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About the unification type of modal logic $\mathbf{K5}$ and its extensions

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1 Introduction

We prove that all extensions of $\mathbf{K45}$ have projective unification and $\mathbf{K5}$ and some of its extensions are of unification type 1. The breakdown of the paper is as follows. Firstly, we prove in Proposition 9 that if \mathbf{L} contains $\mathbf{K45}$ then every formula has extension property in \mathbf{L} . Secondly, generalizing some results obtained in [6], we prove in Proposition 13 that if \mathbf{L} contains $\mathbf{K5}$ then every \mathbf{L} -unifiable formula is \mathbf{L} -filtering. Thirdly, imitating arguments used in [18, 19], we prove in Proposition 20 that if \mathbf{L} contains $\mathbf{K5}$ then every formula having extension property in \mathbf{L} is \mathbf{L} -projective. Fourthly, we prove in Proposition 21 that if \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global¹ then for all substitutions σ , every formula \mathbf{L} -unified by σ is implied by an \mathbf{L} -projective formula based on the variables of the given formula and having σ as one of its \mathbf{L} -unifiers.

2 Syntax and semantics

Let \mathbf{VAR} be a countably infinite set of *variables* (with typical members denoted x, y , etc). The set \mathbf{FOR} of all *formulas* (with typical members denoted φ, ψ , etc) is inductively defined by

- $\varphi := x \mid \perp \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi$.

We adopt the standard rules for omission of parentheses. The Boolean connectives $\top, \wedge, \rightarrow$ and \leftrightarrow and the modal connective \Diamond are defined as usual. For all $\varphi \in \mathbf{FOR}$, let $\text{var}(\varphi)$ be the set of all variables occurring in φ . For all finite $X \subseteq \mathbf{VAR}$, let \mathbf{FOR}_X be the set of all $\varphi \in \mathbf{FOR}$ such that $\text{var}(\varphi) \subseteq X$.

A *substitution* is a triple (X, Y, σ) where $X, Y \subseteq \mathbf{VAR}$ are finite and $\sigma : \mathbf{FOR}_X \rightarrow \mathbf{FOR}_Y$ is a homomorphism. The sets X and Y are respectively its *domain* and its *codomain*. Let \mathbf{SUB} be the set of all substitutions. We say that $(X, Y, \sigma) \in \mathbf{SUB}$ is *variable-free* if $Y = \emptyset$. It is possible to compose two substitutions if the codomain of the first is equal to the domain of the second. The *composition* of $(X, Y, \sigma), (Y, Z, \tau) \in \mathbf{SUB}$ (in symbols $(X, Y, \sigma) \circ (Y, Z, \tau)$) is the substitution (X, Z, ν) such that for all $x \in X, \nu(x) = \tau(\sigma(x))$. When its domain and its codomain can be guessed from the context, the substitution (X, Y, σ) will be simply written σ ². For all finite $X, Y \subseteq \mathbf{VAR}$, let $\mathbf{SUB}_{X,Y}$ be the set of all $\sigma \in \mathbf{SUB}$ such that the domain of σ is X and the codomain of σ is Y .

We say that $\mathbf{L} \subseteq \mathbf{FOR}$ is a *modal logic* if the following conditions hold³: \mathbf{L} contains all tautologies, \mathbf{L} contains the formula $\Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y)$, \mathbf{L} is closed for *modus ponens* (for all $\varphi, \psi \in \mathbf{FOR}$, if $\varphi \rightarrow \psi \in \mathbf{L}$ and $\varphi \in \mathbf{L}$ then $\psi \in \mathbf{L}$), \mathbf{L} is closed for *generalization* (for all $\varphi \in \mathbf{FOR}$, if $\varphi \in \mathbf{L}$ then $\Box\varphi \in \mathbf{L}$),

¹Globality is defined in Section 2.

²However, when we write that two substitutions are equal, this will imply in any case that their domains are equal and their codomains are equal.

³The modal logics considered in this paper are exactly the *normal* modal logics considered in standard textbooks such as [12, 13, 27].

\mathbf{L} is closed for *uniform substitution* (for all $\varphi \in \mathbf{FOR}$, if $\varphi \in \mathbf{L}$ then for all substitutions (X, Y, σ) , if $\text{var}(\varphi) \subseteq X$ then $\sigma(\varphi) \in \mathbf{L}$). For all modal logics \mathbf{L} and for all $\varphi \in \mathbf{FOR}$, we write $\mathbf{L} \oplus \varphi$ for the least modal logic containing \mathbf{L} and φ . The following modal logics — and their extensions — are considered in this paper: $\mathbf{K} \oplus \Diamond x \rightarrow \Box \Diamond x$ (denoted **K5**), $\mathbf{K5} \oplus \Box x \rightarrow \Box \Box x$ (denoted **K45**), $\mathbf{K5} \oplus \Box x \rightarrow x$ (denoted **S5**), \mathbf{K} denoting the least modal logic⁴. We say that a modal logic \mathbf{L} is *consistent* if $\mathbf{L} \neq \mathbf{FOR}$. From now on in this paper, **let \mathbf{L} be a consistent modal logic**. Let $\equiv_{\mathbf{L}}$ be the equivalence relation on \mathbf{FOR} defined for all $\varphi, \psi \in \mathbf{FOR}$, by $\varphi \equiv_{\mathbf{L}} \psi$ if and only if $\varphi \leftrightarrow \psi \in \mathbf{L}$. We shall say that \mathbf{L} is *locally tabular* if for all finite $X \subseteq \mathbf{VAR}$, $\equiv_{\mathbf{L}}$ possesses finitely many equivalence classes on \mathbf{FOR}_X .

Proposition 1. *If \mathbf{L} contains **K5** then \mathbf{L} is locally tabular.*

We say that $\varphi \in \mathbf{FOR}$ is *\mathbf{L} -derivable* from $\Gamma \subseteq \mathbf{FOR}$ (in symbols $\Gamma \vdash_{\mathbf{L}} \varphi$) if there exists $n \geq 1$ and there exists $\varphi_1, \dots, \varphi_n \in \mathbf{FOR}$ such that $\varphi_n = \varphi$ and for all $k \in \{1, \dots, n\}$, at least one of the following 4 conditions holds: (i) $\varphi_k \in \mathbf{L}$, (ii) $\varphi_k \in \Gamma$, (iii) there exists $i, j \in \{1, \dots, n\}$ such that $i, j < k$ and $\varphi_i = \varphi_j \rightarrow \varphi_k$, (iv) there exists $i \in \{1, \dots, n\}$ such that $i < k$ and $\varphi_k = \Box \varphi_i$. Substitutions being completely defined by the restrictions to their domains, it is possible to compare two substitutions by means of these restrictions if their domains are equal. Let $\simeq_{\mathbf{L}}$ be the equivalence relation on \mathbf{SUB} defined for all $(X, Y, \sigma), (X, Z, \tau) \in \mathbf{SUB}$, by $(X, Y, \sigma) \simeq_{\mathbf{L}} (X, Z, \tau)$ if and only if for all $x \in X$, $\sigma(x) \leftrightarrow \tau(x) \in \mathbf{L}$ ⁵. Let $\preceq_{\mathbf{L}}$ be the quasi-order on \mathbf{SUB} defined for all $(X, Y, \sigma), (X, Z, \tau) \in \mathbf{SUB}$, by $(X, Y, \sigma) \preceq_{\mathbf{L}} (X, Z, \tau)$ if and only if there exists $(Z, T, v) \in \mathbf{SUB}$ such that for all $x \in X$, $\sigma(x) \leftrightarrow v(\tau(x)) \in \mathbf{L}$ ⁶.

