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MODEL THEORY OF DIVISIBLE ABELIAN CYCLICALLY ORDERED GROUPS AND MINIMAL C. O. G.

G. LELOUP

ABSTRACT. We make available some results about model theory cyclically ordered groups. We start with a classification of complete theories of divisible abelian cyclically ordered groups. Most of this paper is dedicated to the study of the model theory of torsion-free divisible abelian cyclically ordered groups, since we first show that the theory of divisible abelian cyclically ordered groups with non-trivial torsion elements is complete. Then we look at the cyclically ordered groups where the only parametrically definable subsets are finite unions of singletons and open intervals, and those where the definable subsets are finite union of singletons and c-convex subsets, where being c-convex is the analogue of being convex in the linearly ordered case.

¹ Keywords: cyclically ordered groups, first-order theory, minimality, spines of Schmitt, circular orders.

1. INTRODUCTION.

Cyclically ordered groups continue to interest mathematicians since they arise naturally in lower-dimensional topology (see, for example [1]) and in studying groups which act on a circle (e. g. [3]). In these papers, they are called circularly ordered groups. By studying the topological structure of the space of circular orders on a group, it was proved that this space is either finite or uncountable ([4]). This generalizes a result of François Lucas which proved that there are 2^{\aleph_0} non isomorphic cyclic orders on the additive group $(\mathbb{Q}, +)$ of rational numbers, by giving a description of these cyclic orders. In fact Lucas proved that there are 2^{\aleph_0} elementary classes of discrete divisible cyclic orders on \mathbb{Q} (we will generalize this proposition here). This result was part of a paper that he wanted to publish, however, Lucas died before achieving this task. In this paper he used the spines of Schmitt of ordered abelian groups to study the model theory divisible abelian cyclically ordered groups. He also had studied different notions of minimality concerning cyclically ordered groups. This work may apply to multiplicative groups of fields since the multiplicative group of any field is cyclically orderable ([13, Proposition 5.8]). We make available these results here. The present paper is written in order to make clear which results come from Lucas's preprint. To reach an uninitiated audience we have added cyclically ordered groups reminders. Section 3 is similar to the original work of Lucas, however, the proofs have been drafted in detail and gaps have been filled. Section 4 has been completely overhauled, so that the theorems are written in terms of the cyclically ordered groups, instead of their unwounds, and without requiring the definitions of Schmitt ([19], [20]). Furthermore, we obtain a classification of complete theories of divisible abelian cyclically ordered groups. Section 5 has been revisited, and the proofs have been carefully written. This led to several errors being corrected. Thus the statements of lemmas have been changed. However we attribute to Lucas the authorship of these lemmas.

1.1. Cyclically ordered groups. Let G be a group and R be a ternary relation on G . We say that (G, R) is a *cyclically ordered group*, or that R is a *cyclic order* on G , if the following holds.

R is strict: for every g_1, g_2, g_3 , $R(g_1, g_2, g_3) \Rightarrow g_1 \neq g_2 \neq g_3 \neq g_1$.

R is cyclic: for every g_1, g_2, g_3 , $R(g_1, g_2, g_3) \Rightarrow R(g_2, g_3, g_1)$.

An element g being fixed, $R(g, \cdot, \cdot)$ induces a linear order relation on the set $G \setminus \{g\}$.

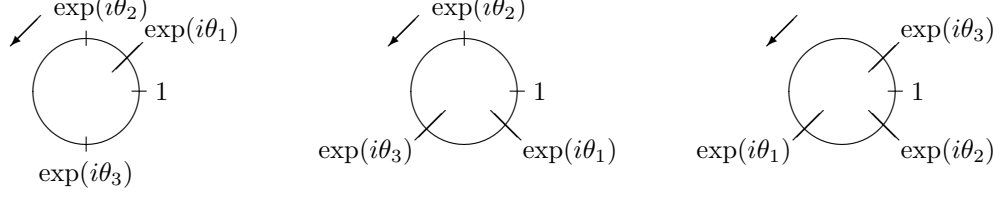
R is compatible: for every g_1, g_2, g_3, h, h' , $R(g_1, g_2, g_3) \Rightarrow R(hg_1h', hg_2h', hg_3h')$ ([16]).

For example, the group of complex numbers of norm 1 is a cyclically ordered group by setting

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$R(\exp(i\theta_1), \exp(i\theta_2), \exp(i\theta_3))$ if and only if either $0 \leq \theta_1 < \theta_2 < \theta_3 < 2\pi$ or $0 \leq \theta_2 < \theta_3 < \theta_1 < 2\pi$ or $0 \leq \theta_3 < \theta_1 < \theta_2 < 2\pi$ (e. g. [22]). This condition is equivalent to saying that $0 \leq \theta_{\sigma(1)} < \theta_{\sigma(2)} < \theta_{\sigma(3)} < 2\pi$, for some permutation σ in the alternating group A_3 of degree 3.



We denote by \mathbb{U} the cyclically ordered group $\{\exp(i\theta) : \theta \in [0, 2\pi[\}$ of unimodular complex numbers, and by $T(\mathbb{U})$ its torsion subgroup. Note that every finite cyclically ordered group is cyclic ([26, Theorem 1], [2, Lemma 3.1]), hence it embeds in $T(\mathbb{U})$. We know that, for every positive integer n , $T(\mathbb{U})$ contains a unique subgroup of n elements, which is the subgroup of n -th roots of 1. We denote it by $T(\mathbb{U})_n$.

The language of cyclically ordered groups is $\{\cdot, R, e, ^{-1}\}$, where the first predicate stands for the group operation, R for the ternary relation, e for the group identity and $^{-1}$ for the inverse function. When we consider abelian cyclically ordered groups we will also use the usual symbols $+$, 0 , $-$. If necessary, the unit element of G will be denoted by e_G (or 0_G) in order to avoid any confusion. For every g_0, g_1, \dots, g_n in G , we will let $R(g_0, g_1, \dots, g_n)$ stand for $\bigvee_{0 \leq i < j < k \leq n} R(g_i, g_j, g_k)$. One can check that this is also equivalent to $\bigvee_{0 < i < n} R(g_0, g_i, g_{i+1})$.

A *c-homomorphism* from a cyclically ordered group (G, R) to a cyclically ordered group (G', R') is a group homomorphism f such that for every g_1, g_2, g_3 in G , if $R(g_1, g_2, g_3)$ holds and $f(g_1) \neq f(g_2) \neq f(g_3) \neq f(g_1)$, then $R'(f(g_1), f(g_2), f(g_3))$ holds (e. g. [10]).

A linearly ordered group is cyclically ordered by the relation given by: $R(g_1, g_2, g_3)$ if, and only if, either $g_1 < g_2 < g_3$ or $g_2 < g_3 < g_1$ or $g_3 < g_1 < g_2$ (e. g. [21], [2]).

The cyclic order defined on a linearly ordered group as above is called a *linear cyclic order*. One also says that the group is *linearly cyclically ordered*. A cyclically ordered group which is not linearly ordered will be called a *nonlinear cyclically ordered group*.

A proper subgroup H of a cyclically ordered group is said to be *c-convex* if it does not contain a non unit element of order 2 and it satisfies:

$$\forall h \in H \forall g \in G (R(h^{-1}, e, h) \& R(e, g, h)) \Rightarrow g \in H \text{ ([10])}.$$

For every proper normal c-convex subgroup H of a cyclically ordered group G , one can define canonically a cyclic order on G/H such that the canonical epimorphism is a c-homomorphism.

If (G, R) is a nonlinear cyclically ordered group, then it contains a largest c-convex proper subgroup $l(G)$ ([10, Corollaries 3.6 and 4.7, Lemmas 4.2 and 4.6], [24]). It is a normal subgroup of G . The restriction to $l(G)$ of the cyclic order is a linear cyclic order. The subgroup $l(G)$ is called the *linear part* of (G, R) . We denote by $<$ the linear order induced by R on $l(G)$. If the cyclic order is linear, then we let $l(G) = G$.

The cyclically ordered group $G/l(G)$ embeds in a unique way in \mathbb{U} ([10, Lemma 5.1, Theorem 5.3]). We denote by $U(G)$ the image of $G/l(G)$ in \mathbb{U} , and for $g \in G$, $U(g)$ denotes the image of $g \cdot l(G)$.

An abelian group G is *divisible* if for each integer n and each $g \in G$ there is some $h \in G$ satisfying $nh = g$, this h is not always unique.

We say that an abelian cyclically ordered group (G, R) is *discrete* if there exists $g \neq 0$ such that for all h in G we have $\neg R(0, h, g)$. This element will be denoted by 1_G . We say that (G, R) is *dense* if it is not discrete.

1.2. Outline of the content. Section 2 is dedicated to cyclically ordered groups reminders and to properties that we will use in the former sections.

One easily sees that the theory of divisible abelian cyclically ordered groups is not complete,

the cyclic order does not need to be linear and the group can have torsion elements ([9, Corollary 6.13]). In Section 3 we prove the following theorem.

Theorem 1. (Lucas) *The theory TcD of divisible abelian cyclically ordered groups with torsion elements:*

- (1) *is complete and model-complete*
- (2) *has the amalgamation property*
- (3) *admits the quantifiers elimination*
- (4) *has a prime model $T(\mathbb{U})$*
- (5) *is the model completion of the theory of abelian cyclically ordered groups (we prove that each abelian cyclically ordered group admits a c-divisible hull).*

In Section 4 we get a characterization of elementary equivalence of torsion-free divisible abelian cyclically ordered groups by mean of the following family of c-convex subgroups. Let (G, R) be a torsion-free divisible abelian cyclically ordered group. For every prime p we let H_p be the greatest p -divisible convex subgroup of the ordered group $l(G)$.

Proposition 1.1. (Lucas) *Let (G, R) be a torsion-free divisible nonlinear abelian cyclically ordered group, and p be a prime such that the cyclically ordered quotient group G/H_p is discrete. Then $H_p \subsetneq l(G)$ and for every $r \in \mathbb{N} \setminus \{0\}$ there exists an integer $k \in \{1, \dots, p^r - 1\}$ which satisfies: for every $g \in l(G)$, $h \in G$ such that $g + H_p = 1_{G/H_p}$ and $g = p^r h$, we have $U(h) = e^{\frac{2ik\pi}{p^r}}$. The integer k is determined by the first-order theory of (G, R) .*

The integer k defined above is denoted $f_{G,p}(r)$.

Let (G, R) be a torsion-free divisible abelian cyclically ordered group. We denote by $\mathcal{CD}(G)$ the family $\{\{0_G\}, G, H_p : p \text{ prime}\}$. It is equipped with the linear preorder relation \subseteq .

Theorem 2. *Let (G, R) and (G', R') be torsion-free divisible nonlinear abelian cyclically ordered groups.*

- *If, for every prime p , G/H_p is dense, then $(G, R) \equiv (G', R')$ if, and only if, the mapping $H_p \mapsto H'_p$ induces an isomorphism from $\mathcal{CD}(G)$ onto $\mathcal{CD}(G')$.*
- *If, for some prime p , G/H_p is discrete, then $(G, R) \equiv (G', R')$ if, and only if, the following holds:
the mapping $H_p \mapsto H'_p$ induces an isomorphism from $\mathcal{CD}(G)$ onto $\mathcal{CD}(G')$,
for every prime p , G/H_p is discrete if, and only if, G'/H'_p is discrete, and if this holds, then the mappings $f_{G,p}$ and $f_{G',p}$ are equal.*

Following Lucas, the proof of this theorem is based on the works of Schmitt ([19], [20]).

Then we describe the cyclic orders on the additive group \mathbb{Q} (Proposition 4.18). This gives rise to a description of the functions $f_{G,p}$ (Proposition 4.28) and the following theorem.

Theorem 3. *Let Γ be a divisible abelian group.*

- (1) *It is cyclically orderable if, and only if, either it is torsion-free or its torsion subgroup $T(\Gamma)$ is isomorphic to $T(\mathbb{U})$.*
- (2) *If Γ is isomorphic to $T(\mathbb{U})$, then it admits uncountably many cyclic orders, but they are pairwise c-isomorphic.*
- (3) *If Γ contains non-torsion elements, and its torsion subgroup is isomorphic to $T(\mathbb{U})$, then it admits uncountably many pairwise non-c-isomorphic cyclic orders, all of them are pairwise elementarily equivalent.*
- (4) *If Γ is torsion-free, then there are uncountably many pairwise non elementarily equivalent cyclic orders on Γ .*

Next, we prove that there is a one-to-one correspondence between the set of partitions of the set of prime numbers, equipped with an arbitrary linear order, and the set of families $\mathcal{CD}(G)$, where G is a divisible abelian cyclically ordered group.

Finally, the minimality is studied in Section 5. We start with definitions.

Definitions 1.2. Let (G, R) be a cyclically ordered group. If I is a subset of G , then I is an *open interval* if there are $g \neq g'$ in G such that $I = \{h \in G : R(g, h, g')\}$. We denote this open interval by $I(g, g')$.

A subset J is *c-convex* if either it has only one point, or, for each $g \neq g'$ in J , either $I(g, g') \subseteq J$ or $I(g', g) \subseteq J$.

Remark 1.3. • A *c-convex proper subgroup of a cyclically ordered group is a proper subgroup which is a c-convex subset.*

- A *c-convex subset is not necessarily a singleton or an open interval: the linear part of a cyclically ordered group is a c-convex subset and if it is nontrivial, then it is not an open interval. Indeed, if $g \notin l(G)$, then any open interval bounded by g contains an element g' such that $U(g') = U(g)$, hence $g' \notin l(G)$. If $g \in l(G)$, then g doesn't belong to the open intervals bounded by itself.*
- *The bounded open intervals of $(l(G), <)$ are open intervals of (G, R) , since for h and $g < g'$ in $l(G)$ we have $g < h < g' \Leftrightarrow R(g, h, g')$.*

Definitions 1.4. Let (G, R) be an infinite cyclically ordered group. We say that (G, R) is:

- (1) *cyclically minimal* if each definable subset of G is a finite union of singletons and open intervals.
- (2) *strongly cyclically minimal* if for each (G', R') which is elementary equivalent to (G, R) , (G', R') is cyclically minimal.
- (3) *weakly cyclically minimal* if each definable subset of G is a finite union of c-convex subsets.

These notions correspond respectively to the notions of o-minimal, strongly o-minimal and weakly o-minimal in the linearly ordered case. In the linearly ordered case the three notions coincide and are satisfied in (G, R) if, and only if, (G, R) is abelian and divisible. For the equivalence between o-minimal and strongly o-minimal see [12] and [15]. The equivalence with weakly o-minimal follows from [7] corollary 1.3.

Clearly if (G, R) is strongly cyclically minimal, then it is cyclically minimal, and if it is cyclically minimal, then it is weakly cyclically minimal.

Note that in a finite structure every subset is a finite union of singletons.

Theorem 4. ([14, Theorem 5.1]) *A cyclically ordered group is cyclically minimal if, and only if, it is abelian, divisible and has torsion elements.*

Corollary 1.5. (Lucas) *A cyclically ordered group is strongly cyclically minimal if, and only if, it is cyclically minimal.*

Proof. Indeed, the theory of divisible abelian cyclically ordered groups with torsion elements is complete. \square

The characterization of weakly cyclically minimal cyclically ordered groups requires to define the construction of cyclically ordered groups by mean of the lexicographic product ([21]).

Let (G, R) be a cyclically ordered group and Γ be a linearly ordered group. One can define a cyclic order on $G \times \Gamma$ as follows: $R'((g_1, x_1), (g_2, x_2), (g_3, x_3))$ holds if, and only if,

- either $g_1 = g_2 = g_3$ and $x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)}$ for some σ in the alternating group A_3 of degree 3
- or $g_{\sigma(1)} = g_{\sigma(2)} \neq g_{\sigma(3)}$ and $x_{\sigma(1)} < x_{\sigma(2)}$ for some $\sigma \in A_3$
- or $R(g_1, g_2, g_3)$.

Definition 1.6. Let (G, R) be a cyclically ordered group and Γ be a linearly ordered group. We let $G \overrightarrow{\times} \Gamma$ denote the cyclically ordered group defined above. It is called the *lexicographic product* of (G, R) and (Γ, \leq) .

Theorem 5. (Lucas) *A cyclically ordered group (G, R) is weakly cyclically minimal if, and only if, either (G, R) is abelian and divisible with torsion elements, or $l(G)$ is infinite, abelian and divisible and there exist a positive integer n such that $G \simeq T(\mathbb{U})_n \overrightarrow{\times} l(G)$, where $T(\mathbb{U})_n$ is the subgroup of n -th roots of 1 in $T(\mathbb{U})$.*

2. PROPERTIES OF CYCLICALLY ORDERED GROUPS.

Before to list properties of cyclically ordered groups, we point out that in [23] an equivalent definition of cyclically ordered groups by mean of a mapping from $G \times G \times G$ to $\{-1, 0, 1\}$ appears. Later, the conditions of this definition are made simpler and the cyclic orderings are called circular orderings. A *compatible circular order* on G is a mapping c from $G \times G \times G$ to $\{-1, 0, 1\}$ which is a cocycle, that is for all g_1, g_2, g_3, g_4 in G we have $c(g_2, g_3, g_4) - c(g_1, g_3, g_4) + c(g_1, g_2, g_4) - c(g_1, g_2, g_3) = 0$ (e. g. [11]). Now, one can check that R is a compatible cyclic order on G if, and only if, the mapping c defined by $c(g_1, g_2, g_3) = 1$ if $R(g_1, g_2, g_3)$ holds, $c(g_1, g_2, g_3) = -1$ if $R(g_3, g_2, g_1)$ holds, and $c(g_1, g_2, g_3) = 0$ if $g_1 = g_2$ or $g_2 = g_3$ or $g_3 = g_1$, is a compatible circular order on G . The reader can find a similar proof in [2, Lemma 2.3]. Many papers about circularly ordered groups do not refer to the classical ones about cyclically ordered groups (except [4] for example), and several results are re-proved.

2.1. Wound-round cyclically ordered groups. If (Γ, \leq) is a linearly ordered group and $z \in \Gamma$, $z > e$, is a central and cofinal element of Γ , then the quotient group $\Gamma/\langle z \rangle$ can be cyclically ordered by setting $R(x_1 \cdot \langle z \rangle, x_2 \cdot \langle z \rangle, x_3 \cdot \langle z \rangle)$ if, and only if, there are x'_1, x'_2, x'_3 such that

$$x_1 \cdot \langle z \rangle = x'_1 \cdot \langle z \rangle, x_2 \cdot \langle z \rangle = x'_2 \cdot \langle z \rangle, x_3 \cdot \langle z \rangle = x'_3 \cdot \langle z \rangle \text{ and}$$

$$\text{either } e \leq x'_1 < x'_2 < x'_3 < z \text{ or } e \leq x'_2 < x'_3 < x'_1 < z \text{ or } e \leq x'_3 < x'_1 < x'_2 < z.$$

With this cyclic order, $\Gamma/\langle z \rangle$ is called the *wound-round* cyclically ordered group associated to Γ and z .

If $\Gamma = \mathbb{Z}$ and n is a positive integer, then we get a natural cyclic order on the group $\mathbb{Z}/n\mathbb{Z}$. This cyclically ordered group is c -isomorphic to the the subgroup $T(\mathbb{U})_n$ of n -th roots of 1 in the cyclically ordered group $T(\mathbb{U})$.

Assume that Γ is a lexicographic product of linearly ordered groups $\mathbb{Z} \overrightarrow{\times} \Gamma_1$, and $z = (1, e)$. For any $x \in \Gamma_1$ and $n \in \mathbb{Z}$, we have $(0, e) < (n, x) < (1, e) = z \Leftrightarrow n = 0$ and $x > e$ or $n = 1$ and $x < e$. So, one can check that the cyclically ordered group $\Gamma/\langle z \rangle$ is linearly cyclically ordered, and isomorphic to the ordered group Γ_1 .

A theorem of Rieger ([16], [8, Theorem 21 p. 64]) states that for every cyclically ordered group (G, R) there exists a linearly ordered group $(uw(G), \leq_R)$ and a positive element $z_G \in uw(G)$ which is central and cofinal such that (G, R) is c -isomorphic to the cyclically ordered group $uw(G)/\langle z_G \rangle$. The structure $(uw(G), \leq_R, z_G)$ is uniquely defined, up to isomorphism. The group $(uw(G), \leq_R)$ is called the *unwound* of (G, R) .

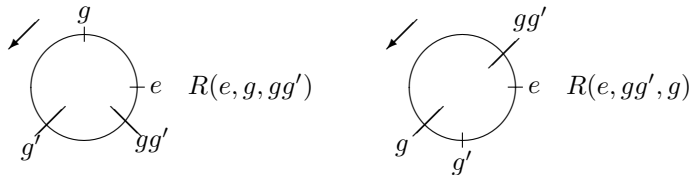
For example, the unwound of \mathbb{U} is \mathbb{R} , the unwound of (G, R) is isomorphic to \mathbb{Z} if, and only if, G is finite.

For further purposes, we give the definition of the order relation and of the group operation on $uw(G)$.

The underlying set of $uw(G)$ is $\mathbb{Z} \times G$.

The relation \leq_R is defined by $(n, g) \leq_R (n', g')$ if, and only if, either $n < n'$ or $[n = n'$ and $(g = e$ or $R(e, g, g')$ or $g = g')$].

Turning to the group operation, $(n, g) \cdot (n', g')$ is equal to $(n + n', gg')$ if either $g = e$ or $g' = e$ or $R(e, g, gg')$ holds, and it is equal to $(n + n' + 1, gg')$ if $g \neq e$ and $gg' = e$ or $R(e, gg', g)$ holds.



Furthermore, $z_G = (1, e)$.

For $x \in uw(G)$, we denote by \bar{x} its image in $uw(G)/\langle z_G \rangle \simeq G$.

There is a set embedding of G in $\{0\} \times G$ such that for any g, g' in $G \setminus \{e\}$, $(uw(G), \leq_R)$ satisfies $(0, g) <_R (0, g')$ if, and only if, (G, R) satisfies $R(e, g, g')$.

We end this subsection with a lemma which will be useful in the proof of Proposition 2.14 and in subsection 4.3. For every abelian group Γ and every prime p , we denote by $[p]\Gamma$ the number

of classes modulo p in Γ (without distinguishing between infinites). Note that Γ being divisible is equivalent to $[p]\Gamma = 1$ for every prime p .

Lemma 2.1. ([13, Lemma 4.16]) *Let (G, R) be an abelian cyclically ordered group. If G contains a p -torsion element, then $[p]\text{uw}(G) = [p]G$. Otherwise, $[p]\text{uw}(G) = p[p]G$.*

2.2. Linear part of a cyclically ordered group. Recall that if (G, R) is a nonlinear cyclically ordered group, then its linear part $l(G)$ is its largest c -convex proper subgroup and if the cyclic order is linear, then we let $l(G) = G$.

Let (G, R) be a cyclically ordered group. An element g is said to be *positive* if $R(e, g, g^2)$ holds. This condition is equivalent to $R(e, g, g^{-1})$. We denote by P the union of $\{e\}$ and of the set of positive elements of (G, R) ([23]). We say that P is the *positive cone* of (G, R) . Both of P and P^{-1} are c -convex subsets of G , and $g \in P \cup P^{-1}$ if, and only if, either $g = e$ or $g^2 \neq e$. The set $P \cap l(G)$ is the positive cone of the linear order on $l(G)$ ([24, p. 547]).

One can prove that the map from G in $\text{uw}(G)$ defined by $f(g) = (0, g)$ if $g \in P$ and $f(g) = (-1, g)$ if not, when restricted to a c -convex proper subgroup H , is a group homomorphism and an order isomorphism between H and $f(H)$. Furthermore, $f(H)$ is a convex subgroup of $\text{uw}(G)$.

Notation 2.2. Let H be a c -convex proper subgroup of a cyclically ordered group (G, R) . We let $H_{uw} = f(H)$.

