0 \cup \cdots \cup P_{l-1}]$ would be a counterexample. Let us start by the following claim.

Let x be a vertex in P_i where $0 \leq i < l$, and y be a neighbor of x in P_l. We say that xy is an *internal edge* (see Figure 6) if one of the following holds:

- $i = l - 1$ and y is an internal vertex of Box_x.
- $i < l-1$, x is an ancestor of $x' \in V(P_{l-1})$, x' has type 1 or 2, y is in $\text{Box}_{x'}$ and y is neither the leftmost neighbor of x in $\text{Box}_{x'}$ nor the rightmost neighbor of x in $\text{Box}_{x'}$.

Claim 1. *H contains no internal edge.*

Proof of Claim 1. Let xy be an internal edge as in the definition and suppose for a contradiction that xy is an edge of H. Let $Q = y \dots z$ be the path of H

that is included in P_l and that is maximal with respect to this property. Let z' be the ancestor of z that is in H (it exists by the maximality of Q).

Suppose first that x is in P_{l-1}. We then set $x = u$ and observe that u has type 0, 1 or 2 (see Figure 6). If u has type 0, then since uy is internal edge, y is either in the zone O_u, or is among the three rightmost vertices of zone $E_{u',u}$, or is among the three leftmost vertices of zone $E_{u,u''}$ (where u' and u'' are the left and the right neighbors of u respectively in P_{l-1} as shown in Figure 6). Since no internal vertex of Q is adjacent to u because H is a hole, we have $zu \in E(G)$ and $z' = u$. So, $H = uyQzu$ and H has odd length by the axiom (B7), a contradiction. If u has type 1 (and ancestor v as represented in Figure 6), the proof is similar (note in this case that $z' \neq v$ for otherwise the triangle uvy would be in H, a contradiction).

If u has type 2 (and ancestors v, w as represented in Figure 6) the proof is similar with some additional situations. For instance, it can be that y is the rightmost vertex of the leftmost zone $Z = E_u$. In this case, z can be either the leftmost vertex of the zone $E_{v,w}$ that is next to Z, or the leftmost vertex of the zone O_u that is closest to Z. In the first case, $z' = v$ or $z' = w$ (say $z' = v$ up to symmetry), so $H = uyQzvu$ and H has odd length by (B7); in the second case, $H = uyQzu$ and H has again odd length by (B7), a contradiction. Similar situations are when y is the leftmost vertex of the leftmost zone O_u, when y is the rightmost vertex of the rightmost zone O_u and when y is the leftmost vertex of the rightmost zone E_u. We omit the details of each situation.

Suppose now that x is not in P_{l-1}. Since x has neighbor in P_l, x is the ancestor of some vertex u from P_{l-1}. If u is of type 1 with ancestor v, then $x = v$. We observe that y must be the middle vertex of the zone $O_{u,v}$. Hence, $H = vyQzv$ and H has odd length by (B7), a contradiction.

So, u has type 2 and ancestors v, w. Up to symmetry, we may assume that $x = v$. As in the previous cases, whatever the place of y in Box_u, we must have either $H = vyQzv$, or $H = vyQzuv$, or $H = vyQzwv$ (when y is the rightmost vertex of $O_{u,v}$ and z is the leftmost vertex of $O_{u,w}$). In all cases, H has odd length, a contradiction. This proves Claim 1.

Now let $P = s \dots t$ be a subpath of H in P_l such that P is inclusion-wise maximal. So both s and t have an ancestor that is in H. If P contains a single vertex (i.e., $s = t$), then s must have two ancestors, say, s_1 and s_2, which are adjacent by (B3) of Construction 3.6. Thus $\{s, s_1, s_2\}$ forms a triangle in H, which is not possible. So P contains at least two vertices and $s \neq t$. Let u and v be ancestors of s and t respectively, such that $u, v \in V(H)$ (possibly $u = v$, or $uv \in E(G)$).

Recall that all layers are viewed as oriented from left to right. We suppose that s and t appear in this order, from left to right, along P_l.

Claim 2. *For every vertex $p \in V(P_{l-1})$, $N(p) \cap V(P_l) \not\subseteq V(P)$.*

Proof of Claim 2. Suppose that $p \in V(P_{l-1})$ and $N(p) \cap V(P_l) \subseteq V(P)$. So, $p \notin V(H)$. Note that p is an internal vertex of P_{l-1}, for otherwise, s or t is an

end of P_l and has degree 2, while having two neighbors in $V(H) \cap V(G_{l,k} \setminus p)$, a contradiction.

By (B5), ancestors of p (if any) and the neighbors of p in P_{l-1} must also have neighbors in P. Thus, all of such vertices do not belong to H because P is a subpath of H. By Lemma 3.9, the path $\textsc{Box}_p = p' \dots p''$ has an even length. Indeed $\textsc{Box}_p$ consists of two shared parts (each of odd length) and one private part (of even length). It yields that $\textsc{Box}_p$ and $p'pp''$ have the same parity, and hence replacing $\textsc{Box}_p$ in H with $p'pp''$ yields an even hole with length strictly less than the length of H, a contradiction to the minimality of H. This proves Claim 2.

Claim 3. *Exactly one of u and v is in P_{l-1}.*

Proof of Claim 3. Suppose that both u and v are not in P_{l-1}. Since u and v have neighbors in P, each of them has a neighbor in P_{l-1} (where such neighbors also have some neighbor in P). Let u' and v' be the respective neighbors of u and v in P_{l-1}.

If $u' = v'$, then u' is a type 2 vertex in P_{l-1}, with ancestors u and v, hence u and v are adjacent. So, H is a hole of form $usPtvu$, and it has odd length by construction. Therefore $u' \neq v'$, and by construction, the interior of $u'P_{l-1}v'$ must contain a vertex w of type 0. It yields that $N_{P_l}(w)$ is all contained in P, a contradiction to Claim 2.

Suppose now that both u and v are in P_{l-1}. By Claim 2, no vertex of P_{l-1} has all its neighbors in P. So the interior of $uP_{l-1}v$ contains at most two vertices.

If $u = v$, then by (B7) P is of odd length, and since $V(H) = \{u\} \cup V(P)$, H is also of odd length, a contradiction. Similarly if $uv \in E(G)$, then by (B7), P is of even length, $V(H) = \{u, v\} \cup V(P)$, and H has odd length, again a contradiction.

If the interior of $uP_{l-1}v$ contains a single vertex, then let w be this vertex. Let w_1 (resp. w_2) be the neighbor of w in P that is closest to s (resp. t). Note that by (B5), $s = w_1, t = w_2$ because both u and v are adjacent to w in P_{l-1}. So, sPt is the private part of $\textsc{Box}_w$, and by Lemma 3.9, it has even length, as uwv. Moreover, by Claim 1, $\{s\} = V(E_{u,w}) \cap V(H)$ and $\{t\} = V(E_{w,v}) \cap V(H)$. Also, if w has an ancestor, then such an ancestor must have neighbors in P, and hence it does not belong to H. Altogether, we see that $N_H(w) \subseteq V(usPtv)$. So, replacing $usPtv$ in H with uwv returns an even hole with length strictly less than the length of H, a contradiction to the minimality of H.

So the interior of $uP_{l-1}v$ contains two vertices. We let $uP_{l-1}v = uww'v$, and w_1 (resp. w_2') be the neighbor of w (resp. w') in P that is closest to s (resp. t). By (B5), $s = w_1, t = w_2'$. So, sPt is edgewise partitioned into the private part of w, the part shared between w and w', and the private part of w'. By Lemma 3.9, sPt has therefore odd length. In particular, the length of $usPtv$ has the same parity as the length of $uww'v$. Moreover, by Claim 1, $\{s\} = V(E_{u,w}) \cap V(H)$ and $\{t\} = V(E_{w',v}) \cap V(H)$. Also, if w or w' has an ancestor, then such an ancestor must have neighbors in P, and hence it does not

belong to H. Altogether, we see that $N_H(\{w, w'\}) \subseteq V(usPtv)$. So, replacing $usPtv$ in H with $uww'v$ returns an even hole that is shorter than H, again a contradiction to the minimality of H. This proves Claim 3.

By Claim 3 and up to symmetry, we may assume that $u \in V(P_{l-1})$ and $v \notin V(P_{l-1})$. So, v has a neighbor v' in P_{l-1} such that $t \in \text{Box}_{v'}$. Note that $v' \notin H$, because by construction v' has some neighbor in P. Hence, $v' \neq u$ (because $u \in H$). If the path $uP_{l-1}v'$ has length at least three, then some vertex in the interior of $uP_{l-1}v'$ contradicts Claim 2.

If $uP_{l-1}v'$ has length two, so $uP_{l-1}v' = uwv'$ for some vertex $w \in V(P_{l-1})$, then w is of type 0 because v' is not of type 0. Hence, P is edgewise partitioned into the private part of w, the part of $\text{Box}_{v'}$ shared between w and v' and the left escape of v in $\text{Box}_{v'}$. Let w' be the rightmost vertex of the shared zone $E_{w,v'}$. By Lemma 3.9, $usPw'$ has even length, as uww'. Moreover, by Claim 1, $\{s\} = V(E_{u,w}) \cap V(H)$ and since w has type 0, we see that $N_H(w) \subseteq V(usPw')$. So, replacing $usPw'$ in H with uww' returns an even hole with length strictly less than the length of H, a contradiction to the minimality of H.