Proposition 2. *If \mathbf{L} is locally tabular then for all finite $X, Y \subseteq \mathbf{VAR}$, $\simeq_{\mathbf{L}}$ possesses finitely many equivalence classes on $\mathbf{SUB}_{X,Y}$.*

A *frame* is a couple (W, R) where W is a non-empty set and R is a binary relation on W ⁷. In a frame (W, R) , for all $s \in W$, let $R(s) = \{t \in W : sRt\}$ and for all $U \subseteq W$, let $R(U) = \{t \in W : \text{there exists } s \in U \text{ such that } sRt\}$. We say that a frame (W, R) is generated from $s \in W$ if for all $t \in W$, there exists $n \geq 0$ and there exists $u_0, \dots, u_n \in W$ such that $u_0 = s$, $u_n = t$ and for all $i \in \{1, \dots, n\}$, $u_{i-1}Ru_i$. A *valuation* on a frame (W, R) is a function assigning to each variable a subset of W . Given a frame (W, R) and a valuation V on (W, R) , the *satisfiability* of $\varphi \in \mathbf{FOR}$ at $s \in W$ (in symbols $(W, R), V, s \models \varphi$) is inductively defined as follows:

- $(W, R), V, s \models x$ if and only if $s \in V(x)$,
- $(W, R), V, s \not\models \perp$,
- $(W, R), V, s \models \neg \varphi$ if and only if $(W, R), V, s \not\models \varphi$,
- $(W, R), V, s \models \varphi \vee \psi$ if and only if either $(W, R), V, s \models \varphi$, or $(W, R), V, s \models \psi$,
- $(W, R), V, s \models \Box \varphi$ if and only if for all $t \in W$, if sRt then $(W, R), V, t \models \varphi$.

We say that a formula φ is *valid* in a frame (W, R) (in symbols $(W, R) \models \varphi$) if for all valuations V on (W, R) and for all $s \in W$, $(W, R), V, s \models \varphi$. We say that \mathbf{L} is *valid* in a frame (W, R) (in symbols $(W, R) \models \mathbf{L}$) if for all $\varphi \in \mathbf{L}$, $(W, R) \models \varphi$. For all frames (W, R) , for all substitutions (X, Y, σ) and for all valuations V on (W, R) , let V^σ be the valuation on (W, R) such that for all $x \in \mathbf{VAR}$, if $x \in X$ then $V^\sigma(x) = \{s \in W : (W, R), V, s \models \sigma(x)\}$ else $V^\sigma(x) = V(x)$ ⁸.

⁴Obviously, **K45** contains **K5**. Moreover, as is well-known [22, Chapter 3], **S5** contains **K45**.

⁵Obviously, for all $(X, Y, \sigma), (X, Z, \tau) \in \mathbf{SUB}$, if $(X, Y, \sigma) \simeq_{\mathbf{L}} (X, Z, \tau)$ then for all $\varphi \in \mathbf{FOR}_X$, $\sigma(\varphi) \leftrightarrow \tau(\varphi) \in \mathbf{L}$.

⁶Obviously, for all $(X, Y, \sigma), (X, Z, \tau) \in \mathbf{SUB}$, if $(X, Y, \sigma) \preceq_{\mathbf{L}} (X, Z, \tau)$ then there exists $(Z, T, v) \in \mathbf{SUB}$ such that for all $\varphi \in \mathbf{FOR}_X$, $\sigma(\varphi) \leftrightarrow v(\tau(\varphi)) \in \mathbf{L}$. Moreover, for all $(X, Y, \sigma), (X, Z, \tau) \in \mathbf{SUB}$, if $(X, Y, \sigma) \simeq_{\mathbf{L}} (X, Z, \tau)$ then $(X, Y, \sigma) \preceq_{\mathbf{L}} (X, Z, \tau)$.

⁷We assume the reader is at home with the relational semantics of modal logics. For more on this, see [12, 13, 27].

⁸Such definition is standard [3, 15, 18, 19].

Proposition 3. *Let (W, R) be a frame, (X, Y, σ) be a substitution and V be a valuation on (W, R) . For all $\varphi \in \mathbf{FOR}_X$ and for all $s \in W$, $(W, R), V^\sigma, s \models \varphi$ if and only if $(W, R), V, s \models \sigma(\varphi)$.*

Proposition 4. *Let (W, R) be a frame such that $(W, R) \models \mathbf{L}$. If \mathbf{L} contains **K5** then for all $s \in W$, if (W, R) is generated from s then exactly one of the following 3 conditions holds: (i) $W = \{s\}$ and $R = \emptyset$, (ii) $R = W \times W$, (iii) there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $s \notin B$, $W = \{s\} \cup B$ and $R = (\{s\} \times A) \cup (B \times B)$. If \mathbf{L} contains **K45** then for all $s \in W$, if (W, R) is generated from s then exactly one of the following 3 conditions holds: (iv) $W = \{s\}$ and $R = \emptyset$, (v) $R = W \times W$, (vi) there exists $A \subseteq W$ such that $A \neq \emptyset$, $s \notin A$, $W = \{s\} \cup A$ and $R = (\{s\} \times A) \cup (A \times A)$.*

Let \mathcal{S} be a frame (W, R) such that $\text{Card}(W) = 1$ and $R = \emptyset$. For all $m \geq 1$, let \mathcal{T}_m be a frame (W, R) such that $\text{Card}(W) = m$ and $R = W \times W$. For all $m \geq 1$ and for all $n \geq 0$, let $\mathcal{U}_{(m,n)}$ be a frame (W, R) such that there exists $s \in W$ and there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $s \notin B$, $W = \{s\} \cup B$, $R = (\{s\} \times A) \cup (B \times B)$, $\text{Card}(A) = m$ and $\text{Card}(B) = m + n$.

Proposition 5. *If \mathbf{L} contains **K5** then exactly one of the following conditions holds: (i) for all $m \geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$, (ii) for all $m \geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \not\models \mathbf{L}$, (iii) there exists $m \geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n \geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$, (iv) there exists $m \geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n \geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $\mathcal{S} \not\models \mathbf{L}$, (v) for all $m \geq 1$, $\mathcal{T}_m \not\models \mathbf{L}$.*

We say that \mathbf{L} is *global* if for all $m, m' \geq 1$ and for all $n' \geq 0$, if $m = m' + n'$ and $\mathcal{T}_m \models \mathbf{L}$ then $\mathcal{U}_{(m',n')} \models \mathbf{L}$. For all positive integers l , let $\varphi_l = \bigwedge \{ \diamond \diamond x_k : 0 \leq k \leq l \} \rightarrow \bigvee \{ \diamond \diamond (x_i \wedge x_j) : 0 \leq i < j \leq l \}$.