The epimorphism from $\text{uw}(G)$ to $\text{uw}(G)/\langle z_G \rangle \simeq G$, when restricted to H_{uw} , is the inverse of the isomorphism f from H onto H_{uw} . It follows that the set of c -convex proper subgroups of (G, R) is linearly ordered by inclusion and when R is nonlinear we have an order isomorphism between the inclusion-ordered set of c -convex proper subgroups of (G, R) and the inclusion-ordered set of proper convex subgroups of $(\text{uw}(G), \leq_R)$. Therefore, the set of c -convex proper subgroups is closed under arbitrary unions and intersections. For each $g \in G$ there is a smallest c -convex proper subgroup of (G, R) containing g . Note that $l(G)_{uw}$ is the largest proper convex subgroup of $(\text{uw}(G), \leq_R)$, since it is the largest convex subgroup which doesn't contain the cofinal element z_G . Hence $\text{uw}(G)/l(G)_{uw}$ embeds in \mathbb{R} .

Recall that for every proper normal c -convex subgroup H of a cyclically ordered group G , one can define canonically a cyclic order on G/H such that the canonical epimorphism is a c -homomorphism. The linear part of G/H is isomorphic to $l(G)/H$, and its unwound is isomorphic to $\text{uw}(G)/H_{uw}$.

A cyclically ordered group (G, R) is said to be *c -archimedean* if for every g and every h there is an integer n such that $R(e, g^n, h)$ is not satisfied ([22], [2, Section 3.3]).

Fact 2.3. (1) *(G, R) is c -archimedean if, and only if, it can be embedded in \mathbb{U} . This in turn holds if, and only if, R is not linear and G has no nontrivial c -convex proper subgroup ([21, p. 162]).*

(2) *(G, R) is nonlinearly cyclically ordered and is c -archimedean if, and only if, its unwound $(\text{uw}(G), \leq_R)$ is archimedean. This follows from (1) and the order isomorphism between the inclusion-ordered set of c -convex proper subgroups of (G, R) and the inclusion-ordered set of convex subgroups of $(\text{uw}(G), \leq_R)$, in the nonlinear case.*

Let g, h be elements of a cyclically ordered group (G, R) . If $1 \neq U(g) \neq U(h) \neq 1$, then $R(e, g, h)$ holds in (G, R) if, and only if, $R(1, U(g), U(h))$ holds in $U(G)$. Now, if $U(g) = 1 \neq U(h)$ (so $g \in l(G)$), then $R(e, g, h)$ holds if, and only if, $g > e$ in $l(G)$, and $R(e, h, g)$ holds if, and only if, $g < e$ in $l(G)$ ([9, Remark 5.6]).

2.3. Dense, discrete, divisible abelian cyclically ordered groups. One can prove that an abelian cyclically ordered group (G, R) is dense if, and only if,

$$\forall (g_1, g_2) \in G \times G \quad g_1 \neq g_2 \Rightarrow (\exists h \in G \quad R(g_1, h, g_2)).$$

Let (G, R) and (G', R') be two abelian cyclically ordered groups such that (G', R') is a sub-structure of (G, R) . We say that (G', R') is *dense in* (G, R) if

$$\forall (g_1, g_2) \in G \times G \quad (\exists h \in G \quad R(g_1, h, g_2)) \Rightarrow (\exists h' \in G' \quad R(g_1, h', g_2)).$$

The group \mathbb{U} is dense, a subgroup of \mathbb{U} is dense if, and only if, it is infinite (this is also equivalent to saying that it is dense in \mathbb{U}). In particular, each divisible subgroup of \mathbb{U} is dense.

For every positive integer n , $T(\mathbb{U})_n$ is discrete, and $1_{T(\mathbb{U})_n} = e^{2i\pi/n}$. We give the proof of the following lemma for completeness.

Lemma 2.4. *The abelian cyclically ordered group (G, R) is discrete if, and only if, its unwound $(\text{uw}(G), \leq_R)$ is a discrete linearly ordered group. More generally, let H be a c -convex subgroup of (G, R) . Then the cyclically ordered group G/H is discrete if, and only if, the linearly ordered group $\text{uw}(G)/H_{\text{uw}}$ is discrete.*

Proof. If $l(G)$ is nontrivial, then one can see that (G, R) is discrete if, and only if, $l(G)$ is a discrete linearly ordered group. If this holds, then 1_G is the smallest positive element of $l(G)$. Since $l(G)$ embeds in a convex subgroup of $(\text{uw}(G), \leq_R)$, it follows that (G, R) is discrete if, and only if, $(\text{uw}(G), \leq_R)$ is discrete. If $l(G)$ is trivial, then, by Fact 2.3, (G, R) embeds in \mathbb{U} . Now, the unwound of \mathbb{U} is \mathbb{R} and the unwound of (G, R) is isomorphic to \mathbb{Z} if, and only if, G is finite. It follows that (G, R) is discrete if, and only if, its unwound is a discrete linearly ordered group.

Now, let H be a c -convex subgroup of G . We saw before Fact 2.3 that the unwound of G/H is $\text{uw}(G)/H_{\text{uw}}$. Therefore G/H is discrete if, and only if, $\text{uw}(G)/H_{\text{uw}}$ is discrete. \square

Remark 2.5. \bullet *A divisible abelian cyclically ordered group is not necessarily dense, as shows the following example of Lucas. Let G be the wound-round $(\mathbb{Q} \overrightarrow{\times} \mathbb{Z}) / \langle (\alpha, 1) \rangle$ with α positive, where $\overrightarrow{\times}$ denotes the lexicographic product. The ordered group $\mathbb{Q} \overrightarrow{\times} \mathbb{Z}$ is discrete with smallest positive element $(0, 1)$. By Lemma 2.4, (G, R) is discrete. Let $n \in \mathbb{N} \setminus \{0\}$ and $(x, m) \in \mathbb{Q} \overrightarrow{\times} \mathbb{Z}$. We have:*

$$n \left(\frac{x + (n - m)\alpha}{n}, 1 \right) = (x + (n - m)\alpha, n) = (x, m) + (n - m)(\alpha, 1).$$

Hence (x, m) is n -divisible modulo $(\alpha, 1)$. So (G, R) is divisible.

- \bullet *It can happen that (G, R) is divisible and $\text{uw}(G)$ is not. This holds in the previous example. One can also take $\Gamma = \mathbb{Q} + \mathbb{Z}\pi$ in \mathbb{R} and $G = \Gamma / \langle \pi \rangle$.*
- \bullet *If (G, R) is divisible, then so is $U(G)$ (since $h^n = g \Rightarrow U(h)^n = U(g)$). Therefore, if (G, R) c -archimedean and divisible, then $G \simeq U(G)$ is either dense or trivial. Hence it is not discrete.*

Proposition 2.6. *A divisible abelian cyclically ordered group (G, R) is not torsion-free if, and only if, its torsion subgroup is c -isomorphic to the group $T(\mathbb{U})$ of torsion elements of \mathbb{U} .*

Proof. If $T(G)$ is isomorphic to $T(\mathbb{U})$, then G is not torsion-free. Conversely, by [26, Proposition 3], $T(G)$ c -embeds in $T(\mathbb{U})$. If G contains a torsion element $g \neq 0$, then, since (G, R) is divisible, for any $n \in \mathbb{N} \setminus \{0, 1\}$, G contains an element of torsion n . This proves that $T(G) \simeq T(\mathbb{U})$. \square

Proposition 2.7. ([13, Proposition 2.26]) *Let $n \in \mathbb{N} \setminus \{0\}$. An abelian cyclically ordered group (G, R) is n -divisible and contains a subgroup which is c -isomorphic to $T(\mathbb{U})_n$ (the subgroup of $(T(\mathbb{U}), \cdot)$ generated by a primitive n -th root of 1) if and only if $\text{uw}(G)$ is n -divisible.*

So an abelian cyclically ordered group (G, R) which is n -divisible and contains a subgroup which is c -isomorphic to $T(\mathbb{U})_n$ is said to be c - n -divisible. It is called c -divisible if it is divisible and it contains a subgroup c -isomorphic to the group $T(\mathbb{U})$ of torsion elements of \mathbb{U} . This is equivalent to $\text{uw}(G)$ being divisible.

Lemma 2.8. (Lucas) *Let $n \in \mathbb{N} \setminus \{0, 1\}$. An abelian cyclically ordered group (G, R) has an element of finite order n if, and only if, z_G is n -divisible within $\text{uw}(G)$.*

Proof. Assume that there is $x \in \text{uw}(G)$ such that $nx = z_G$. Then in G we have $n\bar{x} = 0_G$. Now, for every $n' \in \{1, \dots, n-1\}$ we have $0_{\text{uw}(G)} < n'x < z_G$, hence $n'\bar{x} \neq 0_G$. Conversely, assume that there is $x \in \text{uw}(G)$ such the order of \bar{x} is n , i.e. $n\bar{x} = 0_G$ and, for every $n' \in \{1, \dots, n-1\}$, $n'\bar{x} \neq 0_G$. So there is $k \in \mathbb{Z}$ such that $nx = kz_G$. Let d be the gcd of n and k , and say, $n = dn'$, $k = dk'$. Therefore, we have $n'x = k'z_G$, so $n'\bar{x} = 0_G$. By minimality of n , we have $n' = n$, so $d = 1$. Therefore, there exist u, v in \mathbb{Z} such that $uk + vn = 1$. Hence $z_G = ukz_G + vnz_G = unx + vnz_G = n(ux + vz_G)$. Consequently, z_G is n divisible. \square

Corollary 2.9. *Let (G, R) be a divisible abelian cyclically ordered group. Then, either (G, R) is c -divisible, or z_G is not divisible by any integer within $\text{uw}(G)$.*

Proof. Follows from Lemma 2.8 and Proposition 2.6. \square

Lemma 2.10. (Lucas) *If a divisible abelian cyclically ordered group has an element whose order is finite, then it is dense.*

Proof. By Lemma 2.8 and Corollary 2.9, (G, R) is c -divisible. So $(\text{uw}(G), \leq_R)$ is divisible (Proposition 2.7). Hence $(\text{uw}(G), \leq_R)$ is dense. By Lemma 2.4, (G, R) is dense. \square

Corollary 2.11. *A discrete divisible abelian cyclically ordered group is torsion-free.*

2.4. C-regular cyclically ordered groups. We start with the definition of regular linear groups. They were introduced in [18], [25], but later equivalent definitions were given ([5], [6]). Let n be a positive integer. An abelian linearly ordered group is n -regular if it satisfies $\forall x_1, \dots, \forall x_n \ 0 < x_1 < \dots < x_n \Rightarrow \exists y \ (x_1 \leq ny \leq x_n)$.

A useful criterion is the following. A linearly ordered abelian group (Γ, \leq) is n -regular if, and only if, for each proper convex subgroup C , the quotient group Γ/C is n -divisible ([5]). The group (Γ, \leq) being n -regular is also equivalent to: for each prime number p dividing n , (Γ, \leq) is p -regular ([18], [25]). We say that the group (Γ, \leq) is *regular* if it is n -regular for each n . If (Γ, \leq) is divisible or archimedean, then it is regular.

If $(\text{uw}(G), \leq_R)$ is regular, then we say that (G, R) is *c-regular*. This is also equivalent to: for every c -convex proper subgroup C , G/C is divisible ([13, Theorem 3.5]).

Let (G, R) be a divisible abelian cyclically ordered group. If (G, R) is linearly cyclically ordered, then it is a regular linearly ordered group, since it is divisible. By [13, Corollary 3.15], if (G, R) is nontrivial and linearly ordered, then it is not c -regular. It follows that (G, R) is linearly ordered if, and only if, it is regular (as linearly ordered group) and not c -regular. This is first-order definable. If this holds, then it is elementarily equivalent to the linearly ordered group \mathbb{Q} ([17, Theorem 4.3.2]).

A dense divisible nonlinear abelian cyclically ordered group is not necessarily c -regular, as shows the following example of Lucas. Let (G, R) be the wound-round $((\mathbb{Q} + \pi\mathbb{Z}) \overrightarrow{\times} \mathbb{Q}) / ((\pi, 0))$. It is divisible. We have $\text{uw}(G) = (\mathbb{Q} + \pi\mathbb{Z}) \overrightarrow{\times} \mathbb{Q}$. Now, $((\mathbb{Q} + \pi\mathbb{Z}) \overrightarrow{\times} \mathbb{Q}) / (\{0\} \overrightarrow{\times} \mathbb{Q})$ is isomorphic to $\mathbb{Q} + \pi\mathbb{Z}$, so it is not divisible. Hence $\text{uw}(G)$ is not regular, therefore, (G, R) is not c -regular.

Note that a c -divisible cyclically ordered group is dense and c -regular, since $\text{uw}(G)$ is divisible.

Now, the next theorem shows that the theory of divisible abelian cyclically ordered groups with torsion elements is complete.

Theorem 2.12. *Two dense c -regular divisible abelian cyclically ordered groups are elementarily equivalent if, and only if, they have c -isomorphic torsion subgroups. These torsion subgroups are either trivial or c -isomorphic to $T(\mathbb{U})$. So, there are two elementary classes of dense c -regular divisible abelian cyclically ordered groups: the torsion-free dense c -regular divisible abelian cyclically ordered groups and the c -divisible abelian cyclically ordered groups.*

Proof. This is a consequence of [13, Theorem 1.9], which states that any two dense c -regular abelian cyclically ordered groups are elementarily equivalent if, and only if, they have c -isomorphic torsion subgroups and, for every prime p , the same number of classes of congruence modulo p . Now, it follows from Lemma 2.10 that a c -divisible abelian cyclically ordered group is dense. The fact that $T(G)$ is either trivial or c -isomorphic to $T(\mathbb{U})$ follows from Proposition 2.6. \square

Our Theorem 2 involves dense and discrete divisible abelian cyclically ordered groups. However, the discrete case also follows from [13]. We turn to this case in the remainder of this subsection.

Lemma 2.13. *Let (G, R) be a discrete abelian nonlinear cyclically ordered group. Then either G is finite, or $U(1_G) = 1$.*

Proof. If $U(G)$ is infinite, then it is a dense cyclically ordered group. Therefore, if $l(G) = \{0_G\}$, then (G, R) is discrete if, and only if, it is finite. If $l(G)$ is infinite, then $1_G \in l(G)$ (see proof of Lemma 2.4). Hence $U(1_G) = 1$. \square

Proposition 2.14. (*Lucas*) *Each discrete divisible abelian cyclically ordered group is c-regular.*

Proof. Let (G, R) be a discrete divisible abelian cyclically ordered group and $n \in \mathbb{N} \setminus \{0\}$. The cyclic order is not linear, since every linearly ordered divisible abelian group is dense. Since G is torsion-free (Corollary 2.11), and divisible, by Lemma 2.1 we have $[p]\text{uw}(G) = p$. Since $(0, 1_G)$ is the smallest positive element of $(\text{uw}(G), \leq_R)$, the classes $(0, 0_G), (0, 1_G), (0, 2 \cdot 1_G), \dots, (0, (p-1) \cdot 1_G)$ are pairwise distinct. Therefore, the classes of $\text{uw}(G)$ modulo p are the classes of $(0, 0_G), (0, 1_G), (0, 2 \cdot 1_G), \dots, (0, (p-1) \cdot 1_G)$. Consequently, $\text{uw}(G)/\langle(0, 1_G)\rangle$ is p -divisible. Since $\langle(0, 1_G)\rangle$ is the smallest convex subgroup of $(\text{uw}(G), \leq_R)$, it follows that the quotient of $\text{uw}(G)$ by any proper convex subgroup is p -divisible, and that $(\text{uw}(G), \leq_R)$ is p -regular. Consequently (G, R) is c-regular. \square

The classification of complete theories of discrete c-regular abelian cyclically ordered groups follows from [13, Theorem 4.33 and Corollary 4.34]. If (G, R) is a c-regular discrete abelian cyclically ordered group, then for every p and n there exists a unique k such that the formula $\exists g R(0_G, g, 2g, \dots, (p^n - 1)g) \ \& \ p^n g = k1_G$ holds. The element g in this formula is such that the jj 's where $1 \leq j \leq p^n - 1$ “do not turn full the circle”, but $p^n g$ “turns full the circle”, and it is equal to 1_G . Let us denote this k by $\varphi_{G,p}(n)$.

For every c-regular discrete abelian cyclically ordered group (G, R) and every prime p there exists a unique mapping φ_p from $\mathbb{N} \setminus \{0\}$ to $\{0, \dots, p-1\}$ such that for every $n \in \mathbb{N} \setminus \{0\}$, $\varphi_{G,p}(n) = \varphi_p(1) + p\varphi_p(2) + \dots + p^{n-1}\varphi_p(n)$ ([13, Lemma 4.29]). The sequence of the φ_p 's is called the *characteristic sequence* of (G, R) .

For every sequence of φ_p 's there exists a discrete c-regular abelian cyclically ordered group (G, R) such that this sequence is the characteristic sequence of (G, R) ([13, Proposition 4.36]).

Finally, (G, R) is divisible if, and only if, for every p we have $\varphi_p(1) \neq 0$ ([13, Proposition 4.36]).

Theorem 2.15. ([13, Corollary 4.34]) *Any two discrete divisible abelian cyclically ordered groups are elementarily equivalent if, and only if, they have the same characteristic sequences.*

Since there are 2^{\aleph_0} distinct characteristic sequences of discrete divisible abelian cyclically ordered groups, there are 2^{\aleph_0} elementary classes of discrete divisible abelian cyclically ordered groups. This was originally proved by Lucas, using constructions of cyclic orders on \mathbb{Q} .

Note that if $\varphi_p(1) \neq 0$, then p does not divide $\varphi_{G,p}(n)$, hence $\varphi_{G,p}(n)$ has an inverse modulo p^n . We see that the function $f_{G,p}$ satisfies a similar property. In order to keep the consistency of the paper, we start by proving the existence of $f_{G,p}(n)$.

Lemma 2.16. *Let (G, R) be a torsion-free dense divisible abelian cyclically ordered group. Assume that there is a prime p such that G/H_p is discrete.*

- (1) *We have $H_p \subsetneq l(G)$, and for every $n \in \mathbb{N} \setminus \{0\}$ there exists an integer $f_{G,p}(n) \in \{1, \dots, p^n - 1\}$ which satisfies: for every $g \in l(G)$, $h \in G$ such that $g + H_p = 1_{G/H_p}$ and $g = p^n h$, we have $U(h) = e^{\frac{2if_{G,p}(n)\pi}{p^n}}$.*
- (2) *For every positive integers m, n , $f_{G,p}(n)$ is the remainder of the euclidean division of $f_{G,p}(m+n)$ by p^n . It follows that p and $f_{G,p}(n)$ are coprime.*

Proof. (1) By the definition of H_p we have $H_p \subseteq l(G)$. Now, $G/l(G) \simeq U(G)$, which is dense since it is divisible and nontrivial. Hence if G/H_p is discrete, then $H_p \neq l(G)$ (if (G, R) is discrete, then $l(G)$ is nontrivial (Remark 2.5) and $H_p = \{0_G\}$).

Now, we assume that G/H_p is discrete. Let $g \in G$ be such that $g + H_p = 1_{G/H_p}$, and h be the unique element of G such that $g = p^n h$. Note that $1_{G/H_p}$ belongs to the positive cone of G/H_p , hence g belongs to the positive cone of G . Since $H_p \subsetneq l(G)$, the cyclically ordered group G/H_p is not c-archimedean, and its linear part is isomorphic to $l(G)/H_p$. Since $1_{G/H_p}$ belongs to its linear part, we have $g \in l(G)$, and $g > 0_G$ in $l(G)$. Since $p^n h \in l(G)$, we have $U(p^n h) = 1$. Hence $U(h)$ is a p^n -th root of 1 in \mathbb{U} . Assume that $h \in l(G)$. Since H_p is a convex subgroup of $l(G)$ and $p^n h \notin H_p$, we have $h \notin H_p$. By properties of linearly ordered groups we have $0_G < h < g$. This contradicts the minimality of $g + H_p$. Hence $h \notin l(G)$, that is $U(h) \neq 1$. It follows that there exists $k \in \{1, \dots, p^n - 1\}$ such that $U(h) = e^{\frac{2ik\pi}{p^n}}$.

Let $g' \in G$ be such that $g' + H_p = 1_{G/H_p}$, and h' be the unique element of G such that

$g' = p^n h'$. Then $g - g' \in H_p$, and $p^n(h - h') = g - g'$. Since H_p is p -divisible and $h - h'$ is the unique element of G such that $p^n(h - h') = g - g'$, $h - h'$ belongs to H_p . Therefore $U(h - h') = 1$, hence $U(h') = U(h + h' - h) = U(h) + U(h' - h) = U(h)$. It follows that k doesn't depend of the choice of g . We let $f_{G,p}(n) = k$.

(2) Let $g \in G$ such that $g + H_p = 1_{G/H_p}$, m, n in $\mathbb{N} \setminus \{0\}$ and h, h' be the elements of G such that $g = p^n h = p^{m+n} h'$. Then $h = p^m h'$. It follows that $U(h) = U(p^m h')$, so $\frac{2f_{G,p}(n)\pi}{p^n}$ is congruent to $\frac{p^{m+2} f_{G,p}(m+n)\pi}{p^{m+n}}$ modulo 2π . This in turn is equivalent to saying that $f_{G,p}(m+n) - f_{G,p}(n)$ belongs to $p^n \mathbb{Z}$. Say, $f_{G,p}(m+n) = ap^n + f_{G,p}(n)$. Since $f_{G,p}(n) < p^n$ and $f_{G,p}(m+n) > 0$, we have $a > 0$. Hence $f_{G,p}(n)$ is the remainder of the euclidean division of $f_{G,p}(m+n)$ by p^n .

By Proposition 1.1, $f_{G,p}(n) \geq 1$, hence, since $f_{G,p}(1) \in \{1, \dots, p-1\}$, p and $f_{G,p}(1)$ are coprime. If $n \geq 2$, then p^{n-1} divides $f_{G,p}(n) - f_{G,p}(1)$. Therefore, p does not divide $f_{G,p}(n)$. \square

If (G, R) is a discrete divisible abelian cyclically ordered group, then for every prime p we have $H_p = \{0_G\}$. Hence $G/H_p \simeq G$. The the proof of [13, Proposition 4.36], shows that for every p, n the integer $\varphi_{G,p}(n)$ is the inverse of $f_{G,p}(n)$ modulo p^n . Now, Theorem 2.15 can be reformulated in the following way.

Theorem 2.17. *Let (G, R) and (G', R') be two discrete divisible abelian cyclically ordered groups. Then they are elementarily equivalent if, and only if, for every prime p the functions $f_{G,p}$ and $f_{G',p}$ are equal.*

3. DIVISIBLE ABELIAN CYCLICALLY ORDERED GROUPS WITH TORSION ELEMENTS.

Recall that a divisible abelian cyclically ordered group contains a torsion element if, and only if, its torsion subgroup is c -isomorphic to $(T(\mathbb{U}), \cdot)$ (Proposition 2.6). This is also equivalent to $\text{uw}(G)$ being divisible. If this holds, then we say that it is c -divisible.

Proof of Theorem 1. (1) The fact that TcD is complete follows from Theorem 2.12. Now, Let (G_1, R_1) and (G_2, R_2) be two c -divisible abelian cyclically ordered groups such that G_1 is a subgroup of G_2 . Recall that by Lemma 2.10 a c -divisible abelian cyclically ordered group is dense. By [13, Theorem 1.8], (G_1, R_1) is an elementary subextension of (G_2, R_2) if, and only if, G_1 is pure in G_2 and, for every prime p , $[p]G_1 = [p]G_2$. Now, if (G_1, R_1) and (G_2, R_2) are c -divisible, then G_1 is pure in G_2 , and for every prime p we have $[p]G_1 = [p]G_2 = 1$. It follows that (G_1, R_1) is an elementary substructure of (G_2, R_2) . So, TcD is model complete.

