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On a set-valued Young integral with applications to differential inclusions

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Abstract

We present a new Aumann-like integral for a Hölder multifunction with respect to a Hölder signal, based on the Young integral of a particular set of Hölder selections. This restricted Aumann integral has continuity properties that allow for numerical approximation as well as an existence theorem for an abstract stochastic differential inclusion. This is applied to concrete examples of first order and second order stochastic differential inclusions directed by fractional Brownian motion.

Keywords: set-valued integral, Aumann integral, Young integral

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1. Introduction

Consider $d, e \in \mathbb{N}^*$, a β -Hölder continuous signal $w : [0, T] \rightarrow \mathbb{R}^d$ with $\beta \in (0, 1)$, and an α -Hölder continuous multifunction F defined on $[0, T]$ with convex compact values in the set $M_{e,d}(\mathbb{R})$ of linear mappings from \mathbb{R}^d to \mathbb{R}^e , where $\alpha \in (0, 1)$ and $\alpha + \beta > 1$. The purpose of this paper is to define a set-valued Young integral of F with respect to w , of the form

$$\int_0^T F(s) dw(s) = \left\{ \int_0^T f(s) dw(s) ; f \in \mathcal{S}(F) \right\} \quad (1)$$

in a nontrivial way, but with a small enough set of selections $\mathcal{S}(F)$, so as to get algebraic and topological properties (convexity, boundedness, compactness, continuity, etc.) allowing to give a sense and establish the existence of solutions to several types of differential inclusions.

There have been many different approaches to set-valued integration with respect to a nonnegative σ -additive measure μ . The most popular one is due to Aumann [4], based on Lebesgue integrals of selections:

$$\int_0^T F(s) d\mu(s) = \left\{ \int_0^T f(s) d\mu(s) ; f \in \mathcal{S}_{L^1(\mu)}(F) \right\} \quad (2)$$

where $\mathcal{S}_{L^1(\mu)}(F)$ denotes the set of all μ -integrable selections of F . In the case of multifunctions with convex compact values, other approaches such as Hukuhara's [15] or Debreu's [10] are restricted to multifunctions with compact convex values and use the cone structure of the space of convex compact sets.

The concept of Aumann integral has been applied in several papers to integration of multivalued stochastic processes using classical stochastic calculus and a definition of the form (1), where w is a Brownian motion, or more generally a semimartingale, e.g., [18, 22, 23]. Despite this similarity, the setting of stochastic calculus is quite different from ours, since the stochastic integral needs a probability space to make sense. In this context, some variants have been developed, the main one by Jung and Kim [16], which is the decomposable hull of an integral of the form (1), has been extensively studied by Polish mathematicians from Zielona Góra [19, 20, 17, 24, 25], to cite but a few papers and a book.

In the case of a deterministic signal w with possibly infinite variation, Michta and Motyl [26, 27] are the only references so far defining a set-valued Young integral à la Aumann of the form (1), for convex as well as nonconvex-valued multifunctions. In their approach, the set of selections $\mathcal{S}(F)$ is large, namely, in the case of our setting, $\mathcal{S}(F)$ is the set of all α -Hölder continuous selections of F . Of course this is a natural definition, and the authors obtain basic expected properties on the set-valued integral: nonemptiness, convexity and regularity of the integral with respect to $\mathcal{S}(F)$ (not F). In our approach, the set of selections is smaller:

$$\mathcal{S}_{\alpha,r}(F) := \{f \text{ selection of } F : \|f\|_{\alpha,T} \leq r\}.$$

The "tuning parameter" $r > 0$ controls both the α -Hölder seminorm of F and of the considered selections. This allows to establish the compactness of the integral, to get the upper semicontinuity of the integral with respect to F , and then to establish the existence of solutions to some differential inclusions. Note that our integral converges to that of Michta and Motyl [26] when the tuning parameter r goes to $+\infty$. However, our integral is always compact-valued, whereas that of [26] may be unbounded, see Example 3.11 below.

On differential inclusions driven by α -Hölder continuous signals, let us mention Bailleul *et al.* [6]. In this paper, the authors establish the existence of solutions to a differential inclusion using the approach of Aubin and Cellina [2]. Let us also cite Levakov and Vas'kovskii [21] who mix pathwise integration with respect to fractional Brownian motion with Itô's integral with respect to standard Brownian motion, following Guerra and Nualart [1]. These works on differential inclusions implicitly use an Aumann type set-valued integral of the form (1).

As an application of our set-valued Young integral, we are able to define a stochastic set-valued integral with respect to the fractional Brownian motion (fBm) of Hurst index $H > 1/2$, and then to establish the existence of solutions to several types of stochastic differential inclusions driven by the fBm.

This paper is organized as follows. Section 2 recalls preliminary definitions and results on Steiner's selections and on the Young integral for point-valued functions. Section 3 deals with the set-valued Young integral. Finally, Section 4 provides a fixed-point theorem for functionals of the set-valued Young integral which allows, in particular, to get the existence of solutions to several types of differential inclusions.

2. Preliminaries

2.1. Notations and basic definitions

Let $d \geq 1$ be an integer.

1. The set of nonempty closed subsets of \mathbb{R}^d is denoted by $\mathcal{P}_f(\mathbb{R}^d)$. The semi-Hausdorff distance on $\mathcal{P}_f(\mathbb{R}^d)$ is denoted by d_H : for all $(A, B) \in \mathcal{P}_f(\mathbb{R}^d)^2$

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B) ; \sup_{b \in B} d(b, A) \right\} = \sup_{x \in \mathbb{R}^d} |d(x, A) - d(x, B)| \quad (3)$$

(see, e.g., [7]). For every $A \in \mathcal{P}_f(\mathbb{R}^d)$, we denote $\|A\|_{d_H} := d_H(A, \{0_{\mathbb{R}^d}\}) = \sup\{\|a\| ; a \in A\}$.

2. Let $\mathcal{P}_{ck}(\mathbb{R}^d)$ be the space of nonempty, convex and compact subsets of \mathbb{R}^d . For any $C \in \mathcal{P}_{ck}(\mathbb{R}^d)$, the *support function* of C is the map

$$\delta^*(\cdot, C) : \begin{cases} \mathbb{R}^d & \rightarrow \mathbb{R} \\ l & \mapsto \max_{x \in C} \langle l, x \rangle. \end{cases}$$

We have, for all $A, B \in \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$,

$$d_{\text{H}}(A, B) = \sup_{\ell \in \mathbb{R}^d, \|\ell\|=1} |\delta^*(\ell, A) - \delta^*(\ell, B)|.$$

For any $C \in \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$ and $\ell \in \mathbb{R}^d$, consider

$$Y(\ell, C) := \{c \in C : \langle \ell, c \rangle = \delta^*(\ell, C)\}.$$

If $Y(\ell, C)$ contains exactly one element, it is denoted by $y(\ell, C)$ and called *exposed point* of C with *exposing direction* ℓ . The set of all exposing directions for a given point of C is denoted by \mathcal{T}_C . For all $A, B \in \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$, the *Demyanov distance* between A and B is defined by

$$d_{\text{D}}(A, B) := \sup\{\|y(\ell, A) - y(\ell, B)\| ; \ell \in \mathcal{T}_A \cap \mathcal{T}_B\}.$$

Note that $d_{\text{D}}(A, B) \geq d_{\text{H}}(A, B)$ for all $A, B \in \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$, see, e.g., [29]. On \mathbb{R}^d , which we identify with the subset of singletons of $\mathcal{P}_{\text{ck}}(\mathbb{R}^d)$, both distances d_{D} and d_{H} coincide with the distance induced by $\|\cdot\|$.

3. For any $\alpha \in]0, 1[$, $C^{\alpha\text{-H}\ddot{o}l}([0, T]; \mathcal{P}_{\text{ck}}(\mathbb{R}^d))$ (respectively $C_{\text{D}}^{\alpha\text{-H}\ddot{o}l}([0, T]; \mathcal{P}_{\text{ck}}(\mathbb{R}^d))$) is the space of α -Hölder continuous maps from $[0, T]$ into $\mathcal{P}_{\text{ck}}(\mathbb{R}^d)$, where $\mathcal{P}_{\text{ck}}(\mathbb{R}^d)$ is endowed with the Hausdorff distance (respectively, the Demyanov distance). In the sequel, $C^{\alpha\text{-H}\ddot{o}l}([0, T]; \mathcal{P}_{\text{ck}}(\mathbb{R}^d))$ is equipped with the α -Hölder norm $N_{\alpha, T}(\cdot) := \|\cdot\|_{\infty, T} + \|\cdot\|_{\alpha, T}$, where, for $F \in C^{\alpha\text{-H}\ddot{o}l}([0, T]; \mathcal{P}_{\text{ck}}(\mathbb{R}^d))$,

$$\|F\|_{\alpha, T} = \sup \left\{ \frac{d_{\text{H}}(F(s), F(t))}{|t - s|^{\alpha}} ; s, t \in [0, T] \text{ and } s < t \right\},$$

and (with inconsistent but convenient notations)

$$\|F\|_{\infty, T} = \sup_{t \in [0, T]} \|F(t)\|_{d_{\text{H}}}.$$

We denote by $\overline{B}_{\alpha, \mathcal{P}_{\text{ck}}}(f, \delta)$ the closed ball of center f and radius δ in the space $C^{\alpha\text{-H}\ddot{o}l}([0, T]; \mathcal{P}_{\text{ck}}(\mathbb{R}^d))$.

Similarly, the space $C_{\text{Dem}}^{\alpha\text{-H}\ddot{o}l}([0, T]; \mathcal{P}_{\text{ck}}(\mathbb{R}^d))$ is equipped with the α -Hölder norm $N_{\alpha, T, \text{Dem}}(\cdot) := \|\cdot\|_{\infty, T} + \|\cdot\|_{\alpha, T, \text{Dem}}$, where d_{H} is replaced by d_{D} in the above definition.