Hence, $uP_{l-1}v'$ has length one: $uP_{l-1}v' = uv'$. So, P is the left escape of v in $\text{Box}_{v'}$. By Lemma 3.9, P has even length. By Claim 1, $\{s\} = V(E_{u,v'}) \cap V(H)$. Recall that $v' \notin H$. If $N_H(v') \subseteq V(usPtv)$, then replacing $usPtv$ in H with $uv'v$ returns an even hole with length strictly less than the length of H, a contradiction to the minimality of H.

So, v' has neighbors in H that are not in $usPtv$. Note that if v' is of type 2, the ancestor of v' that is different from v is not in H (because it is adjacent to t and to v). Also, by Claim 1, the neighbors of v' in $E_{u,v'} \setminus s$ are not in H.

We denote by v'' the right neighbor of v' in P_{l-1}. Note that v'' has type 0, since v' has type 1 or 2. Let s' and t' be vertices such that $t'P_l s'$ is the right escape of v in $\text{Box}_{v'}$, t' is adjacent to v, and s' is adjacent to v''. Note that s' is the leftmost vertex of $E_{v',v''}$ and t' is the rightmost vertex of the zone $O_{v',v}$ (when v' is of type 1) or of the rightmost zone $E_{v,w}$ (when v' is of type 2, and w is the other ancestor of v').

Let us see which vertex can be a neighbor of v' in $H \setminus usPtv$. We already know it cannot be an ancestor of v' or be in $E_{u,v'} \setminus s$. Suppose it is v''. Then, H must contain two edges incident to v'', and none of them can be an internal edge by Claim 1. Note that $s'v''$ must be an edge of H, for otherwise, the two only available edges are $v''v'''$ and $v''s''$ (where v''' is the right neighbor of v'' in P_{l-1} and s'' is the rightmost neighbor of v'' in P_l), and this yields a contradiction because $v'''s'' \in E(G)$. Since $s'v'' \in E(H)$, H goes through the path $R = usPtvt'P_l s'v''$. This path has even length, and contains all vertices of $N_H(v')$. So, we may replace R by $uv'v''$ in H, to obtain an even hole that contradicts the minimality of H. Now we know that $v'' \notin V(H)$.

Since v' has a neighbor in $H \setminus usPtv$, and since this neighbor is not an ancestor of v', is not v'', and is not in $E_{u,v'}$, it must be in $\text{Box}_{v'} \setminus (V(P) \cup E_{u,v'})$. By Claim 1, the only way that H can contain some vertex of $\text{Box}_{v'} \setminus (V(P) \cup E_{u,v'})$ is if H goes through the edge vt', in particular through the right escape of v in $\text{Box}_{v'}$. Let t'' be the rightmost vertex of $E_{v',v''}$. Hence, H must go

through the path $S = usPtvt'P_lt''$ (see Figure 7). This path has even length, and contains all vertices of $N_H(v')$. So, we may replace S by $uv't''$ in H, to obtain an even hole that contradicts the minimality of H.

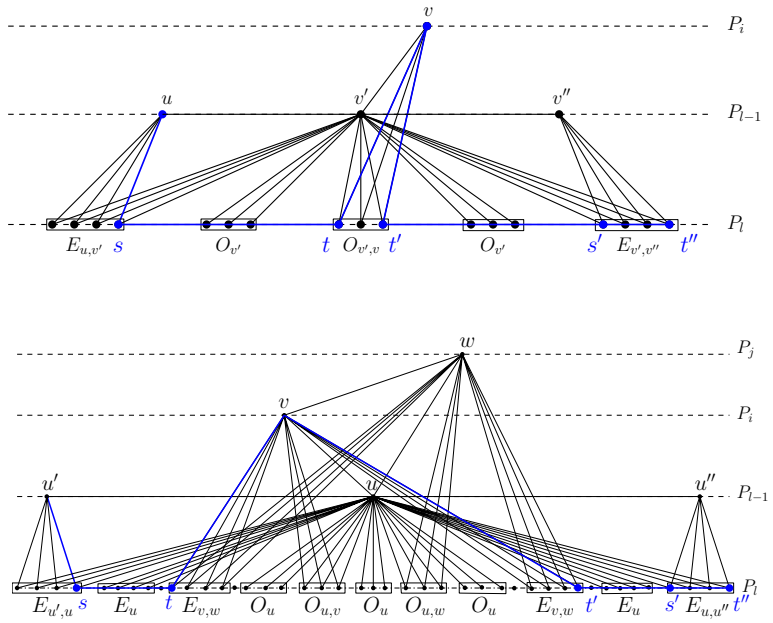


Figure 7: The proof of Theorem 3.10: in blue is the path $S = usPtvt'P_lt''$, when v' is of type 1 (top) and when v' is of type 2 (bottom)

□

Let us now prove that every ehf-layered-wheel is pyramid-free.

Theorem 3.11. *For every integer $l \geq 1$, $k \geq 4$, every (l,k)-ehf-layered-wheel $G_{l,k}$ is pyramid-free.*

Proof. Recall that all layers are viewed as oriented from left to right. For a contradiction, suppose that an ehf-layered-wheel $G_{l,k}$ contains a pyramid $\Pi = 3PC(\Delta, x)$. (Here we denote by Δ the triangle of Π, and call the apex the only vertex of degree 3 in $\Pi \setminus \Delta$ which in this case is the vertex x.) Suppose that l is minimal, and under this assumption that Π contains the minimum number of vertices among all pyramids in $G_{l,k}$. Clearly $l \geq 3$, and layer P_l contains some vertex of Π, for otherwise $G_{l,k}[P_0 \cup \cdots \cup P_{l-1}]$ would be a counterexample.

The next claim is trivially correct, so we omit the proof.

Claim 1. *Any hole in Π contains the apex and two vertices of Δ.*

Claim 2. *If a vertex of Δ is in P_l, then it is not in the interior of some zone.*

Proof of Claim 2. Suppose that some vertex a of Δ is in P_l and is in the interior of some zone Z. Then a is of type 2. If $Z = E_{u',u}$ for some $u', u \in P_{l-1}$, then $\Delta = auu'$, and we see that the left or the right neighbor of a in P_l is in Π. Let $Q = a \dots b$ be the subpath of P_l that contains a, that is included in Π, and that is maximal with respect to these properties. We see that b is adjacent to u and u', so that Π contains a diamond, a contradiction. The proof is the same for all other kinds of zones (namely $E_{u,u''}$, $E_{v,w}$, $O_{u,v}$, or $O_{u,w}$). This proves Claim 2.

Claim 3. *The apex x is not in P_l.*

Proof of Claim 3. Let us see that $x \in P_l$ yields a contradiction. Since x has degree 3 in Π, it is a vertex of type 1 or 2, so it belongs to some zone.

Suppose first that x is in the interior of some zone Z. If $Z = E_{u,u'}$ for some $u, u' \in P_{l-1}$, then since x has degree 3 in Π and is not in a triangle of Π, we see that the two neighbors of x in P_l are in Π. Also, exactly one ancestor y of x must be in Π. Let Q be the subpath of P_l that contains x, that is included in Π, and that is maximal with respect to these properties. We see that the ends of Q are adjacent to y, so that Q and y form a cycle with a unique chord in a pyramid, while not containing a triangle, a contradiction. When Z is another zone, say $E_{u,u''}$, O_u, $O_{u,v}$, etc, the proof is exactly the same.

Suppose now that x is an end of some zone Z. Again, the two neighbors of x in P_l and an ancestor u of x are in Π. So, Π contains the path Q from x to the vertex y with ancestor u that is next to x along Z. Note that y is in the interior of Z. So, Q and u form a hole of Π. Apart from x, y, and u, every vertex of H has degree 2, so uy is an edge of Δ, a contradiction to Claim 2. This proves Claim 3.

Claim 4. *If $u \in P_{l-1}$ has type 0 or 1 and is in Π, then no internal vertex of $\textsc{Box}_u$ is in Π.*

Proof of Claim 4. Suppose $a \in \Pi$ is an internal vertex of $\textsc{Box}_u$. Let Q be the subpath of P_l that contains a, is included in Π, and maximal with respect to this property. Since a is an internal vertex of $\textsc{Box}_u$ and u has type 0 or 1, Q and u form a hole H, that must contain the apex. Since by Claim 3, the apex is not in P_l, it must be u, and since every internal vertex of Q has degree 2, the two neighbors of u in H are in Δ, a contradiction since they are non-adjacent.

This proves Claim 4.

Claim 5. *No vertex of Δ is in P_l.*

Proof of Claim 5. Suppose for a contradiction that a is a vertex of Δ in P_l. So a has type 2, and in particular, it is not an end of P_l. As every internal vertex of P_l, a is in the interior of some box $\textsc{Box}_u$. If u is of type 0 or 1, it must be part of Δ, so a contradicts Claim 4. Hence, u is of type 2.