(2) Let (G, R) be a c -divisible abelian cyclically ordered group (so it is dense). We already saw that the ordered group $\text{uw}(G)/l(G)_{\text{uw}}$ embeds in \mathbb{R} . Since $\text{uw}(G)$ is abelian torsion-free and divisible, it is a \mathbb{Q} -vector space. Hence there is a subspace A of $\text{uw}(G)$ such that the restriction to A of the canonical epimorphism $x \mapsto x + l(G)_{\text{uw}}$ is one-to-one, and so $\text{uw}(G) = A \oplus l(G)_{\text{uw}}$. Note that A is also a divisible abelian subgroup. The canonical epimorphism is order preserving because $l(G)_{\text{uw}}$ is a convex subgroup of $\text{uw}(G)$. Hence its restriction to A is order preserving, and every positive element of A is greater than $l(G)_{\text{uw}}$. It follows that the ordered group $\text{uw}(G)$ is isomorphic to the lexicographic products $A \vec{\times} l(G)_{\text{uw}}$ and $(\text{uw}(G)/l(G)_{\text{uw}}) \vec{\times} l(G)_{\text{uw}}$. Let (G_1, R_1) and (G_2, R_2) be two c -divisible abelian cyclically ordered groups, φ_1 be a c -embedding of (G, R) in (G_1, R_1) and φ_2 be a c -embedding of (G, R) in (G_2, R_2) . These c -embeddings induce embeddings $\varphi_{1\text{uw}}$ of $(\text{uw}(G), \leq_R)$ in $(\text{uw}(G_1), \leq_{R_1})$ and $\varphi_{2\text{uw}}$ of $(\text{uw}(G), \leq_R)$ in $(\text{uw}(G_2), \leq_{R_2})$, where $\varphi_{1\text{uw}}(z_G) = z_{G_1}$ and $\varphi_{2\text{uw}}(z_G) = z_{G_2}$. The restriction of $\varphi_{1\text{uw}}$ (resp. $\varphi_{2\text{uw}}$) to $l(G)$ is an embedding of $l(G)$ in $l(G_1)$ (resp. $l(G_2)$). Since the theory of divisible abelian linearly ordered groups has the amalgamation property, there are an abelian linearly ordered group L and embeddings Φ_1 of $l(G_1)$ in L and Φ_2 of $l(G_2)$ in L such that the restrictions of $\Phi_1 \circ \varphi_{1\text{uw}}$ and $\Phi_2 \circ \varphi_{2\text{uw}}$ are equal. Now, since $\text{uw}(G_1)/l(G_1)_{\text{uw}}$ and $\text{uw}(G_2)/l(G_2)_{\text{uw}}$ are archimedean, there is a unique embedding Ψ_1 (resp. Ψ_2) of $\text{uw}(G_1)/l(G_1)_{\text{uw}}$ (resp. $\text{uw}(G_2)/l(G_2)_{\text{uw}}$) in \mathbb{R} such that $\Psi_1(z_{G_1} + l(G_1)_{\text{uw}}) = 1$ (resp. $\Psi_2(z_{G_2} + l(G_2)_{\text{uw}}) = 1$). Hence we get embeddings of $(\text{uw}(G_1), \leq_{R_1})$ and $(\text{uw}(G_2), \leq_{R_2})$ in $\mathbb{R} \vec{\times} L$ such that z_{G_1} and z_{G_2} has the same image Z , which is positive and cofinal. These embeddings induce c -embeddings φ'_1 of (G_1, R_1) in $(\mathbb{R} \vec{\times} L)/\langle Z \rangle$ and φ'_2 of (G_2, R_2) in $(\mathbb{R} \vec{\times} L)/\langle Z \rangle$ such that $\varphi'_1 \circ \varphi_1 = \varphi'_2 \circ \varphi_2$.

(3) Since TcD is model-complete and has the amalgamation property, it admits the quantifiers

elimination.

(4) Let (G, R) be a c -divisible abelian cyclically ordered group. Then its torsion subgroup is c -isomorphic to $T(\mathbb{U})$. Hence $T(\mathbb{U})$ embeds in (G, R) , and this embedding is elementary, since TcD is model complete.

(5) Clearly, every model of TcD is an abelian cyclically ordered group. Since TcD is model-complete, to prove that it is the model completion of the theory of abelian cyclically ordered groups, it remains to prove that every abelian cyclically ordered group embeds in a unique way in a minimal c -divisible one. Let (G, R) be an abelian cyclically ordered group, and $\overline{\text{uw}(G)}$ be the divisible hull of $\text{uw}(G)$. Then $(G, R) \simeq \text{uw}(G)/\langle z_G \rangle$ c -embeds in the c -divisible abelian cyclically ordered group $\overline{\text{uw}(G)}/\langle z_G \rangle$. Now, if (G, R) c -embeds in a c -divisible abelian cyclically ordered group (G', R') , then $(\text{uw}(G), \leq_R)$ embeds in the divisible linearly ordered group $(\text{uw}(G'), \leq_{R'})$. Hence $\overline{\text{uw}(G)}$ embeds in $\text{uw}(G')$. Therefore $\overline{\text{uw}(G)}/\langle z_G \rangle$ c -embeds in $\text{uw}(G')/\langle z_{G'} \rangle \simeq G'$. It follows that $\overline{\text{uw}(G)}/\langle z_G \rangle$ is the c -divisible hull of (G, R) . \square

4. TORSION-FREE DIVISIBLE ABELIAN CYCLICALLY ORDERED GROUPS.

By [17, Theorem 4.3.2], every linearly ordered divisible abelian group is elementarily equivalent to the ordered group $(\mathbb{Q}, +)$. Therefore, if (G, R) is a torsion-free divisible abelian linearly cyclically ordered group, then it is elementarily equivalent to the linearly cyclically ordered group $(\mathbb{Q}, +)$. In this section we look at the nonlinear cyclically ordered case.

4.1. The preordered family $\mathcal{CD}(G)$. We recall that if (G, R) is a torsion-free divisible abelian nonlinear cyclically ordered group, then for every prime p , H_p denotes the greatest p -divisible convex subgroup of $l(G)$. The family $\{\{0_G\}, G, H_p : p \text{ prime}\}$ is denoted by $\mathcal{CD}(G)$.

Remark 4.1. (1) *The group H_p is a p -divisible c -convex proper subgroup of (G, R) , and the ordered convex subgroup H_p of $l(G)$ is isomorphic to the proper convex subgroup $(H_p)_{\text{uw}}$ of $\text{uw}(G)$.*

(2) *The subgroup $(H_p)_{\text{uw}}$ is the greatest p -divisible convex subgroup of $\text{uw}(G)$, since z_G is not divisible by any integer within $\text{uw}(G)$ (Corollary 2.9).*

(3) *The quotient cyclically ordered group G/H_p is dense (resp. discrete) if, and only if, $\text{uw}(G)/(H_p)_{\text{uw}}$ is a dense (resp. discrete) ordered group (this follows from Lemma 2.4).*

We now prove a lemma which will also be crucial in the study of weakly cyclically minimal cyclically ordered groups, for replacing a lemma of Lucas that was false (it brought a contradiction). For g in a cyclically ordered group, we let $\theta(g)$ be the element of $[0, 2\pi[$ such that $U(g) = \exp(i\theta(g))$.

Lemma 4.2. *Let (G, R) be a torsion-free divisible abelian nonlinear cyclically ordered group. Then for every prime p the sets $\{U(g) : g \in G, \exists h \in G, (R(0_G, h, g, -g) \text{ or } R(-g, g, h, 0_G)) \text{ and } ph = g\}$ and $\{U(g) : g \in G, \forall h \in G, (R(0_G, h, g, -g) \text{ or } R(-g, g, h, 0_G)) \Rightarrow ph \neq g\}$ are both dense in \mathbb{U} .*

Proof. By Remark 2.5, $U(G)$ is divisible, hence it is dense in \mathbb{U} . Let $A = \{h \in G : 0 < \theta(h) < \frac{\pi}{p}\}$. Then $U(A)$ is dense in the subset $\{e^{i\theta} : 0 < \theta < \frac{\pi}{p}\}$ of \mathbb{U} , hence $U(pA)$ is dense in $\{e^{i\theta} : 0 < \theta < \pi\}$. Let $h \in A$ and $g = ph$. Since $0 < \theta(h) < p\theta(h) = \theta(g) < \pi$, both of g and h belong to the positive cone of (G, R) , and g satisfies $\exists h \in G, R(0_G, h, g, -g)$ and $ph = g$ (recall that $R(0_G, h, g, -g)$ stands for $R(0_G, h, g)$ and $R(0_G, g, -g)$, see Subsection 1.1). In the same way, $U(p(-A))$ is dense in $\{e^{i\theta} : \pi < \theta < 2\pi\}$ and every $g \in p(-A)$ satisfies $\exists h \in G, R(-g, g, h, 0_G)$ and $ph = g$. Hence $\{U(g) : g \in G, \exists h \in G, (R(0_G, h, g, -g) \text{ or } R(-g, g, h, 0_G)) \text{ and } ph = g\}$ is dense in \mathbb{U} .

Let $A' = \{h \in G : 2\pi - \frac{2\pi}{p} < \theta(h) < 2\pi - \frac{\pi}{p}\}$. Then $U(A')$ is dense in the set $\{e^{i\theta'} : \frac{2(p-1)\pi}{p} < \theta' < \frac{(2p-1)\pi}{p}\}$. Now, $2(p-1)\pi$ is congruent to 0 modulo 2π , and $(2p-1)\pi$ is congruent to π . Hence $U(pA')$ is dense in $\{e^{i\theta'} : 0 < \theta' < \pi\}$. Let $g \in pA'$ and $h \in A'$ be such that $g = ph$. Since g belongs to the positive cone of G , and h does not, we have $R(0, g, -g)$, and $R(0, g, h)$. Since G is torsion-free, this element h such that $g = ph$ is unique. Hence g belongs to the set $\{U(g) : g \in G, \forall h \in G, (R(0_G, h, g, -g) \text{ or } R(-g, g, h, 0_G)) \Rightarrow ph \neq g\}$. In the same

way, $U(p(-A'))$ is a subset of $\{U(g) : g \in G, \forall h \in G, (R(0_G, h, g, -g) \text{ or } R(-g, g, h, 0_G)) \Rightarrow ph \neq g\}$, and $U(p(-A'))$ is dense in $\{e^{i\theta} : \pi < \theta < 2\pi\}$. Hence $\{U(g) : g \in G, \forall h \in G, (R(0_G, h, g, -g) \text{ or } R(-g, g, h, 0_G)) \Rightarrow ph \neq g\}$ is dense in \mathbb{U} . \square

In order to show that the family $\mathcal{CD}(G)$ depends on the first-order theory of G , we start with a lemma and we prove Proposition 1.1.

Lemma 4.3. *Let (G, R) be a torsion-free divisible abelian nonlinear cyclically ordered group.*

- (1) *For every prime p , H_p is definable by the formula $g = 0_G$
or $(R(0_G, g, -g) \text{ and } \forall g' \in G (g' = g \text{ or } R(0_G, g', g)) \Rightarrow \exists h' R(0_G, h', g') \text{ and } g' = ph')$
or $(R(-g, g, 0_G) \text{ and } \forall g' \in G (g' = g \text{ or } R(g, g', 0_G)) \Rightarrow \exists h' R(g', h', 0_G) \text{ and } g' = ph')$.*
- (2) *The inclusion $H_p \subseteq H_q$ depends on the first-order theory of (G, R) .*
- (3) *The cyclically ordered group G/H_p being discrete is determined by the first-order theory of (G, R) .*

Proof. Since G is torsion-free, for every g in G there is only one h in G such that $ph = g$.

(1) The formula of this statement can be reformulated as $g = 0_G$ or (*) g belongs to the positive cone P of (G, R) and every g' in $I(0_G, g)$ is p -divisible within $I(0_G, g')$, or (**) $g \in -P$ and every g' in $I(g, 0_G)$ is p -divisible within $I(g', 0_G)$. Let H be the set of elements which satisfy this formula. For $g \neq 0_G$ in H , we have either $g \in P$ and $I(0_G, g) \subseteq H$, or $g \in -P$ and $I(g, 0_G) \subseteq H$, and clearly, $g \in H \Leftrightarrow -g \in H$. Therefore, H is equal to the union of the c -convex subsets $\{-g\} \cup I(-g, g) \cup \{g\}$ where $g \in (P \cap H) \setminus \{0_G\}$. It follows that H is a c -convex subset of G .

We prove that $H \subseteq l(G)$. Let $g \notin l(G)$, then $U(g) \neq 1$. By Lemma 4.2, the set $U(G \setminus H)$ is dense in $U(G)$. Hence there exist h, h' in $G \setminus H$ such that $R(1, U(h), U(g), U(h'))$ holds in \mathbb{U} . Therefore, we have $R(0_G, h, g, h')$. So $I(0_G, g) \not\subseteq H$ and $I(g, 0_G) \not\subseteq H$, which proves that $g \notin H$. Therefore $H \subseteq l(G)$.

Now, it follows from (*) and (**) that H is the greatest p -divisible c -convex subset of $l(G)$. In particular, $H_p \subseteq H$. Therefore, in order to prove that $H = H_p$, it is sufficient to prove that H is a subgroup of $l(G)$. The set H is nonempty since it contains 0_G . Assume that $0_G < g \in l(G)$ and every $h \in l(G)$ such that $0_G < h \leq g$ is p -divisible within $l(G)$. Then every $h \in l(G)$ such that $-g < h < 0_G$ also is p -divisible within $l(G)$, so $-g \in H$. In the same way, if $0_G > g \in l(G)$, then $-g \in l(G)$. Let g, g' in H . If $g' < 0_G < g$, then $g' < g + g' < g$, so $g + g' \in H$, since H is convex. If $0_G < g' < g$, then every $h \in l(G)$ such that $g < h < g + g'$ can be written as $h = g' + (h - g')$, where $h - g'$ is p -divisible within $l(G)$ since $0_G < h - g' < g$. Therefore, h is p -divisible within $l(G)$, which proves that $g + g' \in H$. In the same way, if $g < g' < 0_G$, then $g + g' \in H$. So, we proved that H is a subgroup of $l(G)$. Consequently, $H = H_p$.

(2) Since H_p and H_q are definable, the inclusion $H_p \subseteq H_q$ also is definable.

(3) The quotient group G/H_p being discrete is definable by the formula $\exists g \in G R(0_G, g, -g) g \notin H_p$ and $\forall h \in G R(0_G, h, g) \Rightarrow (h \in H_p \text{ or } g - h \in H_p)$. \square

For each positive integer n , the formula:

$$\text{argbound}_n(g) = R(0_G, g, 2g, \dots, ng) \ \& \ \neg R(0_G, g, 2g, \dots, 2ng, (n+1)g)$$

is satisfied in (G, R) exactly by those elements g such that either $\theta(g) = \frac{2\pi}{n+1}$ & $0_G < (n+1)g$ (in $l(G)$), or $\frac{2\pi}{n+1} < \theta(g) < \frac{2\pi}{n}$, or $\theta(g) = \frac{2\pi}{n}$ & $0_G > ng$ (in $l(G)$) (see [9, Lemmas 6.9 and 6.10]).

Proof of Proposition 1.1. The first part of this proposition has been proved in (1) of Lemma 2.16. It remains to prove that the function $f_{G,p}$ is determined by the first-order theory of (G, R) . By Proposition 4.3, the quotient group G/H_p being discrete is definable.

If $p = 2$ and $r = 1$, then $f_{G,2}(1) = 1$. In the following, we assume that $p > 2$, or $p = 2$ and $r > 1$. By (2) of Lemma 2.16, p and $f_{G,p}(r)$ are coprime. Hence $f_{G,p}(r)$ has a unique inverse modulo p^r in $\{1, \dots, p^r - 1\}$. Now, the first-order theory of (G, R) determines $f_{G,p}(r)$ if, and only if, it determines its inverse modulo p^r . Let l in $\{1, \dots, p^r - 1\}$. Note that l being the inverse of $f_{G,p}(r)$

modulo p^r is equivalent to $U(lh) = e^{\frac{2i\pi}{p^r}}$. We prove that this holds if, and only if, $\text{argbound}_{p^{r-1}}(lh)$ holds. With above notations, $\text{argbound}_{p^{r-1}}(lh)$ holds if, and only if,

$$\text{either } \theta(lh) = \frac{2\pi}{p^r} \ \& \ 0_G < p^r lh \text{ (in } l(G)\text{), or } \frac{2\pi}{p^r} < \theta(lh) < \frac{2\pi}{p^r - 1},$$

$$\text{or } \theta(lh) = \frac{2\pi}{p^r - 1} \ \& \ 0_G > (p^r - 1)lh \text{ (in } l(G)\text{)}.$$

Now, $\theta(lh)$ belongs to the set $\left\{0, \frac{2\pi}{p^r}, \dots, \frac{2(p^r-1)\pi}{p^r}\right\}$, and since $p > 2$ we have $\frac{4\pi}{p^r} > \frac{2\pi}{p^r-1}$. Hence if $\text{argbound}_{p^{r-1}}(lh)$ holds, then $\theta(lh) = \frac{2\pi}{p^r} \ \& \ 0_G < p^r lh$. Conversely, assume that $\theta(lh) = \frac{2\pi}{p^r}$. Then 2π divides $\theta(p^r lh)$, so $p^r lh$ belongs to $l(G)$, and $p^r lh = lg > 0_G$. Hence $p^r lh$ is positive in $l(G)$, so $\text{argbound}_{p^{r-1}}(lh)$ holds. \square

Proposition 4.4. *Let (G, R) and (G', R') be divisible abelian cyclically ordered groups such that $(G', R') \equiv (G, R)$. Then the mapping $H_p \mapsto H'_p$ defines an isomorphism from $\mathcal{CD}(G)$ onto $\mathcal{CD}(G')$, for every prime p the cyclically ordered group G/H_p is discrete if, and only if, G'/H'_p is discrete, and if this holds, then $f_{G,p} = f_{G',p}$.*

Proof. This follows from Lemma 4.3 and Proposition 1.1. \square

The converse of this proposition will be proved in Subsection 4.4. First we turn to reminders about model theory of linearly ordered abelian groups.

4.2. The spine of Schmitt of a linearly ordered abelian group. By [9, Theorem 4.1], any two abelian cyclically ordered groups (G, R) and (G', R') are elementary equivalent if, and only if, their Rieger unwounds $(\text{uw}(G), \leq_R, z_G)$ and $(\text{uw}(G'), \leq_{R'}, z_{G'})$ are elementarily equivalent in the language of ordered groups together with a specified element interpreted by z_G :

$$(G, R) \equiv (G', R') \text{ if, and only if, } (\text{uw}(G), \leq_R, z_G) \equiv (\text{uw}(G'), \leq_{R'}, z_{G'}).$$

So, determining the first-order theory of (G, R) is equivalent to determined the first-order theory of $(\text{uw}(G), z_G)$. Lucas based its study of the first-order theory of divisible abelian cyclically ordered groups on Schmitt papers about model theory of linearly ordered groups (cf. [19, p. 15 and the following ones]).

We let (Γ, \leq) be a linearly ordered abelian group.

For every positive integer n , the n -spine of (Γ, \leq) is a family of convex subgroups of Γ :

$$Sp_n(\Gamma) = \{A_n(x) : x \in \Gamma \setminus \{0_\Gamma\}\} \cup \{F_n(x) : x \in \Gamma \setminus n\Gamma\},$$

where the convex subgroups $A_n(x)$ and $F_n(x)$ are defined as follows.

For every $x \in \Gamma$, $F_n(x)$ is the greatest convex subgroup of (Γ, \leq) such that $F_n(x) \cap (x + n\Gamma) = \emptyset$ if any, otherwise $F_n(x) = \emptyset$. By [19, Lemma 2.8, p. 25], $F_n(x) = \emptyset$ is equivalent to $x \in n\Gamma$.

We define $A_n(0_\Gamma) = \emptyset$. For $x \in \Gamma \setminus \{0_\Gamma\}$ we denote by $A(x)$ the greatest convex subgroup of Γ which doesn't contain x , and by $B(x)$ the convex subgroup generated by x . We know that $B(x)/A(x)$ is an archimedean linearly ordered abelian group, so it is regular (regular ordered groups have been defined in Subsection 2.4).

For $n \in \mathbb{N} \setminus \{0\}$ we let $A_n(x)$ be the smallest convex subgroup of (Γ, \leq) such that $B(x)/A_n(x)$ is n -regular.

For x, y in Γ , $A_n(y) \subsetneq A_n(x)$ is definable by the first-order formula

$$|y| < |x| \ \& \ \exists u (|y| \leq u \leq n|x|) \ \& \ \forall v (|v| < n|y| \Rightarrow v - u \notin n\Gamma) \text{ ([19, Lemma 2.6, p. 21])}.$$

(Here, $|y| \leq u$ stands for: $0_\Gamma \leq y \leq u$ or $0_\Gamma < -y \leq u$.)

For $x \in \Gamma \setminus \{0_\Gamma\}$ we also define $B_n(x)$ to be the greatest convex subgroup of Γ such that $B_n(x)/A_n(x)$ is n -regular, and $C_n(x) = B_n(x)/A_n(x)$ (so, $C_n(x)$ is n -regular).

The language of $Sp_n(\Gamma)$ consists in a binary predicate $<_S$ and unary predicates $A, F, Dk, \{\beta_{p,m} : p \text{ prime, } m \in \mathbb{N}\}, \{\alpha_{p,k,m} : p \text{ prime, } k \in \mathbb{N}, m \in \mathbb{N}\}$. These predicates are interpreted as follows.

$$<_S: C_1 <_S C_2 \Leftrightarrow C_2 \subsetneq C_1.$$

$A(C)$: $\exists x \in \Gamma, C = A_n(x)$.

$F(C)$: $\exists x \in \Gamma, C = F_n(x)$.

$Dk(C)$: $\exists x \in \Gamma, C = A_n(x)$ & $C_n(x)$ is discrete.

If C is an abelian group, then for every prime p and non-negative integer r the quotient group $p^r C/p^{r+1}C$ is a $\mathbb{Z}/p\mathbb{Z}$ -vector space. In the following two formulas, \dim denotes the dimension of this vector space. Furthermore, $\text{Tor}_p(C)$ denotes the subgroup of elements of C which torsion is an exponent of p . One can check that the sets $\{y \in \Gamma : F_n(y) \subseteq F_n(x)\}$ and $\{y \in \Gamma : F_n(y) \subsetneq F_n(x)\}$ are subgroups of Γ .

$\beta_{p,m}(C)$: $\exists x \in \Gamma, C = A_n(x)$ and $\lim_{r \rightarrow +\infty} \dim(p^r C_n(x)/p^{r+1}C_n(x)) \geq m$.

$\alpha_{p,k,m}(C)$: $\exists x \in \Gamma, C = A_n(x)$ and $m \leq \dim(\text{Tor}_p(p^k F_n^*(x))/\text{Tor}_p(p^{k+1} F_n^*(x)))$,

where $F_n^*(x) = \frac{\{y \in \Gamma : F_n(y) \subseteq F_n(x)\}}{\{y \in \Gamma : F_n(y) \subsetneq F_n(x)\}}$, or $F_n^*(x) = \{0_\Gamma\}$ if $\{y \in \Gamma : F_n(y) \subsetneq F_n(x)\} = \emptyset$.

Theorem 4.5. ([19, Corollary 3.2, p. 33]). *If Γ' is an other abelian linearly ordered group, then $\Gamma \equiv \Gamma'$ if, and only if, for every integer n we have $Sp_n(\Gamma) \equiv Sp_n(\Gamma')$.*

There is also a relative quantifiers elimination when adding the predicates $M_{n,k}(x)$, $D_{p,r,i}(x)$ and $E_{p,r,i}(x)$ ($i < r$) interpreted as follows.

$M_{n,k}(x)$ holds if, and only if, $C_n(x)$ is discrete and if the class $x_0 + A_n(x)$ of some $x_0 \in \Gamma$ is the smallest positive element of $C_n(x)$, then $x + A_n(x) = k(x_0 + A_n(x))$.