4. For any $s, t \in [0, T]$ such that $t > s$, $\mathfrak{D}_{[s, t]}$ is the set of all dissections of $[s, t]$.

2.2. Steiner point and generalized Steiner selections

Let

$$\begin{aligned} \mathcal{M} &:= \{ \mu \text{ probability measure on } B_{\mathbb{R}^d}(0, 1) \\ &\quad ; \exists \theta \in C^1(B_{\mathbb{R}^d}(0, 1); \mathbb{R}), \mu(dx) = \theta(x)dx \}. \end{aligned}$$

Definition 2.1. The *Steiner point* of $C \in \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$ is defined by

$$\text{St}(C) := \frac{1}{v_d} \int_{\mathcal{T}_C \cap B_{\mathbb{R}^d}(0,1)} y(x, C) dx \quad \text{with} \quad v_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}.$$

It is well known that $\text{St}(C) \in C$ for any $C \in \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$ (see the more general Proposition 2.5 below). Furthermore, we have:

Proposition 2.2 (Lipschitz property). *The map $\text{St} : \mathcal{P}_{\text{ck}}(\mathbb{R}^d) \mapsto \mathbb{R}^d$ is \mathfrak{k}_d -Lipschitz for the Hausdorff distance d_{H} , where the sharp Lipschitz coefficient \mathfrak{k}_d satisfies*

$$\sqrt{\frac{2d}{\pi}} < \mathfrak{k}_d < \sqrt{\frac{2(d+1)}{\pi}}. \quad (4)$$

See [31, 30] for the exact calculation of \mathfrak{k}_d . The estimation (4) can be found in [30].

The Steiner point can be generalized by replacing the probability measure dx/v_d in Definition 2.1 by any element of \mathcal{M} . This is particularly interesting in connection with the Demyanov distance.

Definition 2.3 (generalized Steiner selection). The *Generalized Steiner point* of $C \in \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$, for a measure $\mu \in \mathcal{M}$, is defined by

$$\text{St}_{\mu}(C) := \int_{B_{\mathbb{R}^d}(0,1)} \text{St}(Y(x, C)) \mu(dx).$$

The *generalized Steiner selection* of a multifunction $F : [0, T] \rightarrow \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$ with respect to a measure $\mu \in \mathcal{M}$, is the map $t \in [0, T] \mapsto \text{St}_{\mu}(F(t))$.

Proposition 2.4. *For all $C, C_1, C_2 \in \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$, $\mu_1, \mu_2, \mu \in \mathcal{M}$, $\kappa \in [0, 1]$, $\lambda, \nu \geq 0$, we have*

$$\begin{aligned} \text{St}_{\kappa\mu_1 + (1-\kappa)\mu_2}(C) &= \kappa \text{St}_{\mu_1}(C) + (1-\kappa) \text{St}_{\mu_2}(C), \\ \text{St}_{\mu}(\lambda C_1 + \nu C_2) &= \lambda \text{St}_{\mu}(C_1) + \nu \text{St}_{\mu}(C_2). \end{aligned}$$

See Baier and Farkhi [5, Lemma 4.1] for a proof.

Theorem 2.5. (*Castaing representation*) *For every measurable multifunction $F : [0, T] \rightarrow \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$, there exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ of elements of \mathcal{M} such that, for every $t \in [0, T]$,*

$$F(t) = \overline{\bigcup_{n \in \mathbb{N}} \{\text{St}_{\mu_n}(F(t))\}}.$$

See Dentcheva [11, Theorem 3.4] for a proof.

It is proved in [11] that, for any $\mu \in \mathcal{M}$, the map $\text{St}_{\mu} : \mathcal{P}_{\text{ck}}(\mathbb{R}^d) \mapsto \mathbb{R}^d$ is Lipschitz for the Hausdorff distance d_{H} . However there is no uniform bound on the Lipschitz coefficient with respect to μ . But the Demyanov distance d_{D} can be expressed using generalized Steiner points:

Proposition 2.6 (Demyanov distance and generalized Steiner points). *For every $C_1, C_2 \in \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$,*

$$d_{\text{D}}(C_1, C_2) = \sup_{\mu \in \mathcal{M}} \|\text{St}_{\mu}(C_1) - \text{St}_{\mu}(C_2)\|.$$

See Baier and Farkhi [5, Corollary 4.8] for a proof.

2.3. Young integral for single valued functions

This subsection deals with the definition and some basic properties of Young integral which allow to integrate a map $f \in C^{\alpha\text{-H\"{o}l}}([0, T]; M_{e,d}(\mathbb{R}))$ with respect to w when $\alpha \in]0, 1[$ and $\alpha + \beta > 1$. Here, $M_{e,d}(\mathbb{R})$ denotes the space of real matrices with e rows and d columns.

Theorem 2.7 (Young integral). *Consider $\alpha, \beta \in]0, 1]$ such that $\alpha + \beta > 1$, and two maps*

$$f \in C^{\alpha\text{-H\"{o}l}}([0, T]; M_{e,d}(\mathbb{R})) \text{ and } w \in C^{\beta\text{-H\"{o}l}}([0, T]; \mathbb{R}^d).$$

For every $n \in \mathbb{N}^$ and $D_n = (t_1^n, \dots, t_{m_n}^n) \in \mathfrak{D}_{[0,T]}$ such that $|D_n| \rightarrow 0$ (where $|D_n| = \max_{1 \leq k \leq m_n-1} (t_{k+1}^n - t_k^n)$) the limit*

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{m_n-1} f(t_k^n)(w(t_{k+1}^n) - w(t_k^n))$$

exists and does not depend on the dissection D_n .

Definition 2.8 (Young integral). The limit in Theorem 2.7 is denoted by

$$\int_0^T f(s) dw(s)$$

and called the *Young integral* of f with respect to w on $[0, T]$.

The following theorem provides a bound on Young's integral which is crucial in the sequel.

Theorem 2.9 (Young-Love estimate). *Consider $\alpha, \beta \in]0, 1[$ such that $\alpha + \beta > 1$, and two maps*

$$f \in C^{\alpha\text{-H\"{o}l}}([0, T]; M_{e,d}(\mathbb{R})) \text{ and } w \in C^{\beta\text{-H\"{o}l}}([0, T]; \mathbb{R}^d).$$

There exists a constant $\mathbf{c}_{\alpha,\beta} \geq 1$, depending only on α and β , such that, for all $s, t \in [0, T]$ such that $s < t$,

$$\left\| \int_s^t f(u) dw(u) - f(s)(w(t) - w(s)) \right\| \leq \mathbf{c}_{\alpha,\beta} \|w\|_{\beta,T} \|f\|_{\alpha,T} |t - s|^{\alpha+\beta}.$$

Therefore, for all $s, t \in [0, T]$,

$$\left\| \int_s^t f(u) dw(u) \right\| \leq \mathfrak{c}_{\alpha, \beta} \|w\|_{\beta, T} (\|f\|_{\alpha, T} T^\alpha + \|f\|_{\infty, T}) |t - s|^\beta,$$

and, in particular,

$$\left\| \int_0^\cdot f(s) dw(s) \right\|_{\beta, T} \leq \mathfrak{C}_{\alpha, \beta} \|w\|_{\beta, T} N_{\alpha, T}(f)$$

with $\mathfrak{C}_{\alpha, \beta, T} = \mathfrak{c}_{\alpha, \beta} (T^\alpha \vee 1)$.

See Friz and Victoir [12, Theorem 6.8] for a proof.

3. Set-valued Young integral

This section deals with an Aumann-like integral based on a special subset of selections. We assume the following hypothesis:

- (A) $F \in C^{\alpha\text{-H\"{o}l}}([0, T]; \mathcal{P}_{\text{ck}}(M_{e, d}(\mathbb{R})))$, and $w \in C^{\beta\text{-H\"{o}l}}([0, T]; \mathbb{R}^d)$, with $\alpha, \beta \in]0, 1]$, $\alpha + \beta > 1$.

We shall sometimes consider the subcase obtained by adding the following stronger assumption on F :

- (B) $F \in C_{\text{Dem}}^{\alpha\text{-H\"{o}l}}([0, T]; \mathcal{P}_{\text{ck}}(M_{e, d}(\mathbb{R})))$.

To show that (B) is not contained in (A), we can consider the case when $F(t)$ is the segment of \mathbb{R}^2 with one end at $(0; 0)$ and the other end at $(\sin t, \cos t)$ (see [29, Example 3.3]). Then $d_{\mathbb{D}}(F(t), F(s)) \geq 1$ for $s \neq t$, thus $N_{\alpha, T, \text{Dem}}(F) = +\infty$ for all α , whereas F is Lipschitz for $d_{\mathbb{H}}$.

3.1. Special selections

We now define an appropriate set of selections of F , that will be used to define a set-valued Young integral of F with respect to w .

Let us choose our "tuning parameter" r such that

$$r \geq \mathfrak{k}_{ed} \|F\|_{\alpha, T}, \tag{5}$$

where \mathfrak{k}_{ed} is the constant defined in Proposition 2.2. In the case when (B) is satisfied, we can alternatively take

$$r \geq \|F\|_{\alpha, T, \text{Dem}}. \tag{6}$$

Note that Condition (6) can be less restrictive than (5), when (B) is satisfied. For example, if $F(t)$ has the form $f(t) + C$, where f is single-valued and $C \in \mathcal{P}_{\text{ck}}(\mathbb{R}^d)$ is constant, we have $\|F\|_{\alpha, T, \text{Dem}} = \|F\|_{\alpha, T} \leq \mathfrak{k}_{ed} \|F\|_{\alpha, T}$.

Notation 3.1. In this section, we denote

$$r_{\min} = \min \left(\mathfrak{k}_{ed} \|F\|_{\alpha, T}, \|F\|_{\alpha, T, \text{Dem}} \right). \quad (7)$$

Notation 3.2. We denote by $\mathcal{S}^0(F)$ the set of all measurable selections of F ,

$$\begin{aligned} \mathcal{S}_{\alpha, r}(F) &:= \{f \in \mathcal{S}^0(F) ; \|f\|_{\alpha, T} \leq r\} \text{ and} \\ \mathcal{S}_{\text{St}}(F) &:= \{\text{St}_{\mu}(F(\cdot)) ; \mu \in \mathcal{M}\} \subset \mathcal{S}^0(F). \end{aligned}$$

Remark 3.3 (Dependence on T of $\mathcal{S}_{\alpha, r}(F)$). Let $t \in [0, T[$. If f is an α -Hölder selection of F on $[0, t]$, with $\|f\|_{\alpha, t} \leq r$, there does not necessarily exist a selection $\bar{f} \in \mathcal{S}_{\alpha, r}(F)$ which extends f on $[0, T]$ and such that $\|\bar{f}\|_{\alpha, T} \leq r$. So, $\mathcal{S}_{\alpha, r}(F)$ depends on T , but we chose to keep the relatively light notation $\mathcal{S}_{\alpha, r}(F)$ without stressing this fact.

