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► **To cite this version:**

Majid Alizadeh, Mohammad Ardehir, Philippe Balbiani, Mojtaba Mojtahedi. About the unification type of modal logic $K5$ and its extensions. 2021. hal-03252589

HAL Id: hal-03252589

<https://hal.science/hal-03252589v1>

Preprint submitted on 7 Jun 2021

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About the unification type of modal logic $\mathbf{K5}$ and its extensions

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1 Introduction

We prove that all extensions of $\mathbf{K45}$ have projective unification and $\mathbf{K5}$ and some of its extensions are of unification type 1. The breakdown of the paper is as follows. Firstly, we prove in Proposition 9 that if \mathbf{L} contains $\mathbf{K45}$ then every formula has extension property in \mathbf{L} . Secondly, generalizing some results obtained in [6], we prove in Proposition 13 that if \mathbf{L} contains $\mathbf{K5}$ then every \mathbf{L} -unifiable formula is \mathbf{L} -filtering. Thirdly, imitating arguments used in [18, 19], we prove in Proposition 20 that if \mathbf{L} contains $\mathbf{K5}$ then every formula having extension property in \mathbf{L} is \mathbf{L} -projective. Fourthly, we prove in Proposition 21 that if \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global¹ then for all substitutions σ , every formula \mathbf{L} -unified by σ is implied by an \mathbf{L} -projective formula based on the variables of the given formula and having σ as one of its \mathbf{L} -unifiers.

2 Syntax and semantics

Let \mathbf{VAR} be a countably infinite set of *variables* (with typical members denoted x, y , etc). The set \mathbf{FOR} of all *formulas* (with typical members denoted φ, ψ , etc) is inductively defined by

- $\varphi := x \mid \perp \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi$.

We adopt the standard rules for omission of parentheses. The Boolean connectives $\top, \wedge, \rightarrow$ and \leftrightarrow and the modal connective \Diamond are defined as usual. For all $\varphi \in \mathbf{FOR}$, let $\text{var}(\varphi)$ be the set of all variables occurring in φ . For all finite $X \subseteq \mathbf{VAR}$, let \mathbf{FOR}_X be the set of all $\varphi \in \mathbf{FOR}$ such that $\text{var}(\varphi) \subseteq X$.

A *substitution* is a triple (X, Y, σ) where $X, Y \subseteq \mathbf{VAR}$ are finite and $\sigma : \mathbf{FOR}_X \rightarrow \mathbf{FOR}_Y$ is a homomorphism. The sets X and Y are respectively its *domain* and its *codomain*. Let \mathbf{SUB} be the set of all substitutions. We say that $(X, Y, \sigma) \in \mathbf{SUB}$ is *variable-free* if $Y = \emptyset$. It is possible to compose two substitutions if the codomain of the first is equal to the domain of the second. The *composition* of $(X, Y, \sigma), (Y, Z, \tau) \in \mathbf{SUB}$ (in symbols $(X, Y, \sigma) \circ (Y, Z, \tau)$) is the substitution (X, Z, ν) such that for all $x \in X, \nu(x) = \tau(\sigma(x))$. When its domain and its codomain can be guessed from the context, the substitution (X, Y, σ) will be simply written σ ². For all finite $X, Y \subseteq \mathbf{VAR}$, let $\mathbf{SUB}_{X,Y}$ be the set of all $\sigma \in \mathbf{SUB}$ such that the domain of σ is X and the codomain of σ is Y .

We say that $\mathbf{L} \subseteq \mathbf{FOR}$ is a *modal logic* if the following conditions hold³: \mathbf{L} contains all tautologies, \mathbf{L} contains the formula $\Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y)$, \mathbf{L} is closed for *modus ponens* (for all $\varphi, \psi \in \mathbf{FOR}$, if $\varphi \rightarrow \psi \in \mathbf{L}$ and $\varphi \in \mathbf{L}$ then $\psi \in \mathbf{L}$), \mathbf{L} is closed for *generalization* (for all $\varphi \in \mathbf{FOR}$, if $\varphi \in \mathbf{L}$ then $\Box\varphi \in \mathbf{L}$),

¹Globality is defined in Section 2.

²However, when we write that two substitutions are equal, this will imply in any case that their domains are equal and their codomains are equal.

³The modal logics considered in this paper are exactly the *normal* modal logics considered in standard textbooks such as [12, 13, 27].

\mathbf{L} is closed for *uniform substitution* (for all $\varphi \in \mathbf{FOR}$, if $\varphi \in \mathbf{L}$ then for all substitutions (X, Y, σ) , if $\text{var}(\varphi) \subseteq X$ then $\sigma(\varphi) \in \mathbf{L}$). For all modal logics \mathbf{L} and for all $\varphi \in \mathbf{FOR}$, we write $\mathbf{L} \oplus \varphi$ for the least modal logic containing \mathbf{L} and φ . The following modal logics — and their extensions — are considered in this paper: $\mathbf{K} \oplus \Diamond x \rightarrow \Box \Diamond x$ (denoted **K5**), $\mathbf{K5} \oplus \Box x \rightarrow \Box \Box x$ (denoted **K45**), $\mathbf{K5} \oplus \Box x \rightarrow x$ (denoted **S5**), \mathbf{K} denoting the least modal logic⁴. We say that a modal logic \mathbf{L} is *consistent* if $\mathbf{L} \neq \mathbf{FOR}$. From now on in this paper, **let \mathbf{L} be a consistent modal logic**. Let $\equiv_{\mathbf{L}}$ be the equivalence relation on \mathbf{FOR} defined for all $\varphi, \psi \in \mathbf{FOR}$, by $\varphi \equiv_{\mathbf{L}} \psi$ if and only if $\varphi \leftrightarrow \psi \in \mathbf{L}$. We shall say that \mathbf{L} is *locally tabular* if for all finite $X \subseteq \mathbf{VAR}$, $\equiv_{\mathbf{L}}$ possesses finitely many equivalence classes on \mathbf{FOR}_X .

Proposition 1. *If \mathbf{L} contains **K5** then \mathbf{L} is locally tabular.*

We say that $\varphi \in \mathbf{FOR}$ is *\mathbf{L} -derivable* from $\Gamma \subseteq \mathbf{FOR}$ (in symbols $\Gamma \vdash_{\mathbf{L}} \varphi$) if there exists $n \geq 1$ and there exists $\varphi_1, \dots, \varphi_n \in \mathbf{FOR}$ such that $\varphi_n = \varphi$ and for all $k \in \{1, \dots, n\}$, at least one of the following 4 conditions holds: (i) $\varphi_k \in \mathbf{L}$, (ii) $\varphi_k \in \Gamma$, (iii) there exists $i, j \in \{1, \dots, n\}$ such that $i, j < k$ and $\varphi_i = \varphi_j \rightarrow \varphi_k$, (iv) there exists $i \in \{1, \dots, n\}$ such that $i < k$ and $\varphi_k = \Box \varphi_i$. Substitutions being completely defined by the restrictions to their domains, it is possible to compare two substitutions by means of these restrictions if their domains are equal. Let $\simeq_{\mathbf{L}}$ be the equivalence relation on \mathbf{SUB} defined for all $(X, Y, \sigma), (X, Z, \tau) \in \mathbf{SUB}$, by $(X, Y, \sigma) \simeq_{\mathbf{L}} (X, Z, \tau)$ if and only if for all $x \in X$, $\sigma(x) \leftrightarrow \tau(x) \in \mathbf{L}$ ⁵. Let $\preceq_{\mathbf{L}}$ be the quasi-order on \mathbf{SUB} defined for all $(X, Y, \sigma), (X, Z, \tau) \in \mathbf{SUB}$, by $(X, Y, \sigma) \preceq_{\mathbf{L}} (X, Z, \tau)$ if and only if there exists $(Z, T, v) \in \mathbf{SUB}$ such that for all $x \in X$, $\sigma(x) \leftrightarrow v(\tau(x)) \in \mathbf{L}$ ⁶.

