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Perturbatively conserved higher nonlocal integral invariants of free-surface deep-water gravity waves

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We exhibit a set of six explicit higher nonlocal integral invariants of free-surface deep-water gravity waves conserved in lowest nontrivial orders of perturbation in the amplitude of the surface displacement.

I. INTRODUCTION

Inviscid irrotational deep-water gravity waves in one dimension are a much studied subject in mathematical physics. In 1968, in his pionneering work, Zakharov[1] showed that in a certain well-defined approximation the envelope of a travelling wave satisfies the non-linear Schrödinger equation, which is integrable. A vast literature has followed, which it is impossible to review in this paper. Beyond its conceptual simplicity, a large part of the appeal of the general problem of inviscid irrotational deep-water gravity waves in one dimension comes from the fact that in some sense it seems to come close to integrability[2–4].

Here, by choosing a convenient set of dynamical variables, we give explicit expressions for six previously unknown nonlocal integral invariants beyond energy and momentum, conserved in lowest orders of perturbation in powers of the vertical displacement of the surface. In the linearized approximation two of them generalize momentum and energy conservation with higher spatial derivatives.

In section II we show that the velocity potential at the rest altitude $z=0$, $\mathscr{U}(x,z=0,t)\equiv U(x,t)$ together with the surface position $\eta(x,t)$ of the fluid are convenient variables to study the Euler equations[5] and their conservation laws in perturbation. With these fundamental variables we give perturbative expansions of $\mathscr{U}(x,z,t)$ and its z derivative at the surface where the stream function associated with $\mathscr{U}(x,z,t)$ is introduced and will play an important rôle

$$\mathscr{V}(x,z,t)=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x-x'}{(x-x')^2+z^2}U(x',t)\mathrm{d}x'\,.$$

In section III we give a few mathematical formulas involving several principal value integrals which are necessary to derive the conservation of the nonlocal integral invariants.

In section IV we present these nonlocal integral invariants, beginning with generalizations of energy and momentum, and new ones using $\mathscr{V}(x,z=0,t)\equiv V(x,t)$ already in the lowest order, linear approximation, of the equations of motion.

In the discussion, section V, we comment on the connection with the conjectured integrability of the Euler equations and on the analogy with the φ^4 model in two dimensions.

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II. PERTURBATIVE EXPANSION OF THE EULER EQUATIONS FOR GRAVITY WAVES

The velocity field with horizontal component $u(x,z,t)$ and vertical component $v(x,z,t)$ of an incompressible inviscid fluid of unit mass per unit volume in the gravity field g satisfies the Euler equations of motion[5]

$$\begin{aligned}u_t+uu_x+vu_z&=-p_x\\ v_t+uv_x+vv_z&=-g-p_z\end{aligned}\qquad(1)$$

with p the pressure field.

We consistently restrict ourselves to irrotational motions. In such a motion, the velocity field $u(x,z,t)$, $v(x,z,t)$ derives from a velocity potential $\mathscr{U}(x,z,t)$

$$u=\partial_x\mathscr{U}\,,\qquad v=\partial_z\mathscr{U}\,.$$

and $\mathscr{U}$ is a harmonic function of x and z

$$\partial_x\mathscr{U}=\partial_z\mathscr{V}\,,\qquad\partial_z\mathscr{U}=-\partial_x\mathscr{V}\,.$$

In an infinitely deep fluid, at $z\to-\infty$ the boundary condition is $\mathscr{U}\to0$ and at the surface

$$v(x,\eta(x,t),t)-u(x,\eta(x,t),t)\eta_x(x,t)=\eta_t(x,t)\,.$$

The equations of motion can be derived from the Lagrangian

$$\begin{aligned}\mathscr{L}=\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{\eta}(\mathscr{U}_x^2+\mathscr{U}_z^2)\mathrm{d}z-\frac{1}{2}g\int_{-\infty}^{\infty}\eta^2\mathrm{d}x\\+\int_{-\infty}^{\infty}\mathrm{d}x\lambda\left[\eta_t-\mathscr{U}_z\big(x,\eta(x)\big)+\mathscr{U}_x\big(x,\eta(x)\big)\eta_x\right]\end{aligned}\qquad(2)$$

together with the boundary condition $\mathscr{U}\to0$ at $x\to\pm\infty$. $\lambda(x,t)$ is the Lagrange multiplier which imposes the boundary condition at the surface.