Proposition 6. *If either $\mathbf{L} = \mathbf{K5}$, or $\mathbf{L} = \mathbf{K5} \oplus \diamond \top$, or $\mathbf{L} = \mathbf{K5} \oplus \varphi_l$ for some positive integer l , or $\mathbf{L} = \mathbf{K5} \oplus \varphi_l \oplus \diamond \top$ for some positive integer l , or $\mathbf{L} = \mathbf{K5} \oplus \square \perp$ then \mathbf{L} is global.*

Proposition 7. *If \mathbf{L} contains **K5** and \mathbf{L} is global then either $\mathbf{L} = \mathbf{K5}$, or $\mathbf{L} = \mathbf{K5} \oplus \diamond \top$, or $\mathbf{L} = \mathbf{K5} \oplus \varphi_l$ for some positive integer l , or $\mathbf{L} = \mathbf{K5} \oplus \varphi_l \oplus \diamond \top$ for some positive integer l , or $\mathbf{L} = \mathbf{K5} \oplus \square \perp$.*

Proposition 8. *If \mathbf{L} contains **K5** then for all $\varphi \in \mathbf{FOR}$, if $\varphi \notin \mathbf{L}$ then there exists a finite frame (W, R) , there exists a valuation V on (W, R) and there exists $s \in W$ such that $(W, R) \models \mathbf{L}$, (W, R) is generated from s and $(W, R), V, s \not\models \varphi$.*

For all finite frames (W, R) , for all valuations V on (W, R) , for all $s \in W$ and for all finite $X \subseteq \mathbf{VAR}$, we say that a valuation V' on (W, R) is a *variant* of V with respect to s and X if for all $x \in X$, $V'(x) \setminus \{s\} = V(x) \setminus \{s\}$. We say that $\varphi \in \mathbf{FOR}$ has *extension property* in \mathbf{L} if for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, if $(W, R) \models \mathbf{L}$ and (W, R) is generated from s then there exists a variant V' of V with respect to s and $\text{var}(\varphi)$ such that $(W, R), V', s \models \diamond \square \varphi \rightarrow \varphi$. The following result is essential for the proof of Proposition 23.

Proposition 9. *If \mathbf{L} contains **K45** then for all $\varphi \in \mathbf{FOR}$, φ has extension property in \mathbf{L} .*

For all finite $X \subseteq \mathbf{VAR}$, for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, let $\text{for}_X((W, R), s, V) = \{ \chi \in \mathbf{FOR}_X : (W, R), V, s \models \chi \}$. Obviously, $\text{for}_X((W, R), s, V)$ is an infinite subset of \mathbf{FOR}_X . Nevertheless, when \mathbf{L} is locally tabular, we will treat $\text{for}_X((W, R), s, V)$ as if it is a finite subset of \mathbf{FOR}_X . In that case, $\text{for}_X((W, R), s, V)$ will also denote the conjunction of all formulas in this finite subset.

3 Unification

An \mathbf{L} -*unifier* of $\varphi \in \mathbf{FOR}$ is a substitution $(\text{var}(\varphi), X, \sigma)$ such that $\sigma(\varphi) \in \mathbf{L}$. We write $\Sigma_{\mathbf{L}}(\varphi)$ to mean the set of all \mathbf{L} -unifiers of $\varphi \in \mathbf{FOR}$. We say that $\varphi \in \mathbf{FOR}$ is \mathbf{L} -*unifiable* if $\Sigma_{\mathbf{L}}(\varphi) \neq \emptyset$. Since \mathbf{L} is closed

for uniform substitution, for all **L**-unifiable $\varphi \in \mathbf{FOR}$, $\Sigma_{\mathbf{L}}(\varphi)$ contains variable-free substitutions. We say that an **L**-unifier σ of $\varphi \in \mathbf{FOR}$ is a *most general L-unifier* of φ if for all **L**-unifiers τ of φ , $\tau \preceq_{\mathbf{L}} \sigma$. We say that a set Σ of **L**-unifiers of an **L**-unifiable $\varphi \in \mathbf{FOR}$ is *complete* if for all **L**-unifiers σ of φ , there exists $\tau \in \Sigma$ such that $\sigma \preceq_{\mathbf{L}} \tau$ ⁹. We say that a complete set Σ of **L**-unifiers of an **L**-unifiable $\varphi \in \mathbf{FOR}$ is a *basis* for φ if for all $\sigma, \tau \in \Sigma$, if $\sigma \preceq_{\mathbf{L}} \tau$ then $\sigma = \tau$ ¹⁰.

Proposition 10. *For all L-unifiable $\varphi \in \mathbf{FOR}$ and for all bases Σ, Δ for φ , Σ and Δ have the same cardinality.*

As a consequence of Proposition 10, an important question is the following: when $\varphi \in \mathbf{FOR}$ is **L**-unifiable, is there a basis for φ ? When the answer is “yes”, how large is this basis? For all **L**-unifiable $\varphi \in \mathbf{FOR}$, we say that φ is *of type 1* if there exists a basis for φ with cardinality 1, φ is *of type ω* if there exists a basis for φ with finite cardinality ≥ 2 , φ is *of type ∞* if there exists a basis for φ with infinite cardinality, φ is *of type 0* if there exists no basis for φ ¹¹. We say that **L** is *of type 1* if every **L**-unifiable formula is of type 1, **L** is *of type ω* if every **L**-unifiable formula is either of type 1, or of type ω and there exists an **L**-unifiable formula of type ω , **L** is *of type ∞* if every **L**-unifiable formula is either of type 1, or of type ω , or of type ∞ and there exists an **L**-unifiable formula of type ∞ , **L** is *of type 0* if there exists an **L**-unifiable formula of type 0¹². For all **L**-unifiable $\varphi \in \mathbf{FOR}$, we say that φ is *L-filtering* if for all **L**-unifiers σ, τ of φ , there exists an **L**-unifier ν of φ such that $\sigma \preceq_{\mathbf{L}} \nu$ and $\tau \preceq_{\mathbf{L}} \nu$.

Proposition 11. *Let $\varphi \in \mathbf{FOR}$ be L-unifiable. If φ is L-filtering then φ is either of type 1, or of type 0.*

We say that **L** has *filtering unification* if for all **L**-unifiable $\varphi \in \mathbf{FOR}$, φ is **L**-filtering.

Proposition 12. *If L has filtering unification then L is either of type 1, or of type 0.*

The following result is essential for the proof of Proposition 25.

Proposition 13. *If L contains K5 then for all L-unifiable $\varphi \in \mathbf{FOR}$, φ is L-filtering.*

For all $\varphi \in \mathbf{FOR}$, a substitution $(\text{var}(\varphi), \text{var}(\varphi), \sigma)$ is *L-projective* for φ if for all $x \in \text{var}(\varphi)$, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow \sigma(x)$.

Proposition 14. *Let $\varphi \in \mathbf{FOR}$. Let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models_{\mathbf{L}}, (W, R)$ is generated from s and $(W, R), V, s \models \diamond \Box \varphi$. If L contains K5 then for all L-projective substitutions ν for φ , V^ν is a variant of V with respect to s and $\text{var}(\varphi)$.*

Proposition 15. *Let $\varphi \in \mathbf{FOR}$ and σ be an L-projective substitution for φ . For all $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \sigma(\psi)$.*

Proposition 16. *Let $\varphi \in \mathbf{FOR}$ and σ be an L-projective substitution for φ . For all L-projective substitutions τ for φ , $\sigma \circ \tau$ is L-projective for φ .*

Proposition 17. *Let $\varphi \in \mathbf{FOR}$ and σ be an L-projective substitution for φ . For all L-unifiers τ of φ , $\tau \preceq_{\mathbf{L}} \sigma$.*

For all **L**-unifiable $\varphi \in \mathbf{FOR}$, we say that φ is *L-projective* if there exists an **L**-projective **L**-unifier of φ .