Proposition 2. *If \mathbf{L} is locally tabular then for all finite $X, Y \subseteq \mathbf{VAR}$, $\simeq_{\mathbf{L}}$ possesses finitely many equivalence classes on $\mathbf{SUB}_{X,Y}$.*

A *frame* is a couple (W, R) where W is a non-empty set and R is a binary relation on W ⁷. In a frame (W, R) , for all $s \in W$, let $R(s) = \{t \in W : sRt\}$ and for all $U \subseteq W$, let $R(U) = \{t \in W : \text{there exists } s \in U \text{ such that } sRt\}$. We say that a frame (W, R) is generated from $s \in W$ if for all $t \in W$, there exists $n \geq 0$ and there exists $u_0, \dots, u_n \in W$ such that $u_0 = s$, $u_n = t$ and for all $i \in \{1, \dots, n\}$, $u_{i-1}Ru_i$. A *valuation* on a frame (W, R) is a function assigning to each variable a subset of W . Given a frame (W, R) and a valuation V on (W, R) , the *satisfiability* of $\varphi \in \mathbf{FOR}$ at $s \in W$ (in symbols $(W, R), V, s \models \varphi$) is inductively defined as follows:

- $(W, R), V, s \models x$ if and only if $s \in V(x)$,
- $(W, R), V, s \not\models \perp$,
- $(W, R), V, s \models \neg \varphi$ if and only if $(W, R), V, s \not\models \varphi$,
- $(W, R), V, s \models \varphi \vee \psi$ if and only if either $(W, R), V, s \models \varphi$, or $(W, R), V, s \models \psi$,
- $(W, R), V, s \models \Box \varphi$ if and only if for all $t \in W$, if sRt then $(W, R), V, t \models \varphi$.

We say that a formula φ is *valid* in a frame (W, R) (in symbols $(W, R) \models \varphi$) if for all valuations V on (W, R) and for all $s \in W$, $(W, R), V, s \models \varphi$. We say that \mathbf{L} is *valid* in a frame (W, R) (in symbols $(W, R) \models \mathbf{L}$) if for all $\varphi \in \mathbf{L}$, $(W, R) \models \varphi$. For all frames (W, R) , for all substitutions (X, Y, σ) and for all valuations V on (W, R) , let V^σ be the valuation on (W, R) such that for all $x \in \mathbf{VAR}$, if $x \in X$ then $V^\sigma(x) = \{s \in W : (W, R), V, s \models \sigma(x)\}$ else $V^\sigma(x) = V(x)$ ⁸.

⁴Obviously, **K45** contains **K5**. Moreover, as is well-known [22, Chapter 3], **S5** contains **K45**.

⁵Obviously, for all $(X, Y, \sigma), (X, Z, \tau) \in \mathbf{SUB}$, if $(X, Y, \sigma) \simeq_{\mathbf{L}} (X, Z, \tau)$ then for all $\varphi \in \mathbf{FOR}_X$, $\sigma(\varphi) \leftrightarrow \tau(\varphi) \in \mathbf{L}$.

⁶Obviously, for all $(X, Y, \sigma), (X, Z, \tau) \in \mathbf{SUB}$, if $(X, Y, \sigma) \preceq_{\mathbf{L}} (X, Z, \tau)$ then there exists $(Z, T, v) \in \mathbf{SUB}$ such that for all $\varphi \in \mathbf{FOR}_X$, $\sigma(\varphi) \leftrightarrow v(\tau(\varphi)) \in \mathbf{L}$. Moreover, for all $(X, Y, \sigma), (X, Z, \tau) \in \mathbf{SUB}$, if $(X, Y, \sigma) \simeq_{\mathbf{L}} (X, Z, \tau)$ then $(X, Y, \sigma) \preceq_{\mathbf{L}} (X, Z, \tau)$.

⁷We assume the reader is at home with the relational semantics of modal logics. For more on this, see [12, 13, 27].

⁸Such definition is standard [3, 15, 18, 19].

Proposition 3. *Let (W, R) be a frame, (X, Y, σ) be a substitution and V be a valuation on (W, R) . For all $\varphi \in \mathbf{FOR}_X$ and for all $s \in W$, $(W, R), V^\sigma, s \models \varphi$ if and only if $(W, R), V, s \models \sigma(\varphi)$.*

Proposition 4. *Let (W, R) be a frame such that $(W, R) \models \mathbf{L}$. If \mathbf{L} contains **K5** then for all $s \in W$, if (W, R) is generated from s then exactly one of the following 3 conditions holds: (i) $W = \{s\}$ and $R = \emptyset$, (ii) $R = W \times W$, (iii) there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $s \notin B$, $W = \{s\} \cup B$ and $R = (\{s\} \times A) \cup (B \times B)$. If \mathbf{L} contains **K45** then for all $s \in W$, if (W, R) is generated from s then exactly one of the following 3 conditions holds: (iv) $W = \{s\}$ and $R = \emptyset$, (v) $R = W \times W$, (vi) there exists $A \subseteq W$ such that $A \neq \emptyset$, $s \notin A$, $W = \{s\} \cup A$ and $R = (\{s\} \times A) \cup (A \times A)$.*

Let \mathcal{S} be a frame (W, R) such that $\text{Card}(W) = 1$ and $R = \emptyset$. For all $m \geq 1$, let \mathcal{T}_m be a frame (W, R) such that $\text{Card}(W) = m$ and $R = W \times W$. For all $m \geq 1$ and for all $n \geq 0$, let $\mathcal{U}_{(m,n)}$ be a frame (W, R) such that there exists $s \in W$ and there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $s \notin B$, $W = \{s\} \cup B$, $R = (\{s\} \times A) \cup (B \times B)$, $\text{Card}(A) = m$ and $\text{Card}(B) = m + n$.

Proposition 5. *If \mathbf{L} contains **K5** then exactly one of the following conditions holds: (i) for all $m \geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$, (ii) for all $m \geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \not\models \mathbf{L}$, (iii) there exists $m \geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n \geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$, (iv) there exists $m \geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n \geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $\mathcal{S} \not\models \mathbf{L}$, (v) for all $m \geq 1$, $\mathcal{T}_m \not\models \mathbf{L}$.*

We say that \mathbf{L} is *global* if for all $m, m' \geq 1$ and for all $n' \geq 0$, if $m = m' + n'$ and $\mathcal{T}_m \models \mathbf{L}$ then $\mathcal{U}_{(m',n')} \models \mathbf{L}$. For all positive integers l , let $\varphi_l = \bigwedge \{ \diamond \diamond x_k : 0 \leq k \leq l \} \rightarrow \bigvee \{ \diamond \diamond (x_i \wedge x_j) : 0 \leq i < j \leq l \}$.