One finds that $\lambda(x,t)=\mathscr{U}\big(x,\eta(x),t\big)$ satisfies these equations, which then write:

$$\begin{aligned}\partial_x^2\mathscr{U}+\partial_z^2\mathscr{U}=0\,,\\ \partial_t\lambda+\mathscr{U}_x\big(x,\eta(x)\big)\,\partial_x\lambda-\frac{1}{2}\Big[\mathscr{U}_x\big(x,\eta(x)\big)^2\\+\mathscr{U}_z\big(x,\eta(x)\big)^2\Big]+g\eta=0.\end{aligned}\qquad(3)$$

The harmonic function $\mathscr{U}(x,z,t)$ can be recovered for all $z<0$ from its value at $z=0$ which we call $U(x,t)$ by the formula

$$\mathscr{U}(x,z,t)=-\frac{z}{\pi}\int_{-\infty}^{+\infty}\frac{U(x',t)}{(x-x')^2+z^2}\mathrm{d}x'\qquad(4)$$

and its z-derivative is then

$$-\frac{\partial \mathscr{U}(x,z)}{\partial z} = \frac{1}{\pi}\int_{-\infty}^{+\infty} U(x')\frac{(x-x')^2-z^2}{\big((x-x')^2+z^2\big)^2}\mathrm{d}x'$$

.

We shall also use the harmonic stream function $\mathscr{V}(x,z,t)$ by

$$\mathscr{V}(x,z,t) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{(x-x')}{(x-x')^2+z^2}U(x')\mathrm{d}x'$$

for $z<0$ so that $\mathscr{U}+\mathrm{i}\mathscr{V}$ is a function of $x+\mathrm{i}z$ and $\mathscr{V}(x,0,t)$ is given by a principal value integral which we call $V(x,t)$. One has then

$$\frac{\partial \mathscr{U}(x,z)}{\partial x} = \frac{\partial \mathscr{V}(x,z)}{\partial z} \qquad \frac{\partial \mathscr{U}(x,z)}{\partial z} = -\frac{\partial \mathscr{V}(x,z)}{\partial x}$$

and as $z\to 0_-$, $\partial_z\mathscr{U}(x,z)$ is given by the principal value integral

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{(x-x')}{(x-x')^2+\varepsilon^2}U_{x'}(x')\mathrm{d}x' = -\frac{\partial V(x)}{\partial x}$$

.

Equation (4), valid for $z<0$, does not mean that $\mathscr{U}(x,z)$ is odd in z, but around $z=0$ it can be expanded in powers of z as

$$\mathscr{U}(x,z) = U(x) - zV_x(x) - \frac{1}{2}z^2U_{xx}(x) + \frac{1}{6}z^3V_{xxx}(x) + \mathscr{O}(z^4). \tag{5}$$

From this, one obtains the perturbative expansion of λ by setting $z=\eta(x)$ in this equation.

The usual Hamiltonian, kinetic plus potential energies, is

$$\mathscr{H} = \frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{\eta(x)}(\mathscr{U}_x^2+\mathscr{U}_z^2)\mathrm{d}z + \frac{1}{2}g\int_{-\infty}^{\infty}\eta^2\mathrm{d}x$$

.

Using the equations of motion and the boundary conditions, one finds that this can be reduced to an integral over x only, to give the following expression for the total energy of the motion in terms of the dynamical variables at the surface:

$$E_1 = \int_{-\infty}^{+\infty}\Big[\frac{1}{2}\mathscr{U}\big(x,\eta(x)\big)\mathscr{U}_z\big(x,\eta(x)\big) - \frac{1}{2}\mathscr{U}\big(x,\eta(x)\big)\mathscr{U}_x\big(x,\eta(x)\big)\eta_x(x) + \frac{1}{2}g\,\eta(x)^2\Big]\mathrm{d}x \tag{6}$$

and for the canonical total momentum we have

$$P_1 = \int_{-\infty}^{+\infty}\lambda(x)\eta_x(x)\mathrm{d}x = \int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{\eta(x)}\mathscr{U}_x\mathrm{d}z. \tag{7}$$

In the linearized approximation, the equations of motion reduce to

$$\lambda_t(x,t) = U_t(x,t) = -g\eta(x,t), \tag{8}$$

$$\eta_t(x,t) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{(x-x')}{(x-x')^2+\varepsilon^2}U_{x'}(x')\mathrm{d}x' \tag{9}$$

so that for a plane wave $\exp(\mathrm{i}(kx-\omega t))$ one recovers the usual dispersion law $\omega^2 = g|k|$.

