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Abstract

Continuous-time stochastic processes play an important role in the description of random phenomena, it is therefore of prime interest to study particular variables depending on their paths, like stopping time for example. One approach consists in pointing out explicit expressions of the probability distributions, an other approach is rather based on the numerical generation of the random variables. We propose an algorithm in order to generate the first passage time through a given level of a one-dimensional jump diffusion. This process satisfies a stochastic differential equation driven by a Brownian motion and subject to random shocks characterized by an independent Poisson process. Our algorithm belongs to the family of rejection sampling procedures, also called exact simulation in this context: the outcome of the algorithm and the stopping time under consideration are identically distributed. It is based on both the exact simulation of the diffusion at a given time and on the exact simulation of first passage time for continuous diffusions. It is therefore based on an extension of the algorithm introduced by Herrmann and Zucca [16] in the continuous framework. The challenge here is to generate the exact position of a continuous diffusion conditionally to the fact that the given level has not been reached before. We present the construction of the algorithm and give numerical illustrations, conditions on the recurrence of jump diffusions are also discussed.

Key words and phrases: first passage time, jump diffusion, Girsanov formula, rejection sampling, Bessel process, stochastic algorithm. **2010 AMS subject classifications:** primary: 65C05; secondary: 60J60, 60J76, 60G40.

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Introduction

Describing precisely the time needed by a diffusion process to first overcome a given threshold is a interesting challenge in several fields: economics [21], finance [23, 28], queueing, reliability theory [34], neuroscience [6, 38] and many others. The first passage time can for instance be related to the evaluation of a default risk in mathematical finance. Since this stopping time permits in practice to take important decisions, it is crucial either to obtain an explicit expression of the corresponding probability distribution either to point out efficient algorithms used for the random variable generations. An explicit expression of the density, often based on series expansions, is a source of valuable information. Unfortunately it is available only for particular stochastic models and cannot be used for wide classes of applications. We propose here a numerical approach: the challenge is therefore to generate the first passage times using efficient algorithms.

In many applications, modeling the behaviour of an experimental random value as time elapses consists in choosing a suitable one-dimensional diffusion process $(X_t,\, t \geq 0)$ and the corresponding first passage time through a given threshold L is then defined by

$$\tau_L := \inf\{t \geq 0 :\ X_t \geq L\}. \tag{0.1}$$

For simplicity, we just assume that the starting value $X_0 = y_0 < L$ is deterministic. In the continuous paths framework, the diffusion is represented by the solution of a stochastic differential equation driven by a Brownian motion. Different approaches have been proposed in order to generate τ_L. One way is to use numerical approximations of the paths and to deduce the approximation of the passage time as a by-product. Most of the studies are based on improvements of the classical Euler scheme (see for instance [4], [13], [14]). We also refer,in this framework of time splitting procedure, to the important work [25].

Another way is to use the exact simulation techniques introduced by Beskos and Roberts and based on the Girsanov transformation. Indeed we know exactly the distribution of the Brownian paths and in the way the exact distribution of their passage time. Using the Girsanov transformation permits to build a rejection sampling procedure: one generates a Brownian path and accepts or rejects it with a probability depending on the whole path. Beskos and Roberts [1, 2, 3] first proposed such an approach in order to simulate the diffusion trajectory on a given time interval $[0, \mathbb{T}]$. The procedure requires particular conditions on the drift and diffusion coefficients whose alleviation led to several mathematical improvements. Let us mention in particular the interesting localization argument developed by Chen and Huang [10]. Splitting the state space $\mathbb{R}$ into small intervals permits to observe the diffusion paths inbetween two exit times. Since the trajectory stays in a specific interval during this random time slot, the conditions associated with the exact simulation can be weakened. This does not happen without compensation: instead of using Brownian paths as proposal objects in the rejection sampling, Chen and Huang consider Brownian meanders. The basic elements used in the algorithm are therefore more complicated since they also require a rejection procedure to be generated: the algorithm is finally build as a double layer rejection sampling. Consequently the simulation of first exit times appears as a by-product of the so-called localization argument. Let us also note that Casella and Roberts [7] extended the

initial work of Beskos and Roberts for the simulation of the conditional diffusion value at some fixed time given that the process never exit from an specific interval before. Both approaches are quite different.

In the particular first passage time framework, an alternative approach proposed by Herrmann and Zucca [16] permits to adapt the initial algorithm of Beskos and Roberts in order to generate τ_L in an exact way. The outcome of the algorithm has the same distribution than τ_L, there is no approximation error term. The aim was not to weaken the conditions of the diffusion under consideration but rather to deal directly with an unbounded time interval. In that case, the proposal variable in the rejection procedure was the Brownian first passage time through the given threshold, that is an inverse Gaussian variate, and the weight used in order to accept the proposal depends on a suitable Bessel process, a very convenient process for simulation purposes. Let us also mention that Herrmann and Zucca proposed algorithms based on the Girsanov transformation for the exit time generation [18, 17].

Of course all applications cannot be concerned by one-dimensional continuous diffusion processes. A natural question that arises is to propose a new simulation approach in the jump diffusion framework. These processes are driven by both a Brownian motion and a Poisson random measure, they are indeed special cases of one-dimensional Lévy processes. In finance, for instance, the stock exchange evolution can be represented by a diffusion with jumps. In this particular case, the jumps represent possible events that can occur and produce strong impacts on the asset prices [26].

For such particular stochastic processes, several available results concern the approximation of the diffusion trajectory on some given time interval. The simulation of the first passage time is then obtained as a by-product, similarly to the already used procedure in the continuous case. In particular, for diffusions which are driven by Wiener processes and Poisson random measures, the basic idea of the Euler method can be adapted. Platen [35], Maghsoodi [30, 29] and Gardon [11] introduced explicit time-discretization schemes based on the Itô-Taylor expansion in order to obtain interesting convergence results in the mean-square sense. Several studies permit to understand how to adapt the time grid to the jump times and reduce, by the way, the approximation error. Other kinds of stochastic convergence have been analyzed for the numerical approximation. Let us also mention other challenging research directions: introducing schemes based on semi-implicit Euler–Maruyama methods in [19, 20] or Runge-Kutta methods as in [5] is of particular interest. Our purpose is not to write an exhaustive overview of the literature on numerical analysis since our approach for the generation of the first passage time is not based at all on an approximation procedure. So we prefer to refer to the monograph [36], and the references therein, for additional information on the approximation methods.

The objective of the paper is rather to focus our attention on the so-called exact simulation method. It is important to extend the classical simulation procedure introduced by Beskos and Roberts to that framework. As already presented in the continuous diffusion case, two different approaches can be chosen: one based on the localization argument and using Brownian meanders and the other one based on Bessel processes. The first method has already been adapted to the jump diffusions (see [8], [15] and [37]) and consequently

permits to generate exactly the first exit time from a given interval [12] using this double layer rejection method. We propose here to adapt the second method based on the Bessel processes to the jump diffusions. We propose thereby to avoid both the space splitting and the generation of meanders without looking for the minimal conditions. The algorithm is finally simpler and more efficient but only available for the generation of first passage times.

The study presented here concerns the exact simulation of the first passage time τ_L defined by (0.1) where $(X_t,\, t \geq 0)$ stands for a jump diffusion process. The material is organized as follows: first we end the introduction by defining the jump diffusion as the unique solution of a stochastic differential equation, then we recall the exact simulation techniques introduced for continuous paths in Section 1, proposing new proofs. In Section 2, we focus our attention to the jump diffusion framework and propose an efficient algorithm in order to generate $\tau_L \wedge \mathbb{T}$ where $\mathbb{T} > 0$ is a fixed time. This particular situation permits to deal with the stopped diffusion and consequently to consider a wide class of diffusions. In Section 3, we propose to reduce the study to the particular situation: $\tau_L < \infty$. Of course such context requires additional conditions on the process but permits also to simulate directly τ_L in an exact way. The last section presents numerical illustrations.

Jump diffusions: definition and model reduction

Usually and historically, diffusions with jumps are introduced in the following way: we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a mark space $(\mathcal{E}, B(\mathcal{E}))$ where $\mathcal{E} \subset \mathbb{R} \setminus \{0\}$. This mark space can be interpreted as the space of jumps amplitudes. On this state space we define a $\mathcal{F}$-adapted Poisson measure on $\mathcal{E} \times [0,T]$ for a given T denoted by $p_\phi(dv \times dt)$ whose intensity measure is given by $\phi(dv)dt$, ϕ being a non negative σ-finite measure. We denote by $\lambda = \phi(\mathcal{E})$. This measure permits to generate a sequence of random points $(T_i, \xi_i)_{1 \leq i \leq P_T}$ where $(P_t)_{t \geq 0}$ is the stochastic process counting the number of jumps until time t. This sequence represents each jump time and the amplitude of the corresponding jump.

A jump diffusion X with jump rate j is defined as follows:

$$dX_t = \mu(t, X_{t-})\, dt + \sigma(t, X_{t-})\, dB_t + \int_{\mathcal{E}} j(t, X_{t-}, v) p_\phi(dv \times dt), \quad t \geq 0, \tag{0.2}$$

with the initial position $X_0 = y_0$. Here $(B_t,\, t \geq 0)$ stands for a one-dimensional Brownian motion. Let us just mention that we choose in this work homogeneous Poisson measures, in other words the function ϕ only depends on the space variable. The result can easily be extended to the time-dependent case by using the thinning procedure, see for instance [9] for the description of the general approach.

For the present study, the representation (0.2) is actually not so handy. We prefer the following representation. We start the construction with the jump rate (jump function) $j : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{E} \to \mathbb{R}$. No specific assumption concerning the jump rate is required at the moment. Let us denote by T_i the i-th jump time. We mention that the time spent between two consecutive jumps is exponentially distributed, therefore $T_i = \sum_{k=1}^{i} E_k$ with E_k exponentially distributed random variables with average $1/\lambda$. The initial position of the

diffusion is given by $Y_0 = y_0$. Between two jumps, the stochastic process satisfies a stochastic differential equation:

$$dY_t = \mu(t, Y_t)\, dt + \sigma(t, Y_t)\, dB_t, \quad \text{for } T_i < t < T_{i+1}, \quad i \in \mathbb{N}, \tag{0.3}$$

and the jumps modify the trajectories as follows:

$$Y_{T_i} = Y_{T_i-} + j(T_i, Y_{T_i-}, \xi_i), \quad \forall i \in \mathbb{N}. \tag{0.4}$$

As already defined, $(\xi_i)_{i\geq 1}$ stands for a sequence of independent random variables with distribution function ϕ/λ. This sequence has to be independent of the Brownian motion $(B_t)_{t\geq 0}$. It is quite obvious to observe that $(X_t)_{t\geq 0}$ solution of (0.2) has the same path distribution than $(Y_t)_{t\geq 0}$ defined by (0.3)–(0.4). The aim of our study is to simulate the first passage time of some given level L for this jump diffusion. For our discussion we shall assume that $y_0 < L$ (it is straightforward to deduce the general case). Therefore the second representation plays a crucial role: it suffices to simulate exactly the trajectory of a SDE solution inbetween two successive jumps. The simulation of the first passage time should be based on the following intuitive procedure: on one hand we simulate exactly the trajectory of the stochastic process satisfying the SDE without jump and keep in mind its first passage time through the level L denoted by τ_L. On the other independent hand, we simulate the first exponentially distributed jump time T_1. If $\tau_L \leq T_1$, then τ_L corresponds to the first passage time of the jump diffusion. In the other case, we simulate the position of the diffusion after the first jump using

$$Y_{T_1} = Y_{T_1-} + j(T_1, Y_{T_1-}, \xi_1).$$

We distinguish two likely different cases: if $Y_{T_1} \geq L$ then $\tau_L = T_1$ otherwise we know that $\tau_L > T_1$. The strong Markov property of the jump diffusion permits to start a new jump diffusion with the initial position $Y_{T_1} < L$ and initial time T_1. So we repeat the procedure just presented, inbetween T_1 and T_2, and so on... An important tool for the simulation is therefore the exact generation of continuous diffusion paths. In [16], the authors propose an efficient method based on both a rejection sampling and the Girsanov transformation.