Proposition 6. *If either $\mathbf{L} = \mathbf{K5}$, or $\mathbf{L} = \mathbf{K5} \oplus \diamond \top$, or $\mathbf{L} = \mathbf{K5} \oplus \varphi_l$ for some positive integer l , or $\mathbf{L} = \mathbf{K5} \oplus \varphi_l \oplus \diamond \top$ for some positive integer l , or $\mathbf{L} = \mathbf{K5} \oplus \square \perp$ then \mathbf{L} is global.*

Proposition 7. *If \mathbf{L} contains **K5** and \mathbf{L} is global then either $\mathbf{L} = \mathbf{K5}$, or $\mathbf{L} = \mathbf{K5} \oplus \diamond \top$, or $\mathbf{L} = \mathbf{K5} \oplus \varphi_l$ for some positive integer l , or $\mathbf{L} = \mathbf{K5} \oplus \varphi_l \oplus \diamond \top$ for some positive integer l , or $\mathbf{L} = \mathbf{K5} \oplus \square \perp$.*

Proposition 8. *If \mathbf{L} contains **K5** then for all $\varphi \in \mathbf{FOR}$, if $\varphi \notin \mathbf{L}$ then there exists a finite frame (W, R) , there exists a valuation V on (W, R) and there exists $s \in W$ such that $(W, R) \models \mathbf{L}$, (W, R) is generated from s and $(W, R), V, s \not\models \varphi$.*

For all finite frames (W, R) , for all valuations V on (W, R) , for all $s \in W$ and for all finite $X \subseteq \mathbf{VAR}$, we say that a valuation V' on (W, R) is a *variant* of V with respect to s and X if for all $x \in X$, $V'(x) \setminus \{s\} = V(x) \setminus \{s\}$. We say that $\varphi \in \mathbf{FOR}$ has *extension property* in \mathbf{L} if for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, if $(W, R) \models \mathbf{L}$ and (W, R) is generated from s then there exists a variant V' of V with respect to s and $\text{var}(\varphi)$ such that $(W, R), V', s \models \diamond \square \varphi \rightarrow \varphi$. The following result is essential for the proof of Proposition 23.

Proposition 9. *If \mathbf{L} contains **K45** then for all $\varphi \in \mathbf{FOR}$, φ has extension property in \mathbf{L} .*

For all finite $X \subseteq \mathbf{VAR}$, for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, let $\text{for}_X((W, R), s, V) = \{ \chi \in \mathbf{FOR}_X : (W, R), V, s \models \chi \}$. Obviously, $\text{for}_X((W, R), s, V)$ is an infinite subset of \mathbf{FOR}_X . Nevertheless, when \mathbf{L} is locally tabular, we will treat $\text{for}_X((W, R), s, V)$ as if it is a finite subset of \mathbf{FOR}_X . In that case, $\text{for}_X((W, R), s, V)$ will also denote the conjunction of all formulas in this finite subset.

3 Unification

An \mathbf{L} -*unifier* of $\varphi \in \mathbf{FOR}$ is a substitution $(\text{var}(\varphi), X, \sigma)$ such that $\sigma(\varphi) \in \mathbf{L}$. We write $\Sigma_{\mathbf{L}}(\varphi)$ to mean the set of all \mathbf{L} -unifiers of $\varphi \in \mathbf{FOR}$. We say that $\varphi \in \mathbf{FOR}$ is \mathbf{L} -*unifiable* if $\Sigma_{\mathbf{L}}(\varphi) \neq \emptyset$. Since \mathbf{L} is closed

for uniform substitution, for all **L**-unifiable $\varphi \in \mathbf{FOR}$, $\Sigma_{\mathbf{L}}(\varphi)$ contains variable-free substitutions. We say that an **L**-unifier σ of $\varphi \in \mathbf{FOR}$ is a *most general L-unifier* of φ if for all **L**-unifiers τ of φ , $\tau \preceq_{\mathbf{L}} \sigma$. We say that a set Σ of **L**-unifiers of an **L**-unifiable $\varphi \in \mathbf{FOR}$ is *complete* if for all **L**-unifiers σ of φ , there exists $\tau \in \Sigma$ such that $\sigma \preceq_{\mathbf{L}} \tau$ ⁹. We say that a complete set Σ of **L**-unifiers of an **L**-unifiable $\varphi \in \mathbf{FOR}$ is a *basis* for φ if for all $\sigma, \tau \in \Sigma$, if $\sigma \preceq_{\mathbf{L}} \tau$ then $\sigma = \tau$ ¹⁰.

Proposition 10. *For all L-unifiable $\varphi \in \mathbf{FOR}$ and for all bases Σ, Δ for φ , Σ and Δ have the same cardinality.*

As a consequence of Proposition 10, an important question is the following: when $\varphi \in \mathbf{FOR}$ is **L**-unifiable, is there a basis for φ ? When the answer is “yes”, how large is this basis? For all **L**-unifiable $\varphi \in \mathbf{FOR}$, we say that φ is *of type 1* if there exists a basis for φ with cardinality 1, φ is *of type ω* if there exists a basis for φ with finite cardinality ≥ 2 , φ is *of type ∞* if there exists a basis for φ with infinite cardinality, φ is *of type 0* if there exists no basis for φ ¹¹. We say that **L** is *of type 1* if every **L**-unifiable formula is of type 1, **L** is *of type ω* if every **L**-unifiable formula is either of type 1, or of type ω and there exists an **L**-unifiable formula of type ω , **L** is *of type ∞* if every **L**-unifiable formula is either of type 1, or of type ω , or of type ∞ and there exists an **L**-unifiable formula of type ∞ , **L** is *of type 0* if there exists an **L**-unifiable formula of type 0¹². For all **L**-unifiable $\varphi \in \mathbf{FOR}$, we say that φ is *L-filtering* if for all **L**-unifiers σ, τ of φ , there exists an **L**-unifier ν of φ such that $\sigma \preceq_{\mathbf{L}} \nu$ and $\tau \preceq_{\mathbf{L}} \nu$.

Proposition 11. *Let $\varphi \in \mathbf{FOR}$ be L-unifiable. If φ is L-filtering then φ is either of type 1, or of type 0.*

We say that **L** has *filtering unification* if for all **L**-unifiable $\varphi \in \mathbf{FOR}$, φ is **L**-filtering.

Proposition 12. *If L has filtering unification then L is either of type 1, or of type 0.*

The following result is essential for the proof of Proposition 25.

Proposition 13. *If L contains K5 then for all L-unifiable $\varphi \in \mathbf{FOR}$, φ is L-filtering.*

For all $\varphi \in \mathbf{FOR}$, a substitution $(\text{var}(\varphi), \text{var}(\varphi), \sigma)$ is *L-projective* for φ if for all $x \in \text{var}(\varphi)$, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow \sigma(x)$.

Proposition 14. *Let $\varphi \in \mathbf{FOR}$. Let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models_{\mathbf{L}}$, (W, R) is generated from s and $(W, R), V, s \models \diamond \Box \varphi$. If L contains K5 then for all L-projective substitutions ν for φ , V^ν is a variant of V with respect to s and $\text{var}(\varphi)$.*

Proposition 15. *Let $\varphi \in \mathbf{FOR}$ and σ be an L-projective substitution for φ . For all $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \sigma(\psi)$.*

Proposition 16. *Let $\varphi \in \mathbf{FOR}$ and σ be an L-projective substitution for φ . For all L-projective substitutions τ for φ , $\sigma \circ \tau$ is L-projective for φ .*

Proposition 17. *Let $\varphi \in \mathbf{FOR}$ and σ be an L-projective substitution for φ . For all L-unifiers τ of φ , $\tau \preceq_{\mathbf{L}} \sigma$.*

For all **L**-unifiable $\varphi \in \mathbf{FOR}$, we say that φ is *L-projective* if there exists an **L**-projective **L**-unifier of φ .