⁹Obviously, for all **L**-unifiable $\varphi \in \mathbf{FOR}$, $\Sigma_{\mathbf{L}}(\varphi)$ is a complete set of **L**-unifiers of φ .

¹⁰Obviously, for all complete sets Σ of **L**-unifiers of an **L**-unifiable $\varphi \in \mathbf{FOR}$, Σ is a basis for φ if and only if Σ is a minimal complete set of **L**-unifiers of φ , i.e. for all $\Delta \subseteq \Sigma$, if Δ is a complete set of **L**-unifiers of φ then $\Delta = \Sigma$.

¹¹Obviously, the types 1, ω , ∞ and 0 constitute a set of jointly exhaustive and pairwise distinct situations for each **L**-unifiable $\varphi \in \mathbf{FOR}$.

¹²That is to say, the types 1, ω , ∞ and 0 being ordered by $1 < \omega < \infty < 0$, the unification type of **L** is the greatest one among the types of its unifiable formulas.

Proposition 18. *Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable. If φ is \mathbf{L} -projective then φ is of type 1.*

We say that \mathbf{L} has *projective unification* if for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is \mathbf{L} -projective.

Proposition 19. *If \mathbf{L} has projective unification then \mathbf{L} is of type 1.*

The following result is essential for the proof of Proposition 23.

Proposition 20. *If \mathbf{L} contains $\mathbf{K5}$ then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is \mathbf{L} -projective if and only if φ has extension property in \mathbf{L} .*

The following result is essential for the proof of Proposition 25.

Proposition 21. *If \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$ and for all \mathbf{L} -unifiers σ of φ , there exists $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$ such that $\sigma(\psi) \in \mathbf{L}$, $\psi \rightarrow \varphi \in \mathbf{K}$, ψ is \mathbf{L} -projective.*

4 Extensions of $\mathbf{K5}$

Firstly, let us consider the extensions of $\mathbf{K45}$.

Proposition 22. *If \mathbf{L} contains $\mathbf{K45}$ then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is \mathbf{L} -projective.*

Proposition 23. *If \mathbf{L} contains $\mathbf{K45}$ then \mathbf{L} has projective unification.*

Secondly, let us consider the extensions of $\mathbf{K5}$.

Proposition 24. *If \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is of type 1.*

Proposition 25. *If \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global then \mathbf{L} is of type 1.*

Notice that the line of reasoning leading to Propositions 23 and 25 rules out neither the possibility that all extensions of $\mathbf{K5}$ have projective unification, nor the possibility that some nonglobal extension of $\mathbf{K5}$ is either of type ω , or of type ∞ , or of type 0¹³.

5 Conclusion

A property similar to the extension property has been used by Ghilardi who has proved both in Intuitionistic Logic [18] and in transitive modal logics like $\mathbf{K4}$ and $\mathbf{S4}$ [19] that it is equivalent to the projectivity of formulas. This property has also been considered in [11] where formulas verifying it are called *extendible formulas*. As a matter of fact, Bezhanishvili and de Jongh have provided a complete characterization in Intuitionistic Logic of the set of all extendible formulas with at most 2 variables. However, the question remains unsettled whether a complete characterization in Intuitionistic Logic of the set of all extendible formulas with at least 3 variables can be given. Within the context of extensions of $\mathbf{K5}$, we believe that it is probably easier to give a complete characterization of the set of all formulas verifying the extension property.

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¹³No modal logic is known to be of type ∞ [15, Chapter 5].

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Appendix

Proof of Proposition 1: See [29, Corollary 5].

Proof of Proposition 2: Suppose \mathbf{L} is locally tabular. Let $X, Y \subseteq \mathbf{VAR}$ be finite. For all $x \in X$, let $\simeq_{\mathbf{L}}^x$ be the equivalence relation on $\mathbf{SUB}_{X,Y}$ defined by

- $\sigma \simeq_{\mathbf{L}}^x \tau$ if and only if $\sigma(x) \leftrightarrow \tau(x) \in \mathbf{L}$.

Since \mathbf{L} is locally tabular and Y is finite, for all $x \in X$, $\simeq_{\mathbf{L}}^x$ possesses finitely many equivalence classes on $\mathbf{SUB}_{X,Y}$. Since X is finite and the restriction of $\simeq_{\mathbf{L}}$ to $\mathbf{SUB}_{X,Y}$ is equal to $\bigcap \{\simeq_{\mathbf{L}}^x : x \in X\}$, $\simeq_{\mathbf{L}}$ possesses finitely many equivalence classes on $\mathbf{SUB}_{X,Y}$.

Proof of Proposition 3: By induction on $\varphi \in \mathbf{FOR}_X$.

Proof of Proposition 4: See [28, Sections 2 and 3].

Proof of Proposition 5: See [28, Sections 2 and 3].

Proof of Proposition 6: Left to the reader.

Proof of Proposition 7: Suppose \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global. For the sake of the contradiction, suppose neither $\mathbf{L} = \mathbf{K5}$, nor $\mathbf{L} = \mathbf{K5} \oplus \diamond \top$, nor $\mathbf{L} = \mathbf{K5} \oplus \varphi_l$ for some positive integer l , nor $\mathbf{L} = \mathbf{K5} \oplus \varphi_l \oplus \diamond \top$ for some positive integer l , nor $\mathbf{L} = \mathbf{K5} \oplus \square \perp$. By Proposition 5, we have to consider the following 5 cases.

Case “for all $m \geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$ ”: Since \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global, $\mathbf{L} = \mathbf{K5}$: a contradiction.

Case “for all $m \geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \not\models \mathbf{L}$ ”: Since \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global, $\mathbf{L} = \mathbf{K5} \oplus \diamond \top$: a contradiction.

Case “there exists $m \geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n \geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$ ”: Thus, let l be the greatest positive integer such that $\mathcal{T}_l \models \mathbf{L}$. Since \mathbf{L} contains $\mathbf{K5}$, \mathbf{L} is global and $\mathcal{S} \models \mathbf{L}$, $\mathbf{L} = \mathbf{K5} \oplus \varphi_l$: a contradiction.

Case “there exists $m \geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n \geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $S \not\models \mathbf{L}$ ”: Consequently, let l be the greatest positive integer such that $\mathcal{T}_l \models \mathbf{L}$. Since \mathbf{L} contains **K5**, \mathbf{L} is global and $S \not\models \mathbf{L}$, $\mathbf{L} = \mathbf{K5} \oplus \varphi_l \oplus \diamond\top$: a contradiction.

Case “for all $m \geq 1$, $\mathcal{T}_m \not\models \mathbf{L}$ ”: Hence, $\mathbf{L} = \mathbf{K5} \oplus \square\perp$: a contradiction.

Proof of Proposition 8: Suppose \mathbf{L} contains **K5**. Let $\varphi \in \mathbf{FOR}$ be such that $\varphi \notin \mathbf{L}$. Since \mathbf{L} contains **K5**, by [29, Theorem 3], let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \varphi$. Without loss of generality, by [12, Proposition 2.6 and Theorem 3.14], we can suppose (W, R) is generated from s .