Proposition 3.4. For every $r \geq r_{\min}$, the set $\mathcal{S}_{\alpha, r}(F)$ is nonempty and convex.

Proof. The convexity of $\mathcal{S}_{\alpha, r}(F)$ stems from the convexity of the norm $N_{\alpha, T}$.

If (5) is satisfied, $\mathcal{S}_{\alpha, r}(F)$ is nonempty because it contains $\text{St}(F)$ by Proposition 2.2. Indeed, we have, for $s, t \in [0, T]$,

$$\|\text{St}(F(t)) - \text{St}(F(s))\| \leq \mathfrak{k}_{ed} d_{\text{H}}(F(t), F(s)) \leq \mathfrak{k}_{ed} \|F\|_{\alpha, T} |t - s|^{\alpha},$$

thus $\|\text{St}(F)\|_{\alpha, T} \leq r$.

Similarly, if (B) and (6) are satisfied, we have $\text{St}(F) \subset \mathcal{S}_{\alpha, r}(F)$ by Proposition 2.6. \square

Remark 3.5 (Basic properties of the selections sets).

1. The set $\mathcal{S}_{\text{St}}(F)$ is a nonempty and convex subset of $C^{\alpha\text{-Hö}}([0, T]; M_{e, d}(\mathbb{R}))$ such that, for every $f \in \mathcal{S}(F)$, $N_{\alpha, T}(f) \leq N_{\alpha, T, \text{Dem}}(F)$. Moreover, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{S}_{\text{St}}(F)$ such that, for every $t \in [0, T]$,

$$F(t) = \overline{\bigcup_{n \in \mathbb{N}} \{f_n(t)\}}.$$

Indeed, this follows from Propositions 2.6, 2.5 and 2.4.

2. Consequently, if (B) and (6) are satisfied, we have $\mathcal{S}_{\text{St}}(F) \subset \mathcal{S}_{\alpha, r}(F)$ and there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{S}_{\alpha, r}(F)$ such that, for every $t \in [0, T]$,

$$F(t) = \overline{\bigcup_{n \in \mathbb{N}} \{f_n(t)\}}. \quad (8)$$

Let us establish some topological properties of $\mathcal{S}_{\alpha, r}(F)$ in $\mathbb{L}^2([0, T]; M_{e, d}(\mathbb{R}))$ and in $C^{\alpha\text{-Hö}}([0, T]; M_{e, d}(\mathbb{R}))$.

Proposition 3.6. *For every $r \geq r_{\min}$, the set $\mathcal{S}_{\alpha,r}(F)$ is a bounded and closed subset of $C^{\alpha\text{-Hö}l}([0, T]; M_{e,d}(\mathbb{R}))$. Moreover, for every sequence $(f_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{S}_{\alpha,r}(F)$, there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that, for every $\varepsilon \in]0, \alpha]$, $(f_{n_k})_{k \in \mathbb{N}}$ converges in $C^{(\alpha-\varepsilon)\text{-Hö}l}([0, T]; M_{e,d}(\mathbb{R}))$ to an element of $\mathcal{S}_{\alpha,r}(F)$.*

Proof. Let us prove each property of the set $\mathcal{S}_{\alpha,r}(F)$ stated in Proposition 3.6.

Consider $f \in \mathcal{S}_{\alpha,r}(F)$. Then, $N_{\alpha,T}(f) \leq \|F\|_{\infty,T} + r$. Since r does not depend on f , $\mathcal{S}_{\alpha,r}(F)$ is a bounded subset of $C^{\alpha\text{-Hö}l}([0, T]; M_{e,d}(\mathbb{R}))$.

Consider a sequence $(f_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{S}_{\alpha,r}(F)$ which converges in $C^{\alpha\text{-Hö}l}([0, T]; M_{e,d}(\mathbb{R}))$ (equipped with $N_{\alpha,T}$). Its limit is denoted by f . In particular,

$$\lim_{n \rightarrow \infty} \|f_n(t) - f(t)\| = 0 ; \forall t \in [0, T]. \quad (9)$$

On the one hand,

$$N_{\alpha,T}(f) \leq \sup_{n \in \mathbb{N}} N_{\alpha,T}(f_n) \leq \|F\|_{\infty,T} + r.$$

On the other hand, for every $t \in [0, T]$, since $F(t)$ is a closed subset of $M_{e,d}(\mathbb{R})$, and since $f_n(t) \in F(t)$ for every $n \in \mathbb{N}$, $f(t) \in F(t)$ by the convergence result (9). So, $f \in \mathcal{S}_{\alpha,r}(F)$, and then $\mathcal{S}_{\alpha,r}(F)$ is a closed subset of $C^{\alpha\text{-Hö}l}([0, T]; M_{e,d}(\mathbb{R}))$.

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\mathcal{S}_{\alpha,r}(F)$. By the definition of $\mathcal{S}_{\alpha,r}(F)$,

$$\sup_{n \in \mathbb{N}} \|f_n\|_{\infty,T} \leq \|F\|_{\infty,T} \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|f_n\|_{\alpha,T} \leq r.$$

Therefore, by Friz and Victoir [12, Proposition 5.28], there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that, for every $\varepsilon \in]0, \alpha]$, $(f_{n_k})_{k \in \mathbb{N}}$ converges in $C^{(\alpha-\varepsilon)\text{-Hö}l}([0, T]; M_{e,d}(\mathbb{R}))$ to an element of $\mathcal{S}_{\alpha,r}(F)$. \square

Proposition 3.7. *$\mathcal{S}_{\alpha,r}(F)$ is a bounded, closed and sequentially compact subset of $\mathbb{L}^2([0, T]; M_{e,d}(\mathbb{R}))$.*

Proof. Let us prove each property of the set $\mathcal{S}_{\alpha,r}(F)$ stated in Proposition 3.7.

Consider $f \in \mathcal{S}^0(F)$. For every $t \in [0, T]$,

$$\|f(t)\| \leq \sup\{\|x\| ; x \in F(t)\} = \|F(t)\|_{d_H} \leq \|F\|_{\infty,T}.$$

Then,

$$\|f\|_{L^2([0,T])} \leq T^{1/2} \|f\|_{\infty,T} \leq T^{1/2} \|F\|_{\infty,T}.$$

Therefore, $\mathcal{S}_{\alpha,r}(F) \subset \mathcal{S}^0(F)$ is bounded in $\mathbb{L}^2([0, T]; \mathbb{R}^d)$.

Consider a sequence $(f_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{S}_{\alpha,r}(F)$ converging in $\mathbb{L}^2([0, T]; M_{e,d}(\mathbb{R}))$. Its limit is denoted by f . Then, there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} \|f_{n_k}(t) - f(t)\| = 0 ; \forall t \in [0, T]. \quad (10)$$

On the one hand, for every $t \in [0, T]$, since $F(t)$ is a closed subset of $M_{e,d}(\mathbb{R})$, and since $f_{n_k}(t) \in F(t)$ for every $k \in \mathbb{N}$, we have $f(t) \in F(t)$ by the convergence

result (10). On the other hand, by Proposition 3.6, we can choose $(f_{n_k})_{k \in \mathbb{N}}$ such that, for every $\varepsilon \in]0, \alpha]$, $(f_{n_k})_{k \in \mathbb{N}}$ converges in $C^{(\alpha-\varepsilon)\text{-Hö}}([0, T]; M_{e,d}(\mathbb{R}))$, and then $f \in \mathcal{S}_{\alpha,r}(F)$. Thus, $\mathcal{S}_{\alpha,r}(F)$ is a closed subset of $\mathbb{L}^2([0, T]; M_{e,d}(\mathbb{R}))$.

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\mathcal{S}_{\alpha,r}(F)$. According to Proposition 9, there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that, for every $\varepsilon \in]0, \alpha]$, $(f_{n_k})_{k \in \mathbb{N}}$ converges in $C^{(\alpha-\varepsilon)\text{-Hö}}([0, T]; M_{e,d}(\mathbb{R}))$ to an element of $\mathcal{S}_{\alpha,r}(F)$. Note that

$$\sup_k \|f_{n_k}\|_{\infty, T} \leq \|F\|_{\infty, T} < \infty. \quad (11)$$

Then, the sequence $(f_{n_k})_k$ is uniformly integrable with respect to the Lebesgue measure on $[0, T]$ and $(f_{n_k})_k$ converges to f in $\mathbb{L}^1([0, T]; M_{e,d}(\mathbb{R}))$. Using estimation (11), the convergence holds in $\mathbb{L}^2([0, T]; M_{e,d}(\mathbb{R}))$. Thus, $\mathcal{S}_{\alpha,r}(F)$ is a sequentially compact subset of $\mathbb{L}^2([0, T]; M_{e,d}(\mathbb{R}))$. \square

3.2. Aumann-Young integral

Now, consider $w \in C^{\beta\text{-Hö}}([0, T]; \mathbb{R}^d)$ and let us define a set-valued Young integral with respect to w , using special sets of selections.

Definition 3.8 (Aumann-Young integral). The *Aumann-Young integral* of F with respect to w and parameters α and r is defined by

$$(A_{\alpha,r}) \int_0^T F(t) dw(t) = J_{T,\alpha,r}(F, w) := \left\{ \int_0^T f(t) dw(t) ; f \in \mathcal{S}_{\alpha,r}(F) \right\}.$$

Remark 3.9. Michta and Motyl define a larger Aumann-Young integral in [26]. For convex-valued F , their integral is

$$\begin{aligned} (A_{\alpha,+\infty}) \int_{0,+\infty}^T F(t) dw(t) &= J_{T,\alpha,+\infty}(F, w) \\ &:= \left\{ \int_0^T f(t) dw(t) ; f \in \mathcal{S}_0(F) \cap C^{\alpha\text{-Hö}}([0, T]; M_{e,d}(\mathbb{R})) \right\} \\ &= \bigcup_{r \geq 0} J_{T,\alpha,r}(F, w). \end{aligned}$$

See Example 3.11 below for a comparison with $(A_{\alpha,r}) \int_{0,+\infty}^T F(t) dw(t)$ when $r < \infty$.