Remark 0.1. *In order to simplify the study, we assume that $\sigma(t,x) \equiv \sigma(x)$ and $\mu(t,x) \equiv \mu(x)$. Moreover we consider Lamperti's approach which consists in transforming the SDE*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \quad t \geq 0, \tag{0.5}$$

with initial condition $X_0 = x_0$ into the following equation

$$dZ_t = \alpha(Z_t)dt + dB_t, \quad t \geq 0. \tag{0.6}$$

This well-known transformation corresponds to

$$Z_t = \nu(X_t) = \int_{\xi}^{X_t} \frac{1}{\sigma(x)}\, dx \quad \textit{and} \quad \alpha(x) = \frac{\mu \circ \nu^{-1}(x)}{\sigma \circ \nu^{-1}(x)} - \frac{1}{2}\sigma' \circ \nu^{-1}(x), \quad \forall x \in \mathbb{R}.$$

Of course the jump times are not changed at all by the function ν: we still work with the sequence $(T_i)_{i\geq 1}$ whereas the jump amplitudes are modified. Hence the transport property permits to obtain

$$Z_{T_i} = Z_{T_i-} + \hat{\jmath}(t, Z_{T_i-}, \xi), \quad i \in \mathbb{N}, \tag{0.7}$$

with

$$\hat{\jmath}(t,z,v) := \nu(\nu^{-1}(z) + j(t, \nu^{-1}(z), v)) - \nu(\nu^{-1}(z)).$$

1 Simulation of the first passage time for a stopped continuous diffusion

Let us fix some time parameter $\mathbb{T} > 0$. In this section, we shall focus our attention on the exact simulation of the continuous diffusion paths on the interval $[0, \mathbb{T}]$. Generating a random object in an exact way consists in generating an object using a stochastic algorithm such that both objects have the same distribution. Roberts and Beskos [1] already proposed an efficient algorithm in order to exactly simulate a continuous diffusion path on the interval under consideration: $[0, \mathbb{T}]$. Of course we are not able to generate the whole paths numerically, the exact simulation consists therefore in simulating a sequence of random points belonging to the trajectory of the diffusion.

Herrmann and Zucca [16] proposed an adaptation of the algorithm introduced by Roberts and Beskos in order to exactly generate τ_L, the first passage time through the level L for a continuous diffusion. This approach presented in the next paragraph mainly differs from the localization procedure introduced in [10] and based on the generation of Brownian meanders. In order to deal with jump diffusions (next section), we need also to simulate exactly the couple $(\tau_L \wedge \mathbb{T}, Y_{\tau_L \wedge \mathbb{T}})$ where $(Y_t)_{t\geq 0}$ stands for a continuous diffusion. Our technique differs from the method proposed by Casella and Roberts [7] who studied the exact simulation of killed diffusions introducing conditions on Brownian bridges.

1.1 Diffusion without any jump: the exact simulation method

Let us first consider continuous diffusion processes and propose a numerical approach for the generation of their paths, we recall in this section the famous approach introduced by Beskos and Roberts in order to fix the notations and propose simplified proofs. As already seen in the introduction, we can restrict the study to the reduced model, the generalization being obtained by the Lamperti transformation. That's why we consider, on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the following SDE on the fixed time interval $[0, \mathbb{T}]$:

$$dY_t = \alpha(t, Y_t)\, dt + dB_t, \quad Y_0 = y_0, \tag{1.1}$$

where $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion. The direct generation of a diffusion path is quite a hard task, that's why we need to point out the link between the diffusion and the standard Brownian motion using the famous Girsanov formula. The goal

of this formula is to find a probability space in which the considered diffusion is a Brownian motion. Then we generate a Brownian motion path and do an acceptance/rejection algorithm using the weight enlightened by the Girsanov formula.

The Girsanov transformation: consequences for simulation purposes

First we recall the statement of the Girsanov transformation and the associated Novikov condition (for a reference about Girsanov's transformation, see for instance [32]).

Assumption 1.1 (Novikov's condition)**.** *We say that Novikov's condition is satisfied if*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^{\mathbb{T}}\alpha^2(s,y_0+B_s)\,ds\right)\right]<\infty. \tag{1.2}$$

Let us note that this particular condition is satisfied if the growth of the drift term α is at most linear (see Corollary 5.16 p.200 in [24]): there exists a constant $C_{\mathbb{T}}>0$ such that

$$|\alpha(t,x)|\leq C_{\mathbb{T}}(1+|x|),\quad \forall (t,x)\in[0,\mathbb{T}]\times\mathbb{R}.$$

Then the following transformation holds.

Theorem 1.1. *Assume that α satisfies Novikov's condition. Let us define the martingale $(M_t)_{t\geq 0}$ by*

$$M_t=\exp\left(\int_0^t\alpha(s,y_0+B_s)\,dB_s-\frac{1}{2}\int_0^t\alpha^2(s,y_0+B_s)ds\right),\quad t\leq\mathbb{T}, \tag{1.3}$$

and the measure $\mathbb{Q}$ on $(\Omega,\mathcal{F}_{\mathbb{T}})$:

$$d\mathbb{Q}=M_{\mathbb{T}}\,d\mathbb{P}. \tag{1.4}$$

Then under $\mathbb{Q}$, the stochastic process $\left(B_t-\int_0^t\alpha(s,B_s)\,ds\right)_{0\leq t\leq\mathbb{T}}$ is a one dimensional standard Brownian motion. In other words, $(y_0+B_t)_{0\leq t\leq\mathbb{T}}$ under $\mathbb{Q}$ has the same distribution than $(Y_t)_{0\leq t\leq\mathbb{T}}$ under $\mathbb{P}$.

The Radon-Nikodym derivative pointed out in the previous statement is going to correspond to the weight necessary for the use of an acceptance/rejection sampling. We use the Girsanov formula as follows: let us consider $(Y_t)_{t\geq 0}$, the solution of the SDE (1.1), and f any measurable functional depending on the paths of Y observed on the time interval $[0,\mathbb{T}]$, then

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[f(Y_\cdot)]&=\mathbb{E}_{\mathbb{Q}}[f(y_0+B_\cdot)]=\mathbb{E}_{\mathbb{P}}[f(y_0+B_\cdot)\cdot M_{\mathbb{T}}]\\&=\mathbb{E}_{\mathbb{P}}\left[f(y_0+B_\cdot)\exp\left(\int_0^{\mathbb{T}}\alpha(s,y_0+B_s)dB_s-\frac{1}{2}\int_0^{\mathbb{T}}\alpha^2(s,y_0+B_s)ds\right)\right],\end{aligned}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion under the probability measure $\mathbb{P}$. We introduce two different functions which shall also play a crucial role in the numerical algorithm. Let us define

$$\beta(t,x) := \int_{y_0}^{x} \alpha(t,y)\,dy \quad \text{and} \quad \gamma(t,x) = \frac{\partial \beta}{\partial t}(t,x) + \frac{1}{2}\Big(\frac{\partial \alpha}{\partial x}(t,x) + \alpha^2(t,x)\Big). \tag{1.5}$$

Using Itô's formula applied to the process $(\beta(t, y_0 + B_t))_{t\geq 0}$, we obtain

$$\mathbb{E}_{\mathbb{P}}[f(Y_\cdot)] = \mathbb{E}_{\mathbb{P}}[f(y_0 + B_\cdot)\cdot \hat{M}_{\mathbb{T}}] \quad \text{with} \quad \hat{M}_{\mathbb{T}} := e^{\beta(\mathbb{T}, y_0+B_{\mathbb{T}}) - \int_0^{\mathbb{T}} \gamma(t, y_0+B_s)\,ds}. \tag{1.6}$$

General hypotheses for continuous diffusion processes

We recall that, through all our study, the diffusion (1.1) starts in y_0 satisfying $y_0 < L$. Let us present now different hypotheses concerning the drift coefficient α in (1.1). They permit to describe a typical framework for the introduction of efficient algorithms. The aim is not at this stage to precisely emphasize the most general situation which permits the use of the exact simulation technique, several studies were already introduced in order to successively extend these conditions (see Beskos, Papaspiliopoulos and Roberts [1]...). The aim is rather to focus our attention on a convenient context where the crucial arguments used in the study of the continuous diffusions can easily be translated to jump diffusions. First of all, we need a classical regularity property in order to use Itô's formula.

Assumption 1.2. *The drift coefficient α is a $\mathcal{C}^{1,1}(\mathbb{R}_+ \times \mathbb{R})$-continuous function.*

Of course the regularity property of α immediately implies that β and γ defined by (1.5) are well-defined and continuous functions. According to the simulation goals, we need additional conditions like the boundedness of β or of γ. If the aim is to generate $Y_{\mathbb{T}}$ we need the following

Assumption 1.3. *The function γ defined in* (1.5) *is non negative and satisfies: there exists a constant κ s.t.*

$$0 \leq \gamma(t,x) \leq \kappa, \quad \forall (t,x) \in [0,\mathbb{T}] \times \mathbb{R}. \tag{1.7}$$

Assumption 1.4. *The function β defined in* (1.5) *is upper bounded at time $\mathbb{T}$: there exists a constant $\beta_+ > 0$ s.t.*

$$\beta(\mathbb{T}, x) \leq \beta_+, \quad \forall x \in \mathbb{R}. \tag{1.8}$$

The assumption concerning β can be weaken in particular situations when we just need an integration property.

Assumption 1.5. *The function $\Gamma_{\mathbb{T}} : \mathbb{R} \to \mathbb{R}$ defined by*

$$\Gamma_{\mathbb{T}}(x) := \exp\Big\{\beta(\mathbb{T}, y_0 + x) - \frac{x^2}{2\mathbb{T}}\Big\}, \quad x \in \mathbb{R},$$

with β introduced in (1.5) *is integrable: $\Gamma_{\mathbb{T}} \in L^1(\mathbb{R})$.*

Assumption 1.3, 1.4 and 1.5 essentially concern the simulation of $Y_{\mathbb{T}}$, $\mathbb{T}$ being a fixed time value. If we are rather interested in the first passage time through the level L, we shall focus our attention on the space subset $]-\infty, L]$.

Assumption 1.6. *The function γ defined in* (1.5) *is non negative and satisfies: there exists a constant κ s.t.*

$$0 \leq \gamma(t,x) \leq \kappa, \quad (t,x) \in \mathbb{R}_+ \times]-\infty, L]. \tag{1.9}$$

Assumption 1.7. *The function β defined in* (1.5) *is upper bounded: there exists a constant $\beta_+ > 0$ s.t.*

$$\beta(t,x) \leq \beta_+, \quad \forall (t,x) \in \mathbb{R}_+ \times]-\infty, L]. \tag{1.10}$$

Let us note that all these assumptions take new shapes as soon as the drift coefficient α in equation (1.1) is time homogeneous. Most of the results presented in previous studies ([1], [16]) concern this restrictive context but the generalization is a quite simple task.

The approach developed by Beskos and Roberts

Beskos and Roberts proposed in [1] a simulation approach for the exact generation of diffusion paths on some given interval $[0, \mathbb{T}]$. Their study is based on the Girsanov transformation on one hand and on an acceptance/rejection sampling on the other hand. The procedure therefore consists in the introduction of a Poisson process independent of the diffusion (1.1) and whose realization shall permit to obtain the required weight appearing in the rejection method (1.6). Their approach is not so easy to adapt to a jump diffusion since they do not use at all the Markov property of the diffusion process. That is why we propose an alternative presentation of their result (and the corresponding proof) and laid by that way the foundations of the study in the general jump diffusion context. For a clear and succinct presentation of the issue, we prefer just to introduce the exact simulation of $Y_{\mathbb{T}}$ where $(Y_t)_{t \geq 0}$ corresponds to the solution of (1.1).

EXACT SIMULATION OF $Y_{\mathbb{T}}$ – ALGORITHM $(BR)^1_{\mathbb{T}}$

1. *Let $(G_n)_{n \geq 1}$ be independent random variables with Gaussian distribution $\mathcal{N}(0,1)$*
2. *Let $(E_n)_{n \geq 1}$ be independent exponentially distributed r.v. with average $1/\kappa$.*
3. *Let $(U_n)_{n \geq 1}$ be independent uniformly distributed random variables on $[0,1]$.*

The sequences $(G_n)_{n\geq 1}$, $(E_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ are assumed to be independent.

Initialization: $n = 0$.