Proof of Proposition 9: Suppose \mathbf{L} contains **K45**. Let $\varphi \in \mathbf{FOR}$. Let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$ and (W, R) is generated from s . Since \mathbf{L} contains **K45**, by Proposition 4, we have to consider the following 3 cases.

Case “ $W = \{s\}$ and $R = \emptyset$ ”: Obviously, V is a variant of V with respect to s and $\text{var}(\varphi)$. Moreover, $(W, R), V, s \models \diamond\square\varphi \rightarrow \varphi$.

Case “ $R = W \times W$ ”: Obviously, V is a variant of V with respect to s and $\text{var}(\varphi)$. Moreover, $(W, R), V, s \models \diamond\square\varphi \rightarrow \varphi$.

Case “there exists $A \subseteq W$ such that $A \neq \emptyset$, $s \notin A$, $W = \{s\} \cup A$ and $R = (\{s\} \times A) \cup (A \times A)$ ”: Let $t \in A$. Obviously, $(W, R), V, t \models \diamond\square\varphi \rightarrow \varphi$. Let V' be a valuation on (W, R) such that for all $x \in \text{var}(\varphi)$, if $t \in V(x)$ then $V'(x) = V(x) \cup \{s\}$ else $V'(x) = V(x) \setminus \{s\}$. Obviously, V' is a variant of V with respect to s and $\text{var}(\varphi)$. Moreover, by induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- for all $u \in A$, $(W, R), V, u \models \psi$ if and only if $(W, R), V', u \models \psi$.

In other respect, by induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(W, R), V, t \models \psi$ if and only if $(W, R), V', s \models \psi$.

Since $(W, R), V, t \models \diamond\square\varphi \rightarrow \varphi$, $(W, R), V', s \models \diamond\square\varphi \rightarrow \varphi$.

Proof of Proposition 10: This is a standard result, although we have not been able to find a published proof of it. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable and Σ, Δ be bases for φ . Hence, Σ and Δ are minimal complete sets of \mathbf{L} -unifiers of φ . By the completeness of Σ and Δ , one can readily define functions $f : \Sigma \rightarrow \Delta$ and $g : \Delta \rightarrow \Sigma$ such that $\sigma \preceq_{\mathbf{L}} f(\sigma)$ for each $\sigma \in \Sigma$ and $\delta \preceq_{\mathbf{L}} g(\delta)$ for each $\delta \in \Delta$. By the minimality of Σ and Δ , it easily follows that f and g are injective. Thus, Σ and Δ have the same cardinality.

Proof of Proposition 11: This is a standard result, although we have not been able to find a published proof of it. Suppose φ is \mathbf{L} -filtering. For the sake of the contradiction, suppose φ is neither of type 1, nor of type 0. Hence, φ is either of type ω , or of type ∞ . Thus, let Σ be a basis for φ either with finite cardinality ≥ 2 , or with infinite cardinality. Consequently, let $\sigma, \tau \in \Sigma$ be such that $\sigma \neq \tau$. Since φ is \mathbf{L} -filtering, there exists an \mathbf{L} -unifier v of φ such that $\sigma \preceq_{\mathbf{L}} v$ and $\tau \preceq_{\mathbf{L}} v$. Since Σ is a basis for φ , let $v' \in \Sigma$ be such that $v \preceq_{\mathbf{L}} v'$. Since $\sigma \preceq_{\mathbf{L}} v$ and $\tau \preceq_{\mathbf{L}} v$, $\sigma \preceq_{\mathbf{L}} v'$ and $\tau \preceq_{\mathbf{L}} v'$. Since Σ is a basis for φ , $\sigma = v'$ and $\tau = v'$. Hence, $\sigma = \tau$: a contradiction.

Proof of Proposition 12: By Proposition 11.

Proof of Proposition 13: This result generalizes some results obtained in [6]. Suppose \mathbf{L} contains **K5**. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable. Let $(\text{var}(\varphi), X, \sigma), (\text{var}(\varphi), Y, \tau)$ be \mathbf{L} -unifiers of φ . Hence, $\sigma(\varphi) \in \mathbf{L}$ and $\tau(\varphi) \in \mathbf{L}$. Let $(\text{var}(\varphi), X \cup Y \cup \{z\}, v)$ be the substitution defined for all $x \in \text{var}(\varphi)$, by $v(x) = ((\Box\Box z \wedge (z \vee \Diamond\top)) \wedge \sigma(x)) \vee ((\Diamond\Diamond \neg z \vee (\neg z \wedge \Box\perp)) \wedge \tau(x))$ where z is a new variable, i.e. neither $z \in \text{var}(\varphi)$, nor $z \in X \cup Y$. Obviously, $\sigma \preceq_{\mathbf{L}} v$ and $\tau \preceq_{\mathbf{L}} v$. Moreover, by induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(\Box\Box z \wedge (z \vee \Diamond\top)) \rightarrow (v(\psi) \leftrightarrow \sigma(\psi)) \in \mathbf{L}$,
- $(\Diamond\Diamond \neg z \vee (\neg z \wedge \Box\perp)) \rightarrow (v(\psi) \leftrightarrow \tau(\psi)) \in \mathbf{L}$.

Since $\sigma(\varphi) \in \mathbf{L}$ and $\tau(\varphi) \in \mathbf{L}$, $v(\varphi) \in \mathbf{L}$. Thus, v is an \mathbf{L} -unifier of φ . Consequently, φ is \mathbf{L} -filtering.

Proof of Proposition 14: Suppose \mathbf{L} contains **K5**. Let v be an \mathbf{L} -projective substitution for φ . Hence, for all $x \in \text{var}(\varphi)$, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$. Let $x \in \text{var}(\varphi)$. Thus, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$. Let $t \in W$. Suppose $t \in V^v(x) \setminus \{s\}$. Consequently, $t \neq s$ and by Proposition 3, $(W, R), V, t \models v(x)$. Since $(W, R) \models \mathbf{L}$, (W, R) is generated from s , $(W, R), V, s \models \Diamond\Box\varphi$, \mathbf{L} contains **K5** and $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$, $(W, R), V, t \models x$. Since $t \neq s$, $t \in V(x) \setminus \{s\}$. Reciprocally, suppose $t \in V(x) \setminus \{s\}$. Consequently, $t \neq s$ and $(W, R), V, t \models x$. Since $(W, R) \models \mathbf{L}$, (W, R) is generated from s , $(W, R), V, s \models \Diamond\Box\varphi$, \mathbf{L} contains **K5** and $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$, $(W, R), V, t \models v(x)$. Since $t \neq s$, by Proposition 3, $t \in V^v(x) \setminus \{s\}$. Hence, V^v is a variant of V with respect to s and $\text{var}(\varphi)$.

Proof of Proposition 15: See [1] and [15].

Proof of Proposition 16: See [1] and [15].

Proof of Proposition 17: See [1] and [15].

Proof of Proposition 18: By Proposition 17.

Proof of Proposition 19: By Proposition 18.

Proof of Proposition 20: Suppose \mathbf{L} contains **K5**. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable.