The following proposition provides some basic properties of the set-valued Aumann-Young integral.

Proposition 3.10. *For every $r \geq r_{\min}$, the Aumann-Young integral of F with respect to w and parameters α and r is a nonempty, bounded, closed and convex subset of \mathbb{R}^e .*

Proof. Let us prove each property of the Aumann-Young integral of F with respect to w stated in Proposition 3.10.

Since $\mathcal{S}_{\alpha,r}(F)$ is nonempty (resp. convex), $J_{T,\alpha,r}(F, w)$ is nonempty (resp. convex).

By Theorem 2.9, for every $f \in \mathcal{S}_{\alpha,r}(F)$,

$$\begin{aligned} \left\| \int_0^T f(t) dw(t) \right\| &\leq \mathbf{c}_{\alpha,\beta} T^\beta \|w\|_{\beta,T} (\|f\|_{\alpha,T} T^\alpha + \|f\|_{\infty,T}) \\ &\leq \mathbf{c}_{\alpha,\beta} T^\beta (T^\alpha \vee 1) \|w\|_{\beta,T} (r + \|F\|_{\infty,T}). \end{aligned}$$

Then, the Aumann-Young integral of F with respect to w is a bounded subset of \mathbb{R}^e .

Consider a converging sequence $(j_n)_{n \in \mathbb{N}}$ of elements of $J_{T,\alpha,r}(F, w)$. Its limit is denoted by j . By the definition of $J_{T,\alpha,r}(F, w)$, for every $n \in \mathbb{N}$, there exists $f_n \in \mathcal{S}_{\alpha,r}(F)$ such that

$$j_n = \int_0^T f_n(t) dw(t).$$

By Proposition 3.6, there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that, for any $\varepsilon \in]0, \alpha]$, $(f_{n_k})_{k \in \mathbb{N}}$ converges in $C^{(\alpha-\varepsilon)\text{-H\"{o}l}}([0, T]; M_{e,d}(\mathbb{R}))$ to an element f of $\mathcal{S}_{\alpha,r}(F)$. So, by Theorem 2.9, for $\varepsilon < \alpha + \beta - 1$, for every $k \in \mathbb{N}$,

$$\begin{aligned} \left\| j_{n_k} - \int_0^T f(t) dw(t) \right\| &\leq \mathbf{c}_{\alpha,\beta} \|w\|_{\beta,T} T^\beta (\|f_{n_k} - f\|_{\alpha-\varepsilon,T} T^{\alpha-\varepsilon} + \|f_{n_k} - f\|_{\infty,T}) \\ &\leq \mathbf{c}_{\alpha,\beta} \|w\|_{\beta,T} T^\beta (T^{\alpha-\varepsilon} \vee 1) N_{\alpha-\varepsilon,T}(f_{n_k} - f) \xrightarrow[k \rightarrow \infty]{} 0. \end{aligned}$$

Therefore,

$$j = \int_0^T f(t) dw(t),$$

and then $J_{T,\alpha,r}(F, w)$ is a closed subset of \mathbb{R}^e . \square

Example 3.11 (Comparison with Michta and Motyl's Young integral). In this example, Michta and Motyl's Young integral [26] is equal to \mathbb{R} , whereas, by Proposition 3.10, $J_{T,\alpha,r}(F, w)$ is a compact interval. Assume that $\beta \leq \frac{1}{2}$, and let

$$w(t) = t^{2\beta} \cos(\pi/t) \text{ and } F(t) = [-1, 1], \quad (t \in [0, 1]).$$

We have $N_{\alpha,T}(F) = N_{\alpha,T,\text{Dem}}(F) = 1$ and, for $0 \leq s < t \leq 1$,

$$\begin{aligned} |w(t) - w(s)| &\leq |(t^{2\beta} - s^{2\beta}) \cos(\pi/t)| + |s^{2\beta} (\cos(\pi/t) - \cos(\pi/s))| \\ &\leq (t^{2\beta} - s^{2\beta}) \\ &\quad + s^{2\beta} |\cos(\pi/t) - \cos(\pi/s)|^{1-\beta} |\cos(\pi/t) - \cos(\pi/s)|^\beta \end{aligned}$$

$$\begin{aligned}
&\leq (t^{2\beta} - s^{2\beta}) + 2^{1-\beta} \pi^\beta s^{2\beta} \left(\frac{1}{s^\beta} - \frac{1}{t^\beta} \right) \\
&\leq 2(t^\beta - s^\beta) + 2^{1-\beta} \pi^\beta (t^\beta - s^\beta) \\
&\leq (t-s)^\beta (2 + 2^{1-\beta} \pi^\beta),
\end{aligned}$$

thus $\|w\|_{\beta,1} \leq 2 + 2^{1-\beta} \pi^\beta$.

Let us define a sequence $(f_n)_{n \geq 1}$ of selections of F by

$$f_n(t) = \begin{cases} \sin(\pi/t) & \text{if } \frac{1}{n} \leq t \leq 1 \\ 0 & \text{if } 0 \leq t \leq \frac{1}{n}. \end{cases}$$

Clearly, $\|f_n\|_{\alpha,1} \rightarrow +\infty$ when $n \rightarrow \infty$. Furthermore,

$$\begin{aligned}
\int_0^1 f_n(t) dw(t) &= \int_{1/n}^1 \sin(\pi/t) d(t^{2\beta} \cos(\pi/t)) \\
&= \int_{1/n}^1 \sin(\pi/t) (2\beta t^{2\beta-1} \cos(\pi/t) - \pi t^{2\beta-2} \sin(\pi/t)) dt \\
&= \int_1^n \sin(\pi u) (2\beta u^{1-2\beta} \cos(\pi u) - \pi u^{2-2\beta} \sin(\pi u)) \frac{1}{u^2} du \\
&\leq 2\beta \int_1^n u^{-1-2\beta} - 2 \sum_{k=1}^{n-1} (k+1)^{-2\beta} \int_k^{k+1} \sin^2(\pi u) du \\
&\rightarrow -\infty \text{ when } n \rightarrow \infty.
\end{aligned}$$

By convexity of the Aumann integral and symmetry of F , this shows that the integral of F with respect to w in the sense of Michta and Motyl [26] is the whole line \mathbb{R} .

Proposition 3.12 (Lipschitz continuity result with respect to the driving signal). *For every $r \geq r_{\min}$, the set-valued map*

$$J_{T,\alpha,r}(F, \cdot) : \begin{cases} \mathbf{C}^{\beta\text{-H\"{o}l}}([0, T]; \mathbb{R}^d) & \longrightarrow \mathcal{P}_{\text{ck}}(\mathbb{R}^e) \\ w & \longmapsto J_{T,\alpha,r}(F, w) \end{cases}$$

is Lipschitz continuous when $\mathcal{P}_{\text{ck}}(\mathbb{R}^e)$ is endowed with the Hausdorff distance d_{H} .

Proof. Consider $w^1, w^2 \in \mathbf{C}^{\beta\text{-H\"{o}l}}([0, T]; \mathbb{R}^d)$ and $j^1 \in J_{T,\alpha,r}(F, w^1)$. So, there exists $f^1 \in \mathcal{S}_{\alpha,r}(F)$ such that

$$j^1 = \int_0^T f^1(s) dw^1(s),$$

and by Theorem 2.9,

$$\begin{aligned}
d(j^1, J_{T,\alpha,r}(F, w^2)) &= \inf_{f \in \mathcal{S}_{\alpha,r}(F)} \left\| \int_0^T f(t) dw^2(t) - \int_0^T f^1(t) dw^1(t) \right\| \\
&\leq \left\| \int_0^T f^1(t) d(w^2 - w^1)(t) \right\| + \inf_{f \in \mathcal{S}_{\alpha,r}(F)} \left\| \int_0^T (f - f^1)(t) dw^2(t) \right\| \\
&\leq \mathbf{c}_{\alpha,\beta} T^\beta (T^\alpha \vee 1) \left[N_{\alpha,T}(f^1) \|w^1 - w^2\|_{\beta,T} + \|w^2\|_{\beta,T} \inf_{f \in \mathcal{S}_{\alpha,r}(F)} N_{\alpha,T}(f - f^1) \right].
\end{aligned}$$

Since $f^1 \in \mathcal{S}_{\alpha,r}(F)$, $N_{\alpha,T}(f^1) \leq r + \|F\|_{\infty,T}$ and the second term in the right-hand side of the previous inequality is null. Then

$$d(j^1, J_{T,\alpha,r}(F, w^2)) \leq \mathbf{c}_{\alpha,\beta} T^\beta (T^\alpha \vee 1) (r + \|F\|_{\infty,T}) \|w^1 - w^2\|_{\beta,T}$$

and, by symmetry,

$$d(j^2, J_{T,\alpha,r}(F, w^1)) \leq \mathbf{c}_{\alpha,\beta} T^\beta (T^\alpha \vee 1) (r + \|F\|_{\infty,T}) \|w^1 - w^2\|_{\beta,T}$$

for every $j^2 \in J_{T,\alpha,r}(F, w^2)$. Therefore,

$$\begin{aligned}
&d_{\mathbb{H}}(J_{T,\alpha,r}(F, w^1), J_{T,\alpha,r}(F, w^2)) \\
&= \max \left\{ \sup_{j^1 \in J_{T,\alpha,r}(F, w^1)} d(j^1, J_{T,\alpha,r}(F, w^2)) ; \sup_{j^2 \in J_{T,\alpha,r}(F, w^2)} d(j^2, J_{T,\alpha,r}(F, w^1)) \right\} \\
&\leq \mathbf{c}_{\alpha,\beta} T^\beta (T^\alpha \vee 1) (r + \|F\|_{\infty,T}) \|w^1 - w^2\|_{\beta,T}.
\end{aligned}$$

□

The following proposition provides semicontinuity results for the set-valued Young's integral. Let us first recall the topological superior and inferior limits in Kuratowski's sense for a sequence of sets, see, e.g., [7]. If (A_n) is a sequence of closed subsets of a metric space \mathbb{M} , let us denote

$\text{Li}_{n \rightarrow \infty} A_n$ the set of limits of sequences (x_n) such that $x_n \in A_n$ for every n ,

$\text{Ls}_{n \rightarrow \infty} A_n$ the set of limits of sequences (x_n) such that $x_n \in A_{m_n}$ for every n for some subsequence (A_{m_n}) of (A_n) .

