A hierarchy of modal logics with relative accessibility relations

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Abstract

In this paper we introduce and investigate various classes of multimodal logics based on frames with relative accessibility relations. We discuss their applicability to representation and analysis of incomplete information. We provide axiom systems for these logics and we prove their completeness.

1 Introduction

The original motivation for introducing and investigating modal logics with relative accessibility relations comes from the theory of information systems. However, as it is shown in this paper, a number of standard multimodal logics can be uniformly presented and investigated within the general framework of relative relations as well. Information systems are the collections of information items that have the form of descriptions of some objects in terms of their properties. More formally, by an information system we mean a structure $S = (OB, AT, \{VAL_a : a \in AT\})$ such that OB is a nonempty set

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of objects, AT is a finite nonempty set of attributes, each VAL_a is a nonempty set of values of attribute a. Each attribute is a function $a : OB \to \mathcal{P}(VAL_a)$ that assigns subsets of values to objects. Any set $a(x)$ can be viewed as a set of properties of an object x.
For example, if the attribute a is "colour" and $a(x) = \{green\}$, then x possesses the property of "being green" ; if a is "age" and x is 25 years old, then $a(x) = \{25\}$ and this means that x possesses the property of "being 25 years old" ; if a is "languages spoken" and if a person x speaks, say, Polish (Pl), German (D) and French (F), then $a(x) = \{Pl, D, F\}$ and x possesses the properties of "speaking Polish", "speaking German" and "speaking French". In this setting any set $a(x)$ is referred to as the set of a-properties of object x and its complement $VAL_a \setminus a(x)$ is the set of negative a-properties of x. Consider, for example, the file given below :

$$\begin{array}{cc} & colour \\ \begin{array}{c} o_1 \\ o_2 \\ o_3 \\ o_4 \\ o_5 \end{array} & \begin{pmatrix} green \\ green \\ blue \\ blue \\ red \end{pmatrix} \end{array}$$

In that file, we have $OB = \{o_1, o_2, o_3, o_4, o_5\}$ and $AT = \{colour\}$. Moreover, $VAL_{colour} = \{green, blue, red\}$. Suppose that we are interested in defining a set $X = \{o_2, o_3, o_4\}$ in terms of information provided in the file. The set might be identified with a concept, say, the concept "beautiful objects". We easily observe that the following statement is not true of the given objects : $o \in X$ iff o is either green or blue. The reason being that the information that is available in the file does not enable us to discern between o_1 and o_2. We observe that the attribute "colour" induces a relation in the set of objects (referred to as an indiscernibility relation) that reflects their indistinguishability in terms of colour :

- $(o, o') \in ind(colour)$ iff $colour(o) = colour(o')$.

This suggests that, given an information system $S = (OB, AT, \{VAL_a : a \in AT\})$, the properties of objects available in S induce relationships among the objects. Typically, these relationships have the form of binary relations. They are referred to as information relations. There are two major groups of information relations : the relations that reflect various forms of indistinguishability of objects in terms of their properties and the relations that express distinguishability of the objects. In the following we present few examples of information relations. Let A be a subset of the set AT of attributes. The most familiar family of information relations that can be derived from S is the family of indiscernibility relations :

- strong indiscernibility : $(o, o') \in ind(A)$ iff $a(o) = a(o')$ for all $a \in A$.

These relations are strong indiscernibility relations, they hold between two objects whenever the objects are "the same" with respect to their a-properties for all $a \in A$. Clearly, it is also reasonable to consider weak indiscernibility, that is indistinguishability of objects with respect to some, not necessarily all the properties :

- weak indiscernibility : $(o, o') \in wind(A)$ iff $a(o) = a(o')$ for some $a \in A$.

Another family of useful information relations is the family of similarity relations :

- strong (weak) similarity : $(x, y) \in sim(A)$ $(wsim(A))$ iff $a(x) \cap a(y) \neq \emptyset$ for all (some) $a \in A$.

Indiscernibility and similarity relations exhibit indistinguishability of objects. We might also be interested in deriving information about distinguishability. The following are examples of the families of information relations that reflect differences between the objects :

- strong (weak) diversity : $(x, y) \in div(A)$ $(wdiv(A))$ iff $a(x) \neq a(y)$ for all (some) $a \in A$,
- strong (weak) orthogonality : $(x, y) \in ort(A)$ $(wort(A))$ iff $a(x) \cap a(y) = \emptyset$ for all (some) $a \in A$,
- strong (weak) complementarity : $(x, y) \in com(A)$ $(wcom(A))$ iff $a(y)$ is the complement of $a(x)$ with respect to the set VAL_a, for all (some) $a \in A$.

An information system constitutes an explicit information available in an application domain, while information relations are an implicit information. These relations enable us to identify some aspects of incompleteness of explicit information.
Relational systems consisting of a family of relations on a set are

referred to as frames. By a frame derived from an information system $S = (OB, AT, \{VAL_a : a \in AT\})$ we mean a relational system $K_{S,R} = (OB, \{R(A) : A \subseteq AT\})$, where $\{R(A) : A \subseteq AT\}$ is any of the families of information relations. Observe that relations in these frames depend on the subsets of the set AT. In a general setting these subsets play the role of parameters which provide a means for representing an intensional part of information included in an information system. From a technical point of view, we deal with families of relations indexed with subsets of a set.

Frames with relative accessibility relations have been suggested in Orłowska [20] in the context of a rough set analysis of data, and they were investigated, among others, in Konikowska [15] [16] and Balbiani [2] [3]. Often we are also interested in studying relationships between information relations that belong to different families. Hence, it is also natural to consider frames with families of relations of different types. A great variety of such frames is studied in the literature, see for example Demri [5] [6], Demri and Orłowska [7], Orłowska [18] [19] [21] [23] and Vakarelov [25] [26] [27].

The motivation for using modal logics for reasoning about information relations comes from the methods of data analysis in information systems. In these methods, the modal-like operators are used in the languages for representation of incomplete information. Let $S = (OB, AT, \{VAL_a : a \in AT\})$ be an information system. Given a strong indiscernibility relation $ind(A)$, for A being a subset of AT, and a subset X of OB, the lower A-approximation of X and the upper A-approximation of X are defined as follows :

- $L(A)X = \{x \in OB$: for all $y \in OB$, if $(x, y) \in ind(A)$ then $y \in X\}$.
- $U(A)X = \{x \in OB$: there is $y \in OB$ such that $(x, y) \in ind(A)$ and $y \in X\}$.