$D_{p,r,i}(x)$ holds if, and only if, either $x \in p^i \Gamma$ or there is some $y \in \Gamma$ such that $F_{p^r}(x) \subsetneq F_{p^r}(y)$, $x - y \in p^i \Gamma$ and $F_{p^r}(\frac{x-y}{p^i}) \subseteq F_{p^r}(x)$.

$E_{p,r,k}(x)$ holds if, and only if, there exists $y \in \Gamma$ such that $F_{p^r}(x) = A_{p^r}(y)$ and $C_{p^r}(y)$ is discrete with smallest positive element the class of y and $F_{p^r}(x - ky) \subsetneq F_{p^r}(x)$.

The relative quantifiers elimination is the following.

Theorem 4.6. ([19, Theorem 4.5, p. 66]) *Let $\Psi(Z_1, \dots, Z_m)$ be a formula in the language of ordered groups. Then there exist an integer n , a formula $\varphi_0(X_1, \dots, X_k, Y_1, \dots, Y_k)$ in the language Sp_n , some terms t_1, \dots, t_k with variables Z_1, \dots, Z_m and a quantifier-free formula $\varphi_1(Z_1, \dots, Z_m)$ in above language such that for every abelian ordered group (Γ, \leq) and every sequence $\vec{x} = (x_1, \dots, x_m)$ of elements of Γ we have $(\Gamma, \leq) \models \Psi(\vec{x})$ if, and only if,*

$$Sp_n(\Gamma) \models \varphi_0(A_n(t_1(\vec{x})), \dots, A_n(t_k(\vec{x})), F_n(t_1(\vec{x})), \dots, F_n(t_k(\vec{x}))) \text{ and } (\Gamma, \leq) \models \varphi_1(\vec{x}).$$

4.3. Spines of “almost divisible” abelian groups. If (Γ, \leq) is a divisible linearly ordered abelian group, then for every positive integer n and every $x \in \Gamma$ we have $A_n(x) = \{0_\Gamma\}$, $B_n(x) = \Gamma$ and $F_n(x) = \emptyset$. It follows that for every n the predicates $\alpha_{p,k,m}$ and $\beta_{p,m}$ don't make sense. Now we turn to a less trivial case.

By Lemma 2.1, if G is p -divisible without p -torsion element, then $[p]\text{uw}(G) = p$. So in this subsection we focus on the case where for every prime p we have $[p]\Gamma = p$.

For every linearly ordered abelian group (Γ, \leq) and every prime p , we denote by Γ_p the greatest p -divisible convex subgroup of (Γ, \leq) . The group Γ_p is definable by the formula

$$x \in \Gamma_p \Leftrightarrow \forall y \in \Gamma (0_\Gamma \leq y \leq x \text{ or } x \leq y \leq 0_\Gamma) \Rightarrow \exists z \in \Gamma y = pz.$$

Indeed, Let Γ'_p be the subset defined by this formula. Then, $0_\Gamma \in \Gamma'_p$ and for every $x \in \Gamma$ we have $x \in \Gamma'_p \Leftrightarrow -x \in \Gamma'_p$. So, in the same way as in the proof of (1) of Lemma 4.3, Γ'_p is the greatest p -divisible convex subset of Γ . Finally, the proof of (1) of Lemma 4.3 shows that Γ'_p is a subgroup (in Lemma 4.3, we had $H \subseteq l(G)$, hence we considered a convex subset of a linearly ordered group).

This subsection is dedicated to prove the following proposition.

Proposition 4.7. *Let (Γ, \leq) be a linearly ordered abelian group such that, for every prime p , $[p]\Gamma = p$. Then, the spines of the (Γ, \leq) are determined by the ordered set of Γ_p 's (p prime), and if this set admits a maximal element Γ_p , the property Γ/Γ_p being discrete or dense.*

In the remainder of this subsection, (Γ, \leq) is a linearly ordered abelian group such that, for every prime p , $[p]\Gamma = p$.

Lemma 4.8. *Let $C \subsetneq C'$ be two convex subgroups of (Γ, \leq) and p be a prime. Then C'/C is p -divisible if, and only if, either $\Gamma_p \subsetneq C$, or $C' \subseteq \Gamma_p$.*

Proof. Since $[p]\Gamma = p$ we have: $[p]C = 1$ if, and only if, $C \subseteq \Gamma_p$. Otherwise, $[p]C = p$ (the same holds with C').

Clearly, if $C' \subseteq \Gamma_p$, then C'/C is p -divisible.

Assume that $\Gamma_p \subsetneq C$. Then, there exists $x \in C \setminus pC$ such that the classes of C modulo p are $pC, x + pC, \dots, (p-1)x + pC$. Since C is a convex subgroup of C' , x is not p -divisible within C' (if $x = py$, then $|y| \leq |x|$, hence $y \in C$). Now, $[p]C = [p]C' = p$, so the classes of C' modulo p are $pC', x + pC', \dots, (p-1)x + pC'$. Let $x' \in C'$ and $j \in \{0, \dots, p-1\}$ such that $x' \in jx + pC'$. Then, there is $y' \in C'$ such that $x' - py' = jx$. Since $jx \in C$, we have $x' + C = py' + C$, which proves that C'/C is p -divisible.

Assume that $C \subseteq \Gamma_p \subsetneq C'$. Let $x' \in C'$ such that the class of x' modulo C is p -divisible. Hence, there exists $y' \in C'$ such that $x' - py' \in C$. Now, $C = pC$, hence $x' - py' \in pC \subseteq pC'$, which proves that $x' \in pC'$. Since $C' \neq pC'$, we see that C'/C is not p -divisible. \square

Corollary 4.9. *If there is some $C \subsetneq C'$ in the set $\{\Gamma, \Gamma_p : p \text{ prime}\}$ such that C'/C is discrete, then $C' = \Gamma$, and $C = \max\{\Gamma_p : p \text{ prime}\}$*

Proof. If there exists p_1 and p_2 such that $\{0_G\} \subsetneq \Gamma_{p_1} \subsetneq \Gamma_{p_2}$, then $\Gamma_{p_2}/\Gamma_{p_1}$ is p_2 -divisible. It follows that it is dense, and that Γ/Γ_{p_1} is dense. Consequently, if some group C'/C is discrete, then $C = \max\{\Gamma_p : p \text{ prime}\}$. Since $C' \in \{\Gamma, \Gamma_p : p \text{ prime}\}$, we have $C' = \Gamma$. \square

Note that if $C = \{0_\Gamma\}$ and C'/C is discrete, then C' is discrete. Hence Γ is discrete and for every prime p we have $\Gamma_p = \{0_\Gamma\}$.

Before to prove the next Proposition, we review some properties of the sets $A_n(x)$ and $F_n(x)$.

Lemma 4.10. *Assume that $n = p_1^{r_1} \cdots p_k^{r_k}$, where p_1, \dots, p_k are the primes which divide n and let $x \in \Gamma$.*

- (a) *We have $F_n(x) \subseteq A_n(x)$ and $(F_n(x) = \emptyset \Leftrightarrow x \in n\Gamma)$ ([19, Lemma 2.8, p. 25]). One can check that if $x \notin n\Gamma$, then $F_n(x)$ contains the greatest n -divisible convex subgroup of Γ .*
- (b) *The set $A_n(x)$ is the maximum of the $A_{p_j}(x)$ ([19, Lemma 2.3 p. 19]).*
- (c) *The set $F_n(x)$ is the maximum of the $F_{p_j^{r_j}}(x)$'s, where $1 \leq j \leq k$ ([19, Lemma 2.3 p. 19]).*
If m is an other integer, then $F_{mn}(mx) = F_n(x)$, and if $(m, n) = 1$, then $F_n(x) = F_n(mx)$ ([19, 3 and 4 of Lemma 2.10, p. 26]).

Proposition 4.11. *Let $n = p_1^{r_1} \cdots p_k^{r_k}$ be a positive integer, where p_1, \dots, p_k are the primes which divide n .*

- (1) *$Sp_n(\Gamma) = \{\Gamma_{p_i} : 1 \leq i \leq k\} \cup \{\{0_\Gamma\}\}$, and it does not contain Γ . We let $Sp_n(\Gamma) = \{\{0_\Gamma\} = C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_l\}$, $C_{l+1} = \Gamma$.
 Let $x \in \Gamma \setminus \{0_\Gamma\}$ and j be the integer such that $x \in C_{j+1} \setminus C_j$. Then $A_n(x) = C_j$ and $B_n(x) = C_{j+1}$. In particular, for every $\alpha \in \mathbb{Z} \setminus \{0\}$ we have $A_n(\alpha x) = A_n(x)$ and $B_n(\alpha x) = B_n(x)$.
 If $x \notin n \text{uw}(G)$, then $F_n(x) = \max\{\Gamma_{p_i} : 1 \leq i \leq k, p_i^{r_i} \nmid x\}$.*
- (2) *Let C, C' in $Sp_n(\Gamma)$. Then $A(C), F(C')$ hold if, and only if, C and C' belong to $\{\Gamma_{p_1}, \dots, \Gamma_{p_k}\}$. Furthermore, there exists x such that $C = A_n(x)$ and $C' = F_n(x)$ if, and only if, $C' \subseteq C$.*
- (3) *Let C , in $Sp_n(\Gamma)$. If $Dk(C)$ holds, then $C = C_l$, and for any x such that $A_n(x) = C$ we have $B_n(x) = \Gamma$.*
- (4) *Let $j \in \{2, \dots, l\}$. Then $\beta_{p,0}(C_j)$ holds, for $m \geq 2$, $\beta_{p,m}(C_j)$ does not hold, and $\beta_{p,1}(C_j)$ holds if, and only if, $C_j \subseteq \Gamma_p \subsetneq C_{j+1}$.*
- (5) *Let $C \in Sp_n(\Gamma) \setminus \{\{0_\Gamma\}\}$. If $m > 1$, then $\alpha_{p,k,m}(C)$ does'nt hold. If $m = 0$ then it holds. In the case $m = 1$ it holds if, and only if, p^{k+1} divides n and $\Gamma_p \subseteq C$.*

Proof. (1) Let $x \in \Gamma \setminus \{0_\Gamma\}$ and p be a prime.

Assume that $x \in \Gamma_p$. Then $B(x) \subseteq \Gamma_p$. Hence $A_p(x) = \{0_\Gamma\}$. By (a) of Lemma 4.10 we have $F_p(x) = \emptyset$. If $\Gamma_p \subsetneq C$ for some convex subgroup C , then by Lemma 4.8 C/Γ_p is not p -divisible, so C is not p -regular. Hence $C/A_p(x) = C/\{0_\Gamma\}$ is not p -regular. Therefore, $B_p(x) = \Gamma_p$.

Assume that $x \notin \Gamma_p$. Then, $\Gamma_p \subsetneq B(x)$. The group $B(x)/C$ is p -regular if, and only if, for every C' such that $C \subsetneq C' \subsetneq B(x)$, the group $(B(x)/C)/(C'/C)$ is p -divisible, i.e. $B(x)/C'$ is p -divisible. By Lemma 4.8, this is equivalent to $\Gamma_p \subsetneq C'$. It follows that $A_p(x) = \Gamma_p$, and $B_p(x) = \Gamma$. We deduce from (b) of Lemma 4.10 that $A_n(x)$ is equal to the maximum of the Γ_{p_i} such that $x \notin \Gamma_{p_i}$, $1 \leq i \leq k$ (or to $\{0_\Gamma\}$ if the preceding set is empty).

Assume that $x = p_i^d x'$, where $x' \notin p_i \Gamma$, and let r be a positive integer. By (a) of Lemma 4.10, $\Gamma_{p_i} \subseteq F_{p_i^r}(x') \subseteq A_{p_i^r}(x') = \Gamma_{p_i}$. Hence $F_{p_i^r}(x') = \Gamma_{p_i}$. If p_i^r divides x , then, by (a) of Lemma 4.10, $F_{p_i^r}(x) = \emptyset$. Now, assume that p_i^r doesn't divide x (i.e. $r > d$). By (c) of Lemma 4.10, $F_{p_i^r}(x) = F_{p_i^{r-d} p_i^d}(p_i^d x') = F_{p_i^{r-d}}(x') = \Gamma_{p_i}$. Therefore, if $x \notin n\Gamma$, then, by (c) of Lemma 4.10, $F_n(x) = \max\{\Gamma_{p_i} : 1 \leq i \leq k, p_i^{r_i} \nmid x\}$. Hence $Sp_n(\Gamma) = \{\Gamma_{p_i} : 1 \leq i \leq k\} \cup \{0_\Gamma\}$, that we can write as $\{0_\Gamma\} = C_1 \subsetneq C_2 \subsetneq \dots \subsetneq C_l$. The hypothesis $[p]\Gamma = p$, for every prime, implies that $\Gamma_p \neq \Gamma$. So $\Gamma \notin Sp_n(\Gamma)$, for convenience we let $C_{l+1} = \Gamma$.

We let $C_j = A_n(x)$. Since C_j is the maximum of the Γ_{p_i} such that $x \notin \Gamma_{p_i}$, $1 \leq i \leq k$ (or to $\{0_\Gamma\}$ if the preceding set is empty), we have $x \in C_{j+1} \setminus C_j$. Let C, C' be convex subgroups such that $C_j \subsetneq C \subsetneq C'$, and $i \in \{1, \dots, k\}$. By Lemma 4.8, C'/C is p_i -divisible if, and only if, either $\Gamma_{p_i} \subsetneq C$ or $C' \subseteq \Gamma_{p_i}$. Hence C'/C_j is p_i -regular if, and only if, either $\Gamma_i \subseteq C_j$ or $C' \subseteq \Gamma_{p_i}$. Consequently, C'/C_j is n -regular if, and only if, $C' \subseteq C_{j+1}$. Therefore, $B_n(x) = C_{j+1}$. This proves (1).

(2) Predicates A and F . Since $\emptyset \notin Sp_n(\Gamma)$, it follows from (1) that $A(C), F(C')$ hold if, and only if, C and C' belong to $\{\Gamma_{p_1}, \dots, \Gamma_{p_k}\}$. By (a) of Lemma 4.10, if there exists x such that $A_n(x) = C$ and $F_n(x) = C'$, then $C' \subseteq C$. In order to prove the converse, we assume that $\Gamma_{p_1} \subseteq \dots \subseteq \Gamma_{p_{s-1}} \subsetneq \Gamma_{p_s} = \dots = \Gamma_{p_t} \subsetneq \Gamma_{p_{t+1}} \subseteq \dots \subseteq \Gamma_{p_k}$ and $C = \Gamma_{p_s}$, we denote by B the set $B_n(x)$ for any $x \in \Gamma$ such that $A_n(x) = C$ (this is equivalent to $B = \Gamma_{p_{t+1}}$ if $t < k$ and to $B = \Gamma$ if $t = k$). By the definition of the sets Γ_p , for $i \in \{1, \dots, t\}$ there exists an element $x_i \in B \setminus \Gamma_{p_i} = B \setminus C$ which is not divisible by p_i . We let $x = p_2 \dots p_t x_1 + p_1 p_3 \dots p_t x_2 + \dots + p_1 \dots p_{t-1} x_t$. Then x is not divisible by any of p_1, \dots, p_t . In particular, $x \in B \setminus C$. Hence $A_n(x) = C$ and $F_n(x) = C$. The element $p_s^{r_s} \dots p_t^{r_t} x$ belongs to $B \setminus C$. Hence $A_n(p_s^{r_s} \dots p_t^{r_t} x) = C$, and $F_n(p_s^{r_s} \dots p_t^{r_t} x) = \Gamma_{p_{s-1}}$. In the same way, by multiplying by the other $p_i^{r_i}$'s, $i < s$, we get elements y_i such $A_n(y_i) = C$ and $F_n(y_i) = \Gamma_{p_i}$.

(3) Predicate Dk . If C is not the maximum of the groups Γ_p , then $Dk(C)$ does not hold (Corollary 4.9). Assume that C is the maximum of the Γ_p 's. For every $x \in \Gamma$ such that $C = A_n(x)$, x does not belong to any Γ_p . Therefore, $B_n(x) = \Gamma$.

(4) Predicates $\beta_{p,m}(C_j)$. Let $x \in \Gamma$ such that $C_j = A_n(x)$ (i.e. $x \in C_{j+1} \setminus C_j$). If $C_n(x)$ is not p -divisible, then, $[p]C_n(x) = p$. Hence, for every r , $\dim(p^r C_n(x)/p^{r+1} C_n(x)) = 1$. Otherwise, $\dim(p^r C_n(x)/p^{r+1} C_n(x)) = 0$. It follows that if $m \geq 2$, then $\beta_{p,m}(C_j)$ does not hold and if $m = 0$, then it holds. If $m = 1$, then $\beta_{p,1}(C_j)$ holds if, and only if, $C_n(x)$ is not p divisible. By Lemma 4.8, $C_n(x)$ is not p -divisible if, and only if, $A_n(x) \subseteq \Gamma_p \subsetneq B_n(x)$, that is, $C_i \subseteq \Gamma_p \subsetneq C_{i+1}$.

(5) Predicates $\alpha_{p,k,m}(C)$. We let $x \in \Gamma$ such that $A_n(x) = C$. We assume that $\Gamma_{p_1} \subseteq \dots \subseteq \Gamma_{p_{s-1}} \subsetneq \Gamma_{p_s} = \dots = \Gamma_{p_t} \subsetneq \Gamma_{p_{t+1}} \subseteq \dots \subseteq \Gamma_{p_k}$ and $F_n(x) = \Gamma_{p_s}$. By (1), for $i > t$, $p_i^{r_i}$ divides x , and there is some $i \in \{s, \dots, t\}$ such that $p_i^{r_i}$ doesn't divide x . Let $y \in \Gamma$. Then $F_n(y) \subsetneq F_n(x)$ if, and only if, for every $i \geq s$, $p_i^{r_i}$ divides y . We let $n_1 = p_{t+1}^{r_{t+1}} \dots p_k^{r_k}$ and $n_2 = p_s^{r_s} \dots p_t^{r_t}$. Then $\{y \in \Gamma : F_n(y) \subsetneq F_n(x)\} = n_1 n_2 \Gamma$. In the same way, $\{y \in \Gamma : F_n(y) \subseteq F_n(x)\} = n_1 \Gamma$.

So $F_n^*(x) = n_1 \Gamma / n_1 n_2 \Gamma$. Let $y_0 \in n_1 \Gamma \setminus n_1 n_2 \Gamma$, p be a prime, d be the greatest integer such that p^d divides y_0 , and k be a positive integer. Then $p^k y_0 \in n_1 n_2 \Gamma$ if, and only if, there is $i \in \{s, \dots, t\}$ such that $p = p_i$, $k \geq r_i - d$, and for $i' \neq i$ in $\{s, \dots, t\}$, $p_i^{r_{i'}}$ divides y . Consequently, if $p \notin \{p_s, \dots, p_t\}$, then $F_n^*(x)$ doesn't contain any nontrivial p -torsion element. If this holds, then $\dim(\text{Tor}_p(p^k F_n^*(x))/\text{Tor}_p(p^{k+1} F_n^*(x))) = 0$.

Assume that $p = p_i$ with $i \in \{s, \dots, t\}$. Then $\text{Tor}_{p_i}(F_n^*(x)) = \left(n_1 \frac{n_2}{p_i} \Gamma\right) / (n_1 n_2 \Gamma)$, and

$$\text{Tor}_{p_i}(p_i F_n^*(x)) = \left(n_1 \frac{n_2}{p_i^{r_i-1}} \Gamma\right) / (n_1 n_2 \Gamma).$$

Hence $\dim(\text{Tor}_{p_i}(F_n^*(x))/\text{Tor}_{p_i}(p_i F_n^*(x))) = 1$.

More generally, if $k < r_i$, then $\dim(\text{Tor}_{p_i}(p_i^k F_n^*(x))/\text{Tor}_{p_i}(p_i^{k+1} F_n^*(x))) = 1$.

If $k \geq r_i$, then both of $\text{Tor}_{p_i}(p_i^k F_n^*(x))$ and $\text{Tor}_{p_i}(p_i^{k+1} F_n^*(x))$ are trivial, hence

$\dim(\mathrm{Tor}_{p_i}(p_i^k F_n^*(x))/\mathrm{Tor}_{p_i}(p_i^{k+1} F_n^*(x))) = 0$.

By (1), $F_n(x)$ can be any element of $Sp_n(\Gamma)$ which is a subgroup of C , i.e. any Γ_{p_i} where $i \in \{1, \dots, k\}$ and $\Gamma_{p_i} \subset C$. \square

Now, Proposition 4.7 follows from Proposition 4.11.

4.4. Theories of torsion-free divisible abelian cyclically ordered groups. Let (G, R) be a cyclically ordered group. It follows from Theorems 4.5 and 4.6 that determining the first-order theory of $(\mathrm{uw}(G), z_G)$ is also equivalent to determining $Sp_n(\mathrm{uw}(G))$'s and the type of z_G . So, François Lucas proved that if G is torsion-free and divisible, then the type of z_G is determined by the $Sp_n(\mathrm{uw}(G))$'s and the functions $f_{G,p}$. Our approach here is more explicit and we give the links with the theory of (G, R) . We first state a lemma.

Lemma 4.12. *(Lucas) Let (G, R) be a nonlinear divisible cyclically ordered abelian group. If $C_n(z_G)$ is discrete, then it is not archimedean.*

Proof. Since z_G is cofinal in $(\mathrm{uw}(G), \leq_R)$ and $z_G \in B(z_G) \subseteq B_n(z_G)$, it follows that $B_n(z_G) = \mathrm{uw}(G)$. Hence, the condition “ $C_n(z_G)$ archimedean” is equivalent to $A_n(z_G) = A(z_G) = l(G)_{\mathrm{uw}}$. Hence $C_n(z_G) = \mathrm{uw}(G)/l(G)_{\mathrm{uw}}$. Since (G, R) is nonlinear, we have $U(G) \neq \{1\}$. Now, (G, R) is divisible, hence $U(G)$ is divisible. In particular it is infinite, so it is dense. By Lemma 2.4, $\mathrm{uw}(G)/l(G)_{\mathrm{uw}}$ is dense. \square

Proposition 4.13. *Let (G, R) be a torsion-free divisible nonlinear cyclically ordered abelian group. The spines of Schmitt of $(\mathrm{uw}(G), \leq_R)$ are determined by $\mathcal{CD}(G)$.*

Proof. It follows from Lemma 2.1 that, for every prime p , $[p]\mathrm{uw}(G) = p$. Since $l(G)_{\mathrm{uw}}$ is the greatest convex subgroup of $\mathrm{uw}(G)$ and this group is not p -divisible, we have $\mathrm{uw}(G)_p \subseteq l(G)_{\mathrm{uw}}$. Therefore $\mathrm{uw}(G)_p = (H_p)_{\mathrm{uw}}$. Hence By Proposition 4.7, the spines of $(\mathrm{uw}(G), \leq_R)$ are determined by the set of $(H_p)_{\mathrm{uw}}$'s, and if this set admits a maximal element $(H_p)_{\mathrm{uw}}$, the property $\mathrm{uw}(G)/(H_p)_{\mathrm{uw}}$ being discrete or dense. The preordered subset of the $(H_p)_{\mathrm{uw}}$'s and $\mathcal{CD}(G)$ are isomorphic, and $\mathrm{uw}(G)/(H_p)_{\mathrm{uw}}$ is discrete if, and only if, G/H_p is (Remark 4.1), and this is first-order definable (Lemma 4.3 (3)).