⁹Obviously, for all **L**-unifiable $\varphi \in \mathbf{FOR}$, $\Sigma_{\mathbf{L}}(\varphi)$ is a complete set of **L**-unifiers of φ .

¹⁰Obviously, for all complete sets Σ of **L**-unifiers of an **L**-unifiable $\varphi \in \mathbf{FOR}$, Σ is a basis for φ if and only if Σ is a minimal complete set of **L**-unifiers of φ , i.e. for all $\Delta \subseteq \Sigma$, if Δ is a complete set of **L**-unifiers of φ then $\Delta = \Sigma$.

¹¹Obviously, the types 1, ω , ∞ and 0 constitute a set of jointly exhaustive and pairwise distinct situations for each **L**-unifiable $\varphi \in \mathbf{FOR}$.

¹²That is to say, the types 1, ω , ∞ and 0 being ordered by $1 < \omega < \infty < 0$, the unification type of **L** is the greatest one among the types of its unifiable formulas.

Proposition 18. *Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable. If φ is \mathbf{L} -projective then φ is of type 1.*

We say that \mathbf{L} has *projective unification* if for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is \mathbf{L} -projective.

Proposition 19. *If \mathbf{L} has projective unification then \mathbf{L} is of type 1.*

The following result is essential for the proof of Proposition 23.

Proposition 20. *If \mathbf{L} contains $\mathbf{K5}$ then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is \mathbf{L} -projective if and only if φ has extension property in \mathbf{L} .*

The following result is essential for the proof of Proposition 25.

Proposition 21. *If \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$ and for all \mathbf{L} -unifiers σ of φ , there exists $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$ such that $\sigma(\psi) \in \mathbf{L}$, $\psi \rightarrow \varphi \in \mathbf{K}$, ψ is \mathbf{L} -projective.*

4 Extensions of $\mathbf{K5}$

Firstly, let us consider the extensions of $\mathbf{K45}$.

Proposition 22. *If \mathbf{L} contains $\mathbf{K45}$ then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is \mathbf{L} -projective.*

Proposition 23. *If \mathbf{L} contains $\mathbf{K45}$ then \mathbf{L} has projective unification.*

Secondly, let us consider the extensions of $\mathbf{K5}$.

Proposition 24. *If \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is of type 1.*

Proposition 25. *If \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global then \mathbf{L} is of type 1.*

Notice that the line of reasoning leading to Propositions 23 and 25 rules out neither the possibility that all extensions of $\mathbf{K5}$ have projective unification, nor the possibility that some nonglobal extension of $\mathbf{K5}$ is either of type ω , or of type ∞ , or of type 0¹³.

5 Conclusion

A property similar to the extension property has been used by Ghilardi who has proved both in Intuitionistic Logic [18] and in transitive modal logics like $\mathbf{K4}$ and $\mathbf{S4}$ [19] that it is equivalent to the projectivity of formulas. This property has also been considered in [11] where formulas verifying it are called *extendible formulas*. As a matter of fact, Bezhanishvili and de Jongh have provided a complete characterization in Intuitionistic Logic of the set of all extendible formulas with at most 2 variables. However, the question remains unsettled whether a complete characterization in Intuitionistic Logic of the set of all extendible formulas with at least 3 variables can be given. Within the context of extensions of $\mathbf{K5}$, we believe that it is probably easier to give a complete characterization of the set of all formulas verifying the extension property.

Funding

The preparation of this paper has been supported by *French Ministry of Europe and Foreign Affairs, French Ministry of Higher Education, Research and Innovation and Center for International Scientific Studies and Collaboration (CISSC) of Iranian Ministry of Research, Science and Technology (Project 40903VF)*.

¹³No modal logic is known to be of type ∞ [15, Chapter 5].

Acknowledgement

Special acknowledgement is heartily granted to Çiğdem Gencer (Istanbul Aydın University, Istanbul, Turkey), Maryam Rostamigiv (Toulouse Institute of Computer Science Research, Toulouse, France) and Tinko Tinchev (Sofia University St. Kliment Ohridski, Sofia, Bulgaria) for many stimulating discussions about modal logics and the unification problem.

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Appendix

Proof of Proposition 1: See [29, Corollary 5].

Proof of Proposition 2: Suppose \mathbf{L} is locally tabular. Let $X, Y \subseteq \mathbf{VAR}$ be finite. For all $x \in X$, let $\simeq_{\mathbf{L}}^x$ be the equivalence relation on $\mathbf{SUB}_{X,Y}$ defined by

- $\sigma \simeq_{\mathbf{L}}^x \tau$ if and only if $\sigma(x) \leftrightarrow \tau(x) \in \mathbf{L}$.

Since \mathbf{L} is locally tabular and Y is finite, for all $x \in X$, $\simeq_{\mathbf{L}}^x$ possesses finitely many equivalence classes on $\mathbf{SUB}_{X,Y}$. Since X is finite and the restriction of $\simeq_{\mathbf{L}}$ to $\mathbf{SUB}_{X,Y}$ is equal to $\bigcap \{\simeq_{\mathbf{L}}^x : x \in X\}$, $\simeq_{\mathbf{L}}$ possesses finitely many equivalence classes on $\mathbf{SUB}_{X,Y}$.

Proof of Proposition 3: By induction on $\varphi \in \mathbf{FOR}_X$.

Proof of Proposition 4: See [28, Sections 2 and 3].

Proof of Proposition 5: See [28, Sections 2 and 3].

Proof of Proposition 6: Left to the reader.

Proof of Proposition 7: Suppose \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global. For the sake of the contradiction, suppose neither $\mathbf{L} = \mathbf{K5}$, nor $\mathbf{L} = \mathbf{K5} \oplus \diamond \top$, nor $\mathbf{L} = \mathbf{K5} \oplus \varphi_l$ for some positive integer l , nor $\mathbf{L} = \mathbf{K5} \oplus \varphi_l \oplus \diamond \top$ for some positive integer l , nor $\mathbf{L} = \mathbf{K5} \oplus \square \perp$. By Proposition 5, we have to consider the following 5 cases.

Case “for all $m \geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$ ”: Since \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global, $\mathbf{L} = \mathbf{K5}$: a contradiction.

Case “for all $m \geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \not\models \mathbf{L}$ ”: Since \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global, $\mathbf{L} = \mathbf{K5} \oplus \diamond \top$: a contradiction.

Case “there exists $m \geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n \geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$ ”: Thus, let l be the greatest positive integer such that $\mathcal{T}_l \models \mathbf{L}$. Since \mathbf{L} contains $\mathbf{K5}$, \mathbf{L} is global and $\mathcal{S} \models \mathbf{L}$, $\mathbf{L} = \mathbf{K5} \oplus \varphi_l$: a contradiction.

Case “there exists $m \geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n \geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $S \not\models \mathbf{L}$ ”: Consequently, let l be the greatest positive integer such that $\mathcal{T}_l \models \mathbf{L}$. Since \mathbf{L} contains **K5**, \mathbf{L} is global and $S \not\models \mathbf{L}$, $\mathbf{L} = \mathbf{K5} \oplus \varphi_l \oplus \diamond\top$: a contradiction.