The following hierarchy of definability of sets is obtained in a natural way in terms of the approximations. A subset X of OB is said to be :

- A-definable if $L(A)X = X$ (or equivalently $U(A)X = X$).
- Roughly A-definable if $L(A)X \neq \emptyset$ and $U(A)X \neq OB$.
- Internally A-indefinable if $L(A)X = \emptyset$.
- Externally A-indefinable if $U(A)X = OB$.

- Totally A-indefinable if internally A-indefinable and externally A-indefinable.

A-definability of a set X means that X is the union of some of the equivalence classes of $ind(A)$. In our example $L(colour)X = \{o_3, o_4\}$ and $U(colour)X = \{o_1, o_2, o_3, o_4\}$ and we conclude that X is not definable in terms of colour. Clearly, X cannot be covered with the equivalence classes $\{o_1, o_2\}$, $\{o_3, o_4\}$, $\{o_5\}$ of $ind(colour)$. From the perspective of concept analysis any subset of objects in an information system might be identified with an extension of a concept and any subset of attributes with an intension of a concept.
Following the rough set semantics of vague concepts developed in Orłowska [17] and Read [24], we define the sets of A-positive, A-borderline and A-negative instances of a set X of objects as follows :

- $POS(A)X = L(A)X$.
- $BOR(A)X = U(A)X \setminus L(A)X$.
- $NEG(A)X = OB \setminus U(A)X$.

Elements of $POS(A)X$ definitely, relative to properties corresponding to A, belong to X. Elements of $NEG(A)X$ definitely, up to these properties, do not belong to X. $BOR(A)X$ is a doubtful region, its elements possibly belong to X, but we cannot decide it for certain considering only properties corresponding to A. In other words, as far as indiscernibility $ind(A)$ is concerned, nothing can be said about membership to X of elements from $BOR(A)X$.
The above analysis suggests that it might be useful to define the operators analogous to the lower and upper approximation also with the other information relations. These operators enable us to disclose an interaction between the information relations and subsets of objects. Let $S = (OB, AT, \{VAL_a : a \in AT\})$ be an information system and let $\{R(A) : A \in AT\}$ be a family of information relations derived from S. There are two major groups of operators :

- $[R(A)]X = \{x \in OB :$ for all $y \in OB$, if $(x, y) \in R(A)$ then $y \in X\}$.
- $\langle R(A)\rangle X = \{x \in OB :$ there is $y \in OB$ such that $(x, y) \in R(A)$ and $y \in X\}$.

Clearly, from the logic point of view they are necessity and possibility operators, respectively, and information relations play the role of the accessibility relations that determine these modal operators. Hence, modal logics appear to be a natural formal tool for the analysis of data. However, to represent adequately all the ingredients of information provided in an information system, we need to make the accessibility relations relative. Since the information relations derived from information systems always provide a twofold information, namely, the information which objects are related and the information with respect to which attributes these objects are related, in an abstract setting of modal logics we need the relations that are relative to subsets of a set. In this context, several modal logics have been introduced. Their linguistic basis is the propositional calculus enriched, for every expression Γ in some language of parameters, with the modality $[\Gamma]$. The language of parameters differs from one logic to another.

In the context of the modal logics for information systems introduced by Orłowska [20], one has to consider parameter expressions Γ defined by the atomic parameters and the operation $\cap$. These expressions enable us to represent strong information relations. In the relational semantics of these logics, the accessibility relation R is parametrized in such a way that for every parameter Γ, Δ, $R(\Gamma \cap \Delta) = R(\Gamma) \cap R(\Delta)$.

In the context of DAL, the modal logic for data analysis introduced by Fariñas del Cerro and Orłowska [9], the parameter expressions Γ are built up from atomic expressions with the operations $\cap$ and $\cup^\star$ corresponding to the interpretation of the compound accessibility relations defined with intersection and transitive closure of union. In the relational semantics of these logics, the accessibility relation R is parametrized in such a way that for every parameter Γ, Δ, $R(\Gamma \cap \Delta) = R(\Gamma) \cap R(\Delta)$ and $R(\Gamma \cup^\star \Delta) = R(\Gamma) \cup^\star R(\Delta)$.

In the context of BML, the Boolean modal logic introduced by Gargov, Passy and Tinchev [11] [12], the parameter expressions Γ are built up from atomic expressions with the operations $\cap$, $\cup$ and $\neg$ corresponding to the interpretation of the compound accessibility relations defined with intersection, union and complement. In the relational semantics of these logics, the accessibility relation R is parametrized in such a way that for every parameter Γ, Δ, $R(\Gamma \cap \Delta) = R(\Gamma) \cap R(\Delta)$, $R(\Gamma \cup \Delta) = R(\Gamma) \cup R(\Delta)$ and $R(\neg\Gamma) = \overline{R(\Gamma)}$.

Several modal logics for information systems have been considered, depending on the special properties of the relative accessibility relation. If for every parameter Γ, $R(\Gamma)$ is a relation of equivalence then the accessibility relation is a relation of strong indiscernibility. Such frames

$(W, \{R(\Gamma)\}_\Gamma)$ correspond to the frames of indiscernibility associated to attribute systems. If for every Γ, $R(\Gamma)$ is reflexive and symmetric then the accessibility relation is a relation of strong similarity and the frames $(W, \{R(\Gamma)\}_\Gamma)$ are the frames of similarity associated to attribute systems.
In the section 2 of this paper, we consider several properties of the relative accessibility relations $R(\Gamma)$ where the parameters are defined by the operation $\cap$ in such a way that, in the relational semantics of these logics, $R(\Gamma \cap \Delta) = R(\Gamma) \cap R(\Delta)$ and we prove the axiomatizability of the formulas valid in the corresponding class of frames. In the section 3, we consider an extended language where the parameters are defined by the operations $\cap$ and $\cup$ in such a way that, in the relational semantics of these logics, $R(\Gamma \cap \Delta) = R(\Gamma) \cap R(\Delta)$ and $R(\Gamma \cup \Delta) = R(\Gamma) \cup R(\Delta)$. Therefore, this extended language is a sublanguage of the Boolean modal logic [11] [12], only the parameters of the form $\neg\Gamma$ are missing. We examine the axiomatizability of the set of formulas valid in several classes of frames and we prove the completeness of these axiomatizations. Our proof is based on the techniques of the copying introduced by Vakarelov [26].