By Lemma 4.3 (2), $H_p \subsetneq H_q$ depends on the first-order theory of (G, R) , hence the same holds for the predicate $<_S$ of $Sp_n(\mathrm{uw}(G))$. \square

Lemma 4.14. *Let (G, R) be a torsion-free nonlinear cyclically ordered divisible abelian group and p be a prime such that G/H_p is discrete. Then for every $x \in l(G)_{\mathrm{uw}}$ such that the class $x + (H_p)_{\mathrm{uw}}$ is the smallest positive element of $\mathrm{uw}(G)/(H_p)_{\mathrm{uw}}$ and every $r \in \mathbb{N} \setminus \{0\}$, there exists $y \in \mathrm{uw}(G)$ such that $p^r y = x + f_{G,p}(r)z_G$. Furthermore, if there exists $y' \in \mathrm{uw}(G)$ and $k \in \mathbb{Z}$ such that $p^r y' = x + kz_G$, then k is congruent to $f_{G,p}(r)$ modulo p^r .*

Proof. Let $r \in \mathbb{N} \setminus \{0\}$ and $x \in l(G)_{\mathrm{uw}}$ such that the class $x + (H_p)_{\mathrm{uw}}$ is the smallest positive element of $\mathrm{uw}(G)/(H_p)_{\mathrm{uw}}$. Since G is divisible, its unwound $\mathrm{uw}(G)$ is divisible modulo $\langle z_G \rangle$. Hence there exists $y' \in \mathrm{uw}(G)$ and an integer k such that $p^r y' = x + kz_G$. Let $k = p^r k' + k''$, with $k'' \in \{0, \dots, p^r - 1\}$ be the euclidean division, and $y = y' - k' z_G$. Then $p^r y = x + k'' z_G$. Let $h = \bar{y}$ and $g = \bar{x}$. Then $g = p^r h$, hence, by the definition of $f_{G,p}(r)$, we have $U(h) = e^{\frac{2if_{G,p}(r)\pi}{p^r}}$. In the divisible hull of $\mathrm{uw}(G)$, we have $y = \frac{1}{p^r}x + \frac{k''}{p^r}z_G$. Hence $U(h) = U\left(\frac{k''}{p^r}z_G\right) = e^{\frac{2ik''\pi}{p^r}}$. Consequently, $k'' = f_{G,p}(r)$, and k is congruent to $f_{G,p}(r)$ modulo p^r . \square

The proof of the following theorem is similar to that of an analogous Lucas result, but much read in detail here.

Theorem 4.15. *Let (G, R) be a divisible nonlinear cyclically ordered abelian group. If every G/H_p is dense, then the type of z_G in the language of Schmitt is entirely determined by $\mathcal{CD}(G)$. Otherwise, it is entirely determined by $\mathcal{CD}(G)$ and the mappings $f_{G,p}$, where H_p is the maximum of the set $\{H_q : q \text{ prime}\}$.*

Proof. Let $\Psi(Z)$ be a formula in the language of ordered groups. By Theorem 4.6, there exist an integer n , a formula $\varphi_0(X_1, \dots, X_k, Y_1, \dots, Y_k)$ in the language Sp_n , some terms t_1, \dots, t_k with variables Z and a quantifier-free formula $\varphi_1(Z)$ in the language defined before Theorem 4.6 such that for every abelian ordered group (Γ, \leq) and every x in Γ we have $(\Gamma, \leq) \models \Psi(\vec{x})$ if, and only if,

$$Sp_n(\Gamma) \models \varphi_0(A_n(t_1(x)), \dots, A_n(t_k(x)), F_n(t_1(x)), \dots, F_n(t_k(x))) \text{ and } (\Gamma, \leq) \models \varphi_1(\vec{x}).$$

The only terms with variable z_G look like αz_G , where $\alpha \in \mathbb{Z} \setminus \{0\}$. Set $t_1(z_G) = \alpha_1 z_G, \dots, t_k(z_G) = \alpha_k z_G$. Let (n, α_i) be the gcd of n and α_i . By (1) of Proposition 4.11, we have $A_n(\alpha_i z_G) = A_n(z_G)$. By (c) of Lemma 4.10, $F_n(\alpha_i z_G) = F_{\frac{n}{(n, \alpha_i)}}\left(\frac{\alpha_i}{(n, \alpha_i)} z_G\right) = F_{\frac{n}{(n, \alpha_i)}}(z_G)$. Therefore, φ_0 can be written as

$$\varphi_0\left(A_n(z_G), F_{\frac{n}{(n, \alpha_1)}}(z_G), \dots, F_{\frac{n}{(n, \alpha_k)}}(z_G)\right).$$

Since z_G is cofinal in $\text{uw}(G)$, it follows from (1) of Proposition 4.11 that $A_n(z_G)$ is the greatest $(H_p)_{\text{uw}}$ such that p divides n . Now, since G is torsion-free, by Corollary 2.9, z_G is not divisible by any integer. Hence by (1) of Proposition 4.11 $F_{\frac{n}{(n, \alpha_i)}}(z_G)$ is the greatest $(H_p)_{\text{uw}}$ such that p divides $\frac{n}{(n, \alpha_i)}$. So all these subsets are determined by the preordered set of the $(H_p)_{\text{uw}}$'s.

Now, we look at the quantifier-free formula $\varphi_1(z_G)$ in the language $(0, +, \leq_R, M_{n,k}, D_{p,r,i}, E_{p,r,i})$, where n, k, r, i belong to \mathbb{N} and p is a prime.

(1) The formulas which contain only the symbols $0, +, \leq_R$ can be seen as formulas of \mathbb{Z} . Hence they do not depend on z_G (for example, $\alpha z_G \geq_R (0, 0_G)$ is equivalent to $\alpha \geq 0$).

(2) By Lemma 4.12, if $C_n(z_G)$ is discrete, then it is not archimedean. Since the class of z_G is cofinal, $z_G + A_n(z_G)$ does not belong to the smallest convex subgroup of $C_n(G)$ (which contains the smallest positive element). Hence $M_{n,k}(k' z_G)$ nether holds.

(3) Predicates $D_{p,r,i}(\alpha z_G)$. Since z_G is not divisible by any integer, $z_G \notin p \text{uw}(G)$, and $\alpha z_G \in p^r \text{uw}(G) \Leftrightarrow \alpha \in p^r \mathbb{Z}$. Assume that $\alpha \notin p^r \mathbb{Z}$, say $\alpha = p^s \alpha'$, with $0 \leq s < r$ and $(\alpha', p) = 1$. By (c) of Lemma 4.10, $F_{p^r}(\alpha z_G) = F_{p^{r-s}}(z_G)$. By (1) of Proposition 4.11, $F_{p^{r-s}}(z_G) = (H_p)_{\text{uw}}$. Hence $F_{p^r}(x) \subsetneq F_{p^r}(z_G) \Rightarrow F_{p^r}(x) = \emptyset$, i.e. $x \in p^r \text{uw}(G)$. Let $x \in p^r \text{uw}(G)$. For $i \leq r$ we have:

$$\alpha z_G - x \in p^i \text{uw}(G) \Leftrightarrow \alpha z_G \in p^i \text{uw}(G) \Leftrightarrow \alpha \in p^i \mathbb{Z} \Leftrightarrow i \leq s,$$

which does not depend on (G, R) .

If this holds, then by (1) of Proposition 4.11 we have

$$F_{p^r}\left(\frac{\alpha z_G - x}{p^i}\right) = \emptyset \text{ or } F_{p^r}\left(\frac{\alpha z_G - x}{p^i}\right) = (H_p)_{\text{uw}}.$$

In any case, $F_{p^r}\left(\frac{\alpha z_G - x}{p^i}\right) \subseteq F_{p^r}(z_G)$. So, $D_{p,r,i}(\alpha z_G)$ holds.

Therefore, the formula $D_{p,r,i}(\alpha z_G)$ is determined by $\mathcal{CD}(G)$.

(4) Predicates $E_{p,r,k}(\alpha z_G)$ (there exists some $x \in \text{uw}(G)$ such that $A_{p^r}(x) = F_{p^r}(\alpha z_G)$, $C_{p^r}(x)$ is discrete, the class of x is the smallest positive element, and $F_{p^r}(\alpha z_G - kx) \subsetneq F_{p^r}(\alpha z_G)$). By Corollary 4.9, if $C_{p^r}(x)$ is discrete, then $B_{p^r}(x) = \text{uw}(G)$ and $A_{p^r}(x)$ is the maximum of the set of all the $(H_p)_{\text{uw}}$'s. This is determined by $\mathcal{CD}(G)$ and the set of all primes p such that G/H_p is discrete.

We assume that $\text{uw}(G)/(H_p)_{\text{uw}}$ is discrete. We let x with class in the smallest positive element.

If $\alpha z_G \in p^r \text{uw}(G)$, then we already saw that $F_{p^r}(\alpha z_G) = \emptyset$. It follows that $E_{p,r,k}(\alpha z_G)$ doesn't hold, since the sets $A_{p^r}(x)$ are nonempty. This is determined by $\mathcal{CD}(G)$.

Otherwise, we have $F_{p^r}(\alpha z_G) = (H_p)_{\text{uw}}$, and the last condition is equivalent to $F_{p^r}(\alpha z_G - kx) = \emptyset$. This in turn is equivalent to $\alpha z_G - kx \in p^r \text{uw}(G)$. By Lemma 4.14, there exists $y \in \text{uw}(G)$ such that $p^r y = x + f_{G,p}(r) z_G$. Since z_G is not divisible, we have:

$$\alpha z_G - kx \in p^r \text{uw}(G) \Leftrightarrow \alpha z_G - k f_{G,p}(r) z_G \in p^r \text{uw}(G) \Leftrightarrow \alpha - k f_{G,p}(r) \in p^r \mathbb{Z},$$

which is entirely determined by the function $f_{G,p}$. \square

Now, Theorem 2 follows from Proposition 4.4 and Theorem 4.15.

4.5. Cyclic orders on the additive group \mathbb{Q} of rational numbers. We describe the cyclic orders on \mathbb{Q} , we deduce a characterization of the families of functions $f_{G,p}$, and we prove Theorem 3. Note that there is only one linear order on \mathbb{Q} .

The description of the cyclic orders on \mathbb{Q} will give rise to the characterization of all functions $f_{G,p}$ because of the following proposition.

Proposition 4.16. ([13, Proposition 4.38]) *Let (G, R) be an abelian cyclically ordered group which is not c-archimedean. Then (G, R) is discrete and divisible if, and only if, there exists a discrete cyclic order R' on the group \mathbb{Q} such that (\mathbb{Q}, R') is an elementary substructure of (G, R) .*

Note that in the statement of [13, Proposition 4.38] the word ‘‘c-regular’’ is redundant, since by Proposition 2.14 every discrete divisible abelian cyclically ordered group is c-regular.

The description of the cyclic orders on \mathbb{Q} is based on the following mappings.

Fact 4.17. *For every prime number p , let f_p be a mapping from \mathbb{N} in $\{0, \dots, p-1\}$. Then one can define in a unique way a mapping from $\mathbb{N} \setminus \{0\}$ to \mathbb{N} in the following way. Set $f(1) = 0$. For p prime and $r \in \mathbb{N} \setminus \{0\}$, $f(p^r) = f_p(1) + pf_p(2) + \dots + p^{r-1}f_p(r)$. For $n = p_1^{r_1} \dots p_k^{r_k}$ with r_1, \dots, r_k in $\mathbb{N} \setminus \{0\}$, let $f(n)$ be the unique integer in $\{0, \dots, n-1\}$ which is congruent to every $f(p_i^{r_i})$ modulo $p_i^{r_i}$ (the existence of $f(n)$ follows from the Chinese remainder theorem).*

Proposition 4.18. *Let R be a ternary relation on \mathbb{Q} . Then R is a cyclic order if, and only if, there exist mapping from $\mathbb{N} \setminus \{0\}$ to \mathbb{N} , defined as in Fact 4.17, $\theta \in [0, 2\pi[$ and a non-negative $a \in \mathbb{R}$ such that (\mathbb{Q}, R) is c-isomorphic to the subgroup of the lexicographic product $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$ generated by*

$$\left\{ \left(\exp \left(i \frac{\theta + 2f(n)\pi}{n} \right), \frac{a}{n} \right) : n \in \mathbb{N} \setminus \{0\} \right\}.$$

The proof of this proposition consists of Lemmas 4.19, 4.20, 4.21, 4.22.

Lemma 4.19. (Lucas) *Let R be a nonlinear cyclic order on \mathbb{Q} . The cyclically ordered group (\mathbb{Q}, R) embeds in the lexicographic product $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$.*

Proof. By [21, p. 161], there is a linearly ordered group L such that (\mathbb{Q}, R) embeds in the lexicographic product $\mathbb{U} \overrightarrow{\times} L$. Let $x \mapsto (\varphi_1(x), \varphi_2(x))$ be this embedding. Then φ_1 and φ_2 are group isomorphisms. It follows that they are \mathbb{Q} -linear. In particular, the image of φ_2 is $\mathbb{Q} \cdot \varphi_2(1)$. So either $\mathbb{Q} \cdot \varphi_2(1)$ is trivial, or it is isomorphic to \mathbb{Q} . In any case (\mathbb{Q}, R) embeds in $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$. \square

Lemma 4.20. *Let G be a subgroup of $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$ which is isomorphic to \mathbb{Q} in the language of groups. There exist $\theta \in \mathbb{R}$, $0 \leq a \in \mathbb{Q}$, with $a = 0 \Rightarrow \theta \notin \mathbb{Q} \cdot \pi$, such that G is generated by*

$$\left\{ \left(\exp \left(i \frac{\theta + 2f(n)\pi}{n} \right), \frac{a}{n} \right) : n \in \mathbb{N} \setminus \{0\} \right\},$$

where f is a mapping from $\mathbb{N} \setminus \{0\}$ in \mathbb{N} which satisfies:

(*) for every $n \in \mathbb{N} \setminus \{0\}$, $f(n) \in \{0, \dots, n-1\}$,

(**) for every m, n in $\mathbb{N} \setminus \{0\}$, $f(n)$ is the remainder of the euclidean division of $f(mn)$ by n .

Proof. Let $x \mapsto (\varphi_1(x), \varphi_2(x))$ be the group isomorphism between \mathbb{Q} and G . We pick an $x_0 \neq 0$ in \mathbb{Q} and we let $(\exp(i\theta), a) = (\varphi_1(x_0), \varphi_2(x_0))$. Then G is generated by the set

$$\left\{ \left(\varphi_1 \left(\frac{x_0}{n} \right), \varphi_2 \left(\frac{x_0}{n} \right) \right) : n \in \mathbb{N} \setminus \{0\} \right\}.$$

Now, $\varphi_2 \left(\frac{x_0}{n} \right) = \frac{a}{n}$, and there exists a unique $f(n) \in \{0, \dots, n-1\}$ such that

$$\varphi_1 \left(\frac{x_0}{n} \right) = \exp \left(i \frac{\theta + 2f(n)\pi}{n} \right).$$

Consequently, f satisfies (*).

If $a < 0$, then we can take $-x_0$ instead of x_0 , so we can assume that $a \geq 0$. Now, G is torsion-free, so if $a = 0$, then, for every $m \in \mathbb{Z} \setminus \{0\}$ and every $n \in \mathbb{N} \setminus \{0\}$, $\exp \left(im \frac{\theta + 2f(n)\pi}{n} \right) \neq 1$. This implies that $\theta \notin \mathbb{Q} \cdot \pi$. Conversely, if $a \neq 0$ or $\theta \notin \mathbb{Q} \cdot \pi$, then $\left(\exp \left(im \frac{\theta + 2f(n)\pi}{n} \right), \frac{m}{n} a \right) = (1, 0) \Rightarrow m = 0$.

Let m, n in $\mathbb{N} \setminus \{0\}$. Then $\varphi_1\left(\frac{x_0}{n}\right) = m\varphi_1\left(\frac{x_0}{mn}\right)$. Hence $\frac{\theta+2f(n)\pi}{n} - m\frac{\theta+2f(mn)\pi}{mn} \in 2\pi\mathbb{Z}$. So $f(n) - f(mn) \in n\mathbb{Z}$, where $0 \leq f(n) < n$ and $0 \leq f(mn) < mn$. Hence $f(n)$ is the remainder of the euclidean division of $f(mn)$ by n . This proves that f satisfies (**). \square

Lemma 4.21. *For every $\theta \in \mathbb{R}$, $0 \leq a \in \mathbb{Q}$, with $a = 0 \Rightarrow \theta \notin \mathbb{Q}\cdot\pi$, and every mapping f from $\mathbb{N} \setminus \{0\}$ in \mathbb{N} which satisfies (*) and (**), the set*

$$G = \left\{ \left(\exp\left(im \frac{\theta + 2f(n)\pi}{n} \right), \frac{m}{n} a \right) : m \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\} \right\}$$

is a subgroup of $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$, which is isomorphic to \mathbb{Q} in the language of groups.

Proof. Let m, m' in \mathbb{Z} , n, n' in $\mathbb{N} \setminus \{0\}$ be such that

$$\left(\exp\left(im \frac{\theta + 2f(n)\pi}{n} \right), \frac{m}{n} a \right) = \left(\exp\left(im' \frac{\theta + 2f(n')\pi}{n'} \right), \frac{m'}{n'} a \right).$$

Then $\left(\frac{m}{n} - \frac{m'}{n'}\right)\theta + 2\left(\frac{mf(n)}{n} - \frac{m'f(n')}{n'}\right)\pi \in 2\pi\mathbb{Z}$ and $\left(\frac{m}{n} - \frac{m'}{n'}\right)a = 0$. If $a \neq 0$, then $\frac{m}{n} = \frac{m'}{n'}$. If $a = 0$, then by hypothesis $\theta \notin \mathbb{Q}\pi$. Hence $\left(\frac{m}{n} - \frac{m'}{n'}\right)\theta = 0$, and $\frac{m}{n} = \frac{m'}{n'}$.

Therefore, there is a one-to-one mapping between \mathbb{Q} and G : for $m \in \mathbb{Z}$ and $n \in \mathbb{N} \setminus \{0\}$,

$$\varphi\left(\frac{m}{n}\right) = \left(\exp\left(im \frac{\theta + 2f(n)\pi}{n} \right), \frac{m}{n} a \right).$$

Since $\frac{m}{n} + \frac{m'}{n'} = \frac{mn' + m'n}{nn'}$, to prove that φ is an isomorphism in the language of groups, it suffices to prove that $\frac{(mn' + m'n)f(nn')}{nn'} \cdot 2\pi$ is congruent to $\frac{mn'f(n) + m'nf(n')}{nn'} \cdot 2\pi$ modulo 2π . By (**), the euclidean divisions of $f(nn')$ by n and by n' can be written as $f(nn') = qn + f(n)$ and $f(nn') = q'n' + f(n')$. Therefore

$$\begin{aligned} \frac{mn'f(n) + m'nf(n')}{nn'} \cdot 2\pi &= \frac{(mn' + m'n)f(nn') - nn'(mq + m'q')}{nn'} \cdot 2\pi \\ &= \frac{(mn' + m'n)f(nn')}{nn'} \cdot 2\pi - 2(mq + m'q')\pi. \end{aligned}$$

It follows that φ is an isomorphism in the language of groups, and so its image G is a subgroup of $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$. \square

Lemma 4.22. *There is a one-to-one correspondence between the set of mappings from $\mathbb{N} \setminus \{0\}$ in \mathbb{N} which satisfies (*) and (**), and the set of families $\{f_p : p \text{ a prime}\}$, where the f_p 's are mappings from $\mathbb{N} \setminus \{0\}$ to $\{0, \dots, p-1\}$.*

Proof. For every prime number p , let f_p be a mapping from \mathbb{N} in $\{0, \dots, p-1\}$, and let f be the mapping defined in Fact 4.17. Then f satisfies (*). Let $m = p_1^{r_1} \cdots p_k^{r_k}$ and $n = p_1^{s_1} \cdots p_k^{s_k}$ (where one of r_i or s_i can be equal to 0). We let $i \in \{1, \dots, k\}$. By the construction of $f(p_i^{r_i+s_i})$ and $f(p_i^{s_i})$, it follows that $p_i^{s_i}$ divides $f(p_i^{r_i+s_i}) - f(p_i^{s_i})$. Now, since $f(p_i^{r_i+s_i})$ is congruent to $f(mn)$ modulo $p_i^{r_i+s_i}$ and $f(p_i^{s_i})$ is congruent to $f(n)$ modulo $p_i^{s_i}$, we deduce that $f(mn) - f(n)$ is congruent to $f(p_i^{r_i+s_i}) - f(p_i^{s_i})$ modulo $p_i^{s_i}$. Therefore $p_i^{s_i}$ divides $f(mn) - f(n)$. Consequently, n divides $f(mn) - f(n)$. Since $f(n) < n$, we conclude that $f(n)$ is the remainder of the euclidean division of $f(mn)$ by n . This proves that f satisfies (**).

Conversely, let f be a mapping from $\mathbb{N} \setminus \{0\}$ in \mathbb{N} which satisfies (*) and (**). For every prime p we set $f_p(1) = f(p)$, and for $r \in \mathbb{N} \setminus \{0\}$ we set $f_p(r) = \frac{f(p^r) - f(p^{r-1})}{p^{r-1}}$. By (**), $f_p(r)$ is an integer (it is the quotient of the euclidean division of $f(p^r)$ by p^{r-1}). Now, $f(p^r) < p^r$, hence $\frac{f(p^r) - f(p^{r-1})}{p^{r-1}} < p$, so it belongs to $\{0, \dots, p-1\}$. By construction, $f(p^r) = f_p(1) + pf_p(2) + \cdots + p^{r-1}f_p(r)$. We show that $f(p_1^{r_1} \cdots p_k^{r_k})$ is the unique integer in $\{0, \dots, p_1^{r_1} \cdots p_k^{r_k} - 1\}$ which is congruent to every $f(p_i^{r_i})$ modulo $p_i^{r_i}$. Since $0 \leq f(p_1^{r_1} \cdots p_k^{r_k}) < p_1^{r_1} \cdots p_k^{r_k}$, it suffices to prove that every $p_i^{r_i}$ divides $f(p_1^{r_1} \cdots p_k^{r_k}) - f(p_i^{r_i})$. Now, this follows from (**). \square

Remark 4.23. Let (G, R) be a cyclically ordered group and p be a prime such that G/H_p is discrete. By Lemma 2.16, for every positive integers m, n , $f_{G,p}(n)$ is the remainder of the euclidean division of $f_{G,p}(m+n)$ by p^n and p doesn't divide $f_{G,p}(n)$. Hence in the same way as in the proof of Lemma 4.21 there is a function $f_p : \mathbb{N} \setminus \{0\} \rightarrow \{0, \dots, p-1\}$ such that for every $n \in \mathbb{N} \setminus \{0\}$, $f_{G,p}(n) = f_p(1) + pf_p(2) + \dots + p^{n-1}f_p(n)$. Since p doesn't divide $f_{G,p}(n)$, we have $f_p(1) \neq 0$.

Note that the linear part of the cyclically ordered group $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$ is $\{1\} \overrightarrow{\times} \mathbb{Q}$, so $U(\mathbb{U} \overrightarrow{\times} \mathbb{Q})$ is isomorphic to \mathbb{U} . The unwound of $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$ is isomorphic to $\mathbb{R} \overrightarrow{\times} \mathbb{Q}$. The convex subgroups of this unwound are $\{(0, 0)\}$, $\{0\} \overrightarrow{\times} \mathbb{Q}$, which is isomorphic to $l(\mathbb{U} \overrightarrow{\times} \mathbb{Q})_{uw}$, and $\mathbb{R} \overrightarrow{\times} \mathbb{Q}$. So, if G is a subgroup of $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$, then $\mathcal{CD}(G)$ contains at most three elements.

In the following we let G be a nonlinear subgroup of $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$, which is isomorphic to \mathbb{Q} in the language of groups, and $\theta \in \mathbb{R}$, $0 \leq a \in \mathbb{Q}$, $f : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$ as in Lemma 4.20, and f_p (p prime) be the family of functions defined in Lemma 4.22.

Remark 4.24. If $\theta \in \mathbb{Q}\pi$, then there is $\frac{m}{n} > 0$ in \mathbb{Q} such that $\varphi_1\left(\frac{m}{n}x_0\right) \in 2\pi\mathbb{Z}$. So, by taking $\frac{m}{n}x_0$ instead of x_0 in the proof of Lemma 4.20 we can assume that $\theta = 0$.