Case “for all $m \geq 1$, $\mathcal{T}_m \not\models \mathbf{L}$ ”: Hence, $\mathbf{L} = \mathbf{K5} \oplus \square\perp$: a contradiction.

Proof of Proposition 8: Suppose \mathbf{L} contains **K5**. Let $\varphi \in \mathbf{FOR}$ be such that $\varphi \notin \mathbf{L}$. Since \mathbf{L} contains **K5**, by [29, Theorem 3], let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \varphi$. Without loss of generality, by [12, Proposition 2.6 and Theorem 3.14], we can suppose (W, R) is generated from s .

Proof of Proposition 9: Suppose \mathbf{L} contains **K45**. Let $\varphi \in \mathbf{FOR}$. Let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$ and (W, R) is generated from s . Since \mathbf{L} contains **K45**, by Proposition 4, we have to consider the following 3 cases.

Case “ $W = \{s\}$ and $R = \emptyset$ ”: Obviously, V is a variant of V with respect to s and $\text{var}(\varphi)$. Moreover, $(W, R), V, s \models \diamond\square\varphi \rightarrow \varphi$.

Case “ $R = W \times W$ ”: Obviously, V is a variant of V with respect to s and $\text{var}(\varphi)$. Moreover, $(W, R), V, s \models \diamond\square\varphi \rightarrow \varphi$.

Case “there exists $A \subseteq W$ such that $A \neq \emptyset$, $s \notin A$, $W = \{s\} \cup A$ and $R = (\{s\} \times A) \cup (A \times A)$ ”: Let $t \in A$. Obviously, $(W, R), V, t \models \diamond\square\varphi \rightarrow \varphi$. Let V' be a valuation on (W, R) such that for all $x \in \text{var}(\varphi)$, if $t \in V(x)$ then $V'(x) = V(x) \cup \{s\}$ else $V'(x) = V(x) \setminus \{s\}$. Obviously, V' is a variant of V with respect to s and $\text{var}(\varphi)$. Moreover, by induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- for all $u \in A$, $(W, R), V, u \models \psi$ if and only if $(W, R), V', u \models \psi$.

In other respect, by induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(W, R), V, t \models \psi$ if and only if $(W, R), V', s \models \psi$.

Since $(W, R), V, t \models \diamond\square\varphi \rightarrow \varphi$, $(W, R), V', s \models \diamond\square\varphi \rightarrow \varphi$.

Proof of Proposition 10: This is a standard result, although we have not been able to find a published proof of it. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable and Σ, Δ be bases for φ . Hence, Σ and Δ are minimal complete sets of \mathbf{L} -unifiers of φ . By the completeness of Σ and Δ , one can readily define functions $f : \Sigma \rightarrow \Delta$ and $g : \Delta \rightarrow \Sigma$ such that $\sigma \preceq_{\mathbf{L}} f(\sigma)$ for each $\sigma \in \Sigma$ and $\delta \preceq_{\mathbf{L}} g(\delta)$ for each $\delta \in \Delta$. By the minimality of Σ and Δ , it easily follows that f and g are injective. Thus, Σ and Δ have the same cardinality.

Proof of Proposition 11: This is a standard result, although we have not been able to find a published proof of it. Suppose φ is \mathbf{L} -filtering. For the sake of the contradiction, suppose φ is neither of type 1, nor of type 0. Hence, φ is either of type ω , or of type ∞ . Thus, let Σ be a basis for φ either with finite cardinality ≥ 2 , or with infinite cardinality. Consequently, let $\sigma, \tau \in \Sigma$ be such that $\sigma \neq \tau$. Since φ is \mathbf{L} -filtering, there exists an \mathbf{L} -unifier v of φ such that $\sigma \preceq_{\mathbf{L}} v$ and $\tau \preceq_{\mathbf{L}} v$. Since Σ is a basis for φ , let $v' \in \Sigma$ be such that $v \preceq_{\mathbf{L}} v'$. Since $\sigma \preceq_{\mathbf{L}} v$ and $\tau \preceq_{\mathbf{L}} v$, $\sigma \preceq_{\mathbf{L}} v'$ and $\tau \preceq_{\mathbf{L}} v'$. Since Σ is a basis for φ , $\sigma = v'$ and $\tau = v'$. Hence, $\sigma = \tau$: a contradiction.

Proof of Proposition 12: By Proposition 11.

Proof of Proposition 13: This result generalizes some results obtained in [6]. Suppose \mathbf{L} contains **K5**. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable. Let $(\text{var}(\varphi), X, \sigma), (\text{var}(\varphi), Y, \tau)$ be \mathbf{L} -unifiers of φ . Hence, $\sigma(\varphi) \in \mathbf{L}$ and $\tau(\varphi) \in \mathbf{L}$. Let $(\text{var}(\varphi), X \cup Y \cup \{z\}, v)$ be the substitution defined for all $x \in \text{var}(\varphi)$, by $v(x) = ((\Box\Box z \wedge (z \vee \Diamond\top)) \wedge \sigma(x)) \vee ((\Diamond\Diamond \neg z \vee (\neg z \wedge \Box\perp)) \wedge \tau(x))$ where z is a new variable, i.e. neither $z \in \text{var}(\varphi)$, nor $z \in X \cup Y$. Obviously, $\sigma \preceq_{\mathbf{L}} v$ and $\tau \preceq_{\mathbf{L}} v$. Moreover, by induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(\Box\Box z \wedge (z \vee \Diamond\top)) \rightarrow (v(\psi) \leftrightarrow \sigma(\psi)) \in \mathbf{L}$,
- $(\Diamond\Diamond \neg z \vee (\neg z \wedge \Box\perp)) \rightarrow (v(\psi) \leftrightarrow \tau(\psi)) \in \mathbf{L}$.

Since $\sigma(\varphi) \in \mathbf{L}$ and $\tau(\varphi) \in \mathbf{L}$, $v(\varphi) \in \mathbf{L}$. Thus, v is an \mathbf{L} -unifier of φ . Consequently, φ is \mathbf{L} -filtering.

Proof of Proposition 14: Suppose \mathbf{L} contains **K5**. Let v be an \mathbf{L} -projective substitution for φ . Hence, for all $x \in \text{var}(\varphi)$, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$. Let $x \in \text{var}(\varphi)$. Thus, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$. Let $t \in W$. Suppose $t \in V^v(x) \setminus \{s\}$. Consequently, $t \neq s$ and by Proposition 3, $(W, R), V, t \models v(x)$. Since $(W, R) \models \mathbf{L}$, (W, R) is generated from s , $(W, R), V, s \models \Diamond\Box\varphi$, \mathbf{L} contains **K5** and $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$, $(W, R), V, t \models x$. Since $t \neq s$, $t \in V(x) \setminus \{s\}$. Reciprocally, suppose $t \in V(x) \setminus \{s\}$. Consequently, $t \neq s$ and $(W, R), V, t \models x$. Since $(W, R) \models \mathbf{L}$, (W, R) is generated from s , $(W, R), V, s \models \Diamond\Box\varphi$, \mathbf{L} contains **K5** and $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$, $(W, R), V, t \models v(x)$. Since $t \neq s$, by Proposition 3, $t \in V^v(x) \setminus \{s\}$. Hence, V^v is a variant of V with respect to s and $\text{var}(\varphi)$.

Proof of Proposition 15: See [1] and [15].

Proof of Proposition 16: See [1] and [15].