Clearly, $\text{Li}_{n \rightarrow \infty} A_n \subset \text{Ls}_{n \rightarrow \infty} A_n$. We say that (A_n) converges in Kuratowski's sense to $A \subset \mathbb{M}$ if

$$\text{Ls}_{n \rightarrow \infty} A_n \subset A \subset \text{Li}_{n \rightarrow \infty} A_n.$$

Convergence in Kuratowski's sense is weaker than convergence for the Hausdorff distance, however both convergences are equivalent if \mathbb{M} is compact. Indeed, by, e.g., [8, Theorem 3.1 page 51], the set $\mathcal{P}_k(\mathbb{M})$ of compact subsets of \mathbb{M} is compact for the topology of Hausdorff distance, thus this topology coincide with any weaker separated (T2) topology on $\mathcal{P}_k(\mathbb{M})$. But, if \mathbb{M} is compact, the convergence in Kuratowski's sense is associated with a separated topology (see, e.g., [7, Theorem 5.2.6]).

Proposition 3.13 (Semicontinuity with respect to F). *Let $r \geq r_{\min}$.*

1. *Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of elements of $C^{\alpha\text{-H}\ddot{o}l}([0, T]; \mathcal{P}_{\text{ck}}(M_{e,d}(\mathbb{R})))$ such that*

$$\text{Ls}_{n \rightarrow \infty} F_n(t) \subset F(t) ; \forall t \in [0, T],$$

and that

$$F_n \in \overline{B}_{\alpha, \mathcal{P}_{\text{ck}}(M_{e,d}(\mathbb{R}))}(0, r + \varepsilon_n)$$

for every $n \in \mathbb{N}$, for some sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of elements of \mathbb{R}_+ , converging to 0. Then

$$\text{Ls}_{n \rightarrow \infty} J_{T, \alpha, r + \varepsilon_n}(F_n, w) \subset J_{T, \alpha, r}(F, w).$$

2. *Assume furthermore that*

$$F(t) \subset \text{Li}_{n \rightarrow \infty} F_n(t) ; \forall t \in [0, T].$$

Then

$$J_{T, \alpha, r}(F, w) \subset \text{Li}_{n \rightarrow \infty} J_{T, \alpha, 2r + \varepsilon_n}(F_n, w).$$

Proof. 1. Let us prove that

$$\overline{J} := \text{Ls}_{n \rightarrow \infty} J_{T, \alpha, r + \varepsilon_n}(F_n, w) \subset J_{T, \alpha, r}(F, w).$$

Consider $j \in \overline{J}$. Then, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of elements of the space $C^{\alpha\text{-H}\ddot{o}l}([0, T]; M_{e,d}(\mathbb{R}))$ such that

$$f_n \in \mathcal{S}_{\alpha, r + \varepsilon_n}(F_n) ; \forall n \in \mathbb{N}$$

and

$$j = \lim_{k \rightarrow \infty} \int_0^T f_{n_k}(t) dw(t),$$

where $(f_{n_k})_{k \in \mathbb{N}}$ is a subsequence of $(f_n)_{n \in \mathbb{N}}$. By the definition of $\mathcal{S}_{\alpha, r + \varepsilon_n}(F_n)$, $n \in \mathbb{N}$,

$$\|f_{n_k}(t)\| \leq \sup_{n, s} \|F_n(s)\|_{d_H} \leq \|F\|_{\infty, T} + \sup_{n \in \mathbb{N}} \varepsilon_n < \infty$$

for every $k \in \mathbb{N}$ and $t \in [0, T]$, and

$$\sup_{k \in \mathbb{N}} \|f_{n_k}\|_{\alpha, T} \leq r + \sup_{n \in \mathbb{N}} \varepsilon_n.$$

Then, by Friz and Victoir [12, Proposition 5.28], there exists a subsequence $(f_{m_k})_{k \in \mathbb{N}}$ of $(f_{n_k})_{k \in \mathbb{N}}$ such that, for every $\varepsilon \in]0, \alpha]$, such that $\varepsilon < \alpha + \beta - 1$, $(f_{m_k})_{k \in \mathbb{N}}$ converges in $C^{(\alpha - \varepsilon)\text{-H}\ddot{o}l}([0, T]; \mathbb{R}^d)$ to an element f of $C^{\alpha\text{-H}\ddot{o}l}([0, T]; \mathbb{R}^d)$. So,

$$j = \int_0^T f(t) dw(t).$$

It remains to check that $f \in \mathcal{S}_{\alpha,r}(F)$. For any $t \in [0, T]$, since $f_{m_k}(t) \in F_{m_k}(t)$ for every $k \in \mathbb{N}$, and since f is in particular the pointwise limit of $(f_{m_k})_{k \in \mathbb{N}}$,

$$f(t) = \limsup_{k \rightarrow \infty} f_{m_k}(t) \in \text{Ls}_{k \rightarrow \infty} F_{m_k}(t) = F(t).$$

Moreover, by Friz and Victoir [12, Lemma 5.12],

$$\|f\|_{\alpha,T} \leq \liminf_{k \rightarrow \infty} \|f_{m_k}\|_{\alpha,T} \leq r + \lim_{n \rightarrow \infty} \varepsilon_n = r.$$

Therefore, $j \in J_{T,\alpha,r}(F, w)$.

2. The supplementary hypothesis implies that $(F_n(t))$ converges in Kuratowski's sense to $F(t)$. Furthermore, since (F_n) is bounded in $C^{\alpha\text{-Hö}}([0, T]; \mathcal{P}_{\text{ck}}(M_{e,d}(\mathbb{R})))$, it is bounded for $\|\cdot\|_{\infty,T}$, thus $(F_n(t))$ converges to $F(t)$ for the Hausdorff distance.

Let $j \in J_{T,\alpha,r}(F, w)$, and let $f \in \mathcal{S}_{\alpha,r}(F)$ such that $j = \int_0^T f(t) dw(t)$. Set $f_n(t) = \pi_{F_n(t)}(f(t))$ for every $t \in [0, T]$ and each integer n , where $\pi_{F_n(t)}$ denotes the orthogonal projection on $F_n(t)$. We have, for any $t \in [0, T]$,

$$\|f(t) - f_n(t)\| \leq d_{\text{H}}(F(t), F_n(t)) \rightarrow 0 \text{ when } n \rightarrow \infty,$$

thus (f_n) converges uniformly to f . Furthermore, for any n and for $s, t \in [0, T]$,

$$\begin{aligned} \|f_n(t) - f_n(s)\| &\leq \|\pi_{F_n(t)}(f(t)) - \pi_{F_n(t)}(f(s))\| + \|\pi_{F_n(t)}(f(s)) - \pi_{F_n(s)}(f(s))\| \\ &\leq \|f(t) - f(s)\| + d_{\text{H}}(F_n(t), F_n(s)). \end{aligned}$$

Indeed, it is well known that the projection operator $\pi_{F_n(t)}$ is non expansive (see, e.g., [14, page 118]), and the estimation of $\|\pi_{F_n(t)}(f(s)) - \pi_{F_n(s)}(f(s))\|$ follows from (3). Since $r \geq \|F\|_{\alpha,T}$, we deduce

$$\|f_n\|_{\alpha,T} \leq \|f\|_{\alpha,T} + \|F_n\|_{\alpha,T} \leq 2r + \varepsilon_n.$$

Thus $j_n := \int_0^T f_n(t) dw(t) \in J_{T,\alpha,2r+\varepsilon_n}(F_n, w)$. Furthermore, thanks to [12, Proposition 6.12], we have $\lim_{n \rightarrow \infty} j_n = j$. \square

The preceding result can be improved when the multifunctions F_n are constructed from F using some recipe which can also be applied to their selections. This can be useful for numerical approximations.

Proposition 3.14. *(Time discretization of the multivalued integral) Let (D_n) be a sequence of dissections of $[0, T]$, say, $D_n = (t_0^n, \dots, t_{m_n}^n)$, $0 = t_0 < \dots < t_{m_n}^n = T$, and assume that $|D_n|$ converges to 0, where $|D_n| = \max_{1 \leq i \leq m_n-1} (t_{i+1}^n - t_i^n)$ is the mesh of D_n . For each n , for each $i \in \{1, \dots, m_n - 1\}$ and for any $t \in [t_i^n, t_{i+1}^n]$, set*

$$F_n(t) = \frac{t - t_i^n}{t_{i+1}^n - t_i^n} F(t_i^n) + \frac{t_{i+1}^n - t}{t_{i+1}^n - t_i^n} F(t_{i+1}^n).$$

Then

$$\lim_{n \rightarrow \infty} d_{\text{H}}(J_{T,r}(F_n, w), J_{T,r}(F, w)) = 0.$$

Proof. By uniform continuity of F on $[0, T]$, the sequence (F_n) converges uniformly to F for d_H , and $\|F_n\|_{d_H} \leq \|F\|_{d_H}$ for all n . Furthermore, we have $N_{\alpha, T}(F_n) \leq N_{\alpha, T}(F)$ for all n , see [9].

From Part 1 of Proposition 3.13, we have that $\text{Ls } J_{T, r}(F_n, w) \subset J_{T, \alpha, r}(F, w)$.

Now, let $j \in J_{T, \alpha, r}(F, w)$, and let $f \in \mathcal{S}_{\alpha, r}(F)$ such that $j = \int_0^T f(t) dw(t)$. Define $f_n \in \mathcal{S}_{\alpha, r}(F_n)$ by

$$f_n(t) = \frac{t - t_i^n}{t_{i+1}^n - t_i^n} f(t_i^n) + \frac{t_{i+1}^n - t}{t_{i+1}^n - t_i^n} f(t_{i+1}^n).$$

for each $i \in \{1, \dots, m_n - 1\}$ and for every $t \in [t_i^n, t_{i+1}^n]$. Then (f_n) converges uniformly to f , and we conclude as in the proof of Proposition 3.13 that $J_{T, \alpha, r}(F, w) \subset \text{Li } J_{T, r}(F_n, w)$, thus $(J_{T, r}(F_n, w))_n$ converges to $J_{T, r}(F, w)$ in Kuratowski's sense.