Proposition 4.25. The cyclically ordered group G is c -archimedean if, and only if, $\theta \notin \mathbb{Q}\pi$. If this holds, then it is c -isomorphic to a subgroup G' of $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$ such that $\theta' = \theta$, $f' = f$ and $a' = 0$.

Proof. Assume that $\theta \notin \mathbb{Q}\pi$. Then, $\exp\left(im\frac{\theta + 2f(n)\pi}{n}\right) = 1 \Leftrightarrow m\frac{\theta + 2f(n)\pi}{n} \in 2\pi\mathbb{Z}$. This implies $m\theta \in \mathbb{Q}\pi$, hence $m = 0$. It follows that $\exp\left(im\frac{\theta + 2f(n)\pi}{n}\right) = 1$ if, and only if, $m = 0$. Hence $l(G) = \{(1, 0)\}$ and (G, R) is c -archimedean. Since it is infinite, it is dense.

Assume that $\theta \in \mathbb{Q}\pi$. Recall that this implies $a > 0$. By Remark 4.24 we can assume that $\theta = 0$. Then $l(G)$ contains the $(1, \frac{ma}{n})$'s where $m\frac{f(n)}{n} \in \mathbb{Z}$. In particular, $l(G) \neq \{(1, 0)\}$, so G is not c -archimedean.

Now, assume that $\theta \notin \mathbb{Q}\pi$ and let G' be the subgroup of $\mathbb{U} \overrightarrow{\times} \{0\}$ generated by the set

$$\left\{ \left(\varphi_1\left(\frac{x_0}{n}\right), 0 \right) : n \in \mathbb{N} \setminus \{0\} \right\}.$$

Then $\theta' = \theta$ and $f' = f$. We proved above that φ_1 is one-to-one, hence the groups G and G' are isomorphic. Furthermore, since both of φ_1 and φ_2 are one-to-one, for any x_1, x_2 in \mathbb{Q} we have $(\varphi_1(x_1), \varphi_2(x_1)) \neq (\varphi_1(x_2), \varphi_2(x_2)) \Leftrightarrow x_1 \neq x_2$. Now, let x_1, x_2, x_3 in \mathbb{Q} such that $(\varphi_1(x_1), \varphi_2(x_1)) \neq (\varphi_1(x_2), \varphi_2(x_2)) \neq (\varphi_1(x_3), \varphi_2(x_3)) \neq (\varphi_1(x_1), \varphi_2(x_1))$. This equivalent to $\varphi_1(x_1) \neq \varphi_1(x_2) \neq \varphi_1(x_3) \neq \varphi_1(x_1)$. Hence $R((\varphi_1(x_1), \varphi_2(x_1)), (\varphi_1(x_2), \varphi_2(x_2)), (\varphi_1(x_3), \varphi_2(x_3)))$ holds in $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$ if, and only if, $R(\varphi_1(x_1), \varphi_1(x_2), \varphi_1(x_3))$ holds in \mathbb{U} . Therefore, G and G' are c -isomorphic. \square

By [10, Lemma 5.1, Theorem 5.3], there is one and only one c -embedding of each c -archimedean cyclically ordered group in \mathbb{U} . Hence if $\mathbb{Q}\frac{\theta}{\pi} \cap \mathbb{Q}\frac{\theta'}{\pi} = \emptyset$, then (G, R) and (G', R') are not c -isomorphic. Hence there are 2^{\aleph_0} non isomorphic c -archimedean cyclic orders on \mathbb{Q} . Now, since any c -archimedean cyclically ordered group is c -regular, by Theorem 2.12, all c -archimedean dense torsion-free divisible abelian cyclically ordered groups are elementarily equivalent.

Now, we turn to the non- c -archimedean case. By Lemma 4.20 and Remark 4.24 we can assume that $\theta = 0$ and $a > 0$.

Lemma 4.26. Assume that G is not c -archimedean, and let p be a prime. Then $H_p = l(G)$ if, and only if, f_p is the 0 mapping. Otherwise, $H_p = \{(1, 0)\}$.

Proof. Saying that $H_p = l(G)$ is equivalent to saying that $l(G)$ is a p -divisible group. This is turn is equivalent to saying that for every $r \in \mathbb{N}$ the element $(1, a)$ is p^r -divisible within $l(G)$, that is, there is $\frac{m}{n} \in \mathbb{Q}$ such that $\exp\left(im\frac{2f(n)\pi}{n}\right) = 1$ and $p^r\frac{m}{n}a = a$. Now, $\exp\left(im\frac{2f(n)\pi}{n}\right) = 1$ and $p^r\frac{m}{n}a = a \Leftrightarrow \frac{m}{n} = \frac{1}{p^r}$ and $\exp\left(i\frac{2f(p^r)\pi}{p^r}\right) = 1$. Since $0 \leq f(p^r) \leq p^r - 1$, this in turn is equivalent to $\frac{m}{n} = \frac{1}{p^r}$ and $f(p^r) = 0$. One can prove by induction that $\forall r \in \mathbb{N} \setminus \{0\} f(p^r) = 0$ is equivalent to

$\forall n \in \mathbb{N} \setminus \{0\} f_p(n) = 0$.

Since $l(G)$ is isomorphic to \mathbb{Q} , its only proper subgroup is $\{(1, 0)\}$, hence $H_p \neq l(G) \Leftrightarrow H_p = \{(1, 0)\}$. \square

Corollary 4.27. *For every subset \mathcal{P}' of the set \mathcal{P} of primes we can choose f so that, for every $p \in \mathcal{P}'$, $H_p = l(G)$, and, for every $p \in \mathcal{P} \setminus \mathcal{P}'$, $H_p = \{(1, 0)\}$. In particular, there are 2^{\aleph_0} non elementarily equivalent dense nonlinear cyclic orders on \mathbb{Q} .*

Proof. For every $p \in \mathcal{P}'$ we let f_p be the 0 mapping and for every $p \in \mathcal{P} \setminus \mathcal{P}'$ we let f_p be a function from $\mathbb{N} \setminus \{0\}$ to $\{0, \dots, p-1\}$ such that $f_p(1) \neq 0$. We let f be defined in the same way as in Lemma 4.22. By Lemma 4.26 for every $p \in \mathcal{P}$ we have $H_p = l(G)$ and for every $p \in \mathcal{P} \setminus \mathcal{P}'$ we have $H_p = \{(1, 0)\}$.

Therefore, if \mathcal{P}' is nonempty, then there is p such that $l(G)$ is p -divisible, so it is dense. So the cyclic order is dense. Since there are 2^{\aleph_0} nonempty subsets of \mathcal{P} , by Theorem 2 there are 2^{\aleph_0} pairwise non elementarily equivalent dense nonlinear cyclic orders on \mathbb{Q} . \square

The following provides a necessary and sufficient condition for G being discrete.

Proposition 4.28. (1) *G is discrete if, and only if, for every prime p the mapping f_p is not the zero mapping and there is only a finite number of primes p such that $f_p(1) = 0$. If this holds, then we can choose the real number a such that for every prime p we have $f_p(1) \neq 0$.*
 (2) *Assume that for every prime p we have $f_p(1) \neq 0$. Then, for every prime p and every $n \in \mathbb{N}$, $H_p = \{0\}$ and $f(p^n) = f_{G,p}(n)$.*
 (3) *For every family of functions $f_p : \mathbb{N} \setminus \{0\} \rightarrow \{0, \dots, p-1\}$ with $f_p(1) \neq 0$ (p prime), there is a discrete cyclically ordered group isomorphic to \mathbb{Q} such that the family of $f_{G,p}$ is constructed as in Remark 4.23. Therefore, there are 2^{\aleph_0} pairwise non elementarily equivalent discrete cyclic orders on \mathbb{Q} .*

Proof. (1) Assume that for every prime p we have $f_p(1) \neq 0$. Since for every prime p we have $f_p(1) \neq 0$, the integers $f(p) = f_p(1)$ and p are coprime. For every $r \geq 2$, $f(p^r)$ and p are coprime, since $f(p)$ is the remainder of the euclidean division of $f(p^r)$ by p^{r-1} . Let $p_1 < \dots < p_k$ be primes, r_1, \dots, r_k be positive integers, and $n = p_1^{r_1} \dots p_k^{r_k}$. Further, let u, v in \mathbb{Z} such that $uf(p_1^{r_1}) + vp_1 = 1$, and $q \in \mathbb{N}$ such that $f(n) = qp_1^{r_1} + f(p_1^{r_1})$. Then $1 = uf(n) + p_1(v - qp_1^{r_1-1})$. Therefore $f(n)$ and p_1 are coprime. The same holds with p_2, \dots, p_k , hence $f(n)$ and n are coprime.

Therefore, if $m \frac{2\pi f(n)}{n} \in 2\pi\mathbb{Z}$, then n divides m . Consequently, either $\frac{m}{n}a \leq 0$, or $\frac{m}{n}a \geq a$. It follows that (G, R) is discrete, with smallest positive element $(1, a)$. Conversely, assume that $(1, a)$ is the smallest element of $l(G)$, and let p be a prime. Since $(1, \frac{a}{p}) < (1, a)$, $l(G)$ doesn't contain the element $(1, \frac{a}{p})$ of $\mathbb{U} \times \mathbb{Q}$. Therefore, $e^{i \frac{2\pi f(p)}{p}} \neq 1$, which is equivalent to $f(p) \neq 0$.

Assume that G is discrete. Then for every prime p we have $H_p = \{(1, 0)\}$. By Lemma 4.26, the functions f_p are different from the zero mapping.

Assume that no function f_p is the zero mapping, but that there are infinitely many primes p such that $f_p(1) = 0$. Let n be a positive integer such that $(1, \frac{a}{n})$ belongs to the positive cone of $l(G)$. This is equivalent to $f(n) = 0$. Indeed, since $0 \leq f(n) \leq n-1$, $\frac{2f(n)\pi}{n} \in 2\mathbb{Z}\pi \Leftrightarrow f(n) = 0$. Now, we let p be a prime which does not divide n and such that $f_p(1) = 0$. Since $f(np)$ is congruent to $f(n)$ modulo n and to $f(p) = f_p(1)$ modulo p , we have $f(np) = 0$. Hence $(1, \frac{a}{np}) \in l(G)$, with $(1, 0) < (1, \frac{a}{np}) < (1, \frac{a}{n})$. Therefore $l(G)$ has no smallest positive element, and $l(G)$ is dense. This proves that G is dense.

It remains the case where there is only finitely many primes p_1, \dots, p_k such that, for $i \in \{1, \dots, k\}$, $f_{p_i}(1) = 0$. Since f_{p_i} is not the zero mapping, there is a greatest positive integer r_i such that $f_{p_i}(r_i) = 0$. Let p be a prime and n, r be positive integers such that p^r divides n . Since $f(p^r)$ is the remainder of the euclidean division of $f(n)$ by p^r , if $f(p^r) \neq 0$, then $f(n) \neq 0$. Set $n_0 = p_1^{r_1} \dots p_k^{r_k}$. By construction, we have $f(n_0) = 0$, hence $(1, \frac{a}{n_0}) \in l(G)$. If $\frac{m}{n}$ is a positive rational number such that $\frac{ma}{n} < \frac{a}{n_0}$, then $n > n_0$. Let $n > n_0$. If the primes which divide n belong to $\{p_1, \dots, p_k\}$, then there is $i \in \{1, \dots, k\}$ such that $p_i^{r_i+1}$ divides n . Therefore, $f(n) \neq 0$. If some p

not in $\{p_1, \dots, p_k\}$ divides n , then $f(n) \neq 0$, since $f(p) \neq 0$. So $(1, \frac{ma}{n}) \notin l(G)$. Consequently, $(1, \frac{a}{n_0})$ is the smallest element of $l(G)$. By taking $\frac{m}{n_0}$ instead of a , in the same way as above we get a new function f such that, for every prime p , $f(p) \neq 0$.

(2) By Lemma 4.26, for every prime p we have $H_p = \{0\}$. Recall that $f_{G,p}(n)$ is the element of $\{1, \dots, p^n - 1\}$ such that the element h such that $p^n h = 1_{G/H_p}$ satisfies $U(h) = \exp\left(im \frac{2f_{G,p}(n)\pi}{p^n}\right)$.

From the definition of f , we have $f_{G,p}(n) = f(p^n)$.

(3) We consider a family of functions $f_p : \mathbb{N} \setminus \{0\} \rightarrow \{0, \dots, p-1\}$ with $f_p(1) \neq 0$ (p prime). By Lemma 4.22 we can assume that the function f is generated by the family of f_p 's. Hence for every $n \in \mathbb{N} \setminus \{0\}$ $f(p^n) = f_{G,p}(n)$. Consequently, the family of $f_{G,p}$ is constructed as in Remark 4.23. This implies that there are 2^{\aleph_0} distinct families of functions $f_{G,p}$.

Let (G, R) , (G', R') be discrete cyclically ordered groups isomorphic to \mathbb{Q} such that for some prime p we have $f_{G,p} \neq f_{G',p}$. Then, by Theorem 2, (G, R) and (G', R') are not elementarily equivalent. Hence there are 2^{\aleph_0} pairwise non elementarily equivalent discrete cyclic orders on \mathbb{Q} . \square

Fact 4.29. *Let Γ be a linearly ordered divisible abelian group and (G, R) be a discrete cyclically ordered group which is isomorphic to \mathbb{Q} . Let (G', R') be the lexicographic product $G \overrightarrow{\times} \Gamma$. Then (G', R') is not discrete, for every prime p we have $H'_p = \{0_G\} \times \Gamma$, $G'/H'_p \simeq G \simeq G/H_p$ is discrete and $f_{G',p} = f_{G,p}$. Hence, if some family of functions is the family of $f_{G,p}$'s for some discrete divisible abelian cyclically ordered group, then it is also the family of $f_{G,p}$'s for some dense one.*

Proof of Theorem 3. (1) By [26, Theorem 5.10], Γ is cyclically orderable if, and only if $T(\Gamma)$ embeds in $T(\mathbb{U})$ and $\Gamma/T(\Gamma)$ is orderable. By [8, Corollary 5 on p. 36], any abelian torsion-free group admits a linear order. Now, since $T(\Gamma)$ is divisible, it embeds in $T(\mathbb{U})$ if, and only if, if either it is trivial or it is isomorphic to $T(\mathbb{U})$.

(2) Assume that $\Gamma \simeq T(\mathbb{U})$. By [9, Remark 5.4], every c-archimedean cyclically ordered group c-embeds in a unique way in \mathbb{U} . Hence all cyclic orders on Γ are c-isomorphic. We can construct by induction infinitely many group isomorphisms between $T(\mathbb{U})$ and Γ . For every prime p and positive integer n , we send $e^{2i\pi/p^n}$ to any of the $p-1$ primitive p -th root of the image of $e^{2i\pi/p^{n-1}}$ in Γ . So, this gives that 2^{\aleph_0} isomorphisms between $T(\mathbb{U})$ and Γ , each one gives rise to a cyclic order on Γ .

(4) Assume that Γ is torsion-free. Hence it is a \mathbb{Q} -vector space, and there are two divisible abelian subgroups Γ' and Γ'' such that $\Gamma = \Gamma' \oplus \Gamma''$ and Γ'' is isomorphic to \mathbb{Q} . Recall that any abelian torsion-free group admits a linear order. Hence we can assume that Γ' is linearly ordered. Now, let (G, R) be a cyclically ordered group which is isomorphic to \mathbb{Q} , then Γ is isomorphic to the cyclically ordered group $G' = G \overrightarrow{\times} \Gamma'$. In the same way as in Fact 4.29, if (G, R) is discretely cyclically ordered, then for every prime p we have $f_{G',p} = f_{G,p}$. Since there are 2^{\aleph_0} pairwise different families of $f_{G,p}$'s, there are 2^{\aleph_0} pairwise non elementarily equivalent cyclic orders on Γ .

Note that $H'_p = H_p \overrightarrow{\times} \Gamma'$, hence by Corollary 4.27 we can have 2^{\aleph_0} non isomorphic families of $\mathcal{CD}(G')$. Hence we can have 2^{\aleph_0} non pairwise elementarily equivalent nonlinear cyclic orders such that all the G'/G'_p are dense.

Above construction is not the only possible. For example, we can also send 1 to $e^{i\theta}$, where $\theta \notin 2\pi\mathbb{Z}$. For θ and θ' which are not congruent modulo $2\pi\mathbb{Z}$, we get non-c-isomorphic cyclic orders. So there uncountably many non-isomorphic such cyclic orders. If Γ is isomorphic to a countable product $\prod_{n \in \mathbb{N}} \Gamma_n$ of groups isomorphic to \mathbb{Q} , we can take embeddings of the Γ_n 's in $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$ such that their images have trivial intersections. Then Γ is isomorphic to the subgroup of $\mathbb{U} \overrightarrow{\times} \mathbb{Q}$ generated by these images.

(3) Since $\Gamma/T(\Gamma)$ is abelian torsion-free and divisible, it is a \mathbb{Q} -vector space. Furthermore, its dimension is positive, since Γ contains non-torsion elements. We let \mathcal{B} be a basis of $\Gamma/T(\Gamma)$ and \mathcal{B}' be a subset of Γ such that the restriction of the canonical epimorphism $\Gamma \rightarrow \Gamma/T(\Gamma)$ induces a one-to-one mapping between \mathcal{B}' and \mathcal{B} . Then, the subspace Γ' generated by \mathcal{B}' is a divisible abelian subgroup of Γ which is isomorphic to $\Gamma/T(\Gamma)$. One can check that $\Gamma = \Gamma' \oplus T(\Gamma)$. We let $\Gamma' = \Gamma'_1 \oplus \Gamma'_2$, where $\dim(\Gamma'_2) = 1$ (hence $\Gamma'_2 \simeq \mathbb{Q}$). We embed Γ'_2 in \mathbb{U} . Since $\Gamma'_2 \cap T(\Gamma) = \{0_\Gamma\}$,

this gives rise to a cyclic order on $\Gamma'_2 \oplus T(\Gamma)$. So the lexicographic product $(\Gamma'_2 \oplus T(\Gamma)) \overrightarrow{\times} \Gamma'_1$ is a cyclically group which is isomorphic to Γ . Now, we can embed Γ'_2 in uncountably many in ways \mathbb{U} so to get non-isomorphic cyclic orders. Hence this gives rise to 2^{\aleph_0} non-isomorphic cyclic orders on Γ . Now, since Γ is divisible and has torsion elements, it is c-divisible (Proposition 2.6). Hence by Theorem 2.12, all these cyclic orders are elementarily equivalent. \square

4.6. The families $\mathcal{CD}(G)$. Note that $\mathcal{CD}(G)$ induces an equivalence relation on the set of all primes, where p and q are equivalent if, and only if, $H_p = H_q$. Hence it induces a partition of this set. This partition is linearly ordered in the following way. We set $\alpha < \beta$ if, and only if, for $p \in \alpha$ and $q \in \beta$ we have $H_p \subsetneq H_q$.

If $H_p = \{0_G\}$ for every prime p , then the partition is trivial; this holds if (G, R) is discrete.

Proposition 4.30. (1) *Every linearly ordered partition of the set of all prime numbers is induced by some $\mathcal{CD}(G)$, where (G, R) is a dense divisible abelian nonlinear cyclically ordered group.*

If this chain has a smallest element α_1 , then we can assume that, for $q \in \alpha_1$, $H_q = \{0_G\}$ or not. If this partition has a greatest element α_0 , then we can assume that for $p \in \alpha_0$ we have $H_p = l(G)$, so G/H_p is dense.

(2) *For every linearly ordered partition A of the set of all prime numbers which has a greatest element α_0 and for every family of functions f_p from $\mathbb{N} \setminus \{0\}$ to $\{0, \dots, p-1\}$ ($p \in \alpha_0$) with $f_p(1) \neq 0$, there is a divisible abelian nonlinear cyclically ordered group (G, R) such that A is the partition induced by the H_p 's, for every $p \in \alpha_0$ G/H_p is discrete and for every $p \in \alpha_0$, $r \in \mathbb{N} \setminus \{0\}$ we have $f_{G,p}(r) = f_p(1) + pf_p(2) + \dots + p^{r-1}f_p(r)$.*

Proof. In Proposition 4.28 we constructed examples where the partition contains only one class, $H_p = \{0_G\}$ for every prime p , and (G, R) either dense or discrete. In the case where it is discrete, we also proved that the family of functions $f_{G,p}$ can be any family satisfying the required conditions. Corollary 4.27, gives an example where the partition contains only one class, $H_p = l(G)$ for every prime p . In the following we assume that the partition contains at least two classes.

(1) Let (A, \leq) be an ordered partition of the set of all primes, which contains at least two classes.

(a) Construction of the unwound (Γ, z) . For $\alpha \in A$ we denote by \mathbb{Q}_α the subgroup of \mathbb{Q} generated by $\left\{ \frac{1}{p^n} : p \notin \alpha, n \in \mathbb{N} \right\}$. The group \mathbb{Q}_α is p divisible if, and only if, $p \notin \alpha$, and if $p \in \alpha$, then 1 is not p -divisible. For every prime p we let $\alpha(p)$ denote the unique $\alpha \in A$ such that $p \in \alpha$.

We denote by $\overleftarrow{\prod}_{\alpha \in A} \mathbb{Q}_\alpha$ the additive group $\prod_{\alpha \in A} \mathbb{Q}_\alpha$ together with the inverse lexicographic order. The nontrivial convex subgroups of $\overleftarrow{\prod}_{\alpha \in A} \mathbb{Q}_\alpha$ have the form

$$\overleftarrow{\prod}_{\alpha \in A_1} \mathbb{Q}_\alpha \overleftarrow{\times} \overleftarrow{\prod}_{\alpha \in A_2} \{0\},$$

where $A_1 < A_2$ is an ordered partition of A . Now, such a group is p divisible if, and only if, $\alpha(p) \in A_2$. So the family of maximal p -divisible convex subgroups of $\overleftarrow{\prod}_{\alpha \in A} \mathbb{Q}_\alpha$ induces the ordered partition (A, \leq) .

Let $z = (z_\alpha)_{\alpha \in A}$ be the element of $\overleftarrow{\prod}_{\alpha \in A} \mathbb{Q}_\alpha$ such that for every α we have $z_\alpha = 1$. Let $\alpha_0 \in A$. The element $x = (x_\alpha)_{\alpha \in A}$ such that $x_{\alpha_0} = 1$ and, for $\alpha \neq \alpha_0$, $x_\alpha = 0$ is denoted by 1_{α_0} .

The *support* of an element $x = (x_\alpha)_{\alpha \in A}$ is the set of $\alpha \in A$ such that $x_\alpha \neq 0$ (the support of 0 is \emptyset).

For $\alpha \in A$, we let Δ_α be the subgroup of elements x of $\overleftarrow{\prod}_{\alpha \in A} \mathbb{Q}_\alpha$ such that for every $\alpha' \neq \alpha$ we have $x_{\alpha'} = 0$. Then $\sum_{\alpha \in A} \Delta_\alpha$ is the subgroup of all elements of $\overleftarrow{\prod}_{\alpha \in A} \mathbb{Q}_\alpha$ with finite support. We denote by Γ_0 the subgroup generated by z and the $\frac{1}{p^n}(z - 1_{\alpha(p)})$'s, where p is prime, $n \in \mathbb{N} \setminus \{0\}$. We set $\Gamma = \Gamma_0 + \sum_{\alpha \in A} \Delta_\alpha$. Note that z is cofinal in Γ . We show that it is not divisible by any prime.