Proof of Proposition 17: See [1] and [15].

Proof of Proposition 18: By Proposition 17.

Proof of Proposition 19: By Proposition 18.

Proof of Proposition 20: Suppose \mathbf{L} contains **K5**. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable.

Suppose φ is \mathbf{L} -projective. For the sake of the contradiction, suppose φ has not extension property in \mathbf{L} . Since φ is \mathbf{L} -projective, let v be an \mathbf{L} -projective \mathbf{L} -unifier of φ . Since φ has not extension property in \mathbf{L} , let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that

- $(W, R) \models \mathbf{L}$,
- (W, R) is generated from s ,
- for all variants V' of V with respect to s and $\text{var}(\varphi)$, $(W, R), V', s \not\models \Diamond\Box\varphi \rightarrow \varphi$.

Obviously, V is a variant of V with respect to s and $\text{var}(\varphi)$. Hence, $(W, R), V, s \not\models \Diamond\Box\varphi \rightarrow \varphi$. Thus, $(W, R), V, s \models \Diamond\Box\varphi$. Since v is an \mathbf{L} -projective \mathbf{L} -unifier of φ , $v(\varphi) \in \mathbf{L}$. Moreover, since \mathbf{L} contains **K5**, $(W, R) \models \mathbf{L}$, (W, R) is generated from s and $(W, R), V, s \models \Diamond\Box\varphi$, by Proposition 14, V^v is a variant of V with respect to s and $\text{var}(\varphi)$. Consequently, $(W, R), V^v, s \not\models \Diamond\Box\varphi \rightarrow \varphi$. Hence, $(W, R), V^v, s \models \varphi$. Thus, by Proposition 3, $(W, R), V, s \models v(\varphi)$. Since $(W, R) \models \mathbf{L}$, $v(\varphi) \notin \mathbf{L}$: a contradiction.

Suppose φ has extension property in \mathbf{L} . For the sake of the contradiction, suppose φ is not \mathbf{L} -projective. Since $\varphi \in \mathbf{FOR}$ is \mathbf{L} -unifiable, let σ be a variable-free \mathbf{L} -unifier of φ . Consequently, $\sigma(\varphi) \in \mathbf{L}$. Let $(\text{var}(\varphi), \emptyset, \tau)$ be a variable-free substitution. Let $(\text{var}(\varphi), \text{var}(\varphi), \epsilon_\tau)$ be the substitution such that

- for all $x \in \text{var}(\varphi)$, $\epsilon_\tau(x) = ((\varphi \wedge \Box\Box\varphi) \wedge x) \vee ((\neg\varphi \vee \Diamond\Diamond\neg\varphi) \wedge \tau(x))$.

The following fact can be easily proved: for all $x \in \text{var}(\varphi)$, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow \epsilon_\tau(x)$. Hence, ϵ_τ is \mathbf{L} -projective for φ . By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_\tau(\psi)$,
- $\neg\varphi \vee \Diamond\Diamond\neg\varphi \vdash_{\mathbf{L}} \tau(\psi) \leftrightarrow \epsilon_\tau(\psi)$.

Thus, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon_\tau(\varphi)$ and $\neg\varphi \vee \Diamond\Diamond\neg\varphi \vdash_{\mathbf{L}} \tau(\varphi) \leftrightarrow \epsilon_\tau(\varphi)$. Since \mathbf{L} contains **K5**, if $\tau(\varphi) \in \mathbf{L}$ then $\Box\Box\varphi \rightarrow (\varphi \rightarrow \epsilon_\tau(\varphi)) \in \mathbf{L}$ and $\Diamond\Diamond\neg\varphi \rightarrow \epsilon_\tau(\varphi) \in \mathbf{L}$. In that case, $\varphi \rightarrow \epsilon_\tau(\varphi) \in \mathbf{L}$ and $\Box\Box\varphi \vee \epsilon_\tau(\varphi) \in \mathbf{L}$ — which implies that $\Box\Box\epsilon_\tau(\varphi) \in \mathbf{L}$. Consequently, for all variable-free substitutions $(\text{var}(\varphi), \emptyset, \tau)$, if $\tau(\varphi) \in \mathbf{L}$ then $\Box\Box\epsilon_\tau(\varphi) \in \mathbf{L}$. Since $\sigma(\varphi) \in \mathbf{L}$, $\Box\Box\epsilon_\sigma(\varphi) \in \mathbf{L}$. Let $l \geq 1$ and $((\text{var}(\varphi), \emptyset, \tau_1), \dots, (\text{var}(\varphi), \emptyset, \tau_l))$ be an enumeration of the set of all variable-free substitutions $(\text{var}(\varphi), \emptyset, \tau)$ such that for all $x \in \text{var}(\varphi)$, either $\tau(x) = \top$, or $\tau(x) = \perp$ ¹⁴. Let $\epsilon = \epsilon_\sigma \circ \epsilon_{\tau_1} \circ \dots \circ \epsilon_{\tau_l} \circ \epsilon_\sigma$. Since $\Box\Box\epsilon_\sigma(\varphi) \in \mathbf{L}$, $\Box\Box\epsilon(\varphi) \in \mathbf{L}$. Moreover, since $\epsilon_\sigma, \epsilon_{\tau_1}, \dots, \epsilon_{\tau_l}$ are \mathbf{L} -projective for φ , by Proposition 16, ϵ is \mathbf{L} -projective for φ . Since φ is not \mathbf{L} -projective, $\epsilon(\varphi) \notin \mathbf{L}$. Since \mathbf{L} contains **K5**, by [29, Theorem 3], let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \epsilon(\varphi)$. Without loss of generality, by [12, Proposition 2.6 and Theorem 3.14], we can suppose (W, R) is generated from s . Since $\Box\Box\epsilon(\varphi) \in \mathbf{L}$ and $(W, R) \models \mathbf{L}$, $(W, R), V, s \models \Box\Box\epsilon(\varphi)$. Since $(W, R), V, s \not\models \epsilon(\varphi)$, $R \neq W \times W$. Since \mathbf{L} contains **K5**, $(W, R) \models \mathbf{L}$ and (W, R) is generated from s , by Proposition 4, we have to consider the following 2 cases.

Case “ $W = \{s\}$ and $R = \emptyset$ ”: By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon(\psi)$.

Hence, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon(\varphi)$. Since $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \epsilon(\varphi)$, $(W, R), V, s \not\models \varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\neg\varphi \vee \Diamond\Diamond\neg\varphi \vdash_{\mathbf{L}} \sigma(\psi) \leftrightarrow \epsilon_\sigma(\psi)$.

Thus, $\neg\varphi \vee \Diamond\Diamond\neg\varphi \vdash_{\mathbf{L}} \sigma(\varphi) \leftrightarrow \epsilon_\sigma(\varphi)$. Since $\sigma(\varphi) \in \mathbf{L}$, $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \varphi$, $(W, R), V, s \models \epsilon_\sigma(\varphi)$. Hence, by Proposition 3, $(W, R), V^{\epsilon_\sigma}, s \models \varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_1}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\psi)) \dots)$.

Consequently, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon_{\tau_1}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots)$. Since $(W, R) \models \mathbf{L}$ and $(W, R), V^{\epsilon_\sigma}, s \models \varphi$, $(W, R), V^{\epsilon_\sigma}, s \models \epsilon_{\tau_1}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots)$. Hence, by Proposition 3, $(W, R), V, s \models \epsilon(\varphi)$: a contradiction.