Since the sequence $(J_{T, r}(F_n, w))_n$ is bounded for $\|\cdot\|_{d_H}$, it is relatively compact for the Hausdorff distance, we deduce that it converges to $J_{T, r}(F, w)$ for d_H . \square

Let us conclude this section by investigating the indefinite Aumann-Young integral

$$\begin{aligned} t \mapsto (A_{\alpha, r}) \int_0^t F(s) dw(s) &= J_{t, \alpha, r}(F, w) \\ &:= \left\{ \int_0^T f(s) \mathbf{1}_{[0, t]}(s) dw(s) ; f \in \mathcal{S}_{\alpha, r}(F) \right\}. \end{aligned}$$

Remark 3.15 (Dependence on T of the indefinite integral). Since $(A_{\alpha, r}) \int_0^t F(s) dw(s)$ is built using elements of $\mathcal{S}_{\alpha, r}(F)$, it follows from Remark 3.3 that our indefinite integral depends on T . More accurate but rather heavy notations could be $(A_{\alpha, T, r}) \int_0^t F(s) dw(s) = J_{t, \alpha, T, r}(F, w)$.

Remark 3.16. The previous results on $J_{T, \alpha, r}(F, w)$ remain true for $J_{t, \alpha, r}(F, w)$, by the same arguments.

Proposition 3.17 (Continuity of the indefinite Aumann-Young integral). *The set-valued map $t \in [0, T] \mapsto J_{t, \alpha, r}(F, w)$ is β -Hölder continuous with constant $\mathfrak{C}_{\alpha, \beta, T}(\|F\|_{\infty, T} + r)\|w\|_{\beta, T}$ when $\mathcal{P}_{\text{ck}}(\mathbb{R}^e)$ is endowed with the Hausdorff distance d_H , where $\mathfrak{C}_{\alpha, \beta, T}$ is the constant defined in Theorem 2.9.*

Proof. Consider $s, t \in [0, T]$ with $s < t$, and $j_t \in J_{s, \alpha, r}(F, w)$. So, there exists $f_t \in \mathcal{S}_{\alpha, r}(F)$ such that

$$j_t = \int_0^t f_t(u) dw(u),$$

and by Theorem 2.9,

$$d(j_t, J_{s, \alpha, r}(F, w)) = \inf_{f \in \mathcal{S}_{\alpha, r}(F)} \left\| \int_0^t f_t(u) dw(u) - \int_0^s f(u) dw(u) \right\|$$

$$\begin{aligned} & \leq \left\| \int_s^t f_t(u) dw(u) \right\| + \inf_{f \in \mathcal{S}_{\alpha,r}(F)} \left\| \int_0^s (f - f_t)(u) dw(u) \right\| \\ & \leq \mathfrak{C}_{\alpha,\beta}(T^\alpha \vee 1) \|w\|_{\beta,T} \left[N_{\alpha,T}(f_t) |t - s|^\beta + T^\beta \inf_{f \in \mathcal{S}_{\alpha,r}(F)} N_{\alpha,T}(f - f_t) \right]. \end{aligned}$$

Since $f_s \in \mathcal{S}_{\alpha,r}(F)$, we have $N_{\alpha,T}(f_t) \leq (\|F\|_{\infty,T} + r)$ and the second term in the right-hand side of the previous inequality is null. Then,

$$d(j_t, J_{s,\alpha,r}(F, w)) \leq \mathfrak{C}_{\alpha,\beta}(T^\alpha \vee 1) \|w\|_{\beta,T} (\|F\|_{\infty,T} + r) |t - s|^\beta$$

and, by symmetry,

$$d(j_t, J_{s,\alpha,r}(F, w)) \leq \mathfrak{C}_{\alpha,\beta}(T^\alpha \vee 1) \|w\|_{\beta,T} (\|F\|_{\infty,T} + r) |t - s|^\beta$$

for every $j_t \in J_{t,\alpha,r}(F, w)$. Therefore,

$$\begin{aligned} & d_{\text{H}}(J_{s,\alpha,r}(F, w), J_{t,\alpha,r}(F, w)) \\ & = \max \left\{ \sup_{j_s \in J_{s,\alpha,r}(F, w)} d(j_s, J_{t,\alpha,r}(F, w)) ; \sup_{j_t \in J_{t,\alpha,r}(F, w)} d(j_t, J_{s,\alpha,r}(F, w)) \right\} \\ & \leq \mathfrak{C}_{\alpha,\beta,T} \|w\|_{\beta,T} (\|F\|_{\infty,T} + r) |t - s|^\beta. \end{aligned}$$

□

Corollary 3.18 (Upper bound for $\|J_{\cdot,\alpha,r}(F, w)\|_{\alpha,T}$ and $N_{\alpha,T}(J_{\cdot,\alpha,r}(F, w))$).
Assume that $r \geq r_{\min}$, and let $\rho_w(T, r) := \mathfrak{C}_{\alpha,\beta,T} (\|F\|_{\infty,T} + r) \|w\|_{\beta,T} T^{\beta-\alpha}$.
Then

$$\|J_{\cdot,\alpha,r}(F, w)\|_{\alpha,T} \leq \rho_w(T, r)$$

and

$$N_{\alpha,T}(J_{\cdot,\alpha,r}(F, w)) \leq (1 + T^\alpha) \rho_w(T, r).$$

Proof. Since $\alpha < \beta$, the first inequality is an immediate consequence of Proposition 3.17 and the obvious inequality $\|F\|_{\alpha,T} \leq T^{\beta-\alpha} \|F\|_{\beta,T}$ for any $F \in C^{\beta\text{-Hö}}([0, T]; \mathcal{P}_{\text{ck}}(\mathbb{R}^e))$. The second inequality follows, using that $\int_0^0 f(s) dw(s) = 0$ for any $f \in \mathcal{S}_{\alpha,r}(F)$. □

4. Existence of fixed-points for functionals of Aumann-Young's integral and applications to differential inclusions

The main theorem of this section, derived from Kakutani-Fan-Glicksberg's theorem thanks to the results of Section 3, deals with the existence of fixed-points for functionals of the Aumann-Young integral. We provide some applications to differential inclusions, especially differential inclusions driven by a fractional Brownian motion.

In the sequel, $\alpha, \beta \in]0, 1[$, $\alpha + \beta > 1$ and $\alpha < \beta$. This implies that $\beta > \frac{1}{2}$. As usual, $T > 0$ and $w \in C^{\beta\text{-Hö}}([0, T]; \mathbb{R}^d)$.

4.1. Fixed point theorem

Let us first recall the notions of fixed-points and of upper semicontinuity for multifunctions.

Definition 4.1. Consider a set S and a multifunction $F : S \rightrightarrows S$. An element x of S is a *fixed-point* of F if $x \in F(x)$.

We now give a definition of upper semicontinuity for multifunctions in a particular case which is sufficient for our needs (see [3, Definition 1.4.1 and Proposition 1.4.8]).

Definition 4.2. Let \mathbb{X} be a metric space, let \mathbb{Y} be a compact metric space. Let $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a multifunction with closed values, and let S be a closed subset of \mathbb{X} , with $S \subset \text{Dom } F := \{x \in \mathbb{X} ; F(x) \neq \emptyset\}$. The multifunction F is said to be *upper semicontinuous* on S if, for every sequence $(x_n, y_n)_{n \in \mathbb{N}}$ of elements of $S \times \mathbb{Y}$ and for every $(x, y) \in S \times \mathbb{Y}$ such that (x_n, y_n) converges to (x, y) and $y_n \in F(x_n)$ for every n , we have $y \in F(x)$.

Let us now set the scene for the fixed point theorem. Let S be a convex compact subset of $C^{\alpha\text{-Hö}}([0, T]; \mathbb{R}^e)$, and let $\Phi : [0, T] \times \mathbb{R}^e \rightarrow \mathcal{P}_{\text{ck}}(M_{\ell, d}(\mathbb{R}))$ be continuous for the Hausdorff distance, with $\ell \in \mathbb{N}^*$. Assume that there exists $R > 0$ such that

$$\Phi(\cdot, x(\cdot)) \in \overline{B}_{\alpha, \mathcal{P}_{\text{ck}}(M_{\ell, d}(\mathbb{R}))}(0, R) ; \forall x \in S.$$

Let

$$r \geq \sup_{x \in S} \min \left(\mathfrak{k}_{e\ell} \|\Phi(\cdot, x(\cdot))\|_{\alpha, T}, \|\Phi(\cdot, x(\cdot))\|_{\alpha, T, \text{Dem}} \right),$$

so that the map

$$\Phi_w : \begin{cases} S & \longrightarrow C^{\alpha\text{-Hö}}([0, T]; \mathcal{P}_{\text{ck}}(\mathbb{R}^\ell)) \\ x & \longmapsto (A_{\alpha, r}) \int_0^\cdot \Phi(s, x(s)) dw(s) \end{cases}$$

is well-defined. With the notations of Corollary 3.18, we have

$$\|\Phi_w(x)\|_{\alpha, T} \leq \rho_w(T, r)$$

for all $x \in S$.

Theorem 4.3 (Fixed point theorem). *Let S , Φ , r and Φ_w as above, let $\alpha' \in]0, \alpha[$, and let*

$$\Psi : C^{\alpha\text{-Hö}}([0, T]; \mathcal{P}_{\text{ck}}(\mathbb{R}^\ell)) \rightrightarrows C^{\alpha\text{-Hö}}([0, T]; \mathbb{R}^e),$$

a multifunction such that $\text{Dom } \Psi$ contains $\mathbb{B} := \{\Phi_w(x) ; x \in S\}$, and which satisfies the following conditions:

1. For every sequence $(F_n)_{n \in \mathbb{N}}$ of elements of \mathbb{B} such that there exists $F \in \mathbb{B}$ satisfying

$$\text{Ls}_{n \rightarrow \infty} F_n(t) \subset F(t) ; \forall t \in [0, T],$$

if $\psi_n \in \Psi(F_n)$ converges in $C^{\alpha'-\text{Hö}l}([0, T]; \mathbb{R}^e)$ to $\psi \in C^{\alpha-\text{Hö}l}([0, T]; \mathbb{R}^e)$, then $\psi \in \Psi(F)$.