Suppose φ is \mathbf{L} -projective. For the sake of the contradiction, suppose φ has not extension property in \mathbf{L} . Since φ is \mathbf{L} -projective, let v be an \mathbf{L} -projective \mathbf{L} -unifier of φ . Since φ has not extension property in \mathbf{L} , let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that

- $(W, R) \models \mathbf{L}$,
- (W, R) is generated from s ,
- for all variants V' of V with respect to s and $\text{var}(\varphi)$, $(W, R), V', s \not\models \Diamond\Box\varphi \rightarrow \varphi$.

Obviously, V is a variant of V with respect to s and $\text{var}(\varphi)$. Hence, $(W, R), V, s \not\models \Diamond\Box\varphi \rightarrow \varphi$. Thus, $(W, R), V, s \models \Diamond\Box\varphi$. Since v is an \mathbf{L} -projective \mathbf{L} -unifier of φ , $v(\varphi) \in \mathbf{L}$. Moreover, since \mathbf{L} contains **K5**, $(W, R) \models \mathbf{L}$, (W, R) is generated from s and $(W, R), V, s \models \Diamond\Box\varphi$, by Proposition 14, V^v is a variant of V with respect to s and $\text{var}(\varphi)$. Consequently, $(W, R), V^v, s \not\models \Diamond\Box\varphi \rightarrow \varphi$. Hence, $(W, R), V^v, s \models \varphi$. Thus, by Proposition 3, $(W, R), V, s \models v(\varphi)$. Since $(W, R) \models \mathbf{L}$, $v(\varphi) \notin \mathbf{L}$: a contradiction.

Suppose φ has extension property in \mathbf{L} . For the sake of the contradiction, suppose φ is not \mathbf{L} -projective. Since $\varphi \in \mathbf{FOR}$ is \mathbf{L} -unifiable, let σ be a variable-free \mathbf{L} -unifier of φ . Consequently, $\sigma(\varphi) \in \mathbf{L}$. Let $(\text{var}(\varphi), \emptyset, \tau)$ be a variable-free substitution. Let $(\text{var}(\varphi), \text{var}(\varphi), \epsilon_\tau)$ be the substitution such that

- for all $x \in \text{var}(\varphi)$, $\epsilon_\tau(x) = ((\varphi \wedge \Box\Box\varphi) \wedge x) \vee ((\neg\varphi \vee \Diamond\Diamond\neg\varphi) \wedge \tau(x))$.

The following fact can be easily proved: for all $x \in \text{var}(\varphi)$, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow \epsilon_\tau(x)$. Hence, ϵ_τ is \mathbf{L} -projective for φ . By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_\tau(\psi)$,
- $\neg\varphi \vee \Diamond\Diamond\neg\varphi \vdash_{\mathbf{L}} \tau(\psi) \leftrightarrow \epsilon_\tau(\psi)$.

Thus, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon_\tau(\varphi)$ and $\neg\varphi \vee \Diamond\Diamond\neg\varphi \vdash_{\mathbf{L}} \tau(\varphi) \leftrightarrow \epsilon_\tau(\varphi)$. Since \mathbf{L} contains **K5**, if $\tau(\varphi) \in \mathbf{L}$ then $\Box\Box\varphi \rightarrow (\varphi \rightarrow \epsilon_\tau(\varphi)) \in \mathbf{L}$ and $\Diamond\Diamond\neg\varphi \rightarrow \epsilon_\tau(\varphi) \in \mathbf{L}$. In that case, $\varphi \rightarrow \epsilon_\tau(\varphi) \in \mathbf{L}$ and $\Box\Box\varphi \vee \epsilon_\tau(\varphi) \in \mathbf{L}$ — which implies that $\Box\Box\epsilon_\tau(\varphi) \in \mathbf{L}$. Consequently, for all variable-free substitutions $(\text{var}(\varphi), \emptyset, \tau)$, if $\tau(\varphi) \in \mathbf{L}$ then $\Box\Box\epsilon_\tau(\varphi) \in \mathbf{L}$. Since $\sigma(\varphi) \in \mathbf{L}$, $\Box\Box\epsilon_\sigma(\varphi) \in \mathbf{L}$. Let $l \geq 1$ and $((\text{var}(\varphi), \emptyset, \tau_1), \dots, (\text{var}(\varphi), \emptyset, \tau_l))$ be an enumeration of the set of all variable-free substitutions $(\text{var}(\varphi), \emptyset, \tau)$ such that for all $x \in \text{var}(\varphi)$, either $\tau(x) = \top$, or $\tau(x) = \perp$ ¹⁴. Let $\epsilon = \epsilon_\sigma \circ \epsilon_{\tau_1} \circ \dots \circ \epsilon_{\tau_l} \circ \epsilon_\sigma$. Since $\Box\Box\epsilon_\sigma(\varphi) \in \mathbf{L}$, $\Box\Box\epsilon(\varphi) \in \mathbf{L}$. Moreover, since $\epsilon_\sigma, \epsilon_{\tau_1}, \dots, \epsilon_{\tau_l}$ are \mathbf{L} -projective for φ , by Proposition 16, ϵ is \mathbf{L} -projective for φ . Since φ is not \mathbf{L} -projective, $\epsilon(\varphi) \notin \mathbf{L}$. Since \mathbf{L} contains **K5**, by [29, Theorem 3], let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \epsilon(\varphi)$. Without loss of generality, by [12, Proposition 2.6 and Theorem 3.14], we can suppose (W, R) is generated from s . Since $\Box\Box\epsilon(\varphi) \in \mathbf{L}$ and $(W, R) \models \mathbf{L}$, $(W, R), V, s \models \Box\Box\epsilon(\varphi)$. Since $(W, R), V, s \not\models \epsilon(\varphi)$, $R \neq W \times W$. Since \mathbf{L} contains **K5**, $(W, R) \models \mathbf{L}$ and (W, R) is generated from s , by Proposition 4, we have to consider the following 2 cases.

Case “ $W = \{s\}$ and $R = \emptyset$ ”: By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon(\psi)$.

Hence, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon(\varphi)$. Since $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \epsilon(\varphi)$, $(W, R), V, s \not\models \varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\neg\varphi \vee \Diamond\Diamond\neg\varphi \vdash_{\mathbf{L}} \sigma(\psi) \leftrightarrow \epsilon_\sigma(\psi)$.

Thus, $\neg\varphi \vee \Diamond\Diamond\neg\varphi \vdash_{\mathbf{L}} \sigma(\varphi) \leftrightarrow \epsilon_\sigma(\varphi)$. Since $\sigma(\varphi) \in \mathbf{L}$, $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \varphi$, $(W, R), V, s \models \epsilon_\sigma(\varphi)$. Hence, by Proposition 3, $(W, R), V^{\epsilon_\sigma}, s \models \varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_1}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\psi)) \dots)$.

Consequently, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon_{\tau_1}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots)$. Since $(W, R) \models \mathbf{L}$ and $(W, R), V^{\epsilon_\sigma}, s \models \varphi$, $(W, R), V^{\epsilon_\sigma}, s \models \epsilon_{\tau_1}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots)$. Hence, by Proposition 3, $(W, R), V, s \models \epsilon(\varphi)$: a contradiction.