We denote by $P = a \dots p$ a subpath of Π included in $\textsc{Box}_u$ and maximal with this property. We will now analyze every possible zone where a belongs to, and we will see that each of the cases yields a contradiction.

Suppose first that a is in a shared zone Z. If $Z = E_{u',u}$, and therefore, $\Delta = au'u$, then by Claim 2, a is the rightmost vertex of $E_{u',u}$ (since it is in

the interior of Box_u). Since u' is of type 0 (because u is of type 2), Claim 4 applied to $\text{Box}_{u'}$ implies that a is the only vertex of Π in $E_{u',u}$, so p must be the leftmost vertex of the zone E_u that is next to $E_{u',u}$. So P and u form a hole H of Π, and since u is in Δ, the apex x must be in P_l, a contradiction to Claim 3. The proof is similar when $Z = E_{u,u''}$.

If $Z = O_{u,v}$, then $\Delta = auv$, and by Claim 2, a is either the leftmost or the rightmost vertex of $O_{u,v}$. If a is the leftmost vertex of $O_{u,v}$, then p is either the rightmost vertex of the zone O_u (that is on the left of $O_{u,v}$) or p is the vertex of type 2 next to a along $O_{u,v}$. In either case, P and u form a hole H of Π, and since u is in the triangle of Π, the apex x must be in P_l, a contradiction to Claim 3. If a is the rightmost vertex of $O_{u,v}$, the proof is similar. By symmetry, the case when $Z = O_{u,w}$ yields the same contradiction.

When a is the rightmost vertex of the leftmost zone $E_{v,w}$ (that is between E_u and O_u when oriented from left), we have $\Delta = avw$ and so $u \notin \Pi$. The proof is again the same, with a hole H that goes through v. The case when a is the leftmost vertex of the rightmost zone $E_{v,w}$ (that is between O_u and E_u when oriented from left) can be done in the similar way.

We are left with the case when a is the leftmost vertex of the leftmost zone $E_{v,w}$, or the rightmost vertex of the rightmost zone $E_{v,w}$. These two cases are symmetric, so we may assume that a is the leftmost vertex of the leftmost zone $E_{v,w}$.

It then follows that $\Delta = avw$. Note that $u \notin \Pi$ because a pyramid has only one triangle. If P goes in the interior of the zone $E_{v,w}$, then Π contains a diamond, a contradiction. So, P goes through the zone E_u that is left to $E_{v,w}$ and contains the rightmost vertex of $E_{u',u}$. Furthermore, there are two cases: P contains the zone $E_{u',u}$ (so p is the leftmost vertex of $E_{u',u}$ and $u' \notin \Pi$), or P contains only the rightmost vertex or $E_{u',u}$ (so p is the rightmost vertex of $E_{u',u}$ and $u' \in \Pi$). In the first case, we remove $P \setminus p$ from Π and put instead the edge up; in the second case, we remove P from Π completely and put the edge uu'. We obtain a pyramid (with triangle uvw) that is of smaller size than Π — a contradiction, unless u has some neighbor in $\Pi \setminus (P \cup \{u', v, w\})$. Hence, we now suppose such a neighbor z exists.

Let q be the leftmost vertex of the leftmost zone O_u (that is first met when traversing the layer from left to right), and r be the rightmost vertex of $E_{u,u''}$. Consider the path $Q = qP_lr$. Observe that z is in $Q \cup \{u''\}$ because $Q \cup \{u''\}$ contains all possible neighbors of u in $\Pi \setminus (P \cup \{u', v, w\})$.

Suppose that some vertex of Q is in Π. Let z' be the vertex of Π in Q that is the closest to q along Q. Note that by Claim 3, z' has degree 2 in Π. Since z' is the closest vertex to q, it has a neighbor in $\Pi \setminus Q$. In particular, z' is a type 1 or type 2 vertex, and exactly one of its ancestor is in Π. Since $u \notin \Pi$, such an ancestor is v or w, or possibly u'' if $u'' \in \Pi$ (and only one of them). If $z' \in O_{u,v}$, or $z' \in O_{u,w}$, or $z' \in E_{v,w}$, then there exists a vertex $z'' \in Q$ such that $vz'P_lz''v$, or $wz'P_lz''w$, or $vz'P_lz''wv$ is a hole of Π, which in either case contradicts Claim 3. So $z' \in E_{u,u''}$ and the ancestor of z' in Π must be u'' (in particular $u'' \in \Pi$). But then, the right neighbor of z' in P_l is an internal vertex of $\text{Box}_{u''}$ that belongs to Π, a contradiction to Claim 4. Therefore, $\Pi \cap Q = \emptyset$.

This means that $z = u''$. Note that the neighbors of u'' in Π cannot contain u (because $u \notin \Pi$), cannot be in $E_{u,u''}$ (because $E_{u,u''}$ is subpath of Q), cannot be in the interior of $\text{Box}_{u''}$ (because u'' has type 0 and by Claim 4), so they are precisely the right neighbor u''' of u'' in P_{l-1} and the rightmost vertex b of $E_{u'',u'''}$. But then, $u''u'''b$ is a triangle in Π, a contradiction. This proves Claim 5.

The rest of the proof is quite similar to the proof of Theorem 3.10, that ehf-layered-wheel contains no even hole.

Let $P = s \dots t$ be a subpath of Π in P_l such that P is inclusion-wise maximal (and s, t appear in this order from left to right). By Claims 3 and 5, every vertex of P has degree 2 in Π. Moreover by the maximality of P, each of s and t has an ancestor which is also in Π. Note that $s \neq t$, for otherwise s would be of type 2, and together with its ancestors, it forms a triangle, which contradicts Claim 5. Let u and v be the ancestors of s and t respectively, such that $u, v \in V(\Pi)$. By Claims 1, 3, and 5, $u \neq v$ and $uv \notin E(G)$.

Claim 6. *For every vertex $p \in V(P_{l-1})$, $N(p) \cap V(P_l) \not\subseteq V(P)$.*

Proof of Claim 6. Suppose that $p \in V(P_{l-1})$ and $N(p) \cap V(P_l) \subseteq V(P)$. So, $p \notin V(\Pi)$. Note that p is an internal vertex of P_{l-1}, for otherwise, s or t is an end of P_l and has degree 2, while having two neighbors in $V(\Pi) \cap V(G_{l,k} \setminus p)$, a contradiction.

By (B5), ancestors of p (if any) and the neighbors of p in P_{l-1} must also have neighbors in P. Thus, all of such vertices do not belong to Π because P is a subpath of Π. Hence, replacing $\text{Box}_p = p' \dots p''$ in Π with $p'pp''$ yields a pyramid with strictly less vertices than Π, a contradiction to the minimality of Π. This proves Claim 6.

Let a be a vertex in P_i for some $0 \leq i < l$, and p be a neighbor of a in P_l. We say that ap is an *internal edge* if one of the following holds:

- $i = l - 1$ and p is an internal vertex of Box_a.
- $i < l - 1$, a is an ancestor of some $a' \in V(P_{l-1})$, a' has type 1 or 2, p is in $\text{Box}_{a'}$ and p is neither the leftmost neighbor of a in $\text{Box}_{a'}$ nor the rightmost neighbor of a in $\text{Box}_{a'}$.

Claim 7. *No internal edge is an edge of Π.*

Proof of Claim 7. Suppose that $p \in P_l$ is the end of an internal edge e that is also an edge of Π. If the other end of e is in P_{l-1}, we set $e = pu$ and observe that p is in the interior of Box_u. Otherwise, the other end of e is in P_i, with $i < l - 1$, we set $e = px$ and observe that x has a neighbor u in P_{l-1}. Again, p is an internal vertex of Box_u. Observe that x is either v or w as represented on Figure 6.

By Claims 3 and 5, p has degree 2 in Π, so p has a unique neighbor in $\Pi \cap P_l$. Let $P = p \dots p'$ be the subpath of P_l included in Π, containing p, and maximal with respect to this property.

It can be checked in Figure 6 that P together with u, v, w, uv, uw, or vw forms a hole, that contains the apex and two vertices of Δ (by Claim 1), a contradiction to Claims 3 and 5. This proves Claim 7.

Claim 8. *Exactly one of u and v is in P_{l-1}.*

Proof of Claim 8. Suppose that both u and v are not in P_{l-1}. Since u and v have neighbors in P, each of them has a neighbor u' and v' respectively in P_{l-1}, such that $s \in \text{Box}_{u'}$ and $t \in \text{Box}_{v'}$.

If $u' = v'$, then u' is a type 2 vertex in P_{l-1}, with ancestors u and v, hence u and v are adjacent. It then follows that $usPtvu$ is a hole of Σ, so it must contains the apex and two vertices of Δ, contradicting Claim 3 or Claim 5, since u and v are the only vertices of the hole that are not in P_l.

Since u' and v' are vertices with ancestors, by construction, the interior of $u'P_{l-1}v'$ contains a vertex w of type 0. It yields that $N_{P_l}(w)$ is all contained in P, a contradiction to Claim 6.