2 Modal logics $L2$

In this section we present a class $L2$ of relative modal logics where the parameters are defined by the operation $\cap$ in such a way that, in the relational semantics of these logics, $R(\Gamma \cap \Delta) = R(\Gamma) \cap R(\Delta)$.

2.1 Language

The linguistic basis of any modal logic from class $L2$ is the language of the classical propositional calculus enriched with modal operators. Each modal operator is denoted by an expression that represents a set of parameters. Let $APAR$ be a nonempty set of *atomic parameters*. The set $CPAR$ of the *complex parameters* is defined by induction in the following way :

- Every atomic parameter is a complex parameter,
- For every $\Gamma, \Delta \in CPAR$, $\Gamma \cap \Delta \in CPAR$.

Let *set* be the mapping of $CPAR$ into 2_f^{APAR} — the set of the finite subsets of $APAR$ — defined by induction in the following way :

- For every atomic parameter α, $set(\alpha) = \{\alpha\}$,
- For every $\Gamma, \Delta \in CPAR$, $set(\Gamma \cap \Delta) = set(\Gamma) \cup set(\Delta)$.

In what follows, $X \subseteq_f Y$ means that X is a finite subset of Y. Observe that :

Theorem 1 *For every $P \subseteq_f APAR$, there is $\Gamma \in CPAR$ such that $set(\Gamma) = P$.*

Let $\sqsubseteq$ be the binary relation on $CPAR$ defined in the following way :

- For every $\Gamma, \Delta \in CPAR$, $\Gamma \sqsubseteq \Delta$ iff $set(\Gamma) \subseteq set(\Delta)$.

If the expressions Γ and Δ are considered as Boolean formulas of the classical propositional calculus, then one can easily prove that :

- $\Gamma \sqsubseteq \Delta$ iff $\Delta \to \Gamma$ is classically valid.

Let $AFOR$ be a nonempty set of *atomic formulas*. The set $CFOR$ of the *complex formulas* is defined by induction in the following way :

- Every atomic formula is a complex formula,
- For every $A \in CFOR$, $\neg A \in CFOR$,
- For every $A, B \in CFOR$, $A \wedge B \in CFOR$,
- For every $\Gamma \in CPAR$ and for every $A \in CFOR$, $[\Gamma]A \in CFOR$.

For every $\Gamma \in CPAR$ and for every $A \in CFOR$, let $\langle\Gamma\rangle A = \neg[\Gamma]\neg A$.

2.2 Semantical study

A *frame for $L2$* is a relational structure of the form $\mathcal{F} = (PAR, OB, R)$ where :

- PAR is a nonempty set of *parameters*,
- OB is a nonempty set of *objects*,
- R is a mapping of 2_f^{PAR} — the set of the finite subsets of PAR — into the set of the binary relations on OB such that, for every $P, Q \subseteq_f PAR$, $R(P \cup Q) \subseteq R(P) \cap R(Q)$.

Throughout section 2, by "frame" we always mean a frame for $L2$. $\mathcal{F}$ is *standard* when, for every $P, Q \subseteq_f PAR$, $R(P \cup Q) = R(P) \cap R(Q)$. Let $m, n, j, k \geq 0$. $\mathcal{F}$ is $^{mn}_{jk}$*-normal* when, for every $P \subseteq_f PAR$ and for every $x, y, z \in OB$, if $x\ R(P)^m\ y$ and $x\ R(P)^j\ z$ then there is $t \in OB$ such that $y\ R(P)^n\ t$ and $z\ R(P)^k\ t$ where :

- $R(P)^0 = Id_{OB}$,
- $R(P)^{s+1} = R(P)^s; R(P)$, where ; is the composition of relations.

For example :

- $\mathcal{F}$ is $^{01}_{00}$-normal when each $R(P)$ is reflexive.
- $\mathcal{F}$ is $^{11}_{00}$-normal when each $R(P)$ is symmetric.
- $\mathcal{F}$ is $^{01}_{20}$-normal when each $R(P)$ is transitive.

A mapping m of $APAR$ into 2_f^{PAR} and of $AFOR$ into 2^{OB} is called *assignment on* $\mathcal{F}$. The pair $\mathcal{M} = (\mathcal{F}, m)$ is called *model on* $\mathcal{F}$. $\models_{\mathcal{M}} A$ — the *truth in* $\mathcal{M}$ of a formula A — is defined in the following way :

- For every $A \in CFOR$, $\models_{\mathcal{M}} A$ iff $\tilde{m}(A) = OB$.

where $\tilde{m}$ is the mapping of $CPAR$ into 2_f^{PAR} and of $CFOR$ into 2^{OB} defined by induction in the following way :

- For every atomic parameter α, $\tilde{m}(\alpha) = m(\alpha)$,
- For every $\Gamma, \Delta \in CPAR$, $\tilde{m}(\Gamma \cap \Delta) = \tilde{m}(\Gamma) \cup \tilde{m}(\Delta)$,
- For every atomic formula A, $\tilde{m}(A) = m(A)$,
- For every $A \in CFOR$, $\tilde{m}(\neg A) = OB \setminus \tilde{m}(A)$,
- For every $A, B \in CFOR$, $\tilde{m}(A \wedge B) = \tilde{m}(A) \cap \tilde{m}(B)$,
- For every $\Gamma \in CPAR$ and for every $A \in CFOR$, $\tilde{m}([\Gamma]A) = \{x \in OB :$ for every $y \in OB$, if $x\ R(\tilde{m}(\Gamma))\ y$ then $y \in \tilde{m}(A)\}$.