Case “there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $s \notin B$, $W = \{s\} \cup B$ and $R = (\{s\} \times A) \cup (B \times B)$ ”: For all $i \leq l$, let $\epsilon_i = \epsilon_{\tau_i} \circ \dots \circ \epsilon_{\tau_1} \circ \epsilon_\sigma$. Thus, for all $i \leq l$, if $i = 0$ then $\epsilon_i = \epsilon_\sigma$ else $\epsilon_i = \epsilon_{\tau_i} \circ \epsilon_{i-1}$. Since $\Box\Box\epsilon_\sigma(\varphi) \in \mathbf{L}$ and $(W, R) \models \mathbf{L}$, $(W, R), V, s \models \Box\Box\epsilon_\sigma(\varphi)$. Consequently, by Proposition 3, $(W, R), V^{\epsilon_\sigma}, s \models \Box\Box\varphi$. Hence, $(W, R), V^{\epsilon_\sigma}, s \models \Diamond\Diamond\varphi$. Since φ has extension property in \mathbf{L} , $(W, R) \models \mathbf{L}$ and (W, R) is generated from s , let V' be a variant of V^{ϵ_σ} with respect to s and $\text{var}(\varphi)$ such that $(W, R), V', s \models \Diamond\Diamond\varphi \rightarrow \varphi$. Since $(W, R), V^{\epsilon_\sigma}, s \models \Diamond\Diamond\varphi$, let $t \in A$ be such that $(W, R), V^{\epsilon_\sigma}, t \models \Box\varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

¹⁴Obviously, $l = 2^{\text{Card}(\text{var}(\varphi))}$.

- for all $u \in A$, $(W, R), V^{\epsilon\sigma}, u \models \psi$ if and only if $(W, R), V', u \models \psi$.

Since $t \in A$ and $(W, R), V^{\epsilon\sigma}, t \models \Box\varphi$, $(W, R), V', t \models \Box\varphi$. Since $t \in A$, $(W, R), V', s \models \Diamond\Box\varphi$. Since $(W, R), V', s \models \Diamond\Box\varphi \rightarrow \varphi$, $(W, R), V', s \models \varphi$. Let $j \in \{1, \dots, l\}$ be such that for all $x \in \text{var}(\varphi)$,

- if $(W, R), V', s \models x$ then $\tau_j(x) = \top$,
- if $(W, R), V', s \not\models x$ then $\tau_j(x) = \perp$.

Since $(W, R), V, s \not\models \epsilon(\varphi)$, $(W, R), V, s \not\models \epsilon_{j-1}(\epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots))$. Thus, by Proposition 3, $(W, R), V^{\epsilon_{j-1}}, s \not\models \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\varphi)) \dots)$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\psi)) \dots)$.

Consequently, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\psi)) \dots)$. Since $(W, R) \models \mathbf{L}$ and $(W, R), V^{\epsilon_{j-1}}, s \not\models \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\psi)) \dots)$, $(W, R), V^{\epsilon_{j-1}}, s \not\models \varphi \wedge \Box\Box\varphi$. By induction on $i \leq l$, the reader may easily verify that

- for all $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$ and for all $u \in B$, $(W, R), V^{\epsilon_i}, u \models \psi$ if and only if $(W, R), V^{\epsilon\sigma}, u \models \psi$.

By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(W, R), V^{\epsilon_j}, s \models \psi$ if and only if $(W, R), V', s \models \psi$.

Since $(W, R), V', s \models \Diamond\Box\varphi$ and $(W, R), V', s \models \varphi$, $(W, R), V^{\epsilon_j}, s \models \Diamond\Box\varphi$ and $(W, R), V^{\epsilon_j}, s \models \varphi$. Since \mathbf{L} contains **K5** and $(W, R) \models \mathbf{L}$, $(W, R), V^{\epsilon_j}, s \models \varphi \wedge \Box\Box\varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_{j+1}}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\psi)) \dots)$.

Hence, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_{j+1}}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\psi)) \dots)$. Since $(W, R) \models \mathbf{L}$ and $(W, R), V^{\epsilon_j}, s \models \varphi \wedge \Box\Box\varphi$, $(W, R), V^{\epsilon_j}, s \models \epsilon_{\tau_{j+1}}(\dots \epsilon_{\tau_l}(\epsilon_\sigma(\psi)) \dots)$. Thus, by Proposition 3, $(W, R), V, s \models \epsilon(\varphi)$: a contradiction.

Proof of Proposition 21: Suppose \mathbf{L} contains **K5** and \mathbf{L} is global. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable and $(\text{var}(\varphi), X, \sigma)$ be an \mathbf{L} -unifier of φ . Hence, $\sigma(\varphi) \in \mathbf{L}$. Let $\Gamma_{\text{var}(\varphi)}(\sigma)$ be the set of all formulas of the form $\text{for}_{\text{var}(\varphi)}((W, R), s, V^\sigma)$ where (W, R) is a finite frame, V is a valuation on (W, R) and $s \in W$ are such that $(W, R) \models \mathbf{L}$ and (W, R) is generated from s . Obviously, $\Gamma_{\text{var}(\varphi)}(\sigma)$ is a finite set of infinite subsets of $\mathbf{FOR}_{\text{var}(\varphi)}$. Nevertheless, since \mathbf{L} contains **K5**, by Proposition 1, \mathbf{L} is locally tabular and we will treat $\Gamma_{\text{var}(\varphi)}(\sigma)$ as if it is a finite set of finite subsets of $\mathbf{FOR}_{\text{var}(\varphi)}$. Indeed, knowing that for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, if $(W, R) \models \mathbf{L}$ and (W, R) is generated from s then $\text{for}_{\text{var}(\varphi)}((W, R), s, V^\sigma)$ also denotes the conjunction of the formulas that $\text{for}_{\text{var}(\varphi)}((W, R), s, V^\sigma)$ contains, we will treat $\Gamma_{\text{var}(\varphi)}(\sigma)$ as if it is a finite subset of $\mathbf{FOR}_{\text{var}(\varphi)}$. Let ψ be the disjunction of all formulas in this finite subset. Obviously,

- (*) for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, if $(W, R) \models \mathbf{L}$ and (W, R) is generated from s then $(W, R), V^\sigma, s \models \psi$.

Suppose $\sigma(\psi) \notin \mathbf{L}$. Since \mathbf{L} contains **K5**, by Proposition 8, let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$, (W, R) is generated from s and $(W, R), V, s \not\models \sigma(\psi)$. Thus, by (*), $(W, R), V^\sigma, s \models \psi$. Consequently, by Proposition 3, $(W, R), V, s \models \sigma(\psi)$: a contradiction.

Suppose $\psi \rightarrow \varphi \notin \mathbf{K}$. Hence, let (W', R') be a frame, V' be a valuation on (W', R') and $s' \in W'$ be such that $(W', R'), V', s' \models \psi$ and $(W', R'), V', s' \not\models \varphi$. Thus, there exists a finite frame (W, R) , there exists a valuation V on (W, R) and there exists $s \in W$ such that $(W, R) \models \mathbf{L}$ and

$(W', R'), V', s' \models \text{for}_{\text{var}(\varphi)}((W, R), s, V^\sigma)$. Since $(W', R'), V', s' \not\models \varphi$, $(W, R), V^\sigma, s \not\models \varphi$. Consequently, by Proposition 3, $(W, R), V, s \not\models \sigma(\varphi)$. Since $(W, R) \models \mathbf{L}$, $\sigma(\varphi) \notin \mathbf{L}$: a contradiction.