Suppose now that both u and v are in P_{l-1}. By Claim 6, no vertex of P_{l-1} has all its neighbors in P. So the interior of $uP_{l-1}v$ contains at most two vertices.

If $u = v$, then $usPtu$ is a hole of Π. Since u is the only vertex in the hole that is not in P_l, by Claim 1 P contains the apex or a vertex of Δ, a contradiction to Claims 3 or 5. Similarly if $uv \in E(G)$, then $usPtvu$ is a hole of Π, this again yields a contradiction.

If the interior of $uP_{l-1}v$ contains a single vertex, then let w be this vertex. Let w_1 (resp. w_2) be the neighbor of w in P that is closest to s (resp. t). It follows by construction, that $s = w_1, t = w_2$ (because both u and v are adjacent to w in P_{l-1}). By Claim 7, $\{s\} = V(E_{u,w}) \cap V(\Pi)$ and $\{t\} = V(E_{w,v}) \cap V(\Pi)$. Also, if w has an ancestor, then such an ancestor must have neighbors in P, and hence it does not belong to Π. Altogether, we see that $N_\Pi(w) \subseteq V(usPtv)$. So, replacing $usPtv$ in Π with uwv returns a pyramid with less vertices than Π, a contradiction to the minimality of Π.

So the interior of $uP_{l-1}v$ contains two vertices. We let $uP_{l-1}v = uww'v$, and w_1 (resp. w_2') be the neighbor of w (resp. w') in P that is closest to s (resp. t). Similar as in the previous case, we know that $s = w_1, t = w_2'$; and by Claim 7, $\{s\} = V(E_{u,w}) \cap V(\Pi)$ and $\{t\} = V(E_{w',v}) \cap V(\Pi)$. Also, if w or w' has an ancestor, then such an ancestor must have neighbors in P, and hence it does not belong to Π. Altogether, we see that $N_\Pi(\{w, w'\}) \subseteq V(usPtv)$. So, replacing $usPtv$ in Π with $uww'v$ returns a pyramid with less vertices than Π, again a contradiction to the minimality of Π. This proves Claim 8.

By Claim 8 and up to symmetry, we may assume that $u \in V(P_{l-1})$ and $v \notin V(P_{l-1})$. So, v has a neighbor v' in P_{l-1} such that $t \in \text{Box}_{v'}$. Note that $v' \neq u$, for otherwise $usPtvu$ is a hole of Σ, so it contains the apex and two vertices of Δ, a contradiction to Claim 3 or Claim 5 (because u and v are the only vertices of the hole that are not in P_l). Furthermore, note that the path $uP_{l-1}v'$ has length at most two, for otherwise some vertex in the interior of $uP_{l-1}v'$ contradicts Claim 6.

Suppose that $uP_{l-1}v'$ has length two, so $uP_{l-1}v' = uwv'$ for some vertex $w \in V(P_{l-1})$. Then w is of type 0 because v' is not of type 0. Let w' be the rightmost vertex of the shared zone $E_{w,v'}$. By Claim 7, $\{s\} = V(E_{u,w}) \cap V(\Pi)$ and since w has type 0, we see that $N_\Pi(w) \subseteq V(usPw')$. So, replacing $usPw'$ in Π with uww' returns a pyramid with less vertices than Π, a contradiction to the minimality of Π.

Hence, $uP_{l-1}v'$ has length one, i.e. $uP_{l-1}v' = uv'$. By Claim 7, $\{s\} = V(E_{u,v'}) \cap V(\Pi)$. Observe that P is the left escape of v in $\text{Box}_{v'}$. So, P goes through the zone $O_{v'}$ (when v' has type 1) or through the zone $E_{v'}$ (when v' has type 2). In particular $v' \notin \Pi$.

If $N_\Pi(v') \subseteq V(usPtv)$, then replacing $usPtv$ in Π with $uv'v$ returns a pyramid with less vertices than Π, a contradiction to the minimality of Π. So, v' has neighbors in Π that are not in $usPtv$. Note that if v' is of type 2, the ancestor of v' that is different from v is not in Π (because it is adjacent to t and to v, but $t \notin \Delta$ by Claim 5).

We denote by v'' the right neighbor of v' in P_{l-1}. Note that v'' has type 0, since v' has type 1 or 2. Let s' and t' be vertices such that $t'P_ls'$ is the right escape of v in $\text{Box}_{v'}$, t' is adjacent to v and s' is adjacent to v''. Note that s' is the leftmost vertex of $E_{v',v''}$ and t' is the rightmost vertex of the zone $O_{v',v}$ (when v' is of type 1) or of the rightmost zone $E_{v,w}$ (when v' is of type 2, and w is the other ancestor of v').

Let us see which vertex can be a neighbor of v' in $\Pi \setminus usPtv$. We already know it cannot be an ancestor of v' or be a vertex of $E_{u,v'} \setminus s$. Suppose it is v''. Then, Π must contain two edges incident to v'', and none of them can be an internal edge by Claim 7. Note that $s'v''$ must be an edge of Π, for otherwise, the two only available edges are $v''v'''$ and $v''s''$ (where v''' is the right neighbor of v'' in P_{l-1} and s'' is the rightmost neighbor of v'' in P_l), and this is a contradiction because $v'''s'' \in E(G)$. Since $s'v'' \in E(\Pi)$, Π goes through the path $R = usPtvt'P_ls'v''$. This path contains all vertices of $N_\Pi(v')$. Note that $v \notin \Delta$, because if so, one of t or t' should be in Δ, a contradiction to Claim 5. But v can be the apex. If v is not the apex, we may replace R by $uv'v''$ in Π, to obtain a pyramid that contradicts the minimality of Π. If v is the apex, then we may replace $R \setminus v$ by $uv'v''$ in Π, to obtain a pyramid with apex v' that contradicts the minimality of Π. Now we know that $v'' \notin V(\Pi)$.

Since v' has a neighbor in $\Pi \setminus usPtv$, and since this neighbor is not an ancestor of v', is not v'', and is not in $E_{u,v'}$, it must be in $\text{Box}_{v'} \setminus (V(P) \cup E_{u,v'})$. By Claim 7, the only way that Π can contain some vertex of $\text{Box}_{v'} \setminus (V(P) \cup E_{u,v'})$ is that Π goes through the edge vt', in particular through the right escape of v in $\text{Box}_{v'}$ and through the zone $E_{v',v''}$. Let t'' be the rightmost vertex of $E_{v',v''}$. Hence, Π must go through the path $S = usPtvt'P_lt''$. This path contains all vertices of $N_\Pi(v')$. Note that $v \notin \Delta$, because if so, one of t or t' should be in Δ, a contradiction to Claim 5. But v can be the apex. If v is not the apex, we may replace S by $uv't''$ in Π, to obtain a pyramid that contradicts the minimality of Π. If v is the apex, then we may replace $S \setminus v$ by $uv't''$ in Π, to obtain a pyramid with apex v' that contradicts the minimality of Π. □

Treewidth and cliquewidth

For any $l \geq 0$, ttf-layered-wheels and ehf-layered-wheels on $l+1$ layers contain K_{l+1} as a minor. To see this, note that each vertex in layer P_i, $i < l$, has neighbors in all layers $i+1, \ldots, l$ (see Lemma 3.2 and Lemma 3.7). Hence, by contracting each layer into a single vertex, a complete graph on $l+1$ vertices is obtained. Since when H is a minor of G we have $\mathrm{tw}(H) \leq \mathrm{tw}(G)$ and since for $l \geq 1$, a complete graph on l vertices has treewidth $l-1$, we obtain the following.

Theorem 3.12. *For any $l \geq 0$, ttf-layered-wheels and ehf-layered-wheels on $l+1$ layers have treewidth at least l.*

Gurski and Wanke [16] proved that the treewidth is in some sense equivalent to the cliquewidth when some complete bipartite graph is excluded as a subgraph. Let us state and apply this formally (thanks to Sang-il Oum for pointing this out to us).

Theorem 3.13 (Gurski and Wanke [16])**.** *If a graph G contains no $K_{3,3}$ as a subgraph, then* $\mathrm{tw}(G) \leq 6\,\mathrm{cw}(G) - 1$.

Lemma 3.14. *A layered wheel (ttf or ehf) contains no $K_{3,3}$ as a subgraph.*

Proof. Suppose that a ttf-layered-wheel G contains $K_{3,3}$ as a subgraph. Then, either it contains a theta (if $K_{3,3}$ is an induced subgraph of G) or it contains a triangle (if $K_{3,3}$ is not an induced subgraph of G). In both cases, there is contradiction.