Direct calculations would lead to the conclusion that :

Theorem 2 *Let $\mathcal{F} = (PAR, OB, R)$ be a frame. Let m be an assignment on $\mathcal{F}$. For every $\Gamma, \Delta \in CPAR$, $R(\tilde{m}(\Gamma \cap \Delta)) \subseteq R(\tilde{m}(\Gamma)) \cap R(\tilde{m}(\Delta))$. If $\mathcal{F}$ is standard then, for every $\Gamma, \Delta \in CPAR$, $R(\tilde{m}(\Gamma \cap \Delta)) = R(\tilde{m}(\Gamma)) \cap R(\tilde{m}(\Delta))$.*

Moreover :

Theorem 3 *Let $\mathcal{F} = (PAR, OB, R)$ be a frame. Let m be an assignment on $\mathcal{F}$. For every $\Gamma, \Delta \in CPAR$, if $set(\Gamma) \subseteq set(\Delta)$ then $\tilde{m}(\Gamma) \subseteq \tilde{m}(\Delta)$ and $R(\tilde{m}(\Delta)) \subseteq R(\tilde{m}(\Gamma))$.*

$\models_{\mathcal{F}} A$ — the *truth in* $\mathcal{F}$ of a formula A — is defined in the following way :

- For every $A \in CFOR$, $\models_{\mathcal{F}} A$ iff, for every model $\mathcal{M}$ on $\mathcal{F}$, $\models_{\mathcal{M}} A$.

Let Ω be a nonempty set of frames. $\models_{\Omega} A$ — the *validity in* Ω of a formula A — is defined in the following way :

- For every $A \in CFOR$, $\models_{\Omega} A$ iff, for every $\mathcal{F} \in \Omega$, $\models_{\mathcal{F}} A$.

Let

- K_{L2} be the set of all frames,
- K_{L2}^S be the set of all standard frames,
- $K_{L2}(m,n,j,k)$ be the set of all ${}^{mn}_{jk}$-normal frames,
- $K_{L2}^S(m,n,j,k)$ be the set of all standard ${}^{mn}_{jk}$-normal frames,
- $S5_{L2}$ be the set of all frames with equivalence relations (each $R(P)$ is an equivalence relation),
- $S5_{L2}^S$ be the set of all standard frames with equivalence relations.

2.3 Axiomatic presentation

Let Ω be a class of frames. By the logic of Ω we mean the set of formulas of $CFOR$ that are valid in Ω. For the sake of brevity we denote this logic by Ω as well. Together with the classical tautologies, all the instances of the following schemata are axioms of K_{L2} :

- $[\Gamma](A \to B) \to ([\Gamma]A \to [\Gamma]B)$, for every $\Gamma \in CPAR$,
- $[\Gamma]A \to [\Delta]A$, for every $\Gamma, \Delta \in CPAR$ such that $\Gamma \sqsubseteq \Delta$.

Together with the modus ponens, all the instances of the following schema are rules of K_{L2} :

- From A infer $[\Gamma]A$, for every $\Gamma \in CPAR$.

In a standard way we define the notions of proof and derivability in the logic Ω.

2.4 Completeness

It is easy to see that the following soundness theorem holds for logic K_{L2} :

Theorem 4 *For every $A \in CFOR$, if $\vdash_{K_{L2}} A$ then $\models_{K_{L2}} A$.*

Let OB be the set of the maximal consistent sets of formulas. Let R be the mapping of 2_f^{APAR} into the set of the binary relations on OB defined in the following way :

- For every $P \subseteq_f APAR$ and for every $x, y \in OB$, $x\ R(P)\ y$ iff, for every $\Gamma \in CPAR$ and for every $A \in CFOR$, if $set(\Gamma) \subseteq P$ and $[\Gamma]A \in x$ then $A \in y$.

Let it be proved that the structure of the form $\mathcal{F} = (APAR, OB, R)$ is a frame :

$\mathcal{F}$ **is a frame** Let $P, Q \subseteq_f APAR$, let $x, y \in OB$ be such that $x\ R(P \cup Q)\ y$. Therefore, for every $\Gamma \in CPAR$ and for every $A \in CFOR$, if $set(\Gamma) \subseteq P \cup Q$ and $[\Gamma]A \in x$ then $A \in y$. Since $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$, then, for every $\Gamma \in CPAR$ and for every $A \in CFOR$, if $set(\Gamma) \subseteq P$ and $[\Gamma]A \in x$ then $set(\Gamma) \subseteq P \cup Q$ and $A \in y$ and if $set(\Gamma) \subseteq Q$ and $[\Gamma]A \in x$ then, similarly, $set(\Gamma) \subseteq P \cup Q$ and $A \in y$. Consequently, $x\ R(P)\ y$ and $x\ R(Q)\ y$. Therefore, $R(P \cup Q) \subseteq R(P) \cap R(Q)$.

Let m be an assignment on $\mathcal{F}$ defined in the following way :

- For every atomic parameter α, $m(\alpha) = \{\alpha\}$,
- For every atomic formula A, $m(A) = \{x \in OB : A \in x\}$.

Let $\mathcal{M} = (\mathcal{F}, m)$. The proof is done by induction on Γ that, for every $\Gamma \in CPAR$, $\tilde{m}(\Gamma) = set(\Gamma)$:

Basis For every atomic parameter α, $\tilde{m}(\alpha) = m(\alpha) = \{\alpha\} = set(\alpha)$.

Hypothesis There is $\Gamma, \Delta \in CPAR$ such that $\tilde{m}(\Gamma) = set(\Gamma)$ and $\tilde{m}(\Delta) = set(\Delta)$.

Step $\tilde{m}(\Gamma \cap \Delta) = \tilde{m}(\Gamma) \cup \tilde{m}(\Delta) =$, by the hypothesis, $set(\Gamma) \cup set(\Delta) = set(\Gamma \cap \Delta)$.

The proof is done by induction on A that, for every $A \in CFOR$, $\tilde{m}(A) = \{x \in OB : A \in x\}$.

Basis For every atomic formula A, $\tilde{m}(A) = m(A) = \{x \in OB : A \in x\}$.

Hypothesis There is $A \in CFOR$ such that $\tilde{m}(A) = \{x \in OB : A \in x\}$.

Step For every $\Gamma \in CPAR$, let $x \in OB$ be such that $[\Gamma]A \in x$. Consequently, for every $y \in OB$, if $x\ R(\tilde{m}(\Gamma))\ y$ then $A \in y$

and, by the hypothesis, $y \in \widetilde{m}(A)$. Therefore, $x \in \widetilde{m}([\Gamma]A)$. For every $\Gamma \in CPAR$, let $x \in OB$ be such that $[\Gamma]A \notin x$. Let y be a maximal consistent set of formulas containing $\{\neg A\} \cup \{B \in CFOR : [\Gamma]B \in x\}$. Direct calculations would lead to the conclusion that $x\ R(\widetilde{m}(\Gamma))\ y$. Therefore, $x \notin \widetilde{m}([\Gamma]A)$.