III. MATHEMATICAL FORMULAS

When expanding in perturbation the equation of motion (3) for λ and the boundary condition at the surface

$$\eta_t(x,t) - \mathscr{U}_z\big(x,\eta(x),t\big) + \mathscr{U}_x\big(x,\eta(x),t\big)\eta_x(x,t) = 0, \tag{10}$$

one encounters products of several principal value integrals the evaluation of some of which can be found in textbooks on the Hilbert transform, for example

$$\lim_{\varepsilon_1\to 0,\varepsilon_2\to 0}\int_{-\infty}^{+\infty}\frac{x-x'}{(x-x')^2+\varepsilon_1^2}\frac{x-x''}{(x-x'')^2+\varepsilon_2^2}\,\mathrm{d}x. \tag{11}$$

This is straighforward by contour integration at infinity in the complex plane, and in the limit of the epsilons going to zero gives $\pi^2\delta(x'-x'')$.

Here are other useful identities which can be obtained easily by the same method. For clarity we have not mentioned explicitly that in these identities all the epsilons are independent from one another, and that they are just there to remind us that we are actually dealing in the end with principal value integrals.

$$\frac{x-x'}{(x-x')^2+\varepsilon^2}\frac{x-x''}{(x-x'')^2+\varepsilon^2} = \frac{x-x'}{(x-x')^2+\varepsilon^2}\frac{x'-x''}{(x'-x'')^2+\varepsilon^2} + \frac{x-x''}{(x-x'')^2+\varepsilon^2}\frac{x''-x'}{(x''-x')^2+\varepsilon^2} + \pi^2\delta(x-x')\delta(x-x''), \tag{12}$$

$$\frac{x-x'}{(x-x')^2+\varepsilon^2}\frac{x-x''}{(x-x'')^2+\varepsilon^2} = \frac{1}{2}\frac{x-x'}{(x-x')^2+\varepsilon^2}\frac{x'-x''}{(x'-x'')^2+\varepsilon^2} + \frac{\pi^2}{2}\delta(x-x')\delta(x'-x'') \tag{13}$$

when this second equation is multiplied by an expression symmetric in x and x'', and

$$\int_{-\infty}^{+\infty}\frac{x-x'}{(x-x')^2+\varepsilon^2}\frac{x-x''}{(x-x'')^2+\varepsilon^2}\frac{x-x'''}{(x-x''')^2+\varepsilon^2}\mathrm{d}x=\pi^2\Big(\frac{x'-x''}{(x'-x'')^2+\varepsilon^2}\delta(x'-x''')+\frac{x''-x'''}{(x''-x''')^2+\varepsilon^2}\delta(x''-x')+\frac{x'''-x'}{(x'''-x')^2+\varepsilon^2}\delta(x'''-x'')\Big). \tag{14}$$

IV. NEW CONSERVED QUANTITIES

In many known integrable one-space-one-time dynamical systems, the higher conserved quantities appear in the weak field limit as bilinear expressions involving higher derivatives of the dynamical variables which generalize energy and momentum densities. For deep-water waves it is first natural to start with such generalizations. In lowest order the first generalization of momentum conservation would thus be

$$P_2=\int_{-\infty}^{+\infty}\lambda_x(x)\eta_{xx}(x)\mathrm{d}x \tag{15}$$

and the first generalization of energy conservation would be

$$E_2=\int_{-\infty}^{+\infty}\Big[\frac{1}{2}\mathscr{U}_x\big(x,\eta(x)\big)\mathscr{U}_{xz}\big(x,\eta(x)\big)+\frac{1}{2}g\,\eta_x(x)^2\Big]\mathrm{d}x. \tag{16}$$

We shall come back later in this section to the extension of these two quantities to higher orders, and first explore another route towards new conserved quantities.

The canonical energy (6) involves $\mathscr{U}_z(x,0)$ which is given by a principal value integral in terms of the variables λ and η. So, it is natural to look for other quantities which would be conserved already in lowest order and would similarly involve nonlocal expressions in terms of λ and η.