Step 1. Set $Z = y_0$, $\mathcal{T} = 0$.

Step 2. While $\mathcal{T} < \mathbb{T}$ do:

- set $n \leftarrow n + 1$
- $Z \leftarrow Z + \sqrt{\min(\mathcal{T} + E_n, \mathbb{T}) - \mathcal{T}}\, G_n$ and $\mathcal{T} \leftarrow \min(\mathcal{T} + E_n, \mathbb{T})$
- If ($\mathcal{T} < \mathbb{T}$ and $\kappa\, U_n < \gamma(\mathcal{T}, Z)$) then go to Step 1.
- If ($\mathcal{T} = \mathbb{T}$ and $U_n e^{\beta_+} > e^{\beta(\mathbb{T},Z)}$) then go to Step 1.

Outcome: the random variable Z.

Proposition 1.2. *Under the assumptions 1.2, 1.3 and 1.4. both the outcome Z of Algorithm $(BR)^1_{\mathbb{T}}$ and $Y_{\mathbb{T}}$, the value at time $\mathbb{T}$ of the diffusion process* (1.1)*, have the same distribution.*

Remark 1.3. *An adaptation of Algorithm $(BR)^1_{\mathbb{T}}$ should permit to obtain more than just the random variable Z which has the same distribution than $Y_{\mathbb{T}}$. Indeed denoting by n_1 the value of n throughout the last visit of Step number one, n_2 the value of the increment variable when the algorithm stops, Z_n (respectively $\mathcal{T}_n$) the successive values of Z (resp. $\mathcal{T}$), we obtain that*

$$\Big\{(0, y_0),\, (\mathcal{T}_{n_1+1}, Z_{n_1+1}), \ldots, (\mathcal{T}_{n_2-1}, Z_{n_2-1}),\, (\mathbb{T}, Z_{n_2})\Big\}$$

is a set of points which has the same distribution than points belonging to the diffusion trajectory.

Proof. Let us denote by $\mathcal{N}$ the number of necessary repetitions of the step number one and let ψ a non negative measurable function. Since the algorithm is based on an acceptance/rejection sampling, we get

$$\mathbb{E}[\psi(Z)] = \frac{\mathbb{E}[\psi(Z)1_{\{\mathcal{N}=1\}}]}{\mathbb{P}(\mathcal{N}=1)} = \frac{\nu(\psi)}{\nu(1)} \quad \text{where} \quad \nu(\psi) := \mathbb{E}[\psi(Z)1_{\{\mathcal{N}=1\}}].$$

We use now the notations $(\mathcal{T}_n)_{n\geq 0}$ and $(Z_n)_{n\geq 0}$ introduced in Remark 1.3 and define the event $A_n := \{\mathcal{T}_{n-1} < \mathbb{T} \leq \mathcal{T}_{n-1} + E_n = \mathcal{T}_n\}$, for any $n \geq 1$, which describes the number of random intervals necessary to cover $[0, \mathbb{T}]$. We observe that

$$\nu(\psi) = \sum_{n\geq 1} \mathbb{E}\Big[\psi(Z_n) 1_{\{\kappa U_1 > \gamma(\mathcal{T}_1, Z_1), \ldots, \kappa U_{n-1} > \gamma(\mathcal{T}_{n-1}, Z_{n-1})\}} 1_{\{U_n \leq e^{\beta(\mathbb{T}, Z_n) - \beta_+}\}} 1_{A_n}\Big].$$

Taking the integral with respect to the independent uniform variates $(U_n)_{n\geq 1}$ leads to

$$\nu(\psi) = \sum_{n\geq 1} \kappa^{1-n} \mathbb{E}\Big[\psi(Z_n)(\kappa - \gamma(\mathcal{T}_1, Z_1)) \ldots (\kappa - \gamma(\mathcal{T}_{n-1}, Z_{n-1})) e^{\beta(\mathbb{T}, Z_n) - \beta_+} 1_{A_n}\Big].$$

We note that $(\mathcal{T}_1, \dots, \mathcal{T}_{n-1})$, given A_n, has the same distribution than $(V^{(1)}, \dots, V^{(n-1)})$ an ordered $(n-1)$-tuple of uniform random variables $(V_1, \dots, V_{n-1})$ on $[0, \mathbb{T}]$. Moreover the probability of the event A_n is linked to the Poisson distribution of parameter $\kappa\mathbb{T}$. Finally $(Z_1, \dots Z_n)$ is a Gaussian vector and has the same distribution than

$$(y_0 + B_{V^{(1)}}, \dots, y_0 + B_{V^{(n-1)}}, y_0 + B_{\mathbb{T}})$$

with $(B_t)_{t\geq 0}$ a standard Brownian motion independent of the $(n-1)$-tuple $(V_1, \dots, V_{n-1})$, since the Brownian motion has Gaussian independent increments. Hence

$$\begin{aligned}
\nu(\psi) &= \sum_{n\geq 1} \mathbb{E}\Big[\psi(Z_n)(\kappa - \gamma(\mathcal{T}_1, Z_1)) \dots (\kappa - \gamma(\mathcal{T}_{n-1}, Z_{n-1}))e^{\beta(\mathbb{T}, Z_n)-\beta_+}\Big| A_n\Big] \frac{\mathbb{T}^{n-1}}{(n-1)!}\, e^{-\kappa\mathbb{T}} \\
&= \sum_{n\geq 1} \mathbb{E}\Big[\psi(y_0 + B_{\mathbb{T}}) \prod_{j=1}^{n-1} (\kappa - \gamma(V^{(j)}, y_0 + B_{V^{(j)}})e^{\beta(\mathbb{T}, y_0+B_{\mathbb{T}})-\beta_+}\Big] \frac{\mathbb{T}^{n-1}}{(n-1)!}\, e^{-\kappa\mathbb{T}} \\
&= \sum_{n\geq 1} \mathbb{E}\Big[\psi(y_0 + B_{\mathbb{T}}) \prod_{j=1}^{n-1} (\kappa - \gamma(V_j, y_0 + B_{V_j}))e^{\beta(\mathbb{T}, y_0+B_{\mathbb{T}})-\beta_+}\Big] \frac{\mathbb{T}^{n-1}}{(n-1)!}\, e^{-\kappa\mathbb{T}}.
\end{aligned}$$

Taking the expectation with respect to the uniformly distributed variates V_j, we have

$$\begin{aligned}
\nu(\psi) &= \sum_{n\geq 1} \mathbb{E}\Big[\psi(y_0 + B_{\mathbb{T}})\Big(\kappa - \frac{1}{\mathbb{T}}\int_0^{\mathbb{T}} \gamma(s, y_0 + B_s)\, ds\Big)^{n-1} e^{\beta(\mathbb{T}, y_0+B_{\mathbb{T}})-\beta_+}\Big] \frac{\mathbb{T}^{n-1}}{(n-1)!}\, e^{-\kappa\mathbb{T}} \\
&= \mathbb{E}\Big[\psi(y_0 + B_{\mathbb{T}}) \exp\Big\{\beta(\mathbb{T}, y_0 + B_{\mathbb{T}}) - \beta_+ - \int_0^{\mathbb{T}} \gamma(s, y_0 + B_s)\, ds\Big\}\Big] \\
&= \mathbb{E}[\psi(y_0 + B_{\mathbb{T}}) \cdot \hat{M}_{\mathbb{T}}]\, e^{-\beta_+},
\end{aligned}$$

where $(\hat{M}_t)_{t\geq 0}$ is the martingale defined in (1.6). The martingale property leads to $\nu(1) = \mathbb{E}[\hat{M}_0]e^{-\beta_+} = e^{-\beta_+}$. The Girsanov transformation permits to conclude the proof:

$$\mathbb{E}[\psi(Z)] = \frac{\nu(\psi)}{\nu(1)} = \mathbb{E}[\psi(y_0 + B_{\mathbb{T}}) \cdot \hat{M}_{\mathbb{T}}] = \mathbb{E}[\psi(Y_{\mathbb{T}})].$$

□

Under Assumptions 1.2, 1.3 and 1.4, the algorithm pointed out in Proposition 1.2 has a convenient and intuitive expression. However the boundedness of the function $\beta(\mathbb{T}, \cdot)$ is rather a restrictive assumption and it is important to propose an alternative approach. To that end, Beskos and Roberts proposed an integrability condition for the function $\Gamma_{\mathbb{T}}$, written here in Assumption 1.5. Since $\Gamma_{\mathbb{T}}$ is a non negative function, the integrability condition summarized in the identity $\Gamma_{\mathbb{T}}(\mathbb{R}) := \int_{\mathbb{R}} \Gamma_{\mathbb{T}}(x)\, dx < \infty$ ensures that $\Gamma_{\mathbb{T}}(\cdot)/\Gamma_{\mathbb{T}}(\mathbb{R})$ is a probability distribution function. This crucial property permits to present the following algorithm. Let us just mention that we use the following notation $x_+ = \min(x, 0)$.

EXACT SIMULATION OF $Y_{\mathbb{T}}$ – ALGORITHM $(BR)^2_{\mathbb{T}}$

1. *Let* $(R_n)_{n\geq 1}$ *be independent random variables with density* $\Gamma_{\mathbb{T}}(\cdot)/\Gamma_{\mathbb{T}}(\mathbb{R})$.
2. *Let* $(G_n)_{n\geq 1}$ *be independent random variables with Gaussian distribution* $\mathcal{N}(0,1)$
3. *Let* $(E_n)_{n\geq 1}$ *be independent exponentially distributed r.v. with average* $1/\kappa$.
4. *Let* $(U_n)_{n\geq 1}$ *be independent uniformly distributed random variables on* $[0,1]$.

The sequences $(R_n)_{n\geq 1}$, $(G_n)_{n\geq 1}$, $(E_n)_{n\geq 1}$ *and* $(U_n)_{n\geq 1}$ *are assumed to be independent.*

Initialization: $k=0$, $n=0$.

Step 1. Set $k \leftarrow k+1$ then $Z = y_0$, $W = y_0 + R_k$ and $\mathcal{T} = 0$.

Step 2. While $\mathcal{T} < \mathbb{T}$ do:

- set $n \leftarrow n+1$
- $Z \leftarrow Z + \dfrac{E_n}{\mathbb{T}-\mathcal{T}}\, W + \sqrt{\dfrac{E_n(\mathbb{T}-\mathcal{T}-E_n)_+}{\mathbb{T}-\mathcal{T}}}\; G_n$ and $\mathcal{T} \leftarrow \min(\mathcal{T}+E_n, \mathbb{T})$
- If ($\mathcal{T} < \mathbb{T}$ and $\kappa\, U_n < \gamma(\mathcal{T}, Z)$) then go to Step 1.

Outcome: the random variable W.

The procedure proposed in this second algorithm is mainly different from the first one. Indeed the crucial idea of Algorithm $(BR)^1_{\mathbb{T}}$ is to simulate a Brownian motion on the interval $[0,\mathbb{T}]$ and to accept or reject the trajectory using the weight probability issued from the Girsanov transformation. The acceptance depends strongly on the whole path of the process and leads to the outcome $y_0 + B_{\mathbb{T}}$, the endpoint of the Brownian path. In Algorithm $(BR)^2_{\mathbb{T}}$ the approach is different: we consider a random variable W with the proposal distribution $\Gamma_{\mathbb{T}}(\cdot)/\Gamma_{\mathbb{T}}(\mathbb{R})$ translated by y_0. This variate shall be accepted or rejected using a weight probability based on the whole path of a Brownian bridge starting in y_0 and ending at time $\mathbb{T}$ with the value W. The main difference is therefore to replace the Brownian motion by the Brownian bridge. We obtain the following statement.

Proposition 1.4. *Under the assumptions 1.2, 1.3 and 1.5. both the outcome* W *of Algorithm* $(BR)^2_{\mathbb{T}}$ *and* $Y_{\mathbb{T}}$*, the value at time* $\mathbb{T}$ *of the diffusion process* (1.1)*, have the same distribution.*

The proof of Proposition 1.4 is quite similar to the proof of Proposition 1.2. We refer to the PhD thesis manuscript of Massin [31] for this adapted proof. The conditions concerning the drift coefficient of the diffusion (1.1) can be weaken using the so-called localization procedure [10] but we don't intend to develop such an approach in this study as already mentioned.