Suppose that an ehf-layered-wheel G contains $K_{3,3}$ as a subgraph. If one side of the $K_{3,3}$ is a clique, then G contains a K_4. Otherwise, each side of $K_{3,3}$ contains a non-edge, so G contains $K_{2,2}$, that is isomorphic to a C_4. In both cases, there is contradiction. □

Theorem 3.15. *For any integers $l \geq 2$, $k \geq 4$, the cliquewidth of a layered wheel $G_{l,k}$ is at least $\frac{l+1}{6}$.*

Proof. Follows from Lemma 3.14 and Theorems 3.13 and 3.12. □

Observations and open questions

It should be pointed out that by carefully subdividing, one may obtain bipartite ttf-layered-wheels on any number l of layers. This is easy to prove by induction on l. We just sketch the main step of the proof: when building the last layer, assuming that the previous layers induce a bipartite graph, only the vertices with ancestors are assigned to one side of the bipartition (and only to one side, since a vertex has at most one ancestor in a ttf-layered-wheels). The parity of the paths linking vertices with ancestors can be adjusted to produce a bipartite graph.

It is easy to see that every prism, every theta, and every even wheel contains an even hole. Therefore, by Theorem 3.10 and Theorem 3.11, ehf-layered-wheels

are (prism, pyramid, theta, even wheel)-free, which is not obvious from their definitions. Note that ehf-layered-wheels contain diamonds (recall Conjecture 1.3 we proposed in Section 1).

However, we note that it is possible to modify Construction 3.6 in such a way that we obtain a layered wheel that is even-hole-free but contains a pyramid. Such a construction might be of interest to see what amount of structure one can get in a even-hole-free graphs by studying how the graph attaches to a pyramid. The construction is done by modifying axiom (B5) where the two zones E_u's are obliterated. More specifically, if u is of type 2 (so it is an internal vertex of P_{l-1}), then let $v \in P_i$ and $w \in P_j$, $j \leq i$ be its ancestors. In this case, Box_u is made of only 9 zones, namely $E_{u',u}$, $E_{v,w}$, O_u, $O_{u,v}$, O_u, $O_{u,w}$, O_u, $E_{v,w}$, and $E_{u,u''}$ (see Figure 8). The fact that this modified construction keeps the property of the layered wheel being even-hole-free can be proven similarly as Theorem 3.10. Notice that Lemma 3.9 also remains true for this modified construction. We remark that a corresponding wheel that is even-hole-free (similar to the one in Figure 5) exists considering this modified pattern of zones.

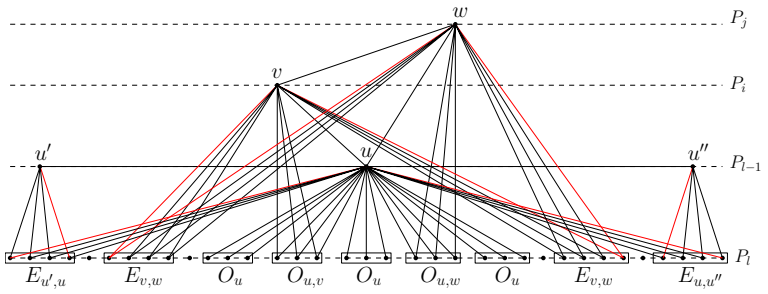


Figure 8: A modified construction of ehf-layered-wheel $G_{l,k}$ which contain pyramids (dashed lines between two vertices in P_l represent paths of odd length)

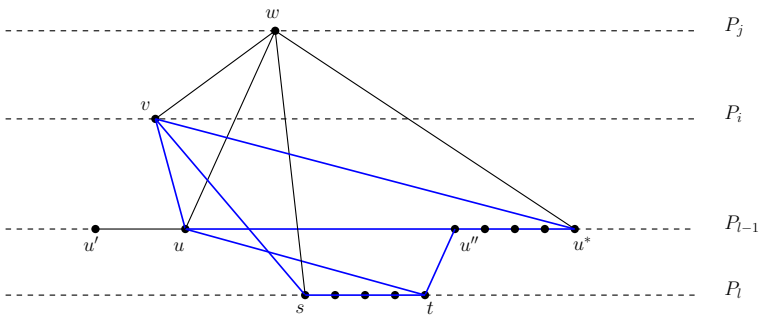


Figure 9: A pyramid (in blue) that is contained in a modified ehf-layered-wheel $G_{l,k}$, for some integers l, k

An example of a pyramid that may be found in such a modified ehf-layered-wheel is given in Figure 9. In the figure, $u \in P_{l-1}$ is a type 2 vertex with ancestors $v \in P_i$ and $w \in P_j$, $j < i$, u^* is a common neighbor of v and w in P_{l-1} such that u and u^* are consecutive common neighbors of v and w in some vw-zone in P_{l-1}, s is the rightmost vertex of a zone labeled $E_{v,w} \subseteq \text{Box}_u$ in P_l; and t is the leftmost vertex of the zone labeled $E_{u,u''} \subseteq \text{Box}_u$ in P_l where $u'' \neq u'$ is adjacent to u in P_{l-1}. The pyramid has triangle utu'' and apex v.

4 Lower bound on rankwidth

In this section, we prove that there exist ttf-layered-wheels and ehf-layered-wheels with arbitrarily large rankwidth. This follows directly from Theorem 3.15 and Lemma 2.1, but by a direct computation, we provide a better bound. Let us first present some useful notion and definition about rankwidth.

For a set X, let 2^X denote the set of all subsets of X. For sets R and C, an (R, C)-matrix is a matrix where the rows are indexed by elements in R and columns are indexed by elements in C. For an (R, C)-matrix M, if $X \subseteq R$ and $Y \subseteq C$, we let $M[X, Y]$ be the submatrix of M where the rows and the columns are indexed by X and Y respectively. For a graph $G = (V, E)$, let A_G denote the adjacency matrix of G over the binary field (i.e., A_G is the (V, V)-matrix, where an entry is 1 if the column-vertex is adjacent to the row-vertex, and 0 otherwise). The *cutrank function* of G is the function $\text{cutrk}_G : 2^V \to \mathbb{N}$, given by

$$\text{cutrk}_G(X) = \text{rank}(A_G[X, V \setminus X]),$$

where the rank is taken over the binary field.

A tree is a connected, acyclic graph. A *leaf* of a tree is a vertex incident to exactly one edge. For a tree T, we let $L(T)$ denote the set of all leaves of T. A tree vertex that is not a leaf is called *internal*. A tree is *cubic*, if it has at least two vertices and every internal vertex has degree 3.

A *rank decomposition* of a graph G is a pair (T, λ), where T is a cubic tree and $\lambda : V(G) \to L(T)$ is a bijection. If $|V(G)| \leq 1$, then G has no rank decomposition. For every edge $e \in E(T)$, the connected components of $T \setminus e$ induce a partition (A_e, B_e) of $L(T)$. The *width of an edge* e is defined as $\text{cutrk}_G(\lambda^{-1}(A_e))$. The *width of* (T, λ), denoted by $\text{width}(T, \lambda)$, is the maximum width over all edges of T. The *rankwidth of* G, denoted by $\text{rw}(G)$, is the minimum integer k, such that there is a rank decomposition of G of width k. (If $|V(G)| \leq 1$, we let $\text{rw}(G) = 0$.) The next lemma follows directly from the definition of the rankwidth.

Lemma 4.1. *Let G be a graph and H be an induced subgraph of G. Then $\text{rw}(H) \leq \text{rw}(G)$.*

A class $\mathcal{C}$ of graphs has *bounded rankwidth* if there exists a constant $k \in \mathbb{N}$, such that every $G \in \mathcal{C}$ satisfies $\text{rw}(G) \leq k$. If such a constant does not exist, then $\mathcal{C}$ has *unbounded rankwidth*. In the following lemmas, we present some

basic properties related to rankwidth. Let T be a tree, we call an edge $e \in E(T)$ *balanced*, if the partition (A_e, B_e) of $L(T)$ satisfies $\frac{1}{3}|L(T)| \leq |A_e|$ and $\frac{1}{3}|L(T)| \leq |B_e|$. The following is well-known (we include a proof for the sake of completeness).

Lemma 4.2. *Every cubic tree has a balanced edge.*

Proof. Let T be a cubic tree with n leaves. We may assume that $n \geq 3$, for otherwise, T is a path of length 1, and the only edge of T is balanced.

Let $e = ab$ be an edge of T such that the set of leaves A_e of the connected component of $T \setminus e$ that contains a, satisfies $|A_e| \geq |L(T)|/3$. Suppose that a and b are chosen subject to the minimality of $|A_e|$. If $|A_e| \leq 2|L(T)|/3$, then e is balanced. Otherwise, $|A_e| > 2|L(T)|/3 \geq 2$ so a has two neighbors a', a'' different from b. Let A' (resp. A'') be the set of leaves of the connected component of $T \setminus aa'$ (resp. $T \setminus aa''$) that contains a' (resp. a''). Since $|A_e| > 2|L(T)|/3$ and $A_e = A' \cup A''$, either $|A'| > |L(T)|/3$ or $|A''| > |L(T)|/3$. Hence, one of A' or A'' contradicts the minimality of $|A_e|$. □

Let us now introduce a notion that is useful to describe how we can represent the structure of layered wheels into a matrix. An $n \times n$ matrix M is *fuzzy triangular* if $m_{1,1} = 1$ and for every $i \in \{2, \ldots, n\}$, $m_{i,i} = 1$ and either $m_{1,i} = m_{2,i} = \cdots = m_{i-1,i} = 0$ or $m_{i,1} = m_{i,2} = \cdots = m_{i,i-1} = 0$.