Therefore :

Theorem 5 *For every $A \in CFOR$, if $\models_{K_{L2}} A$ then $\vdash_{K_{L2}} A$.*

In order to obtain the completeness result for K_{L2} with respect to its standard frames, we apply the method of copying originated in Vakarelov [26] in the context of the modal logics for knowledge representation systems.

2.5 Copying

Let $\mathcal{F} = (PAR, OB, R)$ and $\mathcal{F}' = (PAR, OB', R')$ be frames. Let I be a set of mappings of OB into OB'. I is a *copying of $\mathcal{F}$ into $\mathcal{F}'$* whenever the following conditions are satisfied :

- For every $x' \in OB'$, there is $f \in I$ and there is $x \in OB$ such that $f(x) = x'$,
- For every $f, g \in I$ and for every $x, y \in OB$, if $f(x) = g(y)$ then $x = y$,
- For every $P \subseteq_f PAR$, for every $f \in I$ and for every $x, y \in OB$, $x\ R(P)\ y$ iff there is $g \in I$ such that $f(x)\ R'(P)\ g(y)$.

It is easy to show that :

Theorem 6 *Let $\mathcal{F} = (PAR, OB, R)$ and $\mathcal{F}' = (PAR, OB', R')$ be frames. Let I be a copying of $\mathcal{F}$ into $\mathcal{F}'$. Let m be an assignment on $\mathcal{F}$. Let m' be the assignment on $\mathcal{F}'$ defined in the following way :*

- *For every atomic parameter α, $m'(\alpha) = m(\alpha)$,*
- *For every atomic formula A, $m'(A) = \{f(x) : f \in I$ and $x \in m(A)\}$.*

Then for every $\alpha \in CPAR$, $m'(\alpha) = m(\alpha)$ and for every $A \in CFOR$, $\widetilde{m'}(A) = \{f(x) : f \in I$ and $x \in \widetilde{m}(A)\}$.

2.6 Standard completeness

Let $\mathcal{F} = (PAR, OB, R)$ be a frame. For every $P \subseteq_f PAR$, let $\sigma(P)$ be the mapping of $OB \times OB$ into 2^{OB} defined in the following way : for every $x, y \in OB$, $\sigma(P)(x,y) = \emptyset$ if $x\ R(P)\ y$, otherwise $\sigma(P)(x,y) = OB$. Observe that the set $\mathcal{B} = 2^{OB}$ can be treated as a Boolean ring where :

- $0_{\mathcal{B}} = \emptyset$,
- $1_{\mathcal{B}} = OB$,
- $A +_{\mathcal{B}} B = (A \setminus B) \cup (B \setminus A)$, consequently : $A +_{\mathcal{B}} A = \emptyset$,
- $A \times_{\mathcal{B}} B = A \cap B$.

Therefore, for every $A, B \in 2^{OB}$, there exists exactly one $X \in 2^{OB}$ such that $A +_{\mathcal{B}} X = B$, namely : $X = (A \setminus B) \cup (B \setminus A)$. Let I be the set of the mappings of $2_f^{PAR} \times PAR$ into 2^{OB}. Let $OB' = OB \times I$. For every $P \subseteq_f PAR$, let $R'(P)$ be the binary relation on OB' defined in the following way :

- For every $f, g \in I$ and for every $x, y \in OB$, $(x, f)\ R'(P)\ (y, g)$ iff :
  - For every $O \subseteq_f PAR$ and for every $\alpha \in PAR$, if $\alpha \in O$ and $\alpha \in P$ then $g(O, \alpha) = f(O, \alpha)$ and
  - For every $O \subseteq_f PAR$, $\Sigma_{\alpha \in O} f(O, \alpha) + g(O, \alpha) = \sigma(O)(x, y)$.

Lemma 1 *Let $P \subseteq_f PAR$, let $f \in I$ and let $x, y \in OB$ be such that there is $g \in I$ such that $(x, f)\ R'(P)\ (y, g)$. Then $x\ R(P)\ y$.*

Proof : Let $P \subseteq_f PAR$, let $f \in I$ and let $x, y \in OB$ be such that there is $g \in I$ such that $(x, f)\ R'(P)\ (y, g)$. Consequently, for every $O \subseteq_f PAR$ and for every $\alpha \in PAR$, if $\alpha \in O$ and $\alpha \in P$ then $g(O, \alpha) = f(O, \alpha)$ and, for every $O \subseteq_f PAR$, $\Sigma_{\alpha \in O} f(O, \alpha) + g(O, \alpha) = \sigma(O)(x, y)$. Therefore, for every $\alpha \in PAR$, if $\alpha \in P$ then $g(P, \alpha) = f(P, \alpha)$. Consequently, $\sigma(P)(x, y) = \emptyset$. Therefore, $x\ R(P)\ y$.
$\dashv$

Lemma 2 *Let $P \subseteq_f PAR$, let $f \in I$ and let $x, y \in OB$ be such that $x\ R(P)\ y$. Then there is $g \in I$ such that $(x, f)\ R'(P)\ (y, g)$.*

Proof : Let $P \subseteq_f PAR$, let $f \in I$ and let $x, y \in OB$ be such that $x \; R(P) \; y$. Let Φ be a mapping of 2_f^{PAR} into PAR such that, for every $O \subseteq_f PAR$, if $O \not\subseteq P$ then $\Phi(O) \in O \setminus P$. We have to find a mapping $g \in I$ such that $(x, f) \; R'(P) \; (y, g)$. Let g be the mapping of $2_f^{PAR} \times PAR$ into 2^{OB} such that for every $O \subseteq_f PAR$ and for every $\alpha \in PAR$, the following conditions are satisfied :

- If $\alpha \in O$ and $\alpha \in P$ then $g(O, \alpha) = f(O, \alpha)$,
- If $\alpha \in O$ and $\alpha \notin P$ then either $\alpha = \Phi(O)$ in which case $g(O, \alpha) = \Sigma_{\alpha \in O \setminus P} f(O, \alpha) + \sigma(O)(x, y)$ or $\alpha \neq \Phi(O)$ in which case $g(O, \alpha) = \emptyset$,
- If $\alpha \notin O$ then $g(O, \alpha) = \emptyset$.