The algorithm introduced by Herrmann and Zucca

The procedure of the exact simulation was adapted by Herrmann and Zucca [16] in order to generate the first passage time of continuous diffusion processes. We focus now our attention on the first time passage of the diffusion Y, starting in y_0, through the level L. This algorithm is also based on the Girsanov formula on one hand, and on the other hand, it requires the construction of a skeleton of a 3-dimensional Bessel process.

We first recall that in the particular case of the Brownian motion, the first passage time through the level L denoted by τ_L satisfies $\tau_L \sim (L-y_0)^2/G^2$ where $G \sim \mathcal{N}(0,1)$. The main idea is therefore to first generate a Brownian passage time and secondly to accept or reject this variate proposal using the Girsanov weight. The construction of this algorithm looks very much like the algorithms presented by Beskos and Roberts. The main difference is to replace the Brownian paths (or Brownian bridge paths) appearing in the rejection sampling by Bessel paths. The explanation of such a modification is related to the observation: once the Brownian first passage time τ_L is generated, the Brownian motion constrained to stay under the level L on $[0,\tau_L]$ can be related to a 3-dimensional Bessel process. The algorithm proposal is the following.

EXACT SIMULATION OF τ_L FOR CONTINUOUS DIFFUSIONS – ALGORITHM (*HZ*)

1. *Let $(G_n)_{n\geq 1}$ be independent standard 3-dimensional Gaussian vectors.*
2. *Let $(e_n)_{n\geq 0}$ be independent exponentially distributed r.v. with average $1/\kappa$.*
3. *Let $(V_n)_{n\geq 1}$ be independent uniformly distributed r.v. on $[0,1]$.*
4. *Let $(g_n)_{n\geq 1}$ be independent standard Gaussian r.variables.*

The sequences $(G_n)_{n\geq 1}$, $(e_n)_{n\geq 0}$, $(V_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ are assumed to be independent.

Initialization: $k=0$, $n=0$.

Step 1. $k \leftarrow k+1$, $\delta=(0,0,0)$, $\mathcal{W}=0$, $\mathcal{T}_k \leftarrow (L-y_0)^2/g_k^2$, $\mathcal{E}_0=0$ and $\mathcal{E}_1=e_n$.

Step 2. While $\mathcal{E}_1 \leq \mathcal{T}_k$ do:

- set $n \leftarrow n+1$
- $\delta \leftarrow \dfrac{\mathcal{T}_k-\mathcal{E}_1}{\mathcal{T}_k-\mathcal{E}_0}\delta + \sqrt{\dfrac{(\mathcal{T}_k-\mathcal{E}_1)(\mathcal{T}_k-\mathcal{E}_0)}{\mathcal{T}_k-\mathcal{E}_0}}\, G_n$
- If $\kappa V_n \leq \gamma(\mathcal{E}_1, L- \parallel \mathcal{E}_1(L-y_0)(1,0,0)/\mathcal{T}_k + \delta \parallel)$ then $\mathcal{W} \leftarrow 1$ else $\mathcal{W} \leftarrow 0$
- $\mathcal{E}_0 \leftarrow \mathcal{E}_1$ and $\mathcal{E}_1 \leftarrow \mathcal{E}_1 + e_n$

Step 3. If $\mathcal{W}=0$ then $\mathcal{Y} \leftarrow \mathcal{T}_k$ otherwise go to *Step 1.*

Outcome: the random variable $\mathcal{Y}$.

Proposition 1.5 (Herrmann-Zucca, 2019)**.** *Let us assume that $\tau_L < \infty$ almost surely where τ_L stands for the first passage time of the diffusion* (1.1) *through the level L. Under the assumptions 1.2 and 1.6, both the outcome $\mathcal{Y}$ of Algorithm (HZ) and τ_L have the same distribution.*

The detailed proof of Proposition 1.5 is presented in [16]. We do not present here the sketch of the proof since most of the arguments are quite similar to those pointed out in the proof of Proposition 1.2. Nevertheless we would like to say that Herrmann and Zucca didn't study a time-dependent drift term as appearing in equation (1.1), they focus their attention on the homogeneous case. The statement of Proposition 1.5 is therefore an adaptation of their result to the non-homogeneous case: here the function γ depends both on the time and space variables.

Remark 1.6. *The algorithm (HZ) can be adapted to the particular case of a continuous diffusion process starting at time $\mathbb{T}_0 > 0$ with the value y. In this case, $(Y_t)_{t \geq \mathbb{T}_0}$ is solution of the following stochastic equation:*

$$dY_t = \alpha(t, Y_t)\, dt + dB_t, \quad \forall t \geq \mathbb{T}_0 \quad \textit{and} \quad Y_{\mathbb{T}_0} = y < L.$$

Moreover the definition of the first passage time is slightly modified τ_L becomes the first time after $\mathbb{T}_0$ such that the diffusion hits the level L. The modification of the algorithm consists in replacing y_0 by y, $\gamma(\cdot, \cdot)$ by $\gamma(\mathbb{T}_0 + \cdot, \cdot)$ and adding $\mathbb{T}_0$ to $\mathcal{Y}$. We then denote $(HZ)^{y,L}_{\mathbb{T}_0}$ the corresponding algorithm.

1.2 Stopped continuous diffusion

In the previous section, several procedures of exact simulation have been presented:

- simulation of $Y_{\mathbb{T}}$: the value of the diffusion (1.1) at a fixed time $\mathbb{T}$.
- simulation of the first passage time through the level L for the diffusion, denoted τ_L.

On one hand we are able to generate the position, on the other hand the exit time. In order to complete the description, we introduce a suitable combination of the time and the position which shall play an essential role in the sequel. We propose to build an algorithm which permits to obtain the exact simulation of the random couple $(\tau_L \wedge \mathbb{T}, Y_{\tau_L \wedge \mathbb{T}})$, linked to the stopped diffusion. Here $(Y_t)_{t \geq 0}$ still stands for a continuous diffusion.

Let us first introduce a preliminary result concerning a particular diffusion process, the standard Brownian motion $(B_t)_{t \geq 0}$. Our aim is to generate a random variable which has the same conditional distribution than $B_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$, where $\mathbb{T}$ is fixed and τ_L is the Brownian first passage time (we shall assume that $L > 0$, the other case can be obtained by symmetry arguments). We build the following algorithm.

CONDITIONAL BROWNIAN MOTION GIVEN $\tau_L > \mathbb{T}$ – ALGORITHM $(CBM)^L_{\mathbb{T}}$

1. *Let $(G_n)_{n\geq 1}$ a sequence of independent standard gaussian random variables*
2. *Let $(U_n)_{n\geq 1}$ a sequence of indep. uniformly distributed random variables on $[0,1]$.*

The sequences $(G_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ are assumed to be independent.

Initialization: $n = 1$, $\mathcal{Y} = 0$.
While $\sqrt{\mathbb{T}}\, G_n > L$ or $-\frac{\mathbb{T}}{2L}\ln(U_n) > L - \sqrt{\mathbb{T}}\, G_n$ **do** $n \leftarrow n+1$.
Set $\mathcal{Y} \leftarrow \sqrt{\mathbb{T}} G_n$.
Outcome: The random variable $\mathcal{Y}$.

Proposition 1.7. *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then both the outcome $\mathcal{Y}$ of Algorithm $(CBM)^L_{\mathbb{T}}$ and $B_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$ have the same distribution.*

Proof. The result is based on the classical acceptance/rejection sampling. Let us first describe the conditional distribution of $B_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$ (see for instance Lerche [27])

$$u(\mathbb{T},x)\,dx := \mathbb{P}(\tau_L > \mathbb{T}, B_{\mathbb{T}} \in dx) = \left(\frac{1}{\sqrt{\mathbb{T}}}\,\phi\left(\frac{x}{\sqrt{\mathbb{T}}}\right) - \frac{1}{\sqrt{\mathbb{T}}}\,\phi\left(\frac{x-2L}{\sqrt{\mathbb{T}}}\right)\right) dx,$$

where ϕ denotes the distribution function of a standard Gaussian variate. We introduce Φ the corresponding cumulative distribution. Then the previous expression leads to

$$f_{\mathbb{T}}(x)\,dx := \mathbb{P}(B_{\mathbb{T}} \in dx | \tau_L > \mathbb{T}) = \frac{1}{\sqrt{\mathbb{T}}}\frac{\phi(x/\sqrt{\mathbb{T}}) - \phi((x-2L)/\sqrt{\mathbb{T}})}{\Phi(L/\sqrt{\mathbb{T}}) - \Phi(-L/\sqrt{\mathbb{T}})}\,dx. \tag{1.11}$$

We remark that the following upper-bound is satisfied

$$f_{\mathbb{T}}(x) \leq c\,\frac{\phi(x/\sqrt{\mathbb{T}})}{\sqrt{\mathbb{T}}\Phi(L/\sqrt{\mathbb{T}})}\,1_{]-\infty,L]}(x) =: c\,g_{\mathbb{T}}(x) \quad \text{with} \quad c = \frac{\Phi(L/\sqrt{\mathbb{T}})}{\Phi(L/\sqrt{\mathbb{T}}) - \Phi(-L/\sqrt{\mathbb{T}})}.$$

It is obvious that $g_{\mathbb{T}}(\cdot)$ corresponds to a distribution function: a centered Gaussian distribution of variance $\mathbb{T}$ conditioned to stay under the value L. In the acceptance/rejection procedure, we shall choose $g_{\mathbb{T}}$ as the proposal distribution. So we generate a random variable Z with distribution $g_{\mathbb{T}}$, this variate is then accepted if $c\,U g_{\mathbb{T}}(Z) \leq f_{\mathbb{T}}(Z)$ where U stands for a uniformly distributed random variable, independent of Z. The condition just mentioned is equivalent to

$$U \leq 1 - \exp\left(\frac{2L}{\mathbb{T}}(Z-L)\right).$$

If G stands for a standard Gaussian r.v., then it is obvious to relate the previous condition to

$$\sqrt{\mathbb{T}}\,G \leq L \quad \text{or} \quad -\frac{\mathbb{T}}{2L}\ln(1-U) \geq L - \sqrt{\mathbb{T}}\,G,$$

the acceptance condition appearing in Algorithm $(CBM)^L_{\mathbb{T}}$. □

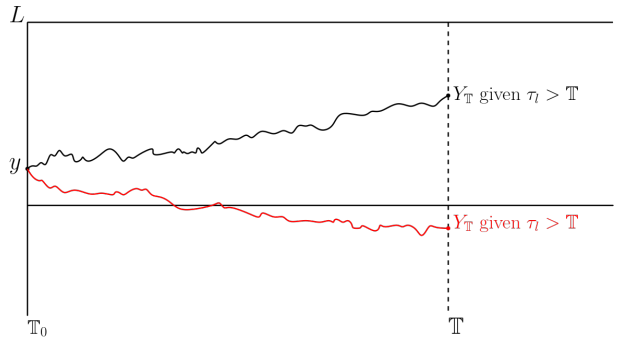

Figure 1: Trajectories of the conditioned diffusion

First we proposed in Proposition 1.7 the generation of the conditional Brownian motion. As already pointed out in the previous section, we can relate distributions concerning the Brownian paths to the diffusion ones using the classical Girsanov transformation. An interesting application of this transformation is therefore to simulate a diffusion value at a fixed time $\mathbb{T}$ given $\tau_L > \mathbb{T}$. In order to get a general statement, we consider a diffusion process starting at time $\mathbb{T}_0 < \mathbb{T}$ with the value $Y_{\mathbb{T}_0} = y$. It is therefore the unique strong solution of the equation:

$$dY_t = \alpha(t, Y_t)\, dt + dB_t, \quad \forall t \geq \mathbb{T}_0, \quad \text{and} \quad Y_{\mathbb{T}_0} = y < L, \tag{1.12}$$

where $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion. The corresponding algorithm is the following (see the illustration in Figure 1).

CONDITIONED DIFFUSION $Y_{\mathbb{T}}$ GIVEN $\tau_L > \mathbb{T}$ – ALGORITHM $(CD)^{y,L}_{\mathbb{T}_0,\mathbb{T}}$

1. *Let $(U_n)_{n\geq 1}$ be independent uniformly distributed random variables on $[0, \kappa]$.*
2. *Let $(E_n)_{n\geq 1}$ be independent exponentially distributed r.v. with average $1/\kappa$.*

The sequences $(U_n)_{n\geq 1}$ and $(E_n)_{n\geq 1}$ are assumed to be independent.