Lemma 4.3. *Every $n \times n$ fuzzy triangular matrix has rank n.*

Proof. Let M be an $n \times n$ fuzzy triangular matrix. We prove by induction on n, that $\mathrm{rank}(M) = n$. For $n = 1$, this trivially holds. Suppose that $n \geq 2$. If $m_{1,n} = m_{2,n} = \cdots = m_{n-1,n} = 0$, we show that rows $r_1, \ldots, r_n$ of M are linearly independent. Let $\lambda_1, \ldots, \lambda_n \in [0, 1]$ be such that $\Sigma_{i=1}^{n} \lambda_i r_i = \mathbf{0}$ (where $\mathbf{0}$ is the zero vector of length n). Since $m_{n,n} = 1$, we have $\lambda_n = 0$. This implies that $\Sigma_{i=1}^{n-1} \lambda_i r'_i = \mathbf{0}$, where r'_i is the row obtained from r_i by deleting its last entry. Since $r'_1, \ldots, r'_{n-1}$ are the rows of an $(n-1) \times (n-1)$ fuzzy triangular matrix, they are linearly independent by the induction hypothesis, so $\lambda_1 = \cdots = \lambda_{n-1} = 0$.

We can prove in the same way that, if $m_{n,1} = m_{n,2} = \cdots = m_{n,n-1} = 0$, then the set of n columns of M is linearly independent. This shows that $\mathrm{rank}(M) = n$. □

Let G be a graph and (X, Y) be a partition of $V(G)$. A path P in G is *separated by* (X, Y) if $V(P) \cap X$ and $V(P) \cap Y$ are both non-empty. Note that when P is separated by (X, Y), there exists a *separating edge* xy of P whose end-vertices are $x \in X$ and $y \in Y$.

Lemma 4.4. *Let (T, λ) be a rank decomposition of width at most r of a layered wheel with layers $P_0, P_1, \ldots, P_l$. Let e be an edge of T, and (X, Y) be the partition of $V(G)$ induced by $T \setminus e$. Then there are at most r paths among $\{P_0, P_1, \ldots, P_l\}$ that are separated by (X, Y).*

Proof. Suppose for a contradiction that $P_{i_1}, \ldots, P_{i_{r+1}}$ are layers that are all separated by (X, Y), where $1 \leq i_1 < \cdots < i_{r+1} \leq l$. For each integer i_j, consider a separating edge $x_{i_j}y_{i_j}$ of P_{i_j} such that $x_{i_j} \in X$ and $y_{i_j} \in Y$. Set $S_X = \{x_{i_1}, \ldots, x_{i_{r+1}}\}$ and $S_Y = \{y_{i_1}, \ldots, y_{i_{r+1}}\}$.

Consider $M[S_X, S_Y]$, the adjacency matrix whose rows are indexed by S_X and columns are indexed by S_Y. The definition of layered wheels (see (A6) and (B7)) says that when two vertices in a layer are adjacent, at most one of them has ancestors. It follows that $M[S_X, S_Y]$ is fuzzy triangular. By Lemma 4.3, $M[S_X, S_Y]$ has rank $r+1$, a contradiction, because

$$\text{width}(T, \lambda) \geq \text{cutrk}_G(X) = \text{rank}(M[X, Y]) \geq \text{rank}(M[S_X, S_Y]) = r + 1.$$

□

We need the following lemma in our proof.

Lemma 4.5 (See [1])**.** *Let G be a graph and (T, λ) be a rank decomposition of G whose width is at most r. Let P be an induced path of G and (X, Y) be the partition of $V(G)$ induced by $T \setminus e$ where $e \in E(T)$. Then each of $P[X]$ and $P[Y]$ contains at most $r+1$ connected components.*

Now we are ready to describe layered wheels for which we can prove that the rankwidth is unbounded. Let us first define some terminology that is used along the proof. Recall Construction 3.1 of ttf-layered-wheels. Let u and v be two vertices that are adjacent in a layer P_i for some $i \in \{1, \ldots, l-1\}$, and they appear in this order (from left to right) along P_i. Let a be the rightmost vertex of Box_u and b be the leftmost vertex of Box_v in P_{i+1}. Let a' (resp. b') be the neighbor of a (resp. b) in $P_{i+1} \setminus \text{Box}_u$ (resp. $P_{i+1} \setminus \text{Box}_v$). The path $a'P_{i+1}b'$ is called the *uv-bridge*. An edge pq in $a'P_{i+1}b'$ is called *the middle edge of the bridge* if the length of the paths $a'P_{i+1}p$ and $qP_{i+1}b'$ are equal.

We have a similar definition for ehf-layered-wheel. For adjacent vertices u and v in P_{i+1}, the uv-bridge in P_{i+1} is the zone labelled $E_{u,v} \subseteq \text{Box}_u \cap \text{Box}_v$ (that we called in the previous section a shared part). Observe that in both layered wheels, every internal vertex of some layer yields two bridges, and each end of a layers yields one bridge. We say that a layered wheel is *special* if every bridge in all layers has odd length (and therefore admits a middle edge). The following lemmas are a direct consequence of Construction 3.1 and Construction 3.6.

Lemma 4.6. *For every integers $l \geq 1$ and $k \geq 4$, there exists a special (l, k)-ttf-layered-wheel.*

Proof. The result follows because by (A6) of Construction 3.1, the path between Box_u and Box_v is of length at least $k-2$. So for any two adjacent vertices in a layer, the uv-bridge can have any odd length, at least $k-4$. □

Lemma 4.7. *For every integers $l \geq 1$ and $k \geq 4$, any ehf-layered-wheel is special.*

Proof. The result follows from the fact that shared parts have odd length (see Lemma 3.9). □

Let $G_{l,k}$ be a layered wheel that is special. Let uv be an edge of some layer P_i, where $1 \leq i < l$, such that u and v appear in this order (from left to right) along P_i. Then we denote by $r_u l_v$ the middle edge of the uv-bridge (again, r_u and l_v appear in this order from left to right).

For any vertex $v \in P_i$, $1 \leq i < l$, the *domain of* v (or the *v-domain*), denoted by $\textsc{Dom}(v)$ is defined as follows:

- if $v \in V(P_0)$, then $\textsc{Dom}(v) = V(P_1)$;
- if v is an internal vertex of P_i, then $\textsc{Dom}(v) = V(l_v P_{i+1} r_v)$;
- if v is the left end of P_i, then $\textsc{Dom}(v) = V(pP_{i+1}r_v)$, where p is the leftmost vertex of $\textsc{Box}_v$; and
- if v is the right end of P_i, then $\textsc{Dom}(v) = V(l_v P_{i+1} q)$, where q is the rightmost vertex of $\textsc{Box}_v$.

Note that for ttf-layered-wheels, $\textsc{Box}_v$ is completely contained in the v-domain, which is not the case for ehf-layered-wheels. We are now ready to describe the layered wheels that we need.

Definition 4.8. *For some integer m, a special layered wheel $G_{l,k}$ is* m-uniform, *if for every vertex $v \in V(P_i)$, $0 \leq i \leq l-1$, $\textsc{Dom}(v)$ contains exactly m vertices.*

Observe that by definition, any m-uniform layered wheel is special.

Lemma 4.9. *For every integers $l \geq 1$ and $k \geq 4$ and M, there exists an integer $m \geq M$ and a ttf-layered-wheel that is m-uniform.*

Proof. We construct an m-uniform ttf-layered-wheel $G_{l,k}$ by adjusting the length obtained in step (A6) of Construction 3.1. □

Lemma 4.10. *For every integers $l \geq 1$ and $k \geq 4$ and M, there exists an integer $m \geq M$ and an ehf-layered-wheel that is m-uniform.*

Proof. We construct an m-uniform ehf-layered-wheel $G_{l,k}$ by adjusting the length obtained in step (B7) of Construction 3.6. □

For a vertex $v \in P_i$, $0 \leq i \leq l$ and an integer $0 \leq d \leq l - i$, the *v-domain of depth d*, denoted by $\textsc{Dom}^d(v)$ is defined as follows.

- $\textsc{Dom}^0(v) = \{v\}$ and $\textsc{Dom}^1(v) = \textsc{Dom}(v)$;
- $\textsc{Dom}^d(v) = \bigcup_{x \in \textsc{Dom}(v)} \textsc{Dom}^{d-1}(x)$ for $d \geq 1$.