In the weak field limit, from (8) and (9, one has

$$\partial_t V(x,t)=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{(x-x')}{(x-x')^2+z^2}\big(-g\eta(x',t)\big)\,\mathrm{d}x' \tag{17}$$

so that

$$\int_{-\infty}^{+\infty}\partial_t V(x,t)\eta(x,t)\,\mathrm{d}x=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{(x-x')}{(x-x')^2+\varepsilon^2}\eta(x,t)\big(-g\eta(x',t)\big)\,\mathrm{d}x\mathrm{d}x' \tag{18}$$

vanishes by antisymmetry in x and x', and

$$\int_{-\infty}^{+\infty}V(x,t)\partial_t\,\eta(x,t)\,\mathrm{d}x=-\int_{-\infty}^{+\infty}V(x,t)\partial_x V(x,t)\,\mathrm{d}x=0 \tag{19}$$

Hence, rather trivially, in the weak field limit

$$\int_{-\infty}^{+\infty}V(x)\eta(x)\,\mathrm{d}x \tag{20}$$

is time independent. In next order, considering the equation of motion (3) for λ and the expansion in powers of z (5), it is more convenient to replace $U(x)$ by λ in the definition of $V(x)$. In the remainder of this section, we adopt this new starting point:

$$V(x)=-\lim_{\varepsilon\to0}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x-x'}{(x-x')^2+\varepsilon^2}\lambda(x')\,\mathrm{d}x'. \tag{21}$$

One must use the identities of the previous section to reduce the cubic terms of the time derivative of (20) with this new definition of V, and one finds that they can be cancelled by a rather simple cubic addition:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}\Big[V(x)\eta(x)+\frac{1}{2}\eta(x)^2\lambda_x(x)\Big]\mathrm{d}x \tag{22}$$

vanishes at third order in the surface variables.

We have pushed the calculation to next order, where it becomes much more involved, and found that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}\Big[V(x)\eta(x)+\frac{1}{2}\eta(x)^2\lambda_x(x)-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{(x-x')^2-\varepsilon^2}{\big((x-x')^2+\varepsilon^2\big)^2}\eta(x')^2\eta(x)U_x(x)\,\mathrm{d}x' \tag{23}$$

$$-\frac{1}{2\pi}\eta(x)^2\int_{-\infty}^{+\infty}\mathrm{d}x'\frac{x-x'}{(x-x')^2+\varepsilon^2}\big(\eta_{x'}(x')U_x(x)-\eta_x(x)U_{x'}(x')\big)\Big]\mathrm{d}x \tag{24}$$

vanishes at fourth order. In this expression, the limit $\varepsilon\to0$ is of course understood.

As the order increases, the number of derivatives increases, and of course also the number of ways to distribute them among the various perturbative quantities, beyond what

seems tractable by hand.

Considering the relative simplicity of this new conserved quantity (20), we try starting points with higher derivatives in it, analogous to the generalizations of energy (16) and momentum (15):

$$\int_{-\infty}^{+\infty} V_x(x)\eta_x(x)\mathrm{d}x. \tag{25}$$

For this quantity we have found that at third order in the dynamical variables

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}\mathrm{d}x\Big[V_x(x)\eta_x(x)-\frac{1}{2}\eta(x)^2\lambda_{xxx}(x)-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{(x-x')}{(x-x')^2+\varepsilon^2}\eta(x)\eta_{x'x'}(x')V_x(x)\mathrm{d}x' \\ -\frac{2}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}x'\eta_x(x)V_x(x)\frac{x-x'}{(x-x')^2+\varepsilon^2}\eta_{x'}(x')\Big] \tag{26}$$

vanishes.

We have even managed to treat a case with two more derivatives:

$$\int_{-\infty}^{+\infty} V_{xx}(x)\eta_{xx}(x)\mathrm{d}x. \tag{27}$$

In the time derivative of this expression, we found that at third order four different terms are involved, which we list:

$$\eta(x)V_x(x)U_{5x}(x),\quad \eta_x(x)V_x(x)U_{4x}(x),\eta_{xx}(x)V_x(x)U_{xxx}(x),\quad \eta_{xxx}(x)V_x(x)U_{xx}(x). \tag{28}$$