Suppose ψ is not **L**-projective. Since **L** contains **K5**, by Proposition 20, ψ has not extension property in **L**. Hence, let (W', R') be a finite frame, V' be a valuation on (W', R') and $s' \in W'$ be such that

- $(W', R') \models \mathbf{L}$,
- (W', R') is generated from s' ,
- for all variants V'' of V' with respect to s' and $\text{var}(\varphi)$, $(W', R'), V'', s' \not\models \Diamond \Box \psi \rightarrow \psi$.

Obviously, V' is a variant of V' with respect to s' and $\text{var}(\varphi)$. Thus, $(W', R'), V', s' \not\models \Diamond \Box \psi \rightarrow \psi$. Consequently, $(W', R'), V', s' \models \Diamond \Box \psi$ and $(W', R'), V', s' \not\models \psi$. Hence, neither $R' = \emptyset$, nor $R' = W' \times W'$. Since **L** contains **K5**, $(W', R') \models \mathbf{L}$ and (W', R') is generated from s' , by Proposition 4, there exists $A', B' \subseteq W'$ such that $A' \neq \emptyset$, $A' \subseteq B'$, $s' \notin B'$, $W' = \{s'\} \cup B'$ and $R' = (\{s'\} \times A') \cup (B' \times B')$. Since $(W', R'), V', s' \models \Diamond \Box \psi$, let $t' \in A'$ be such that $(W', R'), V', t' \models \Box \psi$. Thus, $(W', R'), V', t' \models \psi$. Consequently, there exists a finite frame (W, R) , there exists a valuation V on (W, R) and there exists $t \in W$ such that

- $(W, R) \models \mathbf{L}$,
- (W, R) is generated from t ,
- $(W', R'), V', t' \models \text{for}_{\text{var}(\varphi)}((W, R), t, V^\sigma)$.

Obviously, $(B', B' \times B')$ is the subframe of (W', R') generated from t' . Let $V'_{B'}$ be the restriction of $V'_{B'}$ to B' . Since $(W', R'), V', t' \models \text{for}_{\text{var}(\varphi)}((W, R), t, V^\sigma)$, by [12, Proposition 2.6], $(B', B' \times B'), V'_{B'}, t' \models \text{for}_{\text{var}(\varphi)}((W, R), t, V^\sigma)$. Since $(B', B' \times B')$ and (W, R) are finite, by [12, Theorem 2.24], let $Z \subseteq B' \times W$ be a bisimulation between $(B', B' \times B')$ and (W, R) such that

- $t' Z t$,
- for all $u' \in B'$ and for all $u \in W$, if $u' Z u$ then for all $x \in \text{var}(\varphi)$, $u' \in V'_{B'}(x)$ if and only if $u \in V^\sigma(x)$.

Hence, $R \neq \emptyset$. Since **L** contains **K5**, $(W, R) \models \mathbf{L}$ and (W, R) is generated from t , by Proposition 4, we have to consider the following 2 cases.

Case “ $R = W \times W$ ”: Let $A = \{u \in W : \text{there exists } u' \in A' \text{ such that } u' Z u\}$ and $B = W$. Obviously, $A \neq \emptyset$ and $A \subseteq B$. Moreover, since $(W, R) \models \mathbf{L}$, $(B, B \times B) \models \mathbf{L}$. Let (W_1, R_1) be a finite frame, V_1 be a valuation on (W_1, R_1) and $s_1 \in W_1$ be such that $s_1 \notin B$, $W_1 = \{s_1\} \cup B$, $R_1 = (\{s_1\} \times A) \cup (B \times B)$ and for all $x \in \text{var}(\varphi)$, $V_1(x) = V(x)$. Since **L** is global and $(B, B \times B) \models \mathbf{L}$, $(W_1, R_1) \models \mathbf{L}$. Thus, by (*), $(W_1, R_1), V_1^\sigma, s_1 \models \psi$. Let V'' be a valuation on (W', R') such that for all $x \in \text{var}(\varphi)$, if $s_1 \in V_1^\sigma(x)$ then $V''(x) = V(x) \cup \{s'\}$ else $V''(x) = V(x) \setminus \{s'\}$. Obviously, V'' is a variant of V' with respect to s' and $\text{var}(\varphi)$. Consequently, $(W', R'), V'', s' \not\models \Diamond \Box \psi \rightarrow \psi$. Hence, $(W', R'), V'', s' \not\models \psi$. By induction on $\chi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(W', R'), V'', s' \models \chi$ if and only if $(W_1, R_1), V_1^\sigma, s_1 \models \chi$.

Since $(W', R'), V'', s' \not\models \psi$, $(W_1, R_1), V_1^\sigma, s_1 \models \psi$: a contradiction.

Case “there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $t \notin B$, $W = \{t\} \cup B$ and $R = (\{t\} \times A) \cup (B \times B)$ ”:

Hence, let $t_0 \in A$ be such that $t'Zt_0$. Let Z_0 be the restriction of Z to $W' \times B$ and V_0 be the restriction of V to B . Since $(W, R) \models \mathbf{L}$, by [12, Theorem 3.14], $(B, B \times B) \models \mathbf{L}$. Since $Z \subseteq W' \times W$ is a bisimulation between (W', R') and (W, R) such that $t'Zt_0$ and for all $u' \in W'$ and for all $u \in W$, if $u'Zu$ then for all $x \in \text{var}(\varphi)$, $u' \in V'(x)$ if and only if $u \in V^\sigma(x)$, $Z_0 \subseteq W' \times B$ is a bisimulation between (W', R') and $(B, B \times B)$ such that $t'Z_0t_0$ and for all $u' \in W'$ and for all $u \in B$, if $u'Z_0u$ then for all $x \in \text{var}(\varphi)$, $u' \in V'_0(x)$ if and only if $u \in V_0^\sigma(x)$. Then, proceed as in the case “ $R=W \times W$ ”.

Proof of Proposition 22: By Propositions 9 and 20.

Proof of Proposition 23: By Proposition 22.

Proof of Proposition 24: Suppose \mathbf{L} contains **K5** and \mathbf{L} is global. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable. Let σ be an \mathbf{L} -unifier of φ . Since \mathbf{L} contains **K5** and \mathbf{L} is global, by Proposition 21, let $\psi_\sigma \in \mathbf{FOR}_{\text{var}(\varphi)}$ be such that $\sigma(\psi_\sigma) \in \mathbf{L}$, $\psi_\sigma \rightarrow \varphi \in \mathbf{K}$ and ψ_σ is \mathbf{L} -projective. Hence, let ϵ_σ be an \mathbf{L} -projective \mathbf{L} -unifier of ψ_σ . Let $\Sigma = \{\epsilon_\sigma : \sigma \text{ is an } \mathbf{L}\text{-unifier of } \varphi\}$. By Propositions 17 and 21, Σ is a complete set of \mathbf{L} -unifiers of φ . Let Σ' be the set of substitutions obtained from Σ by keeping only one representative of each equivalence class modulo $\simeq_{\mathbf{L}}$. Since Σ is a complete set of \mathbf{L} -unifiers of φ , Σ' is a complete set of \mathbf{L} -unifiers of φ . Moreover, since \mathbf{L} is locally tabular, by Proposition 2, Σ' is finite. Thus, φ is either of type 1, or of type ω . Since \mathbf{L} contains **K5**, by Propositions 11 and 13, φ is of type 1.

Proof of Proposition 25: By Proposition 24.