It is easy to verify that $(x, f) \; R'(P) \; (y, g)$.
$\dashv$
Observe that, for every $f \in I$, f can be identified with the mapping of OB into OB' defined by : for every $x \in OB$, $f(x) = (x, f)$.

Lemma 3 *The structure $\mathcal{F}' = (PAR, OB', R')$ is a standard frame and I — considered as a set of mappings of OB into OB' — is a copying of $\mathcal{F}$ into $\mathcal{F}'$.*

Proof : The proof is done that $\mathcal{F}'$ is a standard frame and I is a copying of $\mathcal{F}$ into $\mathcal{F}'$.

$\mathcal{F}'$ is a standard frame

- Let $P, Q \subseteq_f PAR$, let $f, g \in I$ and let $x, y \in OB$ be such that $(x, f) \; R'(P \cup Q) \; (y, g)$. Consequently, for every $O \subseteq_f PAR$ and for every $\alpha \in PAR$, if $\alpha \in O$ and $\alpha \in P \cup Q$ then $g(O, \alpha) = f(O, \alpha)$ and, for every $O \subseteq_f PAR$, $\Sigma_{\alpha \in O} f(O, \alpha) + g(O, \alpha) = \sigma(O)(x, y)$. Since $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$, then, for every $O \subseteq_f PAR$ and for every $\alpha \in PAR$, if $\alpha \in O$ and $\alpha \in P$ then $\alpha \in P \cup Q$ and $g(O, \alpha) = f(O, \alpha)$ and if $\alpha \in O$ and $\alpha \in Q$ then, similarly, $\alpha \in P \cup Q$ and $g(O, \alpha) = f(O, \alpha)$. Therefore, $(x, f) \; R'(P) \; (y, g)$ and $(x, f) \; R'(Q) \; (y, g)$. Consequently, $R'(P \cup Q) \subseteq R'(P) \cup R'(Q)$.
- Let $P, Q \subseteq_f PAR$, let $f, g \in I$ and let $x, y \in OB$ be such that $(x, f) \; R'(P) \; (y, g)$ and $(x, f) \; R'(Q) \; (y, g)$. Therefore, for every $O \subseteq_f PAR$ and for every $\alpha \in PAR$, if $\alpha \in O$ and $\alpha \in P$ then $g(O, \alpha) = f(O, \alpha)$ and if $\alpha \in O$ and $\alpha \in Q$ then $g(O, \alpha) = f(O, \alpha)$ and, for every $O \subseteq_f PAR$,

$\Sigma_{\alpha\in O} f(O,\alpha) + g(O,\alpha) = \sigma(O)(x,y)$. Consequently, for every $O \subseteq_f PAR$ and for every $\alpha \in PAR$, if $\alpha \in O$ and $\alpha \in P \cup Q$ then $\alpha \in P$ and $g(O,\alpha) = f(O,\alpha)$ or $\alpha \in Q$ and $g(O,\alpha) = f(O,\alpha)$. Therefore, $(x,f)\ R'(P \cup Q)\ (y,g)$. Consequently, $R'(P \cup Q) = R'(P) \cup R'(Q)$.

I is a copying of $\mathcal{F}$ into $\mathcal{F}'$ This is a direct consequence of the lemmas 1 and 2.

$\dashv$

Consequently, it follows that the completeness theorem with respect to the class of standard frames holds :

Theorem 7 *For every $A \in CFOR$, if $\models_{K^S_{L2}} A$ then $\vdash_{K_{L2S}} A$.*

2.7 Extensions

This subsection presents two extensions — $S5_{L2}$ and $K_{L2}(m,n,j,k)$ — of $L2$.

2.7.1 $S5_{L2}$

Together with the axioms and the rules of K_{L2}, all the instances of the following schemata are axioms of $S5_{L2}$:

- $[\Gamma]A \rightarrow A$, for every $\Gamma \in CPAR$,
- $\langle\Gamma\rangle[\Gamma]A \rightarrow [\Gamma]A$, for every $\Gamma \in CPAR$.

It is easy to see the soundness of logic $S5_{L2}$:

Theorem 8 *For every $A \in CFOR$, if $\vdash_{S5_{L2}} A$ then $\models_{S5_{L2}} A$.*

Moreover, the structure of the form $\mathcal{F} = (APAR, OB, R)$ defined in the subsection 2.4 is a frame with equivalence relations. Therefore we have :

Theorem 9 *For every $A \in CFOR$, if $\models_{S5_{L2}} A$ then $\vdash_{S5_{L2}} A$.*

Let $\mathcal{F} = (PAR, OB, R)$ be a frame with equivalence relations. For every $P \subseteq_f PAR$, let $\sigma(P)$ be the mapping of $OB \times OB$ into 2^{OB} defined in the following way : for every $x, y \in OB$, $\sigma(P)(x,y) = R(P)(x) + R(P)(y)$. Let I be the set of the mappings of $2_f^{PAR} \times PAR$ into 2^{OB}. It can easily be proved that the argument of the subsection 2.6 applies :

Lemma 4 *The structure of the form* $\mathcal{F}' = (PAR, OB', R')$ *defined in the subsection 2.6 is a standard frame with equivalence relations and* I *is a copying of* $\mathcal{F}$ *into* $\mathcal{F}'$*.*

Consequently :

Theorem 10 *For every* $A \in CFOR$*, if* $\models_{S5^S_{L2}} A$ *then* $\vdash_{S5_{L2}} A$*.*

2.7.2 $K_{L2}(m, n, j, k)$

Let $m, n, j, k \geq 0$. Together with the axioms and the rules of K_{L2}, all the instances of the following schema are axioms of $K_{L2}(m, n, j, k)$:

- $\langle\Gamma\rangle^m[\Gamma]^n A \rightarrow [\Gamma]^j\langle\Gamma\rangle^k A$, for every $\Gamma \in CPAR$.