Initialization: $n = 1$.

Step 1. Set $\mathcal{Y} = y$, $\mathcal{T} = \mathbb{T}_0$ and generate $\mathcal{Z} \sim (CBM)^{L-\mathcal{Y}}_{E_n}$.

Step 2. While $\mathcal{T} + E_n < \mathbb{T}$ and $U_n > \gamma(\mathcal{T} + E_n, \mathcal{Y} + \mathcal{Z})$ do

- $\mathcal{T} \leftarrow \mathcal{T} + E_n$
- $n \leftarrow n + 1$
- $\mathcal{Y} \leftarrow \mathcal{Y} + \mathcal{Z}$
- Generate $\mathcal{Z} \leftarrow (CBM)^{L-\mathcal{Y}}_{E_n}$.

Step 3. If $\mathcal{T}+E_n > \mathbb{T}$ then generate $\mathcal{Z} \sim (CBM)_{\mathbb{T}-\mathcal{T}}^{L-\mathcal{Y}}$, set $\mathcal{Y} \leftarrow \mathcal{Y}+\mathcal{Z}$ and $V \sim \mathcal{U}([0,1])$ independent of all other variates otherwise set $n \leftarrow n+1$ and go to *Step 1.*
Step 4. If $V \cdot \exp(\beta_+) > \exp(\beta(\mathbb{T}, \mathcal{Y}))$ then set $n \leftarrow n+1$ and go to *Step 1.*
Outcome: The random variable $\mathcal{Y}$.

Proposition 1.8. *Let us consider $(Y_t)_{t \geq \mathbb{T}_0}$ the diffusion defined by (1.12) and τ_L the associated first passage time through the level L:*

$$\tau_L := \inf\{t \geq \mathbb{T}_0 : \ Y_t \geq L\}. \tag{1.13}$$

Under assumptions 1.2, 1.6 et 1.7, both the outcome $\mathcal{Y}$ of Algorithm $(CD)_{\mathbb{T}_0,\mathbb{T}}^{y,L}$ and $Y_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$ have the same distribution.

Proof. The algorithm $(CD)_{\mathbb{T}_0,\mathbb{T}}^{y,L}$ is clearly based on a rejection sampling. The proof uses therefore similar arguments than those pointed out in Proposition 1.2. Let us denote by $\mathcal{Y}_n$ (respectively $\mathcal{T}_n$ and $\mathcal{Z}_n$) the successive values of $\mathcal{Y}$ (resp. $\mathcal{T}$ and $\mathcal{Z}$). We also introduce a sequence of times $(\mathcal{E}_n)_{n\geq 1}$ defined by $\mathcal{E}_{n+1} = \mathcal{E}_n + E_{n+1}$ with $\mathcal{E}_0 = \mathbb{T}_0$. We finally introduce $\mathcal{N}$ the number of Step 1 used before the algorithm stops. Since the algorithm is a acceptance/rejection sampling, we have for any non negative measurable function ψ:

$$\mathbb{E}[\psi(\mathcal{Y})] = \frac{\mathbb{E}[\psi(\mathcal{Y})1_{\{\mathcal{N}=1\}}]}{\mathbb{P}(\mathcal{N}=1)} = \frac{\nu(\psi)}{\nu(1)} \quad \text{where} \quad \nu(\psi) := \mathbb{E}[\psi(\mathcal{Y})1_{\{\mathcal{N}=1\}}].$$

In the following computations, we denote by A_n the particular event: $\{\mathcal{E}_n \leq \mathbb{T} < \mathcal{E}_{n+1}\}$ and P_n shall correspond to the event:

$$P_n := \{U_1 > \gamma(\mathcal{E}_1, \mathcal{Y}_1 + \mathcal{Z}_1), \ldots, U_n > \gamma(\mathcal{E}_n, \mathcal{Y}_n + \mathcal{Z}_n)\}, \text{ for } n \geq 1 \text{ and } P_0 = \Omega.$$

We therefore obtain that

$$\nu(\psi) = \sum_{n\geq 0} \mathbb{E}\Big[\psi(\mathcal{Y}_{n+2}) 1_{P_n} 1_{\{V\cdot\exp(\beta_+) < \exp(\beta(\mathbb{T}, \mathcal{Y}_{n+2}))\}} 1_{A_n}\Big].$$

Integrating with respect to all uniformly distributed random variables (U_n) and with respect to V leads to

$$\nu(\psi) = \sum_{n\geq 0} \frac{1}{\kappa^n} \mathbb{E}\Big[\psi(\mathcal{Y}_{n+2}) \prod_{k=1}^{n} (\kappa - \gamma(\mathcal{E}_k, \mathcal{Y}_k + \mathcal{Z}_k)) \exp(\beta(\mathbb{T}, \mathcal{Y}_{n+2}) - \beta_+) 1_{A_n}\Big]$$

We note that $(\mathcal{E}_1 - \mathbb{T}_0, \ldots, \mathcal{E}_n - \mathbb{T}_0)$, given A_n, has the same distribution than $(V^{(1)}, \ldots, V^{(n)})$ an ordered n-tuple of uniform random variables $(V_1, \ldots, V_n)$ on $[0, \mathbb{T} - \mathbb{T}_0]$. Moreover the probability of the event A_n is linked to the Poisson distribution of parameter $\kappa(\mathbb{T} - \mathbb{T}_0)$. Finally on the event A_n, $(\mathcal{Y}_1, \mathcal{Y}_2, \ldots \mathcal{Y}_{n+2})$ has the same distribution than

$$(y, y + B_{V^{(1)}}, \ldots, y + B_{V^{(n)}}, y + B_{\mathbb{T}-\mathbb{T}_0}) \text{ given } \tau_{L-y}^B := \inf\{t \geq 0 : \ B_t \geq L - y\} > \mathbb{T} - \mathbb{T}_0$$

with $(B_t)_{t\geq 0}$ a standard Brownian motion independent of the n-tuple $(V_1,\dots,V_n)$. Hence

$$\begin{aligned}\nu(\psi) &= \sum_{n\geq 0}\frac{1}{\kappa^n}\mathbb{E}\Big[\psi(y+B_{\mathbb{T}-\mathbb{T}_0})\prod_{k=1}^{n}\Big(\kappa-\gamma(\mathbb{T}_0+V^{(k)},y+B_{V^{(k)}})\Big)\\ &\qquad\times\exp(\beta(\mathbb{T},y+B_{\mathbb{T}-\mathbb{T}_0})-\beta_+)\Big|\tau^B_{L-y}>\mathbb{T}-\mathbb{T}_0\Big]\frac{\kappa^n(\mathbb{T}-\mathbb{T}_0)^n}{n!}\,e^{-\kappa(\mathbb{T}-\mathbb{T}_0)}\\ &= \sum_{n\geq 0}\mathbb{E}\Big[\psi(y+B_{\mathbb{T}-\mathbb{T}_0})\prod_{k=1}^{n}\Big(\kappa-\gamma(\mathbb{T}_0+V_k,y+B_{V_k})\Big)\\ &\qquad\times\exp(\beta(\mathbb{T},y+B_{\mathbb{T}-\mathbb{T}_0})-\beta_+)\Big|\tau^B_{L-y}>\mathbb{T}-\mathbb{T}_0\Big]\frac{(\mathbb{T}-\mathbb{T}_0)^n}{n!}\,e^{-\kappa(\mathbb{T}-\mathbb{T}_0)}.\end{aligned}$$

Taking the expectation with respect to the uniformly distributed variates V_k leads to

$$\begin{aligned}\nu(\psi) &= \sum_{n\geq 0}\mathbb{E}\Big[\psi(y+B_{\mathbb{T}-\mathbb{T}_0})\Big(\kappa-\frac{1}{\mathbb{T}-\mathbb{T}_0}\int_0^{\mathbb{T}-\mathbb{T}_0}\gamma(\mathbb{T}_0+s,y+B_s)\,ds\Big)^n\\ &\qquad\times\exp(\beta(\mathbb{T},y+B_{\mathbb{T}-\mathbb{T}_0})-\beta_+)\Big|\tau^B_{L-y}>\mathbb{T}-\mathbb{T}_0\Big]\frac{(\mathbb{T}-\mathbb{T}_0)^n}{n!}\,e^{-\kappa(\mathbb{T}-\mathbb{T}_0)}\\ &= \mathbb{E}\Big[\psi(y+B_{\mathbb{T}-\mathbb{T}_0})\exp\Big(-\int_0^{\mathbb{T}-\mathbb{T}_0}\gamma(\mathbb{T}_0+s,y+B_s)\,ds\Big)\\ &\qquad\times\exp(\beta(\mathbb{T},y+B_{\mathbb{T}-\mathbb{T}_0})-\beta_+)\Big|\tau^B_{L-y}>\mathbb{T}-\mathbb{T}_0\Big].\end{aligned}$$

Since $(y+B_t)_{t\geq 0}$ given $\tau^B_{L-y}>\mathbb{T}-\mathbb{T}_0$ has the same distribution than $(B_t)_{t\geq\mathbb{T}_0}$ given both $B_{\mathbb{T}_0}=y$ and $\tau^B_L\circ\theta_{\mathbb{T}_0}>\mathbb{T}$, where θ stands for the translation operator, we obtain

$$\nu(\psi)=\mathbb{E}\Big[\psi(B_{\mathbb{T}})\exp\Big(-\int_{\mathbb{T}_0}^{\mathbb{T}}\gamma(s,B_s)\,ds+\beta(\mathbb{T},B_{\mathbb{T}})-\beta_+\Big)\Big|B_{\mathbb{T}_0}=y,\ \tau^B_L\circ\theta_{\mathbb{T}_0}>\mathbb{T}\Big].$$

Let us now modify the expression under review. We introduce

$$\hat\nu(\psi):=\mathbb{E}\Big[\psi(B_{\mathbb{T}})1_{\{B_t<L,\ \forall t\in[\mathbb{T}_0,\mathbb{T}]\}}\exp\Big(-\int_{\mathbb{T}_0}^{\mathbb{T}}\gamma(s,B_s)\,ds+\beta(\mathbb{T},B_{\mathbb{T}})-\beta(\mathbb{T}_0,y)\Big)\Big|B_{\mathbb{T}_0}=y\Big].$$

It is obvious that

$$\mathbb{E}[\psi(\mathcal{Y})]=\frac{\nu(\psi)}{\nu(1)}=\frac{\hat\nu(\psi)}{\hat\nu(1)}. \tag{1.14}$$

Moreover since $(M_t)_{t\geq 0}$ defined by

$$M_t:=\exp\Big(-\int_{\mathbb{T}_0}^{\mathbb{T}_0+t}\gamma(s,B_s)\,ds+\beta(\mathbb{T}_0+t,B_{\mathbb{T}_0+t})-\beta(\mathbb{T}_0,y)\Big),$$

is the exponential martingale appearing in the Girsanov transformation, we obtain by the change of measure procedure:

$$\hat\nu(\psi)=\mathbb{E}\Big[\psi(Y_{\mathbb{T}})1_{\{Y_t<L,\ \forall t\in[\mathbb{T}_0,\mathbb{T}]\}}\Big],$$

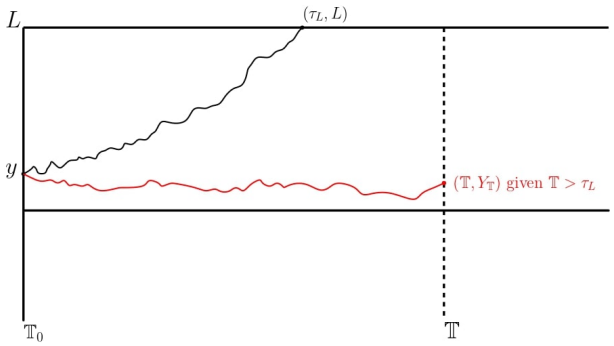

Figure 2: Two typical trajectories of the continuous diffusion.

where $(Y_t)_{t\geq\mathbb{T}_0}$ stands for the diffusion (1.12). The ratio defined by (1.14) then permits to conclude the proof

$$\mathbb{E}[\psi(\mathcal{Y})] = \mathbb{E}[\psi(Y_{\mathbb{T}})|\tau_L > \mathbb{T}],$$

the stopping time τ_L being introduce in the statement (1.13).