An expression whose time derivative could cancel these terms is found to be of the form

$$A\,\eta(x)^2U_{5x}(x)+B\,\eta_x(x)^2U_{xxx}(x)+C\,\eta_{xx}(x)^2U_x(x)+D\,U_{xxx}(x)U_{xx}(x)V_x(x)+E\,U_{4x}(x)U_x(x)V_x(x) \\ +F\,\eta(x)V_x(x)\frac{(x-x')}{(x-x')^2+\varepsilon^2}\eta_{4x'}(x')+G\,\eta_x(x)V_x(x)\frac{(x-x')}{(x-x')^2+\varepsilon^2}\eta_{x'x'x'}(x') \\ +H\,\eta_{xx}(x)V_x(x)\frac{(x-x')}{(x-x')^2+\varepsilon^2}\eta_{x'x'}(x')+J\,\eta_{xxx}(x)V_x(x)\frac{(x-x')}{(x-x')^2+\varepsilon^2}\eta_{x'}(x'). \tag{29}$$

The time derivative of this involves five more terms:

$$\eta_{4x}(x)V_x(x)U_x(x),\ \ \eta_{xx}(x)^2\eta_x(x),\ \ \eta_{4x'}(x')\frac{(x-x')}{(x-x')^2+\varepsilon^2}U_x(x)^2, \\ \eta_{x'x'}(x')\frac{(x-x')}{(x-x')^2+\varepsilon^2}U_{xx}(x)^2,\ \ \eta_x(x)\eta_{x''x''}(x'')\eta_{x'x'}(x')\frac{(x-x')}{(x-x')^2+\varepsilon^2}\frac{(x-x'')}{(x-x'')^2+\varepsilon'^2}. \tag{30}$$

So, to have conservation at third order, we arrive at a system of nine equations in nine unknowns, which has a unique solution:

$$A=\frac{1}{2},\ B=1,\ C=1,\ D=-\frac{6}{g},\ E=\frac{3}{g},\ F=-\frac{1}{\pi},\ G=-\frac{5}{\pi},\ H=-\frac{15}{\pi},\ J=-\frac{9}{\pi}$$

.

There is still another relatively simple candidate for a new conservation law: The starting point (20) which is bilinear in U and η looks like the momentum (7) also bilinear in U and η. It is natural then to investigate the conservation of a quantity which would resemble energy (6). Indeed, one finds immediately that to leading, bilinear, order

$$\int_{-\infty}^{+\infty}\mathrm{d}x\Big[\frac{1}{2}\lambda_x(x)^2+\frac{g}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}x'\eta(x)\frac{(x-x')}{(x-x')^2+\varepsilon^2}\eta_{x'}(x')\Big] \tag{31}$$

is conserved.

In next, trilinear order, the time derivative of this expression gives the relatively simple cubic terms

$$\int_{-\infty}^{+\infty}\mathrm{d}x\Big[-\frac{1}{2}V_x(x)^2\lambda_{xx}(x)-g\eta(x)\eta_{xx}(x)V_x(x) \\ +\frac{g}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}x'\eta(x)U_x(x)\frac{(x-x')}{(x-x')^2+\varepsilon^2}\eta_{x'x'}(x')\Big]. \tag{32}$$

To cancel this, one finds rapidly that one may have to add to (31) no less than five terms

$$\int_{-\infty}^{+\infty} \mathrm{d}x \Big[A\,\eta_x(x)^2 \eta(x) + B\,\eta_x(x) U_x(x) V_x(x) + C\,\eta(x) U_{xx}(x) V_x(x)$$
$$+D \int_{-\infty}^{+\infty} \eta_x(x) \frac{(x-x')}{(x-x')^2+\varepsilon^2} \eta_{x'}(x') \frac{(x'-x'')}{(x'-x'')^2+\varepsilon^2} \eta(x'') \mathrm{d}x' \mathrm{d}x'' + E \int_{-\infty}^{+\infty} \eta_x(x) \frac{(x-x')}{(x-x')^2+\varepsilon^2} \eta_{x'}(x') U_{x'}(x')^2 \mathrm{d}x' \Big]. \quad (33)$$

One obtains five equations in five unknowns, whose unique solution is

$$A = 0, \quad B = -1, \quad C = 0, \quad D = -\frac{g}{\pi^2}, \quad E = -\frac{1}{2\pi}.$$

Let us now come to the generalizations to higher derivatives of energy E_2 (16) and momentum P_2 (15).