Theorem 11 *For every* $A \in CFOR$*, if* $\vdash_{K_{L2}(m,n,j,k)} A$ *then* $\models_{K_{L2}(m,n,j,k)} A$*.*

The structure of the form $\mathcal{F} = (APAR, OB, R)$ defined in the subsection 2.4 is an ${}^{mn}_{jk}$-normal frame. Therefore we have the following completeness theorem for logics $K_{L2}(m, n, j, k)$:

Theorem 12 *For every* $A \in CFOR$*, if* $\models_{K_{L2}(m,n,j,k)} A$ *then* $\vdash_{K_{L2}(m,n,j,k)} A$*.*

In what follows we present examples of logics of the form $K_{L2}(m, n, j, k)$ for which the completeness theorem with respect to the standard frames holds. Let $\mathcal{F} = (PAR, OB, R)$ be an ${}^{mn}_{jk}$-normal frame. For every $P \subseteq_f PAR$, let $\sigma(P)$ be the mapping of $OB \times OB$ into 2^{OB} defined in the following way : for every $x, y \in OB$, $\sigma(P)(x, y) = \emptyset$ if x $R(P)$ y, otherwise $\sigma(P)(x, y) = OB$. Let I be the set of the mappings of $2_f^{PAR} \times PAR$ into 2^{OB}. If $m = 0$, $n = 1$, $j = 0$ and $k = 0$, then the argument of the subsection 2.6 applies :

Lemma 5 *If* $m = 0$*,* $n = 1$*,* $j = 0$ *and* $k = 0$ *then the structure of the form* $\mathcal{F}' = (PAR, OB', R')$ *defined in the subsection 2.6 is a standard reflexive frame and* I *is a copying of* $\mathcal{F}$ *into* $\mathcal{F}'$*.*

Consequently :

Theorem 13 *For every* $A \in CFOR$*, if* $\models_{K^S_{L2}(0,1,0,0)} A$ *then* $\vdash_{K_{L2}(0,1,0,0)} A$*.*

If $m = 1$, $n = 1$, $j = 0$ and $k = 0$ then the argument of the subsection 2.6 applies as well :

Lemma 6 *If $m = 1$, $n = 1$, $j = 0$ and $k = 0$ then the structure of the form $\mathcal{F}' = (PAR, OB', R')$ defined in the subsection 2.6 is a standard symmetric frame and I is a copying of $\mathcal{F}$ into $\mathcal{F}'$.*

Consequently, we have the completeness theorem for logic $K_{L2}(1, 1, 0, 0)$ with respect to its standard frames :

Theorem 14 *For every $A \in CFOR$, if $\models_{K^S_{L2}(1,1,0,0)} A$ then $\vdash_{K_{L2}(1,1,0,0)} A$.*

If $m+j \geq 2$ then the proof of the standard completeness of $K_{L2}(m, 1, j, 0)$ and the proof of the standard completeness of $K_{L2}(m, 0, j, 0)$ are open. If $n + k \geq 2$ then let $OB' = OB \times (I^{OB \times I} \times I)$. For every $P \subseteq_f PAR$, let $R'(P)$ be the binary relation on OB' defined in the following way :

- For every $f_1, g_1 \in I^{OB \times I}$, for every $f_2, g_2 \in I$ and for every $x, y \in OB$, $(x, (f_1, f_2))\ R'(P)\ (y, (g_1, g_2))$ iff :
  - For every $O \subseteq_f PAR$ and for every $\alpha \in PAR$, if $\alpha \in O$ and $\alpha \in P$ then $g_1(x, f_2)(O, \alpha) = f_2(O, \alpha)$,
  - For every $O \subseteq_f PAR$, $\Sigma_{\alpha \in O} f_2(O, \alpha) + g_1(x, f_2)(O, \alpha) = \sigma(O)(x, y)$.

The proofs of the three following lemmas are similar to the proofs of the lemmas 1, 2 and 3 :

Lemma 7 *Let $P \subseteq_f PAR$, let $f_2 \in I$ and let $x, y \in OB$ be such that there is $G \in I$ such that for every $f_1, g_1 \in I^{OB \times I}$ and for every $g_2 \in I$, if $g_1(x, f_2) = G$ then $(x, (f_1, f_2))\ R'(P)\ (y, (g_1, g_2))$. Then $x\ R(P)\ y$.*

Lemma 8 *Let $P \subseteq_f PAR$, let $f_2 \in I$ and let $x, y \in OB$ be such that $x\ R(P)\ y$. There is $G \in I$ such that for every $f_1, g_1 \in I^{OB \times I}$ and for every $g_2 \in I$, if $g_1(x, f_2) = G$ then $(x, (f_1, f_2))\ R'(P)\ (y, (g_1, g_2))$.*

Lemma 9 *The structure of the form $\mathcal{F}' = (PAR, OB', R')$ is a standard ${}^{mn}_{jk}$-normal frame and $I^{OB \times I} \times I$ — considered as a set of mappings of OB into OB' — is a copying of $\mathcal{F}$ into $\mathcal{F}'$.*

Consequently :

Theorem 15 *If $n+k \geq 2$, then for every $A \in CFOR$, if $\models_{K^S_{L2}(m,n,j,k)} A$ then $\vdash_{K_{L2}(m,n,j,k)} A$.*

2.8 Examples

Whereas several extensions of $L2$ have a remote relationship with information systems, several extensions, on the other hand, have already been considered by many authors in the context of the modal logics for information systems.
$S5^S_{L2}$ is the class of standard frames in which each $R(m(\Gamma))$ is a relation of equivalence. In the context of the modal logics for information systems, it is the class of the frames of indiscernibility introduced by Orłowska [18].
One can easily extend the proof of the standard completeness of $K_{L2}(1,1,0,0)$ — the class of the frames where each $R(m(\Gamma))$ is symmetric — to the class of the frames where each $R(m(\Gamma))$ is both reflexive and symmetric. This class is the class of the frames of strong similarity.

3 Modal logics $L3$

In this section we consider a class $L3$ of relative modal logics where the parameters are defined by the operations $\cap$ and $\cup$ in such a way that, in the relational semantics of these logics, $R(\Gamma \cap \Delta) = R(\Gamma) \cap R(\Delta)$ and $R(\Gamma \cup \Delta) = R(\Gamma) \cup R(\Delta)$. We present axiomatization of various classes of these logics and we discuss their completeness.

3.1 Language

The language of any modal logic from class $L3$ is obtained from the language of class $L2$ by adjoining the operation of union to the set of operations acting on parameter expressions. Let $APAR$ be a nonempty set of *atomic parameters*. The set $CPAR$ of the *complex parameters* is defined by induction in the following way :

- Every atomic parameter is a complex parameter,
- For every $\Gamma, \Delta \in CPAR$, $\Gamma \cap \Delta \in CPAR$,
- For every $\Gamma, \Delta \in CPAR$, $\Gamma \cup \Delta \in CPAR$.