2. For every $F \in \mathbb{B}$, $\Psi(F)$ is convex, closed and contained in S .

Then, $\Gamma = \Psi \circ \Phi_w$ has at least one fixed-point in S .

Remark 4.4. By Corollary 3.18, in order that $\text{Dom}(\Psi) \supset \mathbb{B}$, it is sufficient that $\text{Dom}(\Psi) \supset \overline{B}_{\alpha, \mathcal{P}_{\text{ck}}(M_{\ell, d}(\mathbb{R}))}(0, (1 + T^\alpha)\rho_w(T, r))$.

Proof of Theorem 4.3. By Condition 2, $\Gamma(x)$ is closed convex and contained in S for every $x \in S$.

Let us check that Γ is upper semicontinuous. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements of S converging to $x \in S$ in $\mathbb{X} := C^{\alpha'-\text{Hö}l}([0, T]; \mathbb{R}^e)$, and consider $\psi_n \in \Gamma(x_n) = \Psi(\Phi_w(x_n))$ converging to $\psi \in C^{\alpha-\text{Hö}l}([0, T]; \mathbb{R}^e)$ in \mathbb{X} . By Proposition 3.13 (more precisely Remark 3.16) and the Hausdorff continuity of Φ (which gives

$$\text{Ls}_{n \rightarrow \infty} \Phi(t, x_n(t)) \subset \Phi(t, x(t)) ; \forall t \in [0, T]),$$

we have

$$\text{Ls}_{n \rightarrow \infty} \Phi_w(x_n)(t) \subset \Phi_w(x)(t) ; \forall t \in [0, T].$$

Then, by Condition 1, $\psi \in \Psi(\Phi_w(x)) = \Gamma(x)$, which proves the upper semicontinuity.

We deduce by Kakutani-Fan-Glicksberg Theorem [13, Theorem 8.6 of II.§7] that Γ has at least one fixed-point in S . \square

Remark 4.5. Consider $\gamma \in]0, 1 \wedge (\beta/\alpha)]$ $\alpha\gamma + \beta > 1$. The statement of Theorem 4.3 remains true when

$$\Phi(\cdot, x(\cdot)) \in \overline{B}_{\alpha\gamma, \mathcal{P}_{\text{ck}}(M_{\ell, d}(\mathbb{R}))}(0, R) ; \forall x \in S$$

and

$$\Phi_w(x) := (A_{\alpha\gamma, r}) \int_0^\cdot \Phi(s, x(s)) dw(s).$$

For the sake of readability, this result has been detailed in the case $\gamma = 1$, but the proof of Theorem 4.3 remains unchanged for $\gamma \neq 1$, again with $S \subset C^{\alpha-\text{Hö}l}([0, T]; \mathbb{R}^e)$, but replacing α by $\alpha\gamma$ everywhere else.

Let us provide two examples of multifunctions Ψ fulfilling Conditions 1 and 2 of Theorem 4.3. These examples will be the basis for our applications to differential inclusions.

Example 4.6. Assume that T satisfies $\rho_w(T, r) \leq \mathfrak{r}$ with $\mathfrak{r} > 0$. With the notations of Theorem 4.3, let us show that $\Psi_h(\cdot) := h + \{x \in \mathcal{S}_{\alpha, \rho_w(T, r)}(\cdot) : x(0) = 0\}$, with $\ell = e$ and $h \in C^{\alpha\text{-Hö}1}([0, T]; \mathbb{R}^e)$, fulfills Conditions 1 and 2 for $S = S_{h, \mathfrak{r}} := \{x \in \overline{B}_{\alpha, \mathbb{R}^e}(h, \mathfrak{r}) : x(0) = h(0)\}$ which is a convex compact subset of $\mathbb{X} := C^{\alpha'\text{-Hö}1}([0, T]; \mathbb{R}^e)$ thanks to [12, Proposition 5.28]:

1. Let $(F_n)_{n \in \mathbb{N}}$ be a sequence in $\overline{B}_{\alpha, \mathcal{P}_{\text{ck}}(M_{e, d}(\mathbb{R}))}(0, (1 + T^\alpha)\rho_w(T, r))$ such that there exists $F \in \overline{B}_{\alpha, \mathcal{P}_{\text{ck}}(M_{e, d}(\mathbb{R}))}(0, (1 + T^\alpha)\rho_w(T, r))$ satisfying

$$\text{Ls}_{n \rightarrow \infty} F_n(t) \subset F(t) ; \forall t \in [0, T]. \quad (12)$$

Consider also $\psi_n \in \Psi_h(F_n)$ converging in \mathbb{X} to $\psi \in C^{\alpha\text{-Hö}1}([0, T]; \mathbb{R}^e)$. For any $n \in \mathbb{N}$, $(\psi_n - h)(0) = 0$ and, by the definition of $\mathcal{S}_{\alpha, \rho_w(T, r)}(F_n)$, $\psi_n - h \in \mathcal{S}^0(F_n)$ and $N_{\alpha, T}(\psi_n - h) \leq \rho_w(T, r)$. Thanks to (12), $\psi - h \in \mathcal{S}^0(F)$, and since ψ is the limit of ψ_n in \mathbb{X} , $(\psi - h)(0) = 0$ and $N_{\alpha, T}(\psi - h) \leq \rho_w(T, r)$. Therefore, $\psi \in \Psi_h(F)$.

2. For any $F \in \overline{B}_{\alpha, \mathcal{P}_{\text{ck}}(M_{e, d}(\mathbb{R}))}(0, (1 + T^\alpha)\rho_w(T, r))$, since $\mathcal{S}_{\alpha, \rho_w(T, r)}(F)$ is convex (resp. closed) by Proposition 3.4 (resp. Proposition 3.6), $\Psi_h(F)$ is convex (resp. closed). Moreover, since $\rho_w(T, r) \leq \mathfrak{r}$,

$$\begin{aligned} \Psi_h(F) &\subset h + \mathcal{S}^0(F) \cap \{x \in \overline{B}_{\alpha, \mathbb{R}^e}(0, (1 + T^\alpha)\rho_w(T, r)) : x(0) = 0\} \\ &\subset h + \{x \in \overline{B}_{\alpha, \mathbb{R}^e}(0, \mathfrak{r}) : x(0) = 0\} = S_{h, \mathfrak{r}}. \end{aligned}$$

Example 4.7. Assume that $d = \ell = 2$, $e = 1$ and that T satisfies $\rho_w(T, r) \leq \mathfrak{r}$ with $\mathfrak{r} > 0$. Consider $f \in C^{\alpha\text{-Hö}1}([0, T]; \mathbb{R})$. With the notations of Theorem 4.3 and Example 4.6, let us show that $\Psi_{h, f}(\cdot) := h + \{fx_1 - x_2 ; x = (x_1, x_2) \in \Psi_0(\cdot)\}$ fulfills Conditions 1 and 2 of Theorem 4.3 for $S = S_{h, f, \mathfrak{r}} := h + \{fx_1 - x_2 ; x = (x_1, x_2) \in S_{0, \mathfrak{r}}\}$:

1. Let $(F_n)_{n \in \mathbb{N}}$ be a sequence in $\overline{B}_{\alpha, \mathcal{P}_{\text{ck}}(M_{1, 2}(\mathbb{R}))}(0, (1 + T^\alpha)\rho_w(T, r))$, and let $F \in \overline{B}_{\alpha, \mathcal{P}_{\text{ck}}(M_{1, 2}(\mathbb{R}))}(0, (1 + T^\alpha)\rho_w(T, r))$ satisfying

$$\text{Ls}_{n \rightarrow \infty} F_n(t) \subset F(t) ; \forall t \in [0, T]. \quad (13)$$

Consider also $\psi_n \in \Psi_{h, f}(F_n)$ converging in \mathbb{X} to $\psi \in C^{\alpha\text{-Hö}1}([0, T]; \mathbb{R})$. For any $n \in \mathbb{N}$, $\psi_n = h + fx_{1, n} - x_{2, n}$ with $x_n = (x_{1, n}, x_{2, n}) \in \mathcal{S}^0(F_n)$ such that $x_n(0) = 0$ and $N_{\alpha, T}(x_n) \leq \rho_w(T, r)$. Then, by Friz and Victoir [12, Proposition 5.28], there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ converging in \mathbb{X} to $x = (x_1, x_2) \in C^{\alpha\text{-Hö}1}([0, T]; \mathbb{R})$ such that $x(0) = 0$ and $N_{\alpha, T}(x) \leq \rho_w(T, r)$. Moreover, thanks to (13), $x \in \mathcal{S}^0(F)$. So, for every $t \in [0, T]$,

$$\begin{aligned} \psi(t) &= \lim_{n \rightarrow \infty} \psi_n(t) = \lim_{k \rightarrow \infty} \psi_{n_k}(t) \\ &= h(t) + f(t) \lim_{k \rightarrow \infty} x_{1, n_k}(t) - \lim_{k \rightarrow \infty} x_{2, n_k}(t) \\ &= h(t) + f(t)x_1(t) - x_2(t). \end{aligned}$$

Therefore, $\psi \in \Psi_{h, f}(F)$.

2. For any $F \in \overline{B}_{\alpha, \mathcal{P}_{\text{ck}}(M_{1, 2}(\mathbb{R}))}(0, (1 + T^\alpha)\rho_w(T, r))$, since $\Psi_0(F)$ is convex (resp. $\Psi_0(F) \subset S_{0, \mathfrak{r}}$), $\Psi_{h, f}(F)$ is convex (resp. $\Psi_{h, f}(F) \subset S_{h, f, \mathfrak{r}}$). Moreover, the same arguments than in the previous step yield that $\Psi_{h, f}(F)$ is closed.