The group $\sum_{\alpha \in A} \Delta_\alpha$ is generated by the $\frac{1}{m}1_\alpha$'s, where, for $p \in \alpha$, the integers p and m are coprime. So, if $x = \frac{1}{m}1_{\alpha(p)}$, then the denominator of $x_{\alpha(p)}$ is not divisible by p . If $x = \frac{1}{p^n}(z - 1_{\alpha(p)})$,

then $x_{\alpha(p)} = 0$, and $x_\alpha = \frac{1}{p^n}$ for $\alpha \neq \alpha(p)$. Now, the group Γ is generated by z , the $\frac{1}{p^n}(z - 1_{\alpha(p)})$'s and the $\frac{1}{m}1_\alpha$'s, where m and p are coprime. Hence, for any $x \in \Gamma$ and p prime, if $x_{\alpha(p)} \neq 0$, then its denominator is not divisible by p . In particular, z is not divisible by any prime.

(b) We let $G = \Gamma / \langle z \rangle$. This group is torsion-free, since z is not divisible by any prime. We show that the cyclically ordered group G is nonlinear. We saw after the definition of the wound-round (Subsection 2.1) that if G is linearly cyclically ordered, then it is isomorphic to $(\mathbb{Z} \overrightarrow{\times} G) / \langle (1, 0_G) \rangle$, where $\overrightarrow{\times}$ denotes the lexicographic product of linearly ordered groups. By uniqueness of the unwound, it follows that G is linearly cyclically ordered if, and only if, Γ is order isomorphic to $\mathbb{Z} \overrightarrow{\times} l(G)_{uw}$, where $l(G)_{uw}$ is the greatest proper convex subgroup of Γ . Therefore, G is linearly ordered if, and only if, for every cofinal $x \in \Gamma$ there exists a positive integer n such that $x - nz$ belongs to the greatest proper convex subgroup of Γ . If A is infinite, then the greatest convex subgroup of Γ is $\sum_{\alpha \in A} \Delta_\alpha$. For every prime p , $\frac{1}{p}(z - 1_{\alpha(p)})$ is cofinal in Γ . Now, for every positive integer n , $\frac{1}{p}(z - 1_{\alpha(p)}) - nz$ does not belong to $\sum_{\alpha \in A} \Delta_\alpha$, since it has an infinite support. Therefore G is nonlinear. If A is finite, then it has a greatest element α_0 . Then $\Gamma = \bigoplus_{\alpha \in A} \Delta_\alpha$ and its greatest convex subgroup is $\bigoplus_{\alpha \neq \alpha_0} \Delta_\alpha$. We let p be a prime not in α_0 . So $\frac{1}{p}(z - 1_{\alpha(p)})$ is cofinal in Γ . Now, for every positive integer n , $\frac{1}{p}(z - 1_{\alpha(p)}) - nz$ does not belong to $\bigoplus_{\alpha \neq \alpha_0} \Delta_\alpha$. Consequently, G is nonlinear.

(c) We show that G is divisible by proving that for every prime p and every $n \in \mathbb{N} \setminus \{0\}$, each generator of Γ is divisible by $\frac{1}{p^n}$ modulo $\langle z \rangle$. We start with $\frac{1}{m}1_\alpha$ where, for every $p \in \alpha$, m and p are coprime. Clearly, if $p \notin \alpha$, then $\frac{1}{m}1_\alpha$ is divisible by p^n . Assume that $p \in \alpha$. We have

$$1_\alpha = p^n \left(1_\alpha + (p^n - 1) \frac{1}{p^n} (z - 1_\alpha) \right) + (1 - p^n) z.$$

Therefore 1_α is divisible by p^n modulo $\langle z \rangle$. Now, since m and p^n are coprime, there exist integers u and v such that $up^n + vm = 1$. Then $\frac{1}{m}1_\alpha = v1_\alpha + p^n \frac{u}{m}1_\alpha$. Hence $\frac{1}{m}1_\alpha$ is divisible by p^n modulo $\langle z \rangle$.

We turn to $\frac{1}{q^m}(z - 1_{\alpha(q)})$. If $p = q$, then $\frac{1}{q^m}(z - 1_{\alpha(q)}) = p^n \frac{1}{p^{m+n}}(z - 1_{\alpha(p)})$. Assume that p, q are coprime and let u, v be integers such that $up^n + vq^m = 1$. Then,

$$\begin{aligned} & p^n \left(\frac{u}{q^m} (z - 1_{\alpha(q)}) + \frac{v}{p^n} (z - 1_{\alpha(p)}) \right) + v(1_{\alpha(p)} - 1_{\alpha(q)}) = \\ &= \frac{1}{q^m} ((up^n + vq^m)z - up^n 1_{\alpha(q)} - vq^m 1_{\alpha(p)}) + \frac{1}{q^m} (vq^m 1_{\alpha(p)} - vq^m 1_{\alpha(q)}) = \\ &= \frac{1}{q^m} (up^n + vq^m) (z - 1_{\alpha(q)}) = \frac{1}{q^m} (z - 1_{\alpha(q)}). \end{aligned}$$

Since $1_{\alpha(p)}$ and $1_{\alpha(q)}$ are divisible by p^n modulo $\langle z \rangle$, this proves that $\frac{1}{q^m}(z - 1_{\alpha(q)})$ is divisible by p^n modulo $\langle z \rangle$.

Since the unwound Γ of G is dense, G is dense.

(d) Now, we look at the subgroups $(H_p)_{uw}$ of Γ . Since Γ is a subgroup of $\prod_{\alpha \in A} \mathbb{Q}_\alpha$, $(H_p)_{uw}$ is the set of $x \in \Gamma$ such that $\alpha \geq \alpha(p) \Rightarrow x_\alpha = 0$.

Since $\mathcal{CD}(G)$ is isomorphic to the preordered family $\{(0, 0_G)\}, uw(G), (H_p)_{uw} : p \text{ prime}\}$, it defines the ordered partition A .

(e) If the partition has a smallest element α_1 , then for $p \in \alpha_1$ we have $H_p = \{0_G\}$. Then we can take the lexicographical product $G' = G \overrightarrow{\times} \mathbb{Q}$. The partition is the same, but it contains a divisible c-convex proper subgroup. Hence for every prime p , $H'_p \neq \{0\}$.

If the partition has no greatest element, then every G/H_p is dense, since there is no maximal H_p . Assume that the partition has a greatest element α_0 . Then, for $p \in \alpha_0$ the greatest p -divisible convex subgroup of Γ is equal to the subset of elements x such that $x_{\alpha_0} = 0$. Now, this is also the greatest convex subgroup of Γ , and we know that it is isomorphic to $l(G)$. Therefore, $H_p = l(G)$, hence $G/H_p \simeq U(G)$ is divisible, so it is dense.

(2) Let A be a linearly ordered partition of the set of all prime numbers which has a greatest element α_0 and a family of functions f_p from $\mathbb{N} \setminus \{0\}$ to $\{0, \dots, p-1\}$ ($p \in \alpha_0$) with $f_p(1) \neq 0$. For every prime $p \notin \alpha_0$, we set $f_p(1) = 1$, and for every $r > 1$ we set $f_p(r) = 0$. We let f be the function defined in the same way as in the proof of Lemma 4.22. Denote by G_0 the cyclically ordered group

$$\left\{ \left(\exp \left(im \frac{2f(n)\pi}{n} \right), \frac{m}{n} \right) : m \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\} \right\}$$

defined in Lemma 4.21. By Proposition 4.28, G_0 is discrete. By Proposition 4.25, it is not c-archimedean. Since it embeds in $\mathbb{U} \times \mathbb{Q}$, its linear part is isomorphic to a discrete subgroup of \mathbb{Q} . Hence $l(G_0) \simeq \mathbb{Z}$.

(a) Construction of the unwound. We embed the group $\text{uw}(G_0)$ in its divisible closure, and we let Δ'_{α_0} be the subgroup generated by $\text{uw}(G_0)$ and the $\frac{1}{p^n} z_{G_0}$'s, where $p \notin \alpha_0$ is a prime and $n \in \mathbb{N} \setminus \{0\}$. The greatest convex subgroup $l(G_0)_{\text{uw}}$ of $\text{uw}(G_0)$ is isomorphic to $l(G_0)$, so to \mathbb{Z} . Now, z_{G_0} does not belong to this subgroup. Hence the greatest convex subgroup of Δ'_{α_0} is isomorphic to \mathbb{Z} . It follows that Δ'_{α_0} is discrete. We take $\prod_{\alpha \in A \setminus \{\alpha_0\}} \mathbb{Q}_\alpha \times \Delta'_{\alpha_0}$, instead of $\prod_{\alpha \in A} \mathbb{Q}_\alpha$.

For every $\alpha \in A$, we let Δ_α the subgroup of elements x such that $x_{\alpha'} = 0$ for every $\alpha' \neq \alpha$. If $\alpha \neq \alpha_0$, then 1_α is defined as in the first part of this proof. We let 1_{α_0} be the element x such that $\alpha \neq \alpha_0$ implies $x_\alpha = 0$, and $x_{\alpha_0} = z_{G_0}$. We have $\Delta_{\alpha_0} \simeq \Delta'_{\alpha_0}$ and, for $\alpha \neq \alpha_0$, $\Delta_\alpha \simeq \mathbb{Q}_\alpha$.

Let $z = (z_\alpha)_{\alpha \in A}$ the element such that for every $\alpha \neq \alpha_0$ we have $z_\alpha = 1$, and $z_{\alpha_0} = z_{G_0}$. Note that in Δ'_{α_0} the element z_{G_0} is not divisible by any $p \in \alpha_0$, and it is divisible by every p^n such that $p \notin \alpha_0$, and $n \in \mathbb{N} \setminus \{0\}$. We denote by Γ_0 the subgroup generated by z and the $\frac{1}{p^n} (z - 1_{\alpha(p)})$'s, where p is prime, $n \in \mathbb{N} \setminus \{0\}$, and $\alpha(p)$ denotes the unique $\alpha \in A$ such that $p \in \alpha(p)$. We set $\Gamma = \Gamma_0 + \sum_{\alpha \in A} \Delta_\alpha$.

(b) Since z is cofinal in Γ , we can consider the cyclically ordered group $G = \Gamma / \langle z \rangle$. In the same way as in (1) (b), G is nonlinear. We show that G is torsion-free and divisible.

In the same way as in (1) (a), z is not divisible by any integer in Γ , so G is torsion-free. Furthermore, for $\alpha \in A$, the elements $\frac{1}{m} 1_\alpha$ and $\frac{1}{q^m} (z - 1_{\alpha(q)})$ are divisible by every $\frac{1}{p^n}$ modulo $\langle z \rangle$. In order to prove that G is divisible, it remains to look at the elements of Δ_{α_0} . Note that Δ_{α_0} is generated by the $\frac{1}{m} 1_{\alpha_0}$ (where the gcd of m and every $p \in \alpha_0$ is 1) and the elements x such that $x_{\alpha_0} \in \text{uw}(G_0)$ and for $\alpha \neq \alpha_0$ we have $x_\alpha = 0$. So it is sufficient to focus on those x . Since $\text{uw}(G_0)$ is divisible modulo $\langle z_{G_0} \rangle$, there is $y \in \Delta_{\alpha_0}$ and $k \in \mathbb{Z}$ such that $p^n y_{\alpha_0} = x_{\alpha_0} + k z_{G_0}$. Then $p^n y = x + k \cdot 1_{\alpha_0}$. Since 1_{α_0} is divisible by p^n modulo $\langle z \rangle$, so is x .

(c) We look at the maximal p -divisible convex subgroups Γ_p of Γ . If $p \notin \alpha_0$, then they are constructed in the same way as in (1). If $p \in \alpha_0$, then in the same way as in (1) we see that $\Gamma_p \supseteq \Gamma \cap \prod_{\alpha \in \setminus \{\alpha_0\}} \mathbb{Q}_\alpha \times \{0_G\}$. We prove that this inclusion is an equality. Since G_0 is not c-archimedean, $\text{uw}(G_0)$ is not archimedean. Since z_{G_0} is cofinal in $\text{uw}(G_0)$, so are the $\frac{1}{p^n} z_{G_0}$. Therefore, the greatest proper convex subgroup of Δ'_{α_0} is $l(G_0)_{\text{uw}}$, which is isomorphic to \mathbb{Z} . It follows that the greatest proper convex subgroup of Γ is $\Gamma \cap \prod_{\alpha \in \setminus \{\alpha_0\}} \mathbb{Q}_\alpha \times l(G_0)_{\text{uw}}$, and there is no convex subgroup between $\Gamma \cap \prod_{\alpha \in \setminus \{\alpha_0\}} \mathbb{Q}_\alpha \times \{0_G\}$ and $\Gamma \cap \prod_{\alpha \in \setminus \{\alpha_0\}} \mathbb{Q}_\alpha \times l(G_0)_{\text{uw}}$. Since $l(G_0)$ is discrete, the greatest proper convex subgroup is not divisible by any prime. So above inequality is an equality.

(d) Now, $\Gamma = \text{uw}(G)$, hence the greatest proper convex subgroup $l(G)_{\text{uw}}$ of Γ is isomorphic to $l(G)$. Consequently, the partition induced by G on the set of prime numbers is A . Let $p \in \alpha_0$, $q \notin \alpha_0$ and $n \in \mathbb{N} \setminus \{0\}$. Then $\frac{1}{p^n} (z - 1_{\alpha_0}) \in \Gamma_p$ and $\frac{1}{q^n} (z - 1_{\alpha(q)}) - \frac{1}{q^n} \cdot 1_{\alpha_0} \in \Gamma_p$. Clearly, z and the elements of $\sum_{\alpha \in A} \Delta_\alpha$ are congruent modulo Γ_p to an element of Δ_{α_0} . It follows that every element of Γ is congruent modulo Γ_p to an element of Δ_{α_0} . Therefore, $\Gamma / \Gamma_p \simeq \Delta'_{\alpha_0}$ is discrete.

(e) Finally, we turn to the functions $f_{G,p}$, where $p \in \alpha_0$. By the definition of the group G_0 , we have $f_{G_0,p} = f_p$. Let x be the element of Δ_{α_0} such that x_{α_0} is the smallest positive element of Δ'_{α_0} . Then the class $x + \Gamma_p$ is the smallest positive element of Γ / Γ_p . Let $r \in \mathbb{N} \setminus \{0\}$. By Lemma

4.14 there is $y' \in \text{uw}(G_0)$ such that $x_{\alpha_0} = p^r y' + f_p(r)z_{G_0}$. We denote by y the element of Δ_{α_0} such that $y_{\alpha_0} = y'$. Then $x = p^r y + f_p(r)1_{\alpha_0}$. So $x = p^r y - f_p(r)(z - 1_{\alpha_0}) + f_p(r)z$. Since $p \in \alpha_0$, $t = \frac{1}{p^r}(z - 1_{\alpha_0})$ belongs to Γ . Therefore $x = p^r(y - f_p(r)t) + f_p(r)z$. By Lemma 4.14 again, this shows that $f_p(r)$ is congruent to $f_{G,p}(r)$ modulo p^r . Now, since both of $f_p(r)$ and $f_{G,p}(r)$ belong to $\{0, \dots, p^r - 1\}$, we have $f_p(r) = f_{G,p}(r)$. This proves that the function $f_{G,p}$ is equal to f_p . \square

5. CYCLICALLY MINIMAL CYCLICALLY ORDERED GROUPS.

In the case of real numbers, each definable subset is a finite union of intervals, it is definable by a quantifier-free formula in the language of order. The general study of linearly ordered algebraic structures having such a property has been done by A. Pillay and C. Steinhorn as o-minimal structures ([15]). Afterwards M. Dickmann introduced the notion of weakly o-minimal structures ([7]). In a very large context D. Macpherson and S. Steinhorn looked at analogues ([14]). In the case of cyclically ordered structures the analogue is the notion of cyclically minimal structures. They proved that a cyclically ordered group (G, R) is cyclically minimal if, and only if, it is abelian and its unwound is divisible. Lucas proved independently this theorem. He deduced from Section 3 that this condition is sufficient. He proved the converse in several lemmas, which also characterized the weakly cyclically minimal structures. Some of these lemmas contained errors, so we had to make changes in the proof of Lucas.

Since the groups are not necessarily abelian, we take here the multiplicative notation for the group law, however we speak of divisible group and of elements divisible by n (with this convention x is divisible by n if $\exists y \ y^n = x$).

Proposition 5.1. *Each c-divisible abelian cyclically ordered group is cyclically minimal.*

Proof. First, note that a c-divisible abelian cyclically ordered group is infinite since its torsion subgroup is isomorphic to $T(\mathbb{U})$. In [14], D. Macpherson and C. Steinhorn asserted that this follows from the minimality of the divisible abelian linearly ordered groups and the interpretability of a cyclically ordered group in its unwound. This interpretability has been studied more in detail in [9, Lemma 4.3]. Lucas asserted that this result can be obtained using the elimination of quantifiers in c-divisible abelian cyclically ordered groups (Theorem 1). We prove this assertion, that is we show that any quantifier-free formula defines a finite union of intervals and singletons. First, note that a c-divisible abelian cyclically ordered group is infinite, since its torsion subgroup is. A quantifier-free formula is a boolean combination of formulas such that $ax^n = b$ and $R(ax^m, bx^n, cx^p)$. Since, $ax^n = b \Leftrightarrow a = bx^{-n}$, we can assume that $n \geq 1$. The formula $ax^n = b$ is equivalent to $x^n = ba^{-1}$, and it defines the set n -th roots of ba^{-1} , which contain n elements, by properties of the subgroup of torsion elements of a c-divisible abelian cyclically ordered group. If $m = n = p$, then $R(ax^m, bx^n, cx^p)$ is equivalent to $R(a, b, c)$. If $m = n \neq p$, then $R(ax^m, bx^n, cx^p)$ is equivalent to $R(bc^{-1}, x^{p-n}, ac^{-1})$. The cases $m \neq n = p$ and $n \neq p = m$ are similar. If $m \neq n \neq p \neq m$, and $m = \min(m, n, p)$, then $R(ax^m, bx^n, cx^p)$ is equivalent to $R(ab^{-1}, x^{n-m}, cb^{-1}x^{p-m})$, where $n - m > 0$ and $p - m > 0$. The cases $n = \min(m, n, p)$ and $p = \min(m, n, p)$ are similar. So, it remains to prove that the formulas $R(a, x^n, b)$ and $R(a, x^m, bx^n)$ (where in the second case m, n are positive integers) define unions of open intervals.

We let y be the element of the unwound $\Gamma = \text{uw}(G)$ such that x is the image of y in $G = \Gamma / \langle z_G \rangle$ and $e_\Gamma \leq y < z_G$. We also denote by a' (resp. b') the element of $[e_\Gamma, z_G[$ such that its image is a (resp. b). Assuming $n \geq 1$, we let $a^{1/n}$ (resp. $b^{1/n}$, resp. ζ_n) be the image of $(a')^{1/n}$ (resp. $(b')^{1/n}$, resp. $z_G^{1/n}$). Assume that $n \geq 1$. By properties of the unwounds $R(a, x^n, b)$ holds if, and only if, there is $k \in \mathbb{Z}$ such that $e_\Gamma \leq a' < y^n z_G^{-k} < b' < z_G$, or $e_\Gamma \leq y^n z_G^{-k} < b' < a' < z_G$ or $e_\Gamma \leq b' < a' < y^n z_G^{-k} < z_G$. This in turn is equivalent to $(a')^{1/n} z_G^{k/n} < y < (b')^{1/n} z_G^{k/n}$, or $z_G^{k/n} \leq y < (b')^{1/n} z_G^{k/n} < (a')^{1/n} z_G^{k/n}$ or $(b')^{1/n} z_G^{k/n} < (a')^{1/n} z_G^{k/n} < y < z_G^{(1+k)/n}$. Note that $e_\Gamma \leq y < z_G$ and $e_\Gamma < y^n z_G^{-k} < z_G$ imply $0 \leq k \leq n - 1$. It follows that $R(a, x^n, b)$ is equivalent to $\bigvee_{0 \leq k \leq n-1} R(a^{1/n} \zeta_n^k, x, b^{1/n} \zeta_n^k)$, which defines a finite union of intervals.

Assume that $n < 0$. Then $R(a, x^n, b) \Leftrightarrow R(b^{-1}, (x^{-1})^{-n}, a^{-1})$. This is equivalent to saying that x^{-1} belongs to a finite union of open intervals. Now, $x^{-1} \in I(c, d) \Leftrightarrow x \in I(d^{-1}, c^{-1})$. Hence $R(a, x^n, b)$ defines a finite union of open intervals.

We assume that $m > n$. The formula $R(a, x^m, bx^n)$ holds if, and only if, there are integers k, l such that $e_\Gamma \leq a' < y^m z_G^{-k} < b'y^n z_G^{-l} < z_G$ or $e_\Gamma \leq y^m z_G^{-k} < b'y^n z_G^{-l} < a' < z_G$ or $e_\Gamma \leq b'y^n z_G^{-l} < a' < y^m z_G^{-k} < z_G$.

In the same way as above, this implies $0 \leq k \leq m-1$, and $e_\Gamma \leq y < z_G$ and $e_\Gamma < b'y^n z_G^{-l} < z_G$ imply $0 \leq l \leq n$.

- $e_\Gamma \leq a' < y^m z_G^{-k} < b'y^n z_G^{-l} < z_G$ is equivalent to

$$(a')^{1/m} z_G^{k/m} < y, y < (b')^{1/(m-n)} z_G^{(k-l)(m-n)} \text{ and } y < (b')^{-1/n} z_G^{(l+1)/n}.$$

We let $c'_{k,l} = \min\left((b')^{1/(m-n)} z_G^{(k-l)(m-n)}, (b')^{-1/n} z_G^{(l+1)/n}\right)$ and $c_{k,l} = c'_{k,l} \cdot \langle z_G \rangle$.

If $l < n$, then $(b')^{-1/n} z_G^{(l+1)/n} < z_G$, and if $l = n$, then $(b')^{1/(m-n)} z_G^{(k-n)(m-n)} < z_G$. It follows that $c'_{k,l} < z_G$. Hence $e_\Gamma \leq a' < y^m z_G^{-k} < b'y^n z_G^{-l} < z_G$ is equivalent to $R(a^{1/m} \zeta_m^k, x, c_{k,l})$.

- $e_\Gamma \leq y^m z_G^{-k} < b'y^n z_G^{-l} < a' < z_G$ is equivalent to $y = z_G^{k/m}$ or:

$$z_G^{k/m} < y, y < (b')^{1/(m-n)} z_G^{(k-l)(m-n)} \text{ and } y < (a')^{1/n} (b')^{-1/n} z_G^{l/n}.$$

We let $d'_{k,l} = \min\left((b')^{1/(m-n)} z_G^{(k-l)(m-n)}, (a')^{1/n} (b')^{-1/n} z_G^{l/n}\right)$ and $d_{k,l} = d'_{k,l} \cdot \langle z_G \rangle$.

If $l < n$, then $(a')^{1/n} (b')^{-1/n} z_G^{l/n} < z_G$, and if $l = n$, then $(b')^{1/(m-n)} z_G^{(k-n)(m-n)} < z_G$. Therefore $d'_{k,l} < z_G$. Hence $e_\Gamma \leq y^m z_G^{-k} < b'y^n z_G^{-l} < a' < z_G$ is equivalent to $x = \zeta_m^k$ or $R(\zeta_m^k, x, d_{k,l})$.

- $e_\Gamma \leq b'y^n z_G^{-l} < a' < y^m z_G^{-k} < z_G$ is equivalent to $y = (b')^{-1/n} z_G^{l/n}$ or:

$$(b')^{-1/n} z_G^{l/n} < y, y < (a')^{1/n} (b')^{-1/n} z_G^{l/n}, (a')^{1/m} z_G^{k/m} < y \text{ and } y < z_G^{(k+1)/m}.$$

We let $r'_{k,l} = \max\left((b')^{-1/n} z_G^{l/n}, (a')^{1/m} z_G^{k/m}\right)$ and $r_{k,l} = r'_{k,l} \cdot \langle z_G \rangle$.