Observation 4.11. *For every $v \in P_i$ with $0 \leq i \leq l$, and for any $0 \leq d \leq l-i$, we have $\textsc{Dom}^d(v) \subseteq V(P_{i+d})$, where the equality holds when $i = 0$.*

Lemma 4.12. *For every $0 \leq i \leq l$ and $0 \leq d \leq i$, $V(P_i) = \bigcup_{v \in P_{i-d}} \textsc{Dom}^d(v)$. Moreover, for any distinct $u, v \in V(P_{i-d})$, $\textsc{Dom}^d(u) \cap \textsc{Dom}^d(v) = \emptyset$.*

Proof. The statement simply follows by induction on d. □

Lemma 4.13. *For some integers l, k, m, let $G_{l,k}$ be an m-uniform layered wheel. For every $0 \leq i \leq l-1$, $v \in P_i$, and $1 \leq d \leq l-i$, we have $|\textsc{Dom}^d(v)| = m^d$.*

Proof. The statement simply follows from Lemma 4.12 and the m-uniformity: for any vertex v, $|\textsc{Dom}^1(v)| = m$ and $|\textsc{Dom}^d(v)| = m \cdot |\textsc{Dom}^{d-1}(v)|$. □

Lemma 4.14. *For some integers l, k, m, let $G_{l,k}$ be an m-uniform layered wheel. Denote by $G_{i,k}$, the subgraph induced by the first $i+1$ layers $P_0, P_1, \ldots, P_i$ of $G_{l,k}$. Then $|V(G_{i,k})| < \frac{1}{m-1}|V(P_{i+1})|$ for $0 \leq i \leq l-1$.*

Proof. Recall that $V(P_i) = \textsc{Dom}^i(r)$ for every $1 \leq i \leq l$, with $r \in V(P_0)$. So by Lemma 4.13, $|V(P_i)| = m^i$. Moreover, $|V(G_{i,k})| = \Sigma_{d=0}^{i} |\textsc{Dom}^d(r)| = \frac{m^{i+1}-1}{m-1}$. Hence, the result directly follows. □

Lemma 4.15. *Let $l \geq 2$, $k \geq 4$, and $m \geq 15$ be integers, and (T, λ) be a rank decomposition of an m-uniform layered wheel $G_{l,k}$ of width at most r. Let e be a balanced edge in T, and (X, Y) be the partition of $V(G_{l,k})$ induced by e. Then P_l is separated by (X, Y), and each of X and Y contains an induced subpath of P_l, namely P_X and P_Y where:*

$$|V(P_X)|, |V(P_Y)| \geq \left\lfloor \frac{|V(P_l)|}{3.5(r+1)} \right\rfloor.$$

Proof. Let first prove that P_l is separated by (X, Y). By Lemma 4.14, we know that $|V(P_l)| > (m-1)|V(G_{l-1,k})|$ where $G_{l-1,k} = G_{l,k} \setminus P_l$. Since $m - 1 \geq 14$, we have $|V(P_l)| > \frac{14}{15}|V(G_{l,k})|$ Hence, P_l cannot be fully contained in X, for otherwise $|Y| < \frac{1}{15}|V(G_{l,k})|$ that would contradict the fact that (X, Y) is a balanced decomposition. By the same reason, P_l is not fully contained in Y. This proves the first statement.

For the second statement, we will only prove the existence of P_X (for P_Y, the proof is similar). Since e is a balanced edge of T, we have $|X| \geq \frac{1}{3}|V(G_{l,k})|$. Clearly,

$$|V(P_l) \cap X| \geq \frac{1}{3}|V(G_{l,k})| - |V(G_{l-1,k})| = \frac{1}{3}\left(|V(P_l) - 2|V(G_{l-1,k})|\right).$$

By Lemma 4.5, X contains at most $r+1$ connected components of P_l. Hence:

$$\begin{aligned} |V(P_X)| &\geq \frac{|V(P_l) \cap X|}{r+1} \\ &> \frac{|V(P_l)| - \frac{2}{m-1}|V(P_l)|}{3(r+1)} \qquad (1) \\ &= \frac{m-3}{3(m-1)(r+1)} \ |V(P_l)| \\ &\geq \frac{2}{7(r+1)} \ |V(P_l)| \qquad (2) \end{aligned}$$

Inequality (1) is obtained from Lemma 4.14, and (2) follows because $m \geq 15$. □

The following theorem is the main result of this section.

Theorem 4.16. *For $l \geq 2$, $k \geq 4$, there exists an integer m such that the rankwidth of an m-uniform layered wheel $G_{l,k}$ is at least l.*

Proof. Set $M = 15$ and consider an integer m as in Lemma 4.9 (or Lemma 4.10), and let $G_{l,k}$ be m-uniform.

Suppose for a contradiction that $\mathrm{rw}(G_{l,k}) = r$ for some integer $r \leq l-1$. Let (T, λ) be a rank decomposition of $G_{l,k}$ of width r, and e be a balanced edge of T that partition $V(G_{l,k})$ into (X, Y). Let $\mathcal{P} = \{P_0, P_1, \ldots, P_l\}$ be the set of layers in the layered wheel, and $\mathcal{S}$ be the set of paths in $\mathcal{P}$ that are separated by (X, Y). By Lemma 4.4, $|\mathcal{S}| \leq r$.

Note that $P_0 \notin \mathcal{S}$ because it contains a single vertex. So, $\mathcal{P} \setminus \mathcal{S} \neq \emptyset$. Let $P_j \in \mathcal{P} \setminus \mathcal{S}$, i.e., the vertices of P_j are completely contained either in X or Y. Without loss of generality, we may assume that $V(P_j) \subseteq X$.

Claim 1. *There exists some j such that $1 \leq j < l$.*

Proof of Claim 1. Note that $l - r \geq 1$, because $r \leq l-1$. So it is enough to prove that such a $j \geq l-r$ exists. We know that $|\mathcal{S}| \leq r \leq l-1$. If every path $P_j \in \mathcal{P} \setminus \mathcal{S}$ has index $j < l-r$, then $|\mathcal{P} \setminus \mathcal{S}| \leq l-r$. This implies $|\mathcal{S}| \geq (l+1) - (l-r) = r+1$, a contradiction, so the left inequality of the statement holds (the bound is tight when $\mathcal{S} = \bigcup_{l-r+1 \leq i \leq l} \{P_i\}$). Furthermore, by Lemma 4.15, $P_l \in \mathcal{S}$, so for every P_j that satisfies the left inequality, we know that $j < l$. This proves Claim 1.

Now by Lemma 4.15, there exists a subpath P_Y of P_l, such that $V(P_Y) \subseteq Y$ and $|V(P_Y)| \geq \left\lfloor \frac{|V(P_l)|}{3.5(r+1)} \right\rfloor$, with $|V(P_l)| = m^l$ (because $|V(P_l)| = \textsc{Dom}^l(r)$ where $r \in P_0$).

Let P' be the set of vertices in P_j such that $N(v) \cap V(P_Y) \neq \emptyset$ for every $v \in P'$. Note that the order (left to right) of the domains of $V(P_j)$ in layer P_l appear as the order of $V(P_j)$ in P_j, and by Lemma 4.12, for every $v \neq v' \in P_j$, we have $\textsc{Dom}^{l-j}(v) \cap \textsc{Dom}^{l-j}(v') = \emptyset$. So P' induces a path. Moreover, for each vertex $v \in P'$, we can fix a vertex $y_v \in V(P_Y) \cap \textsc{Dom}^{l-j}(v)$, such that

$vy_v \in E(G)$. Thus for any $v \neq v' \in P'$, we have $y_v \neq y_{v'}$, and in particular, $vy_v, v'y_{v'} \in E(G)$ and $v'y_v, vy_{v'} \notin E(G)$. Let us denote $S_X = V(P')$ and $S_Y = \{y_v \mid v \in S_X\}$. Observe that there is a bijection between S_X and S_Y, so $M[S_X, S_Y]$ is the identity matrix of size $|S_X|$.

Furthermore, by Lemmas 4.12 and 4.13, we have $|S_X| \geq \left\lfloor \frac{|V(P_Y)|}{|\text{DOM}^{l-j}(v)|} \right\rfloor = \left\lfloor \frac{|V(P_Y)|}{m^{l-j}} \right\rfloor$. By Claim 1, Lemma 4.15, and taking $m \geq 4l^2$, the following holds.

$$|S_X| \geq \left\lfloor \frac{m^l}{3.5(r+1)m^{l-j}} \right\rfloor \geq \left\lfloor \frac{m^j}{3.5(r+1)} \right\rfloor \geq \left\lfloor \frac{m}{3.5l} \right\rfloor \geq \left\lfloor \frac{3.5l^2}{3.5l} \right\rfloor \geq l$$

which yields a contradiction, because

$$r \geq \text{width}(T, \lambda) \geq \text{cutrk}_{G_{l,k}}(X) = \text{rank}(M[X, Y]) \geq \text{rank}(M[S_X, S_Y]) = |S_X| \geq l.$$

□

5 Upper bound

Layered wheels have an exponential number of vertices in terms of the number of layers l. In Section 3, we have seen that the treewidth of layered wheels is lower-bounded by l. In this section, we give an upper bound of the treewidth of layered wheels. As mentioned in the introduction, we indeed prove a stronger result: the so-called *pathwidth* of layered wheels is upper-bounded by some linear function of l. Since layered wheels $G_{l,k}$ contain an exponential number of vertices in terms of the number of layers, this implies that $\text{tw}(G_{l,k}) = \mathcal{O}(\log |V(G_{l,k})|)$. Beforehand, let us state some useful notions.