□

Finally we are able to produce an algorithm which generates exactly the distribution of the couple $(\tau_L \wedge \mathbb{T}, Y_{\tau_L\wedge\mathbb{T}})$ where $(Y_t)_{t\geq\mathbb{T}_0}$ stands for the continuous diffusion defined in (1.12) and τ_L, the first passage time defined in (1.13) (see the illustration in Figure 2).

STOPPED DIFFUSION: $(\tau_L \wedge \mathbb{T}, Y_{\tau_L\wedge\mathbb{T}})$ – ALGORITHM $(SD)^{y,L}_{\mathbb{T}_0,\mathbb{T}}$

Step 1. Generate $\mathcal{T} \sim (HZ)^{y,L}_{\mathbb{T}_0}$ (defined in Remark 1.6).

Step 2. If $\mathcal{T} < \mathbb{T}$ then set $\mathcal{Y} \leftarrow L$ otherwise generate $\mathcal{Y} \sim (CD)^{y,L}_{\mathbb{T}_0,\mathbb{T}}$ and set $\mathcal{T} \leftarrow \mathbb{T}$.
Outcome: The random couple $(\mathcal{T}, \mathcal{Y})$.

Proposition 1.9. *Let us consider $(Y_t)_{t\geq\mathbb{T}_0}$ the diffusion defined by (1.12) and τ_L the associated first passage time (1.13). Under assumptions 1.2, 1.6 et 1.7, both the outcome $(\mathcal{T}, \mathcal{Y})$ of Algorithm $(SD)^{y,L}_{\mathbb{T}_0,\mathbb{T}}$ and $(\tau_L \wedge \mathbb{T}, Y_{\tau_L\wedge\mathbb{T}})$ have the same distribution.*

Proof. The proof is straightforward. Either τ_L is smaller than $\mathbb{T}$ which corresponds to $\tau_L \wedge \mathbb{T} = \tau_L$ and $Y_{\tau_L\wedge\mathbb{T}} = L$ or τ_L is larger than $\mathbb{T}$ and therefore the distribution of $Y_{\tau_L\wedge\mathbb{T}}$ is the conditional distribution of $Y_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$. □

2 Simulation of the first passage time for stopped jump diffusions

The aim of this section is to generate the first passage time of a jump diffusion. This challenging objective was already considered by Giesecke and Smelov [12] who adapted the

localization approach introduced in [10] to that particular context. We propose here a different method based on the work of Herrmann and Zucca [16] and on the generation of Bessel processes. Let us just recall that the jump diffusion is characterized by the stochastic differential equation between the jump times (0.3) and the jump height described in (0.4). In the introduction, we discussed the possibility to reduce the considered model. Consequently we shall first consider the following reduced model. Let us introduce $(T_n)_{n\geq 1}$ the sequence of jump times. We mention that the time spent between two consecutive jumps is exponentially distributed, therefore $T_n = \sum_{k=1}^{n} E_k$ with $(E_k)_{k\geq 1}$ a sequence of independent exponentially distributed random variables with average $1/\lambda$. The initial position of the diffusion is given by $Y_0 = y_0$ and the jump diffusion under consideration moreover satisfies

$$dY_t = \alpha(t, Y_t)\, dt + dB_t, \quad \text{for } T_n < t < T_{n+1}, \quad n \in \mathbb{N}, \tag{2.1}$$

the jumps modify the trajectories as follows:

$$Y_{T_n} = Y_{T_n-} + j(T_n, Y_{T_n-}, \xi_n), \quad \forall n \in \mathbb{N}, \tag{2.2}$$

where $j : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{E} \to \mathbb{R}$ denotes the jump function and $(\xi_n)_{n\geq 1}$ stands for a sequence of independent random variables with distribution function ϕ/λ (also independent of the Brownian motion $(B_t)_{t\geq 0}$ and of the sequence $(T_n)_{n\geq 0}$). Let us associate to the stochastic process defined by (2.1)–(2.2) the first passage time throught the level L (with $L > y_0$):

$$\tau_L := \inf\{t \geq 0 : \ Y_t \geq L\}. \tag{2.3}$$

Since the jump times and the behaviour of the diffusion process inbetween the jump times are independent, we can use the approach developped in the continuous diffusion case in order to simulate jump diffusions. The crucial argument is that $(t, Y_t)_{t\geq 0}$ is a Markovian stochastic process. Consequently the diffusion paths can be constructed in a piecewise Markovian way. Let us present the algorithm for the generation of the stopped first passage time $\tau_L \wedge \mathbb{T}$ where $\mathbb{T}$ stands for a fixed time.

STOPPED JUMP DIFFUSION ($\tau_L \wedge \mathbb{T}$) – ALGORITHM $(SJD)_{\mathbb{T}}^{y,L}$

1. *Let $(E_n)_{n\geq 1}$ be independent exponentially distributed r.v. with average $1/\lambda$.*
2. *Let $(\xi_n)_{n\geq 1}$ be independent r.v. with distribution function ϕ/λ.*

The sequences $(E_n)_{n\geq 1}$ and $(\xi_n)_{n\geq 1}$ are assumed to be independent.
Initialization. $n = 0$, $\mathcal{T}_s = 0$ (starting time), $\mathcal{T}_f = 0$ (final time), $\mathcal{Y} = y$, $\mathcal{Z} = y$.
Step 1. While ($\mathcal{T}_f < \mathbb{T}$ and $\mathcal{Y} < L$ and $\mathcal{Z} < L$) do

- $n \leftarrow n + 1$

- $\mathcal{T}_s \leftarrow \mathcal{T}_f$
- $\mathcal{T}_f \leftarrow \mathcal{T}_f + E_n$
- Generate $(\mathcal{S}, \mathcal{Z}) \sim (SD)^{\mathcal{Y},L}_{\mathcal{T}_s,\mathcal{T}_f}$
- $\mathcal{Y} \leftarrow \mathcal{Z} + j(\mathcal{T}_f, \mathcal{Z}, \xi_n)$

Step 2.

- If $\mathcal{S} > \mathbb{T}$ then set $\mathcal{S} \leftarrow \mathbb{T}$
- If $\mathcal{S} \leq \mathbb{T}$ and $\mathcal{Z} < L$ then $\mathcal{S} \leftarrow \mathcal{T}_f$

Outcome: The random variable $\mathcal{S}$.

Theorem 2.1. *Let us consider $(Y_t)_{t\geq 0}$ the jump diffusion defined by (2.1)–(2.2) and τ_L the associated first passage time (2.3). Under assumptions 1.2, 1.6 et 1.7, both the outcome $\mathcal{S}$ of Algorithm $(SJD)^{y_0,L}_{\mathbb{T}}$ and $\tau_L \wedge \mathbb{T}$ have the same distribution.*

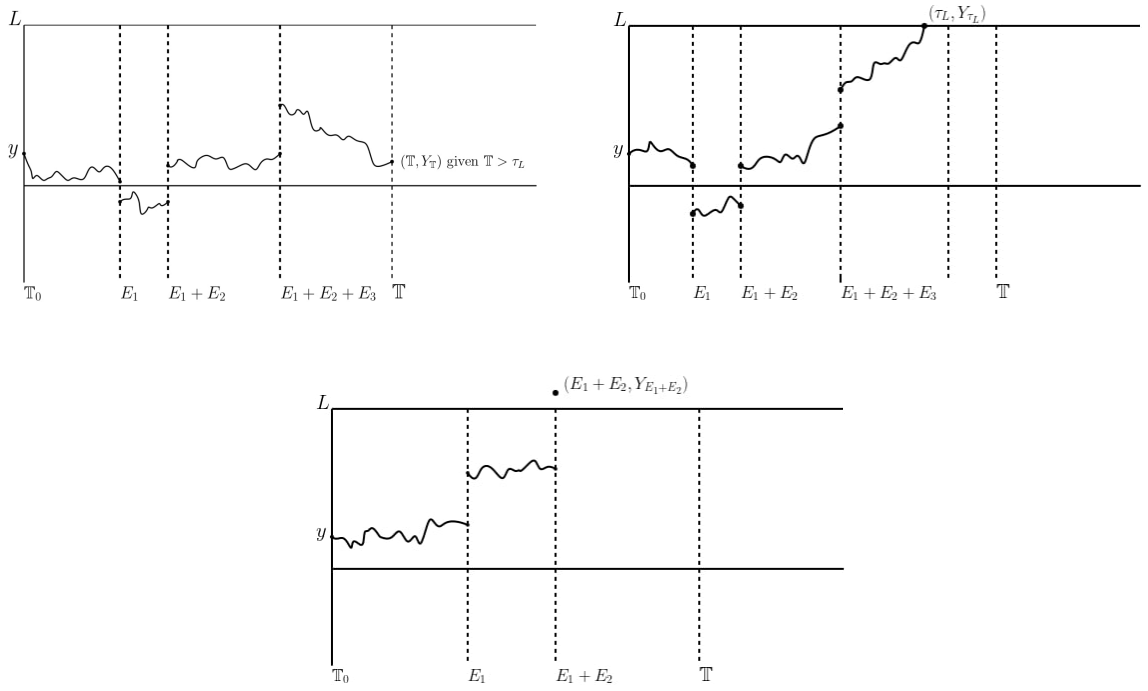

Figure 3: Three typical paths representing different scenarios

Proof. The proof is based on the Markov property of the process $(t, Y_t)_{t\geq 0}$. We start an iterated procedure. First we set $\mathcal{T}_s = 0$ (starting time) and generate a random variable

which represents the first jumping time T_1. This r.v. is denoted $\mathcal{T}_f = E_1$ in the algorithm (final time). Two situations can therefore be considered (see Figure 3).

1. First case: $\mathcal{T}_f > \mathbb{T}$ (this means that Step 1 is only used once). Conditionally to the event *"the first jumping time is larger than* $\mathbb{T}$", the diffusion behaves like a continuous diffusion process on the interval $[0, \mathbb{T}]$ that we shall denote $(\hat{Y}_t)$. Hence $Y_t = \hat{Y}_t$ for any $t < \mathcal{T}_f$. Indeed the jumping times are independent of the Brownian motion driving the diffusive part of the stochastic process. Consequently the results developed in the previous section can be applied. The algorithm $(SD)^{y_0,L}_{\mathcal{T}_s,\mathcal{T}_f}$ permits to obtain the random couple $(\hat{\tau}_L \wedge \mathcal{T}_f, \hat{Y}_{\hat{\tau}_L \wedge \mathcal{T}_f})$ where $\hat{\tau}_L$ stands for the first passage time of the continuous diffusion $(\hat{Y}_t)$ after time $\mathcal{T}_s$. The random couple is denoted by $(\mathcal{S}, \mathcal{Z})$ in Algorithm $(SJD)^{y,L}_{\mathbb{T}}$. We observe then one of the following situation:

   - $\hat{\tau}_L \wedge \mathcal{T}_f > \mathbb{T}$ which is equivalent to the condition $\mathcal{S} > \mathbb{T}$. In such a situation we deduce easily that $\hat{\tau}_L \wedge \mathbb{T} = \mathbb{T} = \tau_L \wedge \mathbb{T}$, since $\mathcal{T}_f > \mathbb{T}$, and therefore it suffices to set $\mathcal{S} = \mathbb{T}$ in the algorithm in order to obtain the distribution identity announced in the statement of Theorem 2.1.
   - $\hat{\tau}_L \wedge \mathcal{T}_f \leq \mathbb{T}$ corresponding to ($\mathcal{S} \leq \mathbb{T}$ and $\mathcal{Z} \geq L$). Then obviously we get $\tau_L \wedge \mathbb{T} = \tau_L = \tau_L \wedge \mathcal{T}_f = \hat{\tau}_L \wedge \mathcal{T}_f$ which is identically distributed as $\mathcal{S}$.