If $k < m-1$, then $z_G^{(k+1)/m} < z_G$, and if $l < n$, then $(a')^{1/n} (b')^{-1/n} z_G^{l/n} < z_G$. So, if $k < m-1$ or $l < n$, then we set $s'_{k,l} = \min\left((a')^{1/n} (b')^{-1/n} z_G^{l/n}, z_G^{(k+1)/m}\right)$, and $r_{k,l} = r'_{k,l} \cdot \langle z_G \rangle$. We have $s'_{k,l} < z_G$. Then $e_\Gamma < b'y^n z_G^{-l} < a' < y^m z_G^{-k} < z_G$ is equivalent to $R(r_{k,l}, x, s_{k,l})$. If $k = m-1$ and $l = n$, then $z_G^{(k+1)/m} = z_G$. $e_\Gamma < b'y^n z_G^{-n} < a' < y^m z_G^{1-m} < z_G$ is equivalent to $r'_{k,l} < y < (a')^{1/n} (b')^{-1/n} z_G^{l/n}$, but we don't know whether $(a')^{1/n} (b')^{-1/n} z_G^{n/n} < z_G$ or not. This inequality is equivalent to $a' < b'$. If $a' < b'$, then we set $s'_{m,n} = (a')^{1/n} (b')^{-1/n} z_G$ and $s_{m,n} = s'_{m,n} \cdot \langle z_G \rangle$. Therefore $e_\Gamma < b'y^n z_G^{-n} < a' < y^m z_G^{1-m} < z_G$ is equivalent to $R(r_{m,n}, x, s_{m,n})$. If $a' \geq b'$, then $e_\Gamma < b'y^n z_G^{-n} < a' < y^m z_G^{1-m} < z_G$ is equivalent to $r'_{m,n} < y < z_G$, which in turn is equivalent to $R(r_{m,n}, x, e_G)$. We set $s_{m,n} = e_G$.

Consequently, $R(a, x^m, bx^n)$ is equivalent to the disjunction for $k \in \{0, \dots, m-1\}$ and $l \in \{0, \dots, n\}$ of: $R(a^{1/m} \zeta_m^k, x, c_{k,l})$ or $x = \zeta_m^k$ or $R(\zeta_m^k, x, d_{k,l})$ or $R(r_{k,l}, x, s_{k,l})$.

Assume that $m < n$. Then $R(a, x^m, bx^n) \Leftrightarrow R(ba^{-1}, (x^{-1})^n, b(x^{-1})^m)$. We proved above that this is equivalent to saying that x^{-1} belongs to a finite union of open intervals and singletons. In the same way as in the case of the formula $R(a, x^n, b)$, it follows that $R(a, x^m, bx^n)$ defines a finite union of open intervals and singletons. \square

Recall that a c-convex subset is a subset J such that either J is a singleton or for every $g \neq g'$ in J , either $I(g, g') \subseteq J$ or $I(g', g) \subseteq J$.

We saw in Remark 1.3 that a c-convex subset is not necessarily a singleton or an open interval : the linear part of a cyclically ordered group is a c-convex subset and if it is nontrivial, then it is neither open interval nor finite unions of open intervals or singleton. There also exist definable c-convex subset which are neither an open intervals nor a finite union of open intervals or singletons. For example the subgroups H_p , and the subsets defined by the formulas argbound_n , before the proof of Proposition 1.1. Indeed, consider the lexicographic product $\mathbb{U} \overrightarrow{\times} D$, where D is an abelian linearly ordered group (see Definition 1.6). In this cyclically ordered group, the formula argbound_n defines the open interval $I\left(\left(\exp\left(\frac{2\pi}{n+1}\right), 0\right), \left(\exp\left(\frac{2\pi}{n}\right), 0\right)\right)$. Now, if G is a subgroup of $\mathbb{U} \overrightarrow{\times} D$ such

that $U(G)$ does not contain $\exp\left(\frac{2\pi}{n+1}\right)$ or $\exp\left(\frac{2\pi}{n+1}\right)$, and $U(G)$ is infinite, then it is not a finite union of open intervals or singletons.

Proposition 5.2. (Lucas) For each $n \in \mathbb{N} \setminus \{0\}$ and each nontrivial divisible abelian linearly ordered group D , the cyclically ordered group $T(\mathbb{U})_n \overrightarrow{\times} D$ is weakly cyclically minimal.

Proof. We denote by (G, R) the cyclically ordered group $T(\mathbb{U})_n \overrightarrow{\times} D$. Since D is nontrivial, G is infinite. Let E be a subset of G . Then E can be written as a disjoint union $E = E_0 \cup \left(e^{i\frac{2\pi}{n}}, e_D\right) E_1 \cup \dots \cup \left(e^{i\frac{2\pi}{n}}, e_D\right)^{n-1} E_{n-1}$, where E_0, E_1, \dots, E_{n-1} are subsets of $l(G) = \{1\} \overrightarrow{\times} D$. Now, in G , the element $\left(e^{i\frac{2\pi}{n}}, e_D\right)$ is definable by the formula $R(e_G, x, \dots, x^{n-1})$ and $x^n = e_G$. Hence all the elements of $T(\mathbb{U})_n \overrightarrow{\times} \{e_D\}$ are definable. The subgroup $l(G)$ is definable by the formula $x = e_G$ or $R(e_G, x, x^2, \dots, x^n)$ or $R(e_G, x^{-1}, x^{-2}, \dots, x^{-n})$. Assume that E is definable by a formula $\varphi(x)$ in the language $\{\cdot, R, e, ^{-1}\}$. Then, for $k \in \{0, 1, \dots, n-1\}$ the set E_k is definable by the formula $x \in l(G)$ and $\varphi\left(\left(e^{i\frac{2\pi}{n}}, e_D\right)^k x\right)$. Now, if x, y, z belong to $l(G)$, then $R(x, y, z)$ is equivalent to either $x < y < z$ or $y < z < x$ or $z < x < y$. Hence E_k is definable in $l(G)$ equipped with the language of ordered groups. Since $l(G)$ is abelian and divisible, it is minimal. Therefore, E_k is a finite union of open intervals. We saw in Remark 1.3 that the bounded open intervals of $(l(G), <)$ are open intervals of (G, R) , hence they are c -convex. Now, every open interval of $(l(G), <)$ is an increasing union of bounded open intervals, so it is c -convex. \square

Now, we look at necessary conditions. In the remainder of this section (G, R) is a cyclically ordered group which is not necessarily abelian. We take Lucas's approach to show that if (G, R) is weakly cyclically minimal, then $l(G)$ is divisible and abelian.

Definition 5.3. If H is a subset of (G, R) and $h \in H$, then we define the c -convex component of H which contains h to be the greatest c -convex subset of (G, R) containing h and contained in H . It is denoted by $C(h, H)$.

Fact 5.4. For g, h in H , we have $g \in C(h, H) \Leftrightarrow h \in C(g, H)$. It follows that $C(g, H) \neq C(h, H) \Leftrightarrow C(g, H) \cap C(h, H) = \emptyset$.

A subset H of G is said to be *symmetric* if for every $g \in G$ we have $g \in H \Leftrightarrow g^{-1} \in H$.

Lemma 5.5. (Lucas) Let H be a symmetric subset of (G, R) which contains e .

- (1) $C(e, H)$ is symmetric.
- (2) $C(e, H) = \{h \in G : I(e, h) \subseteq H\} \cup \{h \in G : I(e, h^{-1}) \subseteq H\}$.
- (3) If H is a subgroup, then $C(e, H)$ is a subgroup of G , and for every $h \in H$ we have $C(h, H) = hC(e, H) = C(e, H)h$.

Proof. (1) Let $h \in C(e, H)$ such that $h \neq h^{-1}$. We prove that $h^{-1} \in C(e, H)$. We have either $I(e, h) \subseteq C(e, H)$ or $I(h, e) \subseteq C(e, H)$.

(a) Assume that $I(e, h) \subseteq C(e, H)$.

If $R(e, h^{-1}, h)$ holds, then $h^{-1} \in C(e, H)$. Assume that $R(e, h, h^{-1})$ holds. Let $t \in I(h^{-1}, e)$. Then we have $R(h^{-1}, t, e)$ so $R(e, t^{-1}, h)$. Hence $t^{-1} \in C(h, H) \subseteq H$. Since H is symmetric, we have $t \in H$. Consequently, $I(h^{-1}, e) \subseteq C(e, H)$. The subset $\{h^{-1}\} \cup I(h^{-1}, e) \cup \{e\}$ is c -convex. It contains e , and is contained in H hence it is contained in $C(e, H)$, so $h^{-1} \in C(e, H)$.

(b) Assume that $I(h, e) \subseteq C(e, H)$.

If $R(h, h^{-1}, e)$ holds, then $h^{-1} \in C(e, H)$. Assume that $R(e, h^{-1}, h)$ holds. Let $t \in I(e, h^{-1})$. Then $R(e, t, h^{-1})$ so $R(e, h, t^{-1})$ and $R(h, t^{-1}, e)$. In the same way as in (a), this proves that $I(h^{-1}, e) \subseteq C(e, H)$, and we conclude $h^{-1} \in C(e, H)$.

(2) Let $h \in C(e, H)$. Since $C(e, H)$ is c -convex, we have either $I(e, h) \subseteq C(e, H)$ or $I(h, e) \subseteq C(e, H)$. Since $C(e, H)$ is symmetric, then $I(h, e) \subseteq C(e, H)$ implies $I(e, h^{-1}) \subseteq C(e, H)$. Now, let $h \in H$. If $I(e, h) \subseteq H$, then $\{e\} \cup I(e, h) \cup \{h\}$ is a c -convex subset of G contained in H , so it is contained in $C(e, H)$. Since H is symmetric, we have $h^{-1} \in H$, hence in the same way if

$I(e, h^{-1}) \subseteq H$, then $h^{-1} \in C(e, H)$, so $h \in C(e, H)$.

(3) Let $h \in H$. For every c-convex subset F of G , hF is c-convex. So $hC(e, H)$ is c-convex and contains h . Hence by the maximality of $C(h, H)$ we have $hC(e, H) \subseteq C(h, H)$. Conversely, let $h' \in C(h, H)$. Then either $I(h, h') \subseteq H$ or $I(h', h) \subseteq H$. Since H is a subgroup, this is equivalent to either $I(e, h^{-1}h') \subseteq H$ or $I(h^{-1}h', e) \subseteq H$. Since $h^{-1}h' \in H$, by (2), this is equivalent to $h^{-1}h' \in C(e, H)$. Therefore, $h' \in hC(e, H)$. Symmetrically, we can prove that $C(h, H) = C(e, H)h$.

Let h, h' in $C(e, H)$. Since $C(h, H)$ is the greatest c-convex subset which contains h , we have $C(h, H) = C(e, H) = C(h', H)$. Furthermore, $e \in C(e, H)$. Hence $h \cdot h' \in h \cdot h' C(e, H) = hC(h', H) = hC(e, H) = C(h, H) = C(e, H)$. This proves that $C(e, H)$ is a subgroup of G . \square

Lemma 5.6. (Lucas) Assume that (G, R) is weakly cyclically minimal and let H be a definable subset. If $H \cap l(G)$ is a subgroup of $l(G)$, then it is a convex subgroup of $l(G)$.

Proof. If $H \cap l(G) = l(G)$, then it is a convex subgroup of $l(G)$. In the following, we assume that $H \cap l(G) \neq l(G)$ and we let $H' = H \cap l(G)$.

(a) We show that for every $h \in H'$, $C(h, H) = C(h, H')$. Let $h \in H'$. Since $H' \subseteq H$, we have $C(h, H') \subseteq C(h, H)$. Now, there is $e < h_1 \in l(G) \setminus H$. Since H' is a subgroup, we have $h_1^{-1}h \in l(G) \setminus H$, $h_1h \in l(G) \setminus H$, and $h_1^{-1}h < h < h_1h$. Since $C(h, H)$ is c-convex, we have $C(h, H) \subseteq (h_1^{-1}h, h_1h) \subseteq l(G)$. Therefore $C(h, H) \subseteq H \cap l(G) = H'$. By the maximality of $C(h, H')$, we have $C(h, H) = C(h, H')$.

(b) By Lemma 5.5 (3), $C(e, H) = C(e, H')$ is a subgroup of G . So, it is a convex subgroup of $l(G)$.

(c) We prove that $H' = C(e, H)$, so it is a convex subgroup of $l(G)$. We assume that $H' \neq C(e, H)$, and we show that the number of the c-convex components of $H \cap l(G)$ is infinite. By (a), $C(e, H) = C(e, H') \subseteq H'$. Hence there is $h \in H' \setminus C(e, H)$. Then $C(h, H) \cap C(e, H) = \emptyset$ (Fact 5.4). Assume that $h > e$ in $l(G)$. Then $e < h < h^2 < \dots < h^n < \dots$. Since $C(e, H)$ is a convex subgroup of $l(G)$, for every $n \in \mathbb{N} \setminus \{0\}$ we have $h^n \notin C(e, H)$. Therefore, $\forall n \in \mathbb{N} \setminus \{0\}$ $C(h^n, H) \cap C(e, H) = \emptyset$. By Lemma 5.5 (3), for every positive integer n we have $h^n C(e, H) = h^n C(e, H') = C(h^n, H') = C(h^n, H)$. Let n, q in $\mathbb{N} \setminus \{0\}$. Therefore, $C(h^{n+q}, H) \cap C(h^n, H) = h^{n+q} C(e, H) \cap h^n C(e, H) = h^n (h^q C(e, H) \cap C(e, H)) = h^n (C(h^q, H) \cap C(e, H)) = \emptyset$. So the $C(h^n, H)$'s are pairwise disjoint. Consequently, H has infinitely many c-convex components. Since H is definable, this contradicts the fact that G is weakly cyclically minimal. The case $h < e$ is similar. \square

Proposition 5.7. (Lucas) If (G, R) is weakly cyclically minimal, then $l(G)$ is abelian and divisible.

Proof. Let g in $l(G)$, and $Com(g)$ be the definable subgroup $Com(g) = \{h : hg = gh\}$. By Lemma 5.6, $Com(g) \cap l(G)$ is a convex subgroup of $l(G)$, and we know that $g \in Com(g) \cap l(G)$. It follows that the convex subgroup generated by g is contained in $Com(g)$. Since this holds for any $g \in l(G)$, we deduce that $l(G)$ is abelian.

For each positive integer n consider

$$ID_n(G) = \{e\} \cup \{g : R(g^{-1}, e, g) \ \& \ \exists h (R(e, h, g) \ \& \ h^n = g)\} \\ \cup \{g : R(g, e, g^{-1}) \ \& \ \exists h (R(g, h, e) \ \& \ h^n = g)\}.$$

It is a definable set. Now, $ID_n(G) \cap l(G) = \{e\} \cup \{g \in l(G) : e < g \ \& \ \exists h \in l(G) \ h^n = g\} \cup \{g \in l(G) : g < e \ \& \ \exists h \in l(G) \ h^n = g\}$ is the subgroup of n -th powers of elements of the abelian group $l(G)$. So by Lemma 5.6 $ID_n(G) \cap l(G)$ is a convex subgroup of $l(G)$. Moreover it is cofinal in $l(G)$, so, for each n , $ID_n(l(G)) \cap l(G) = l(G) \subseteq ID_n(G)$. Hence each element of $l(G)$ is n -divisible within $l(G)$. \square

We turn to the other necessary conditions, starting with a lemma.

Lemma 5.8. (Lucas) Assume that (G, R) is weakly cyclically minimal. Let D be a definable subset of (G, R) and $D' = G \setminus D$. Then $U(D)$ and $U(D')$ are not both dense in \mathbb{U} .

Proof. The definable set D is a finite union of c-convex subsets. Let C be such a c-convex subset, g, h in C and assume that $U(D')$ is dense in \mathbb{U} . We prove that $U(g) = U(h)$. Indeed, otherwise there are h'_1, h'_2 in D' such that $R(U(g), U(h'_1), U(h))$ and $R(U(h), U(h'_2), U(g))$ hold in $U(G)$.

This implies that $R(g, h'_1, h)$ and $R(h, h'_2, g)$ hold in (G, R) , which contradicts the fact that C is c -convex. It follows that $U(D)$ is finite. So it is not dense in \mathbb{U} . Now, D' is definable by the formula $g \notin D$. So in the same way we can prove that if $U(D)$ is dense in \mathbb{U} , then $U(D')$ is finite. \square

Corollary 5.9. *Let (G, R) be a divisible abelian nonlinear cyclically ordered group which is not c -divisible. Then (G, R) is not weakly cyclically minimal.*

Proof. Recall that by Proposition 2.6, for a divisible abelian cyclically ordered group, not being c -divisible is equivalent to being torsion-free. Let p be a prime, $D = \{g \in G, : \exists h \in G, (R(e, h, g, g^{-1}) \text{ or } R(g^{-1}, g, h, e)) \text{ and } h^p = g\}$. By Lemma 4.2, both of $U(D)$ and $U(G \setminus D)$ are dense in \mathbb{U} . It follows that (G, R) is not weakly cyclically minimal. \square

Not that if (G, R) is c -divisible, then the set D defined above is equal to $G \setminus \{e\}$. Hence $G \setminus D$ is finite.

Proposition 5.10. *(Lucas) If (G, R) is weakly cyclically minimal, then it is abelian.*

Proof. For every $g \in G$ we let $Com(g) = \{h \in G : hg = gh\}$ and $Ncom(g) = \{h \in G : hg \neq gh\}$. Note that $Com(g) = \{h \in G : g^{-1}hg = h\} = \{h \in G : ghg^{-1} = h\} = Com(g^{-1})$. We start listing some properties of $Com(g)$ and $Ncom(g)$.

(1) Assume that $Ncom(g) \neq \emptyset$. If $h \in Com(g)$, then $hNcom(g) \subseteq Ncom(g)$. Indeed, if $g^{-1}g'g \neq g'$, then $g^{-1}hg'g = hg^{-1}g'g \neq hg'$. Hence $g' \in Ncom(g) \Rightarrow hg' \in Ncom(g)$. If $h \in Ncom(g)$, then $hCom(g) \subseteq Ncom(g)$. Indeed, let $g' \in Com(g)$. Then $ghg'g^{-1} = ghg^{-1}g' \neq hg'$.

(2) We prove that if (G, R) is not abelian and $U(G)$ contains torsion-free elements, then there is $g_0 \in G$ such that $U(g_0)$ is torsion-free and $Ncom(g_0) \neq \emptyset$. Let $g \in G$ such that $Ncom(g) \neq \emptyset$. Since $Com(g) \cup Ncom(g) = G$, we have $U(Com(g)) \cup U(Ncom(g)) = U(G)$. So $U(Com(g))$ or $U(Ncom(g))$ contains a torsion-free element. We prove that in any case $Ncom(g)$ contains an element g_0 such that $U(g_0)$ is torsion-free. Let $g' \in Ncom(g)$ and $h \in Com(g)$ be such that $U(g')$ is a torsion element and $U(h)$ is torsion-free. Let $g_0 = hg'$. Then, $U(g_0)$ is torsion-free. Since $hNcom(g) \subseteq Ncom(g)$, $g_0 \in Ncom(g)$. So, $gg_0 \neq g_0g$, and $Ncom(g_0) \neq \emptyset$.

(3) Assume that G is weakly cyclically minimal and $U(G)$ contains a torsion-free element. If G is not abelian, then there is g_0 in G such that $Ncom(g_0) \neq \emptyset$ and $U(g_0)$ is torsion-free. Since $Com(g_0)$ contains the subgroup generated by g_0 , $U(Com(g_0))$ is dense in \mathbb{U} . Now, for $h \in Ncom(g_0)$ we have $hCom(g) \subseteq Ncom(g)$. Hence $U(Ncom(g))$ is dense in \mathbb{U} . This contradicts Lemma 5.8. So G is abelian.

(4) Assume that G is weakly cyclically minimal and $U(G) \subseteq T(\mathbb{U})$. By Proposition 5.7 $l(G)$ is divisible. Hence by [10, Lemmas 5.1 and 6.7], G embeds in the lexicographic product $U(G) \overrightarrow{\times} l(G)$. Since $l(G)$ is abelian (Proposition 5.7), $U(G) \overrightarrow{\times} l(G)$ is abelian. Therefore G also is abelian. \square

Proposition 5.11. *(Lucas) If (G, R) is weakly cyclically minimal and not divisible, then $G \cong T(\mathbb{U})_n \overrightarrow{\times} l(G)$, for some $n \in \mathbb{N} \setminus \{0, 1\}$, and $l(G)$ is nontrivial, abelian, divisible.*

Proof. By Propositions 5.7 and 5.10 $l(G)$ is abelian, divisible and G is abelian. Assume that $U(G)$ is infinite. Let n be a positive integer such that (G, R) is not n -divisible, and D_n (resp. D'_n) be the set of elements which are n -divisible (resp. not n -divisible). Since $g^n \in D_n$ for every $g \in G$, $U(D_n)$ is infinite. Therefore it is dense in \mathbb{U} . Let $g' \in D'_n$. Then $g'D_n \subseteq D'_n$. It follows that $U(D'_n)$ is infinite, so it is dense. This contradicts Lemma 5.8. Therefore, $U(G)$ is finite, and it embeds in $T(\mathbb{U})$. Since $l(G)$ is divisible, by [10, Lemmas 5.1 and 6.7], G embeds in the lexicographic product $U(G) \overrightarrow{\times} l(G)$. We show that this embedding is onto. First, note that since (G, R) is weakly cyclically minimal, G is infinite. Hence $l(G)$ is nontrivial. Since $U(G)$ is a finite subgroup of $T(\mathbb{U})$, there exists a positive integer n such that $U(G)$ is c -isomorphic to $T(\mathbb{U})_n$. Since G is not divisible and $\{1\} \overrightarrow{\times} l(G)$ is divisible, we have $n > 1$. We let $g \in G$ be such that $U(g) = e^{2i\pi/n}$. Then $U(g^n) = 1$, hence $g^n \in l(G)$. Since $l(G)$ is divisible (Proposition 5.7) there exists $h \in l(G)$ such that $g^n = h^n$. Now, G is abelian (Proposition 5.10), so $(gh^{-1})^n = e$. Therefore, G contains a subgroup H which is isomorphic to $U(G) \simeq G/l(G)$, hence $G = H \oplus l(G)$. It follows that G is c -isomorphic to $T(\mathbb{U})_n \overrightarrow{\times} l(G)$. \square

Proof of Theorems 4 and 5. Let (G, R) be a cyclically ordered group.

Assume that G is not divisible. By Propositions 5.2 and 5.11, (G, R) is weakly cyclically minimal if, and only if $l(G)$ is nontrivial, abelian, divisible, and there is an integer $n > 1$ such that $G \cong T(\mathbb{U})_n \overrightarrow{\times} l(G)$. Now, this group is not cyclically minimal. Indeed, in this cyclically ordered group, $l(G)$ is definable by the formula $x = e$ or $R(e, x, x^2, \dots, x^n)$ or $R(e, x^{-1}, x^{-2}, \dots, x^{-n})$. So, $l(G)$ is a definable subset which is not a finite union of open interval and singletons.

Now, we assume that G is divisible.

If (G, R) is abelian and c-divisible, then by Proposition 5.1 it is cyclically minimal. In particular, it is weakly cyclically minimal.

If G is not abelian or if (G, R) is abelian non linear and not c-divisible, then, by Proposition 5.10 and by Corollary 5.9, (G, R) is not weakly cyclically minimal. Hence it is no cyclically minimal.

Finally, assume that (G, R) is linearly cyclically ordered. Then $G \simeq l(G) \simeq T(\mathbb{U})_1 \overrightarrow{\times} l(G)$. By Propositions 5.2 and 5.7, (G, R) is weakly cyclically minimal if, and only if, it is abelian and divisible. Now, it is not cyclically minimal. For example, the set $\{g \in G : g > e_G\}$ is defined by the formula $R(e_G, g, g^2)$, but it is not a finite union of bounded intervals and singletons. Note that in a c-divisible abelian cyclically ordered group, $R(e_G, g, g^2) \Leftrightarrow R(e_G, g, \zeta_2)$, where $\zeta_2 \neq e_G$ and $\zeta_2^2 = e_G$. \square

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