Pathwidth

A *path decomposition* of a graph G is defined similarly as a tree decomposition except that the underlying tree is required to be a path. Similarly, the width of the path decomposition is the size of a largest bag minus one, and the pathwidth is the minimum width of a path decomposition of G. The *pathwidth* of a graph G is denoted by $\text{pw}(G)$. As outlined in the introduction, path decomposition is a special case of tree decomposition. We restated the following lemma that was already mentioned in Lemma 2.1.

Lemma 5.1. *For any graph G,* $\text{tw}(G) \leq \text{pw}(G)$.

Let P be a path, and $P_1, \ldots, P_k$ be subpaths of P. The *interval graph* associated to $P_1, \ldots, P_k$ is the graph whose vertex set is $\{P_1, \ldots, P_k\}$ with an edge between any pair of paths sharing at least one vertex. So, interval graphs are intersection graphs of a set of subpaths of a path.

Lemma 5.2 (See Theorem 7.14 of [12])**.** *Let G be a graph, and I be an interval graph that contains G as a subgraph (possibly not induced). Then* $\text{pw}(G) \leq \omega(I) - 1$*, where $\omega(I)$ is the size of the maximum clique of I.*

Now, for every layered wheel $G_{l,k}$, we describe an interval graph $I(G_{l,k})$ such that $G_{l,k}$ is a subgraph of $I(G_{l,k})$. We define the *scope* of a vertex. This is similar to its domain, but slightly different (the main difference is that scopes may overlap while domains do not). For $v \in V(P_i)$, where $0 \leq i \leq l-1$, the scope of v, denoted by $\textsc{Scp}(v)$, is defined as follows.

For a ttf-layered-wheel:

- if $v \in P_0$, $\textsc{Scp}(v) = V(P_1)$;
- if v is in the interior of P_i, then $\textsc{Scp}(v) = V(L) \cup \textsc{Box}_v \cup V(R)$, where L is the uv-bridge and R is the vw-bridge, u and w are the left and the right neighbors of v in P_i respectively;
- if v is the left end of P_i, then $\textsc{Scp}(v) = \textsc{Box}_v \cup V(R)$ where R is the vw-bridge and w is the right neighbor of v in P_i;
- if v is the right end of P_i, then $\textsc{Scp}(v) = V(L) \cup \textsc{Box}_v$, where L is the uv-bridge and u is the left neighbor of v in P_i.

For an ehf-layered-wheel:

- $\textsc{Scp}(v) = \textsc{Box}_v$ for every $v \in P_i$, $0 \leq i \leq l-1$.

For $d \geq 0$, we also define the *depth-d scope* of each vertex in the layered wheel, which will be denoted by $\textsc{Scp}^d(v)$. We define $\textsc{Scp}^0(v) = \{v\}$, and

$$\textsc{Scp}^d(v) = \bigcup_{x \in \textsc{Scp}(v)} \textsc{Scp}^{d-1}(x) \text{ for } 1 \leq d \leq l-i.$$

For a layered wheel $G_{l,k}$, we define the interval graph $I(G_{l,k})$. For every vertex $v \in G_{l,k}$, define path $P(v)$ associated to v as follows:

- if $v \in P_l$ is not the right end of P_l, then $P(v) = vw$ where w is the right neighbor of v;
- if v is the right end of P_l, then $P(v) = \{v\}$;
- if $v \in P_i$ with $i < l$, then $P(v) = P_l\left[\textsc{Scp}^{l-i}(v)\right]$.

Note that $P(v)$ is a subpath of P_l. The graph $I(G_{l,k})$ is the interval graph associated to $\{P(v) \mid v \in V(G_{l,k})\}$.

Lemma 5.3. *For any layered wheel $G_{l,k}$ and the corresponding interval graph $I(G_{l,k})$, $G_{l,k}$ is a subgraph (possibly not induced) of $I(G_{l,k})$.*

Proof. It is clear by definition that there is a bijection between $V(I(G_{l,k}))$ and $V(G_{l,k})$. We show that $E(G_{l,k}) \subseteq E(I(G_{l,k}))$: for any two vertices $u, v \in G_{l,k}$, if $uv \in E(G_{l,k})$ then the corresponding paths $P(u)$ and $P(v)$ share at least one vertex (i.e. $V(P(u)) \cap V(P(v)) \neq \emptyset$).

For $u, v \in P_l$ where u is on the left of v, this property trivially holds, because by definition, $P(u)$ and $P(v)$ both contain v. If $u \in P_i$ for some $i < l$ and $v \in P_l$, then $V(P(v)) \subseteq V(P(u)) = \textsc{Scp}^{l-i}(u)$. The case is similar when $v \in P_i$ for some $i < l$ and $u \in P_l$.

If $u, v \in P_i$ for some $i < l$, then by definition, $\textsc{Scp}(u) \cap \textsc{Scp}(v) \neq \emptyset$ (they both contain the uv-bridge). Let $x \in \textsc{Scp}(u) \cap \textsc{Scp}(v)$. Note that for $1 \leq d \leq l - i$, $\textsc{Scp}^d(u)$ and $\textsc{Scp}^d(v)$ both contain $\textsc{Scp}^{d-1}(x)$. If $u \in P_i$ and $v \in P_j$ where $1 \leq i < j < l$, then $\textsc{Scp}(v) \subseteq \textsc{Scp}^{j-i+1}(u)$. So $\textsc{Scp}^d(v) \subseteq \textsc{Scp}^{d+j-i}(u)$ for every $1 \leq d \leq l - j$. The case is similar when $u \in P_j$ and $v \in P_i$ where $1 \leq i < j < l$. Hence, $V(P(u)) \cap V(P(v)) \neq \emptyset$. □

Theorem 5.4. *For every integers $l \geq 2$ and $k \geq 4$, we have* $\text{tw}(G_{l,k}) \leq \text{pw}(G_{l,k}) \leq 2l$.

Proof. By Lemmas 5.1 (third item), 5.2 and 5.3, it is enough to show that $\omega(I(G_{l,k})) \leq 2l + 1$.

Claim 1. *Let u and v be non-adjacent vertices in P_i for some $1 \leq i \leq l - 1$. Then for any $1 \leq d \leq l - i$, we have $\textsc{Scp}^d(u) \cap \textsc{Scp}^d(v) = \emptyset$.*

Proof of Claim 1. Let u and v be be non-adjacent vertices in P_i, where $1 \leq i \leq l-1$ and without loss of generality, they appear in this order (from left to right) along P_i. We prove the statement by induction on d.

For $d = 1$, it follows from the definition that $\textsc{Scp}^1(u) \cap \textsc{Scp}^1(v) = \emptyset$ for every possible i. Suppose for induction that $\textsc{Scp}^d(u) \cap \textsc{Scp}^d(v) = \emptyset$ for some $1 \leq d \leq l - i - 1$. Note that $\textsc{Scp}^d(u)$ and $\textsc{Scp}^d(v)$ appear in this order along P_{i+d}. Moreover, the right end of $\textsc{Scp}(u)$ and the left end of $\textsc{Scp}(v)$ are also non-adjacent (because they both are vertices with an ancestor). So for any $x \in \textsc{Scp}(u)$ and $y \in \textsc{Scp}(v)$, we have $xy \notin E(G_{l,k})$, It then follows by construction, that for every $d \geq 2$, for any $x \in \textsc{Scp}^d(u)$ and $y \in \textsc{Scp}^d(v)$, we have $xy \notin E(G_{l,k})$, so the induction hypothesis holds for the pair x and y. We need to show that $\textsc{Scp}^{d+1}(u) \cap \textsc{Scp}^{d+1}(v) = \emptyset$. Indeed:

$$\textsc{Scp}^{d+1}(u) \cap \textsc{Scp}^{d+1}(v) = \bigcup_{x \in \textsc{Scp}(u)} \textsc{Scp}^d(x) \cap \bigcup_{y \in \textsc{Scp}(v)} \textsc{Scp}^d(y) = \emptyset,$$

which completes our induction. This proves Claim 1.

Let K be a maximum clique in $I(G_{l,k})$. By definition, for every $u, v \in P_l$ that are non-adjacent, we have $V(P(u)) \cap V(P(v)) = \emptyset$. So no edge exists between P_u and P_v in $I(G_{l,k})$. Similarly for non-adjacent vertices $u, v \in P_i$ where $1 \leq i \leq l - 1$, it follows from Claim 1, that $V(P(u)) \cap V(P(v)) = \emptyset$. Therefore, K contains at most two vertices of every layer P_i, with $1 \leq i \leq l$. Since K may also contain the unique vertex in P_0, then $\omega(I(G_{l,k})) \leq 2l + 1$ as desired. □

The following directly follows.

Corollary 5.5. *For any integers $l \geq 2$ and $k \geq 4$, we have* $\mathrm{tw}(G_{l,k}) = \mathcal{O}(\log |V(G_{l,k})|)$.

Proof. By Lemma 3.2 and Lemma 3.7, we know that $G_{l,k}$ contains at least $c \cdot 3^l$ vertices for some integer $c \geq 3$. Hence by Theorem 5.4, we have $\mathrm{tw}(G_{l,k}) \leq 2l \leq c' \cdot \log |V(G_{l,k})|$ for some constant $c' > 0$. □

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