4.2. Applications to differential inclusions

Let us provide two applications of Theorem 4.3 to differential inclusions. First, let $\Phi : [0, T] \times \mathbb{R}^e \rightarrow \mathcal{P}_{\text{ck}}(M_{e,d}(\mathbb{R}))$ be a multifunction such that, for every $t \in [0, T]$ and every $x \in \mathbb{R}^e$, $\Phi(\cdot, x)$ is α -Hölder continuous with respect to the Hausdorff distance and $\Phi(t, \cdot)$ is Lipschitz continuous with respect to the Hausdorff distance too, that is, there exist $k_1, k_2 > 0$ such that for every $s, t \in [0, T]$ and $x, y \in \mathbb{R}^e$,

$$d_{\text{H}}(\Phi(s, x), \Phi(t, x)) \leq k_1 |t - s|^\alpha \quad \text{and} \quad d_{\text{H}}(\Phi(t, x), \Phi(t, y)) \leq k_2 \|x - y\|. \quad (14)$$

Assume also that Φ is bounded with respect to the Hausdorff distance, that is, there exists $R > 0$ such that

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^e} \sup_{y \in \Phi(t,x)} \|y\| \leq R. \quad (15)$$

Consider $w \in C^{\beta\text{-Hö}}([0, T]; \mathbb{R}^d)$ and an inclusion of the form

$$x(t) \in \xi + (A_{\alpha,r}) \int_0^t \Phi(s, x(s)) dw(s) ; t \in [0, T], \quad (16)$$

where r is large enough and the unknown function x is in $C^{\alpha\text{-Hö}}([0, T]; \mathbb{R}^e)$.

Corollary 4.8 (First order differential inclusion). *Assume that $0 < \alpha < \beta$, $\alpha + \beta > 1$ and $r \geq r_0 := R + k_1 + k_2$. Then, the set of solutions to (16) is nonempty.*

Proof. First, with the notations of Example 4.6, for any $x \in S_{\xi,1}$, the map $s \mapsto \Phi(s, x(s))$ is α -Hölder continuous. Precisely, for every $s, t \in [0, T]$,

$$\begin{aligned} d_{\text{H}}(\Phi(t, x(t)), \Phi(s, x(s))) &\leq d_{\text{H}}(\Phi(t, x(t)), \Phi(s, x(t))) + d_{\text{H}}(\Phi(s, x(t)), \Phi(s, x(s))) \\ &\leq k_1 |t - s|^\alpha + k_2 \|x(t) - x(s)\| \\ &\leq (k_1 + k_2 \|x - \xi\|_{\alpha,T}) |t - s|^\alpha \\ &\leq (k_1 + k_2) |t - s|^\alpha \end{aligned}$$

and then,

$$N_{\alpha,T}(\Phi(\cdot, x(\cdot))) \leq R + k_1 + k_2 = r_0.$$

Let $1 > T_0 > 0$ be such that $(1 + T_0^\alpha) \rho_w(T_0, r_0) \leq 1$ (see Remark 4.4). Applying Theorem 4.3 on $[0, T_0]$, with $r = r_0$, $S = S_{\xi,1}$ and $\Psi = \Psi_\xi$ (see Example 4.6), shows that $\Gamma = \Psi \circ \Phi_w$ has a fixed point on $[0, T_0]$. Since the definition of T_0 is independent of ξ , gluing solutions on successive intervals provides a fixed point for Γ on $[0, T]$, which is thus a solution to (16) on $[0, T]$. \square

Remark 4.9. Consider $\gamma \in]0, 1 \wedge (\beta/\alpha)]$ such that $\alpha\gamma + \beta > 1$. Thanks to Remark 4.5, the statement of Corollary 4.3 remains true when, for every $t \in [0, T]$, $\Phi(t, \cdot)$ is γ -Hölder continuous but not necessarily Lipschitz continuous.

Now, for $e = 1$, consider $w_0 \in C^{\beta\text{-Hö}l}([0, T]; \mathbb{R})$ and a second order inclusion of the form

$$x(t) \in \xi + (A_{\alpha, \rho_{w_0}(T, r)}) \int_0^t \left[(A_{\alpha, r}) \int_0^s \Phi(u, x(u)) dw_0(u) \right] dw_0(s). \quad (17)$$

For any $x \in C^{\alpha\text{-Hö}l}([0, T]; \mathbb{R})$ such that the Aumann-Young integral in Inclusion (17) is well defined, thanks to the integration by parts formula for Young's integral,

$$\begin{aligned} & (A_{\alpha, \rho_{w_0}(T, r)}) \int_0^t \left[(A_{\alpha, r}) \int_0^s \Phi(u, x(u)) dw_0(u) \right] dw_0(s) \\ &= \left\{ \int_0^t \int_0^s \varphi(u) dw_0(u) dw_0(s) ; \varphi \in \mathcal{S}_{\alpha, r}(\Phi(\cdot, x(\cdot))) \right\} \\ &= \left\{ w_0(t) \int_0^t \varphi(s) dw_0(s) - \int_0^t w_0(s) \varphi(s) dw_0(s) ; \varphi \in \mathcal{S}_{\alpha, r}(\Phi(\cdot, x(\cdot))) \right\} \\ &= \left\{ w_0(t) \left(\int_0^t \varphi(s) (dw_0(s), w_0(s) dw_0(s)) \right)_1 \right. \\ &\quad \left. - \left(\int_0^t \varphi(s) (dw_0(s), w_0(s) dw_0(s)) \right)_2 ; \varphi \in \mathcal{S}_{\alpha, r}(\Phi(\cdot, x(\cdot))) \right\}. \end{aligned}$$

Then, proving the existence of solutions to (17) amounts to prove that

$$\Gamma : x \mapsto \Psi_{h, w_0} \left((A_{\alpha, r}) \int_0^\cdot \Phi(s, x(s)) dw(s) \right) \text{ with } w := \left(w_0, \int_0^\cdot w_0(s) dw_0(s) \right)$$

has fixed points.

Corollary 4.10 (Second order differential inclusion). *Assume that $\alpha < \beta$, $\alpha + \beta > 1$ and $r \geq r_{w_0, T} := k_1 + k_2(N_{\alpha, T}(w_0) + 1)$. Then, the set of solutions to (17) is nonempty.*

Proof. First, consider $x_{w_0, \xi} := \xi + w_0 x_1 - x_2$ with $x = (x_1, x_2) \in S_{0,1}$. The map $s \mapsto \Phi(s, x_{w_0, \xi}(s))$ is α -Hölder continuous. Precisely, for every $s, t \in [0, T]$,

$$\begin{aligned} & d_{\mathbb{D}}(\Phi(t, x_{w_0, \xi}(t)), \Phi(s, x_{w_0, \xi}(s))) \\ &\leq d_{\mathbb{D}}(\Phi(t, x_{w_0, \xi}(t)), \Phi(s, x_{w_0, \xi}(t))) \\ &\quad + d_{\mathbb{D}}(\Phi(s, x_{w_0, \xi}(t)), \Phi(s, x_{w_0, \xi}(s))) \\ &\leq k_1 |t - s|^\alpha + k_2 |x_{w_0, \xi}(t) - x_{w_0, \xi}(s)| \\ &\leq k_1 |t - s|^\alpha + k_2 (|w_0(t)(x_1(t) - x_1(s))| \\ &\quad + |(w_0(t) - w_0(s))x_1(s)| + |x_2(t) - x_2(s)|) \\ &\leq (k_1 + k_2(N_{\alpha, T}(w_0) + 1)) |t - s|^\alpha \end{aligned}$$

and then,

$$N_{\alpha, T}(\Phi(\cdot, x(\cdot))) \leq k_1 + k_2(N_{\alpha, T}(w_0) + 1) = r_{w_0, T}.$$

Let $T_0 > 0$ be such that $(1 + T_0^\alpha)\rho_w(T_0, r_{w_0, T}) \leq 1$. By Theorem 4.3 applied on $[0, T_0]$, with $r = r_{w_0, T}$, $S = S_{\xi, w_0, 1}$ and $\Psi = \Psi_{\xi, w_0}$ (see Example 4.7), $\Gamma = \Psi \circ \Phi_w$ has a fixed point, which is thus a solution to (17) on $[0, T_0]$. Since the definition of T_0 is independent of ξ , Γ has a fixed point, which is thus a solution to (17) on $[0, T]$. \square

Let us conclude with applications to stochastic inclusions. Consider a $(d-1)$ -dimensional fractional Brownian motion $B = (B(t))_{t \in [0, T]}$ of Hurst index $H \in (1/2, 1)$, which is a centered Gaussian process such that

$$\mathbb{E}(B_i(s)B_j(t)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})\delta_{i,j}$$

for every $s, t \in [0, T]$ and $i, j \in \{1, \dots, d-1\}$, and let $(\Omega, \mathcal{A}, \mathbb{P})$ be the associated canonical probability space. By the Garcia-Rodemich-Rumsey lemma (see Nualart [28, Lemma A.3.1]), the paths of B are β -Hölder continuous for any $\beta \in (0, H)$. Consider $r > 0$ and $\alpha \in (0, 1)$ such that $\alpha + \beta > 1$ and $\alpha < \beta$. Then, for any measurable map $F : \Omega \rightarrow \overline{B}_{\alpha, \mathcal{P}_{\text{ck}}(M_{e,d}(\mathbb{R}))}(0, r)$, one can define a set-valued stochastic integral of F with respect to B by

$$\left[(A_{\alpha, r}) \int_0^t F(s) dB(s) \right] (\omega) := (A_{\alpha, r}) \int_0^t F(s, \omega) dB(s, \omega) ; \omega \in \Omega, t \in [0, T].$$

This allows to consider the stochastic inclusion

$$X(t) \in \xi + (A_{\alpha, r}) \int_0^t \Phi(s, X(s)) dw(s) ; t \in [0, T], \quad (18)$$

where $\Phi : [0, T] \times \mathbb{R}^e \rightarrow \mathcal{P}_{\text{ck}}(M_{e,d}(\mathbb{R}))$ fulfills Assumptions (14) and (15), and $W(t) := (t, B_1(t), \dots, B_{d-1}(t))$ for every $t \in [0, T]$. By Corollary 4.8, for every $r \geq k_1 + k_2 + R$, Inclusion (18) has at least one pathwise solution. One can also consider the one-dimensional second order stochastic inclusion

$$X(t) \in \xi + (A_{\alpha, \rho_B(T, r)}) \int_0^t \left[(A_{\alpha, r}) \int_0^s \Phi(u, X(u)) dB(u) \right] dB(s) ; t \in [0, T]. \quad (19)$$

By Corollary 4.8, for every $r \geq k_1 + k_2(N_{\alpha, T}(B) + 1)$, Inclusion (19) has at least one pathwise solution.

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