Case “there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $s \notin B$, $W = \{s\} \cup B$ and $R = (\{s\} \times A) \cup (B \times B)$ ”: For all $i \leq l$, let $\epsilon_i = \epsilon_{\tau_i} \circ \dots \circ \epsilon_{\tau_1} \circ \epsilon_\sigma$. Thus, for all $i \leq l$, if $i = 0$ then $\epsilon_i = \epsilon_\sigma$ else $\epsilon_i = \epsilon_{\tau_i} \circ \epsilon_{i-1}$. Since $\Box\Box\epsilon_\sigma(\varphi) \in \mathbf{L}$ and $(W, R) \models \mathbf{L}$, $(W, R), V, s \models \Box\Box\epsilon_\sigma(\varphi)$. Consequently, by Proposition 3, $(W, R), V^{\epsilon_\sigma}, s \models \Box\Box\varphi$. Hence, $(W, R), V^{\epsilon_\sigma}, s \models \Diamond\Diamond\varphi$. Since φ has extension property in \mathbf{L} , $(W, R) \models \mathbf{L}$ and (W, R) is generated from s , let V' be a variant of V^{ϵ_σ} with respect to s and $\text{var}(\varphi)$ such that $(W, R), V', s \models \Diamond\Diamond\varphi \rightarrow \varphi$. Since $(W, R), V^{\epsilon_\sigma}, s \models \Diamond\Diamond\varphi$, let $t \in A$ be such that $(W, R), V^{\epsilon_\sigma}, t \models \Box\varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

¹⁴Obviously, $l = 2^{\text{Card}(\text{var}(\varphi))}$.

- for all $u \in A$, $(W, R), V^{\epsilon\sigma}, u \models \psi$ if and only if $(W, R), V', u \models \psi$.

Since $t \in A$ and $(W, R), V^{\epsilon\sigma}, t \models \Box\varphi$, $(W, R), V', t \models \Box\varphi$. Since $t \in A$, $(W, R), V', s \models \Diamond\Box\varphi$. Since $(W, R), V', s \models \Diamond\Box\varphi \rightarrow \varphi$, $(W, R), V', s \models \varphi$. Let $j \in \{1, \dots, l\}$ be such that for all $x \in \text{var}(\varphi)$,

- if $(W, R), V', s \models x$ then $\tau_j(x) = \top$,
- if $(W, R), V', s \not\models x$ then $\tau_j(x) = \perp$.

Since $(W, R), V, s \not\models \epsilon(\varphi)$, $(W, R), V, s \not\models \epsilon_{j-1}(\epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots))$. Thus, by Proposition 3, $(W, R), V^{\epsilon_{j-1}}, s \not\models \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots)$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\psi)) \dots)$.

Consequently, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots)$. Since $(W, R) \models \mathbf{L}$ and $(W, R), V^{\epsilon_{j-1}}, s \not\models \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots)$, $(W, R), V^{\epsilon_{j-1}}, s \not\models \varphi \wedge \Box\Box\varphi$. By induction on $i \leq l$, the reader may easily verify that

- for all $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$ and for all $u \in B$, $(W, R), V^{\epsilon_i}, u \models \psi$ if and only if $(W, R), V^{\epsilon\sigma}, u \models \psi$.

By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(W, R), V^{\epsilon_j}, s \models \psi$ if and only if $(W, R), V', s \models \psi$.

Since $(W, R), V', s \models \Diamond\Box\varphi$ and $(W, R), V', s \models \varphi$, $(W, R), V^{\epsilon_j}, s \models \Diamond\Box\varphi$ and $(W, R), V^{\epsilon_j}, s \models \varphi$. Since \mathbf{L} contains **K5** and $(W, R) \models \mathbf{L}$, $(W, R), V^{\epsilon_j}, s \models \varphi \wedge \Box\Box\varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_{j+1}}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\psi)) \dots)$.

Hence, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon_{\tau_{j+1}}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots)$. Since $(W, R) \models \mathbf{L}$ and $(W, R), V^{\epsilon_j}, s \models \varphi \wedge \Box\Box\varphi$, $(W, R), V^{\epsilon_j}, s \models \epsilon_{\tau_{j+1}}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots)$. Thus, by Proposition 3, $(W, R), V, s \models \epsilon(\varphi)$: a contradiction.

Proof of Proposition 21: Suppose \mathbf{L} contains **K5** and \mathbf{L} is global. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable and $(\text{var}(\varphi), X, \sigma)$ be an \mathbf{L} -unifier of φ . Hence, $\sigma(\varphi) \in \mathbf{L}$. Let $\Gamma_{\text{var}(\varphi)}(\sigma)$ be the set of all formulas of the form $\text{for}_{\text{var}(\varphi)}((W, R), s, V^\sigma)$ where (W, R) is a finite frame, V is a valuation on (W, R) and $s \in W$ are such that $(W, R) \models \mathbf{L}$ and (W, R) is generated from s . Obviously, $\Gamma_{\text{var}(\varphi)}(\sigma)$ is a finite set of infinite subsets of $\mathbf{FOR}_{\text{var}(\varphi)}$. Nevertheless, since \mathbf{L} contains **K5**, by Proposition 1, \mathbf{L} is locally tabular and we will treat $\Gamma_{\text{var}(\varphi)}(\sigma)$ as if it is a finite set of finite subsets of $\mathbf{FOR}_{\text{var}(\varphi)}$. Indeed, knowing that for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, if $(W, R) \models \mathbf{L}$ and (W, R) is generated from s then $\text{for}_{\text{var}(\varphi)}((W, R), s, V^\sigma)$ also denotes the conjunction of the formulas that $\text{for}_{\text{var}(\varphi)}((W, R), s, V^\sigma)$ contains, we will treat $\Gamma_{\text{var}(\varphi)}(\sigma)$ as if it is a finite subset of $\mathbf{FOR}_{\text{var}(\varphi)}$. Let ψ be the disjunction of all formulas in this finite subset. Obviously,

- (*) for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, if $(W, R) \models \mathbf{L}$ and (W, R) is generated from s then $(W, R), V^\sigma, s \models \psi$.

Suppose $\sigma(\psi) \notin \mathbf{L}$. Since \mathbf{L} contains **K5**, by Proposition 8, let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$, (W, R) is generated from s and $(W, R), V, s \not\models \sigma(\psi)$. Thus, by (*), $(W, R), V^\sigma, s \models \psi$. Consequently, by Proposition 3, $(W, R), V, s \models \sigma(\psi)$: a contradiction.

Suppose $\psi \rightarrow \varphi \notin \mathbf{K}$. Hence, let (W', R') be a frame, V' be a valuation on (W', R') and $s' \in W'$ be such that $(W', R'), V', s' \models \psi$ and $(W', R'), V', s' \not\models \varphi$. Thus, there exists a finite frame (W, R) , there exists a valuation V on (W, R) and there exists $s \in W$ such that $(W, R) \models \mathbf{L}$ and

$(W', R'), V', s' \models \text{for}_{\text{var}(\varphi)}((W, R), s, V^\sigma)$. Since $(W', R'), V', s' \not\models \varphi$, $(W, R), V^\sigma, s \not\models \varphi$. Consequently, by Proposition 3, $(W, R), V, s \not\models \sigma(\varphi)$. Since $(W, R) \models \mathbf{L}$, $\sigma(\varphi) \notin \mathbf{L}$: a contradiction.