2. Second case: $\mathcal{T}_f \leq \mathbb{T}$. Here also the diffusion (Y_t) corresponds to a continuous diffusion $(\hat{Y}_t)$ on the time interval $[\mathcal{T}_s, \hat{\tau}_L \wedge \mathcal{T}_f[$. So the previous section permits to generate $(\hat{\tau}_L \wedge \mathcal{T}_f, \hat{Y}_{\hat{\tau}_L \wedge \mathcal{T}_f})$, denoted by $(\mathcal{S}, \mathcal{Z})$. Of course $\mathcal{S} \leq \mathbb{T}$. We distinguish three different situations:

   - $\mathcal{Z} \geq L$: this means that $\mathcal{S} = \hat{\tau}_L \wedge \mathcal{T}_f = \hat{\tau}_L = \tau_L = \tau_L \wedge \mathbb{T}$ which corresponds to the statement of Theorem 2.1.
   - $\mathcal{Z} < L$ and $\mathcal{Z} + j(\mathcal{T}_f, \mathcal{Z}, \xi_1) \geq L$ which implies that the diffusion doesn't cross the level L before the first jump whereas the first jump permits to observe this first passage. In other words, the first passage time corresponds to the first jump. Hence $\tau_L = \mathcal{T}_f$ (this situation occurs in the algorithm in the second step as both $\mathcal{S} \leq \mathbb{T}$ and $\mathcal{Z} < L$).
   - $\mathcal{Z} < L$ and $\mathcal{Z} + j(\mathcal{T}_j, \mathcal{Z}, \xi_1) < L$. In such a situation, the jump diffusion does not reach any value larger than L on the interval $[\mathcal{T}_s, \mathcal{T}_f] = [0, E_1]$. Indeed $\mathcal{Z} < L$ implies that $\hat{\tau}_L \wedge \mathcal{T}_f = \mathcal{T}_f$, the only possibility to overcome the level L on the time interval $[\mathcal{T}_s, \mathcal{T}_f]$ being to observe a suitable jump at time $\mathcal{T}_f$. Unfortunately such an event cannot happen since $\mathcal{Z} + j(\mathcal{T}_j, \mathcal{Z}, \xi_1) < L$. In conclusion the first passage time is strictly larger than the first jump time. In order to generate the FPT, we propose therefore to start again using the Markov property: we should observe a jump diffusion starting at time $\mathcal{T}_f$ (becoming the starting time $\mathcal{T}_s$) with the value $\mathcal{Z} + j(\mathcal{T}_j, \mathcal{Z}, \xi_1) < L$. In that situation, the algorithm permits to repeat Step 1 with new initial values.

Since $\tau_L \wedge \mathbb{T}$ is finite a.s. only a finite number of repetition of Step 1 is observed, the iterative procedure which is directly associated to the Markov property of the jump diffusion permits then to conclude and to obtain the announced statement. □

Theorem 2.1 concerns the generation of the finite stopping time $\tau_L \wedge \mathbb{T}$ associated to the reduced model (2.1)–(2.2) (we recall that $\mathbb{T}$ is a fixed time). Using the Lamperti transformation presented in the introduction, it is possible to generalize the study (see [31], for more details).

Particular models with $\tau_L < \infty$ a.s.

If the considered first passage time is a.s. finite, we are able to extend the previous algorithm. This new algorithm permits to generate the first passage time τ_L of the considered jump diffusion through the level L: it suffices to set $\mathbb{T} = +\infty$. However it is not so easy in general to determine if such a condition is satisfied. Let us focus our attention on particular assumptions related to the diffusion characteristics (diffusion coefficient, drift term, jump measure) which lead to the required event $\tau_L < \infty$.

In fact the condition can be related to usual tools for studying the transience or recurrence of stochastic processes. The proofs of this kind of results are often based on Lyapunov functions. Wee proposed an interesting study for the recurrence or transience of d-dimensional jump diffusions [39]. We shall just present the arguments in the particular one-dimensional case in order to point out conditions for the first passage time to be a.s. finite.

Let us recall the definition of the jump diffusion (0.2) in situations where the coefficient are time-homogeneous:

$$dX_t = \mu(X_{t-})\,dt + \sigma(X_{t-})\,dB_t + \int_{\mathcal{E}} j(X_{t-}, v) p_\phi(dv \times dt), \quad t \geq 0, \quad X_0 = y_0. \tag{2.4}$$

Here $p_\phi(dv \times dt)$ stands for the Poisson measure of intensity $\phi(dv)dt$. Let us introduce the compensated Poisson measure

$$\hat{p}_\phi(dv \times dt) := p_\phi(dv \times dt) - \phi(dv)dt.$$

Equation (2.4) can easily be rewritten using $\hat{p}_\phi$ instead of p_ϕ just by changing the drift term of the diffusion process $\mu(\cdot)$ by

$$\hat{\mu}(x) = \mu(x) + \int_{\mathcal{E}} j(x, v)\phi(dv), \quad \forall x \in \mathbb{R}.$$

Let us recall that $(X_t,\, t \geq 0)$ has the same distribution than $(Y_t,\ t \geq 0)$ defined by (0.3)–(0.4) with time-homogeneous coefficients. So we shall focus our attention to the first passage time problem for the diffusion $(X_t,\, t \geq 0)$. In the particular case $\mathcal{E} = \mathbb{R}$, we obtain the jump diffusion introduced by Wee:

$$X_t = y_0 + \int_0^t \hat{\mu}(X_{s-})\,ds + \int_0^t \sigma(X_{s-})dB_s + \int_0^t \int_{\mathbb{R}} j(X_{s-}, v)\hat{p}_\phi(dv \times ds), \quad t \geq 0. \tag{2.5}$$

We shall denote by $\mathbb{P}_{y_0}$ the probability distribution of such a solution. In order to state the result concerning the first passage time that is $\mathbb{P}_{y_0}(\tau_L < \infty) = 1$ for $y_0 < L$ (the symmetric case can be handled with similar arguments), we need to introduce several assumptions. We assume in particular that the coefficients are regular and the diffusion nondegenerate.

Assumption 2.2. *Let us suppose that there exists a constant $K > 0$ such that*

$$|\hat{\mu}(x) - \hat{\mu}(y)|^2 + |\sigma(x) - \sigma(y)|^2 + \int_{\mathbb{R}} |j(x,v) - j(y,v)|^2 \phi(dv) \leq K|x-y|^2 \tag{2.6}$$

and

$$|\hat{\mu}(x)|^2 + |\sigma(x)|^2 + \int_{\mathbb{R}} |j(x,v)|^2 \phi(dv) \leq K(1+|x|^2). \tag{2.7}$$

Moreover there exists $\sigma_0 > 0$ such that $\sigma(x) \geq \sigma_0$ for all $x \in \mathbb{R}$.

Under Assumption 2.2, it is well known that (2.5) admits a unique strong solution which is right-continuous with left-hand limits, see for instance Theorem 9.1 in [22] for homogeneous coefficients and Theorem 1.19 in [33] in the general non homogeneous case. Let us introduce two additional assumptions which are crucial in order to obtain almost sure finite times τ_L when the initial value y_0 satisfies $y_0 < L$.

Assumption 2.3. *There exists $r > 0$ such that:*

- *the following bound holds*

$$\sup_{y \leq -r} \int_{\mathbb{R}} \left(\ln \left(\frac{|y + j(y+L+r,v)|}{|y|} \right) \right)^2 \phi(dv) =: \kappa_{L,r} < \infty. \tag{2.8}$$

- *there exist $\epsilon > 0$ and $\eta > 0$ satisfying*

$$y\hat{\mu}(y+L+r) + \int_{\mathbb{R}} \left(y^2 \ln \left(\frac{|y + j(y+L+r,v)|}{|y|} \right) - y j(y+L+r,v) \right) \phi(dv) \\ < \frac{(1-\epsilon)}{2} \sigma^2(y+L+r) - \eta y^2, \tag{2.9}$$

  for all $y \leq -r$.

The first part of Assumption 2.3 requires that the jumps are not too large. The second part corresponds to a kind of competition between the drift, the diffusion coefficient and the jump measure. A careful reading of the proof of Theorem 1 in [39] permits to adapt the statement to the situation just introduced above.

Theorem 2.4. *Let Assumption 2.2 and Assumption 2.3 be satisfied, then $\mathbb{P}_{y_0}(\tau_L < \infty) = 1$ for any $y_0 \leq L$, where τ_L is the first passage time through the level L for the diffusion* (2.5).

We completed this section by emphasizing a particular situation which insures that $\tau_L < \infty$ almost surely. It concerns a time-homogeneous jump diffusion process since the essential tools used in the proof of Theorem 2.4 are based on the generator and on an explicit Lyapunov function. Nevertheless the use of suitable comparison results should permit to point out examples with almost surely finite first passage times associated to non-homogeneous jump diffusions.

3 Numerical illustrations

In this last section, we present two examples of jump diffusion processes and propose simulation experiments which illustrate the algorithms introduced in Section 2. Let us just recall that the diffusion $(Y_t,\, t \geq 0)$ satisfies (0.3)-(0.4) with a determinsistic starting value $Y_0 = y_0$. In all examples and without loss of generality, we shall focus our attention to the case $y_0 < L$ where L stands for the level the diffusion process should overcome (τ_L denotes the first passage time).

Example of a stopped jump diffusion.

We consider the time-homogeneous jump diffusion (0.3)-(0.4) with coefficients $\mu(t,y) = 2 + \sin(y)$ and $\sigma(t,y) = 1$. In other words the diffusion starting in y_0 satisfies the following SDE inbetween the jump times:

$$dY_t = (2 + \sin(Y_t))\, dt + dB_t, \quad \text{for } T_i < t < T_{i+1}, \quad i \in \mathbb{N}, \tag{3.1}$$

and the jumps satisfy

$$Y_{T_i} = Y_{T_i -} + j(T_i, Y_{T_i -}, \xi_i), \quad i \in \mathbb{N}.$$

Let us just recall that $T_i = \sum_{k=1}^{i} E_k$ with E_k exponentially distributed random variables with average $1/\lambda$. We set $\lambda = 1$ in the whole section. Moreover $(\xi_i)_{i\geq 1}$ is a sequence of i.i.d variates with density ϕ which will be precised later on. We fix $\mathbb{T} > 0$ and we aim to generate $\tau_L \wedge \mathbb{T}$ using Algorithm $(SJD)_{\mathbb{T}}^{y,L}$ (see Theorem 2.1) and therefore we need to verify that the assumptions 1.2, 1.6 and 1.7 hold.

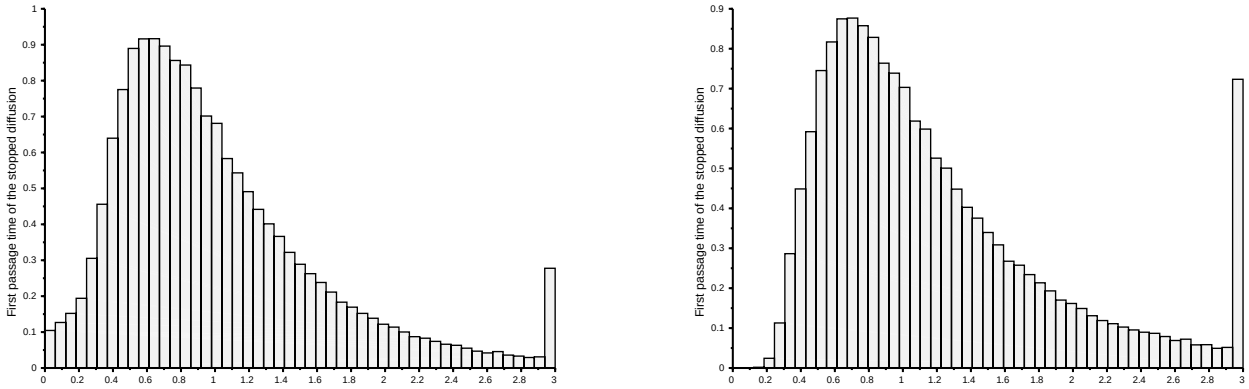

Figure 4: Histograms of the stopping time $\tau_L \wedge \mathbb{T}$ for the jump diffusion (3.1) with $j(t,y,z) = -z\sin(y)$. Here $y_0 = -1$, $L = 1$, $\mathbb{T} = 3$, $\lambda = 1$ and the size of the sample equals 100 000. The noise used for the jump generation corresponds to $\phi(t) = e^{-t}1_{\{t\geq 0\}}$ (left) or $\phi(t) = 2\,1_{[-1/4,1/4]}(t)$ (right).