The time derivative of E_2 involves seven terms at trilinear order:

$$V_{xx} U_x U_{xx}, \quad V_x U_{xx}^2, \quad \eta_x^2 U_{xx}, \quad \eta^2 U_{4x},$$
$$V_x(x) \eta(x) \frac{(x-x')}{(x-x')^2+\varepsilon^2} \eta_{x'x'x'}(x'),$$
$$V_x(x) \eta_x(x) \frac{(x-x')}{(x-x')^2+\varepsilon^2} \eta_{x'x'}(x'),$$
$$V_x(x) \eta_{xx}(x) \frac{(x-x')}{(x-x')^2+\varepsilon^2} \eta_{x'}(x'). \quad (34)$$

There are precisely seven terms whose time derivatives can cancel these:

$$A\,\eta_{xx} U_x^2 + B\,\eta U_{xx}^2 + C\,\eta_{xx} V_x^2 + D\,\eta V_{xx}^2 + E\,\eta(x)^2 \frac{(x-x')}{(x-x')^2+\varepsilon^2} \eta_{x'x'x'}(x')$$
$$+F\,\eta_x(x)^2 \frac{(x-x')}{(x-x')^2+\varepsilon^2} \eta_{x'}(x') + G V_x(x) U_{xx}(x) \frac{(x-x')}{(x-x')^2+\varepsilon^2} \eta_{x'}(x'), \quad (35)$$

and the cancellation occurs for the following values:

$$A = -\frac{1}{8}, \quad B = \frac{5}{8}, \quad C = -\frac{3}{8}, \quad D = \frac{1}{8}, \quad E = -\frac{g}{8\pi}, \quad F = -\frac{19g}{8\pi}, \quad G = \frac{3}{2\pi}.$$

The conservation of P_2 (15) at third order is relatively simple. One finds:

$$\partial_t(\lambda_x \eta_{xx}) = -\frac{3}{2} \eta_x V_{xx}^2 + \frac{3}{2} \eta_x U_{xx}^2$$

(up to total derivatives, of course) and we have found that this can be cancelled by adding just two trilinear terms:

$$\partial_t \Big[\lambda_x \eta_{xx} - \frac{3}{\pi} \eta_x(x) U_{xx}(x) \frac{(x-x')}{(x-x')^2+\varepsilon^2} \eta_{x'}(x') - \frac{3}{2} \eta_x^2 V_{xx} \Big]$$

vanishes at third order up to total derivatives.

In next order, for E_2 as well as for P_2 there are up to seven derivatives to be shared between η, V, U raised to various powers, not to mention one or two principal values, and it seems that some formal manipulation software would be needed to arrive at a reliable conclusion.

V. DISCUSSION

Craig and Worfolk[3], and Dyachenko, Lvov and Zakharov[4] had given rather convincing evidence that at fifth order in the surface displacement the Euler equations (1) are not integrable. However, a more recent publication[6] has revived the conjecture of integrability by studying the analytic structure, poles and cuts, of the complex velocity potential in the upper half-plane above the free surface, using a new non-canonical Hamiltonian structure of the Euler equations[7]. In this publication, new conserved quantities are found. How they could relate to those presented here goes beyond the scope of this paper. An argument for or against integrability could come out of a fifth-order calculation of our six new quantities which could show whether they can or cannot be made time-independent.

An analogy can be made with the relativistic φ^4 theory in 1+1 dimensions

$$\varphi_{tt} - \varphi_{xx} + \varphi - \frac{1}{6}\varphi^3 = 0 \tag{36}$$

as a low-amplitude, not integrable[8], approximation of the integrable sine-Gordon equation

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0. \tag{37}$$

In both systems the total momentum is conserved of course:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} \varphi_x \varphi_t \,\mathrm{d}x = 0,$$

and when trying a generalization with more derivatives as in our equation (15), one finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} \Big[\varphi_{xx}\varphi_{xt} + \frac{1}{8}\big(\varphi^3\varphi_{xt} - \varphi_t^3\varphi_x - \varphi_x^3\varphi_t\big)\Big]\mathrm{d}x$$

vanishes in this quartic approximation in φ, but not beyond, where the higher-order terms of the expansion of the sine are necessary to achieve conservation. These approximate conservation laws could explain the long persistence of coherent structures in deep-water waves in our case and why approximate breathers, weak solutions of (36), can be relevant in condensed matter physics[9].

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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