Suppose ψ is not **L**-projective. Since **L** contains **K5**, by Proposition 20, ψ has not extension property in **L**. Hence, let (W', R') be a finite frame, V' be a valuation on (W', R') and $s' \in W'$ be such that

- $(W', R') \models \mathbf{L}$,
- (W', R') is generated from s' ,
- for all variants V'' of V' with respect to s' and $\text{var}(\varphi)$, $(W', R'), V'', s' \not\models \Diamond \Box \psi \rightarrow \psi$.

Obviously, V' is a variant of V' with respect to s' and $\text{var}(\varphi)$. Thus, $(W', R'), V', s' \not\models \Diamond \Box \psi \rightarrow \psi$. Consequently, $(W', R'), V', s' \models \Diamond \Box \psi$ and $(W', R'), V', s' \not\models \psi$. Hence, neither $R' = \emptyset$, nor $R' = W' \times W'$. Since **L** contains **K5**, $(W', R') \models \mathbf{L}$ and (W', R') is generated from s' , by Proposition 4, there exists $A', B' \subseteq W'$ such that $A' \neq \emptyset$, $A' \subseteq B'$, $s' \notin B'$, $W' = \{s'\} \cup B'$ and $R' = (\{s'\} \times A') \cup (B' \times B')$. Since $(W', R'), V', s' \models \Diamond \Box \psi$, let $t' \in A'$ be such that $(W', R'), V', t' \models \Box \psi$. Thus, $(W', R'), V', t' \models \psi$. Consequently, there exists a finite frame (W, R) , there exists a valuation V on (W, R) and there exists $t \in W$ such that

- $(W, R) \models \mathbf{L}$,
- (W, R) is generated from t ,
- $(W', R'), V', t' \models \text{for}_{\text{var}(\varphi)}((W, R), t, V^\sigma)$.

Obviously, $(B', B' \times B')$ is the subframe of (W', R') generated from t' . Let $V'_{B'}$ be the restriction of $V'_{B'}$ to B' . Since $(W', R'), V', t' \models \text{for}_{\text{var}(\varphi)}((W, R), t, V^\sigma)$, by [12, Proposition 2.6], $(B', B' \times B'), V'_{B'}, t' \models \text{for}_{\text{var}(\varphi)}((W, R), t, V^\sigma)$. Since $(B', B' \times B')$ and (W, R) are finite, by [12, Theorem 2.24], let $Z \subseteq B' \times W$ be a bisimulation between $(B', B' \times B')$ and (W, R) such that

- $t' Z t$,
- for all $u' \in B'$ and for all $u \in W$, if $u' Z u$ then for all $x \in \text{var}(\varphi)$, $u' \in V'_{B'}(x)$ if and only if $u \in V^\sigma(x)$.

Hence, $R \neq \emptyset$. Since **L** contains **K5**, $(W, R) \models \mathbf{L}$ and (W, R) is generated from t , by Proposition 4, we have to consider the following 2 cases.

Case “ $R = W \times W$ ”: Let $A = \{u \in W : \text{there exists } u' \in A' \text{ such that } u' Z u\}$ and $B = W$. Obviously, $A \neq \emptyset$ and $A \subseteq B$. Moreover, since $(W, R) \models \mathbf{L}$, $(B, B \times B) \models \mathbf{L}$. Let (W_1, R_1) be a finite frame, V_1 be a valuation on (W_1, R_1) and $s_1 \in W_1$ be such that $s_1 \notin B$, $W_1 = \{s_1\} \cup B$, $R_1 = (\{s_1\} \times A) \cup (B \times B)$ and for all $x \in \text{var}(\varphi)$, $V_1(x) = V(x)$. Since **L** is global and $(B, B \times B) \models \mathbf{L}$, $(W_1, R_1) \models \mathbf{L}$. Thus, by (*), $(W_1, R_1), V_1^\sigma, s_1 \models \psi$. Let V'' be a valuation on (W', R') such that for all $x \in \text{var}(\varphi)$, if $s_1 \in V_1^\sigma(x)$ then $V''(x) = V(x) \cup \{s'\}$ else $V''(x) = V(x) \setminus \{s'\}$. Obviously, V'' is a variant of V' with respect to s' and $\text{var}(\varphi)$. Consequently, $(W', R'), V'', s' \not\models \Diamond \Box \psi \rightarrow \psi$. Hence, $(W', R'), V'', s' \not\models \psi$. By induction on $\chi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(W', R'), V'', s' \models \chi$ if and only if $(W_1, R_1), V_1^\sigma, s_1 \models \chi$.

Since $(W', R'), V'', s' \not\models \psi$, $(W_1, R_1), V_1^\sigma, s_1 \models \psi$: a contradiction.

Case “there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $t \notin B$, $W = \{t\} \cup B$ and $R = (\{t\} \times A) \cup (B \times B)$ ”:

Hence, let $t_0 \in A$ be such that $t'Zt_0$. Let Z_0 be the restriction of Z to $W' \times B$ and V_0 be the restriction of V to B . Since $(W, R) \models \mathbf{L}$, by [12, Theorem 3.14], $(B, B \times B) \models \mathbf{L}$. Since $Z \subseteq W' \times W$ is a bisimulation between (W', R') and (W, R) such that $t'Zt_0$ and for all $u' \in W'$ and for all $u \in W$, if $u'Zu$ then for all $x \in \text{var}(\varphi)$, $u' \in V'(x)$ if and only if $u \in V^\sigma(x)$, $Z_0 \subseteq W' \times B$ is a bisimulation between (W', R') and $(B, B \times B)$ such that $t'Z_0t_0$ and for all $u' \in W'$ and for all $u \in B$, if $u'Z_0u$ then for all $x \in \text{var}(\varphi)$, $u' \in V'_0(x)$ if and only if $u \in V_0^\sigma(x)$. Then, proceed as in the case “ $R=W \times W$ ”.

Proof of Proposition 22: By Propositions 9 and 20.

Proof of Proposition 23: By Proposition 22.

Proof of Proposition 24: Suppose \mathbf{L} contains **K5** and \mathbf{L} is global. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable. Let σ be an \mathbf{L} -unifier of φ . Since \mathbf{L} contains **K5** and \mathbf{L} is global, by Proposition 21, let $\psi_\sigma \in \mathbf{FOR}_{\text{var}(\varphi)}$ be such that $\sigma(\psi_\sigma) \in \mathbf{L}$, $\psi_\sigma \rightarrow \varphi \in \mathbf{K}$ and ψ_σ is \mathbf{L} -projective. Hence, let ϵ_σ be an \mathbf{L} -projective \mathbf{L} -unifier of ψ_σ . Let $\Sigma = \{\epsilon_\sigma : \sigma \text{ is an } \mathbf{L}\text{-unifier of } \varphi\}$. By Propositions 17 and 21, Σ is a complete set of \mathbf{L} -unifiers of φ . Let Σ' be the set of substitutions obtained from Σ by keeping only one representative of each equivalence class modulo $\simeq_{\mathbf{L}}$. Since Σ is a complete set of \mathbf{L} -unifiers of φ , Σ' is a complete set of \mathbf{L} -unifiers of φ . Moreover, since \mathbf{L} is locally tabular, by Proposition 2, Σ' is finite. Thus, φ is either of type 1, or of type ω . Since \mathbf{L} contains **K5**, by Propositions 11 and 13, φ is of type 1.

Proof of Proposition 25: By Proposition 24.