The first assumption concerns the obvious regularity of $\alpha(t,y) = \mu(t,y)$. The second and third assumptions consist in pointing out bounds for both functions $\gamma(t,y)$ and $\beta(t,y)$

defined by (1.5). Here we observe that

$$\beta(t,y) = \int_0^y (2+\sin(x))\,dx = 2y + 1 - \cos(y) \le 2L + 2 =: \beta_+, \quad \forall y \le L.$$

and

$$0 \le \gamma(t,y) = \frac{(2+\sin(y))^2 + (2+\sin(y))'}{2} = \frac{(2+\sin(y))^2 + \cos(y)}{2} \le 5 =: \kappa, \quad \forall y \le L.$$

Let us also describe the jump function: we choose $j(t,y,z) = -z\sin(y)$. The histograms emphasizing the distribution of the first passage time $\tau_L \wedge \mathbb{T}$ are represented in Figure 4. Let us note that the exact generation is quite time-consuming: the sample associated to the left figure requires about CPU 377 sec while the right one requires about CPU 150 sec.

Example of a jump diffusion which satisfies $\tau_L < \infty$.

Let us conclude the numerical illustrations with a jump diffusion starting in $y_0 < L$ and satisfying $\tau_L < \infty$. We introduce to that end a stochastic process which satisfies on one hand assumptions 2.2 and 2.3 insuring the finiteness of τ_L (see Theorem 2.4) and on the other hand assumptions 1.2, 1.6 et 1.7. All these assumptions permit to use Algorithm $(\mathrm{JD})^{y_0,L}$ for the generation of the stopping time τ_L. We consider $(Y_t)_{t\ge 0}$ the solution of (3.1) between to consecutive jump times. The jumps are associated to a time-homogeneous function $j(y,v) = (L+1-y)v$ as follows:

$$Y_{T_i} = Y_{T_i-} + j(Y_{T_i-}, \xi_i), \quad i \in \mathbb{N}.$$

This particular jump function drives the stochastic process towards the threshold L. Let us just recall that $T_i = \sum_{k=1}^i E_k$ with E_k exponentially distributed random variables with average 1. Moreover $(\xi_i)_{i\ge 1}$ is a sequence of independent uniformly distributed variates with density $\phi(v) = 1_{[0,1]}(v)$. This model satisfies the announced assumptions quite easily. Let us just point out the arguments used for Assumption 2.3. First we observe that, for any $r > 1$, $v \in]0,1[$ and $y \le -r$,

$$\ln(1-v) \le \ln\left(\frac{|y + j(y+L+r,v)|}{|y|}\right) = \ln\left(\frac{|y(1-v) + (1-r)v|}{|y|}\right) \le \ln(1-\frac{v}{r}). \quad (3.2)$$

This inequality leads to the condition (2.8) since

$$\int_{\mathbb{R}} \left(\ln\left(\frac{|y + j(y+L+r,v)|}{|y|}\right)\right)^2 \phi(dv) \le \int_0^1 (\ln(1-v))^2\,dv < \infty.$$

Finally for the condition (2.9), we note on one hand that $y\alpha(y+L+r) \le 0$ for any $y \le -r$ and introduce on the other hand the constant

$$\eta := -\int_0^1 \ln\left(1 - \frac{v}{r}\right) dv.$$

Then the definition of η and (3.2) imply

$$\int_{\mathbb{R}} y^2 \ln\left(\frac{|y+j(y+L+r,v)|}{|y|}\right)\phi(dv) = \int_{\mathbb{R}} y^2 \ln\left(\frac{|y(1-v)+(1-r)v|}{|y|}\right)\phi(dv) \leq -\eta y^2.$$

All the conditions presented in Assumption 2.3 are therefore satisfied. In conclusion, the first passage time τ_L is almost surely finite. We use Algorithm $(\mathrm{JD})^{y_0,L}$ in order to generate a sample of this stopping time: the histogram in Figure 5 describes the probability distribution of the random variables. The generation of a sample of size 100 000 requires a processing time of about CPU 90 sec for the case $y_0 = -1$ and CPU 1 000 sec for the case $y_0 = -3$ (here we used the C++ programming language). This large processing time is strongly related to the nature of the algorithm which is based on an acceptance/rejection procedure.

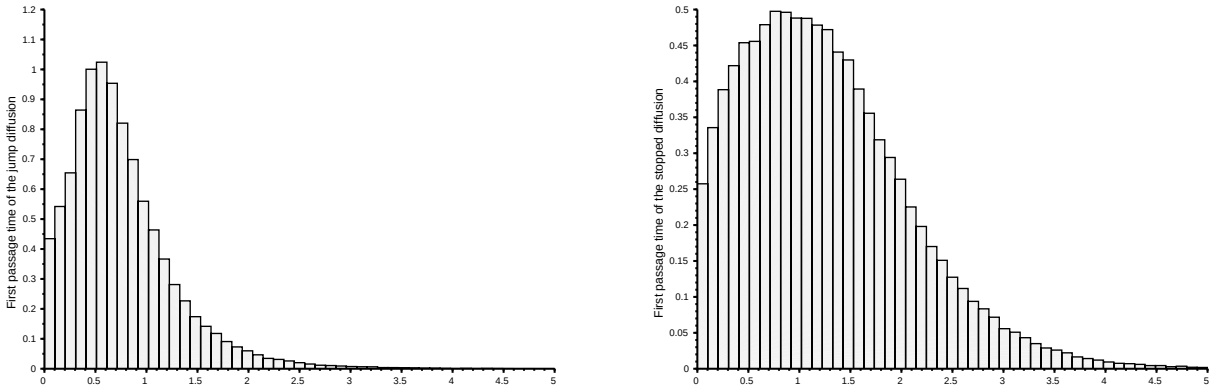

Figure 5: Histograms of the stopping time τ_L for the jump diffusion (3.1) with $j(t,y,z) = (L+1-y)z$. Here $L = 1$, the size of the sample equals 100 000 and $y_0 = -1$or $y_0 = -3$ (right). The noise used for the jump generation corresponds to $\phi(t) = 1_{[0,1]}(t)$.

References

[1] A. Beskos, O. Papaspiliopoulos, and G. O. Roberts. Retrospective exact simulation of diffusion sample paths with applications. *Bernoulli*, 12(6):1077–1098, 2006.

[2] A. Beskos, O. Papaspiliopoulos, and G. O. Roberts. A factorisation of diffusion measure and finite sample path constructions. *Methodology and Computing in Applied Probability*, 10(1):85–104, 2008.

[3] A. Beskos, G. O. Roberts, et al. Exact simulation of diffusions. *The Annals of Applied Probability*, 15(4):2422–2444, 2005.

[4] M. Broadie, P. Glasserman, and S. Kou. A continuity correction for discrete barrier options. *Mathematical Finance*, 7(4):325–349, 1997.

[5] E. Buckwar and M.G. Riedler. Runge–kutta methods for jump–diffusion differential equations. *Journal of Computational and Applied Mathematics*, 236(6):1155–1182, 2011.

[6] A. N. Burkitt. A review of the integrate-and-fire neuron model: I. homogeneous synaptic input. *Biological cybernetics*, 95(1):1–19, 2006.

[7] B. Casella and G. O. Roberts. Exact monte carlo simulation of killed diffusions. *Advances in Applied Probability*, 40(1):273–291, 2008.

[8] B. Casella and G. O. Roberts. Exact simulation of jump-diffusion processes with Monte Carlo applications. *Methodol. Comput. Appl. Probab.*, 13(3):449–473, 2011.

[9] B. Casella and G. O. Roberts. Exact simulation of jump-diffusion processes with monte carlo applications. *Methodology and Computing in Applied Probability*, 13(3):449–473, 2011.

[10] N. Chen and Z. Huang. Localization and exact simulation of Brownian motion-driven stochastic differential equations. *Math. Oper. Res.*, 38(3):591–616, 2013.

[11] A. Gardoń. The order of approximations for solutions of Itô-type stochastic differential equations with jumps. *Stochastic analysis and applications*, 22(3):679–699, 2004.

[12] K. Giesecke and D. Smelov. Exact sampling of jump diffusions. *Oper. Res.*, 61(4):894–907, 2013.

[13] E. Gobet. Weak approximation of killed diffusion using euler schemes. *Stochastic processes and their applications*, 87(2):167–197, 2000.

[14] E. Gobet and S. Menozzi. Stopped diffusion processes: boundary corrections and overshoot. *Stochastic Processes and Their Applications*, 120(2):130–162, 2010.

[15] F. B. Gonçalves and G. O. Roberts. Exact simulation problems for jump-diffusions. *Methodol. Comput. Appl. Probab.*, 16(4):907–930, 2014.

[16] S. Herrmann and C. Zucca. Exact simulation of the first-passage time of diffusions. *J. Sci. Comput.*, 79(3):1477–1504, 2019.

[17] S. Herrmann and C. Zucca. Exact simulation of diffusion first exit times: algorithm acceleration. *arXiv preprint arXiv:2004.02313*, 2020.

[18] S. Herrmann and C. Zucca. Exact simulation of first exit times for one-dimensional diffusion processes. *ESAIM: Mathematical Modelling and Numerical Analysis*, 54(3):811–844, 2020.

[19] D.J. Higham and P.E. Kloeden. Numerical methods for nonlinear stochastic differential equations with jumps. *Numerische Mathematik*, 101(1):101–119, 2005.

[20] D.J. Higham and P.E. Kloeden. Convergence and stability of implicit methods for jump-diffusion systems. *Int. J. Numer. Anal. Model*, 3(2):125–140, 2006.

[21] Q. Hu, Y. Wang, and X. Yang. The hitting time density for a reflected brownian motion. *Computational Economics*, 40(1):1–18, 2012.

[22] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, second edition, 1989.

[23] J. Janssen, O. Manca, and R. Manca. *Applied Diffusion Processes from Engineering to Finance*. John Wiley & Sons, 2013.

[24] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[25] P.E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1992.

[26] S. G. Kou. A jump-diffusion model for option pricing. *Management science*, 48(8):1086–1101, 2002.

[27] H. R. Lerche. *Boundary crossing of Brownian motion*, volume 40 of *Lecture Notes in Statistics*. Springer-Verlag, Berlin, 1986. Its relation to the law of the iterated logarithm and to sequential analysis.

[28] V. Linetsky. Lookback options and diffusion hitting times: a spectral expansion approach. *Finance Stoch.*, 8(3):373–398, 2004.

[29] Y. Maghsoodi. Mean square efficient numerical solution of jump-diffusion stochastic differential equations. *Sankhyā Ser. A*, 58(1):25–47, 1996.

[30] Y. Maghsoodi. Exact solutions and doubly efficient approximations of jump-diffusion Itô equations. *Stochastic Anal. Appl.*, 16(6):1049–1072, 1998.

[31] N. Massin. *First passage time for diffusions*. PhD thesis, Université de Bourgogne, 2021. https://hal.archives-ouvertes.fr/tel-03242974.

[32] B. Øksendal. *Stochastic differential equations*. Universitext. Springer-Verlag, Berlin, 1985. An introduction with applications.

[33] B.K. Øksendal and A. Sulem. *Applied stochastic control of jump diffusions*, volume 498. Springer, 2005.

[34] V. Pieper, M. Dominé, and P. Kurth. Level crossing problems and drift reliability. *Mathematical Methods of Operations Research*, 45(3):347–354, 1997.

[35] E. Platen. An approximation method for a class of Itô processes with jump component. *Litovsk. Mat. Sb.*, 22(2):124–136, 1982.

[36] E. Platen and N. Bruti-Liberati. *Numerical solution of stochastic differential equations with jumps in finance*, volume 64. Springer Science & Business Media, 2010.

[37] M. Pollock, A. M. Johansen, and G. O. Roberts. On the exact and ε-strong simulation of (jump) diffusions. *Bernoulli*, 22(2):794–856, 2016.

[38] L. Sacerdote and M. T. Giraudo. Stochastic integrate and fire models: a review on mathematical methods and their applications. In *Stochastic biomathematical models*, pages 99–148. Springer, 2013.

[39] I.-S. Wee. Recurrence and transience for jump-diffusion processes. *Stochastic Anal. Appl.*, 18(6):1055–1064, 2000.