Let *set* be the mapping of $CPAR$ into $2_f^{2_f^{APAR}}$ — the set of the finite sets of finite subsets of $APAR$ — defined by induction in the following way :

- For every atomic parameter α, $set(\alpha) = \{\{\alpha\}\}$,

- For every $\Gamma, \Delta \in CPAR$, $set(\Gamma \cap \Delta) = set(\Gamma) \cup set(\Delta)$,
- For every $\Gamma, \Delta \in CPAR$, $set(\Gamma \cup \Delta) = \{S \cup T : S \in set(\Gamma)$ and $T \in set(\Delta)\}$.

Direct calculations would lead to the conclusion that :

Theorem 16 *For every $P \subseteq_f 2_f^{APAR}$, there is $\Gamma \in CPAR$ such that $set(\Gamma) = P$.*

Let $\sqsubseteq$ be the binary relation on $CPAR$ defined in the following way :

- For every $\Gamma, \Delta \in CPAR$, $\Gamma \sqsubseteq \Delta$ iff for every $S \in set(\Gamma)$ there is $T \in set(\Delta)$ such that $T \subseteq S$.

As remarked in section 2.1, if the expressions Γ and Δ are considered as Boolean formulas of the classical propositional calculus, then one can easily prove that :

- $\Gamma \sqsubseteq \Delta$ iff $\Delta \rightarrow \Gamma$ is classically valid.

Let $AFOR$ be a nonempty set of *atomic formulas*. The set $CFOR$ of the *complex formulas* is defined by induction in the following way :

- Every atomic formula is a complex formula,
- For every $A \in CFOR$, $\neg A \in CFOR$,
- For every $A, B \in CFOR$, $A \wedge B \in CFOR$,
- For every $\Gamma \in CPAR$ and for every $A \in CFOR$, $[\Gamma]A \in CFOR$.

For every $\Gamma \in CPAR$ and for every $A \in CFOR$, let $\langle\Gamma\rangle A = \neg[\Gamma]\neg A$.

3.2 Semantical study

A *frame for L*3 is a relational structure of the form $\mathcal{F} = (PAR, OB, R)$ where :

- PAR is a nonempty set of *parameters*,
- OB is a nonempty set of *objects*,
- R is a mapping of $2_f^{2_f^{PAR}}$ — the set of the finite sets of finite subsets of PAR — into the set of the binary relations on OB such that, for every $P, Q \subseteq_f 2_f^{PAR}$, $R(P \cup Q) \subseteq R(P) \cap R(Q)$ and $R(\{S \cup T : S \in P$ and $T \in Q\}) = R(P) \cup R(Q)$.

Throughout section 3, by "frame" we always mean a frame for $L3$. $\mathcal{F}$ is *standard* when, for every $P, Q \subseteq_f 2_f^{PAR}$, $R(P \cup Q) = R(P) \cap R(Q)$. A mapping m of $APAR$ into $2_f^{2_f^{PAR}}$ and of $AFOR$ into 2^{OB} is called *assignment on* $\mathcal{F}$. The pair $\mathcal{M} = (\mathcal{F}, m)$ is called *model on* $\mathcal{F}$. $\models_{\mathcal{M}} A$ — the *truth in* $\mathcal{M}$ of a formula A — is defined in the following way :

- For every $A \in CFOR$, $\models_{\mathcal{M}} A$ iff $\widetilde{m}(A) = OB$.

where $\widetilde{m}$ is the mapping of $CPAR$ into $2_f^{2_f^{PAR}}$ and of $CFOR$ into 2^{OB} defined by induction in the following way :

- For every atomic parameter α, $\widetilde{m}(\alpha) = m(\alpha)$,
- For every $\Gamma, \Delta \in CPAR$, $\widetilde{m}(\Gamma \cap \Delta) = \widetilde{m}(\Gamma) \cup \widetilde{m}(\Delta)$,
- For every $\Gamma, \Delta \in CPAR$, $\widetilde{m}(\Gamma \cup \Delta) = \{S \cup T : S \in \widetilde{m}(\Gamma)$ and $T \in \widetilde{m}(\Delta)\}$,
- For every atomic formula A, $\widetilde{m}(A) = m(A)$,
- For every $A \in CFOR$, $\widetilde{m}(\neg A) = OB \setminus \widetilde{m}(A)$,
- For every $A, B \in CFOR$, $\widetilde{m}(A \wedge B) = \widetilde{m}(A) \cap \widetilde{m}(B)$,
- For every $\Gamma \in CPAR$ and for every $A \in CFOR$, $\widetilde{m}([\Gamma]A) = \{x \in OB :$ for every $y \in OB$, if $x\ R(\widetilde{m}(\Gamma))\ y$ then $y \in \widetilde{m}(A)\}$.

It could easily be observed that :

- For every $\Gamma, \Delta \in CPAR$, if $\Gamma \sqsubseteq \Delta$ then $R(\widetilde{m}(\Delta)) \subseteq R(\widetilde{m}(\Gamma))$.

Moreover, direct calculations would lead to the conclusion that :

Theorem 17 *Let $\mathcal{F} = (PAR, OB, R)$ be a frame. Let m be an assignment on $\mathcal{F}$. For every $\Gamma, \Delta \in CPAR$, $R(\widetilde{m}(\Gamma \cap \Delta)) \subseteq R(\widetilde{m}(\Gamma)) \cap R(\widetilde{m}(\Delta))$ and $R(\widetilde{m}(\Gamma \cup \Delta)) = R(\widetilde{m}(\Gamma)) \cup R(\widetilde{m}(\Delta))$. If $\mathcal{F}$ is standard then, for every $\Gamma, \Delta \in CPAR$, $R(\widetilde{m}(\Gamma \cap \Delta)) = R(\widetilde{m}(\Gamma)) \cap R(\widetilde{m}(\Delta))$.*

Moreover :

Theorem 18 *Let $\mathcal{F} = (PAR, OB, R)$ be a frame. Let m be an assignment on $\mathcal{F}$. For every $\Gamma, \Delta, \Lambda \in CPAR$, if, for every $S \in set(\Gamma \cup \Delta)$, there is $T \in set(\Lambda)$ such that $T \subseteq S$ then, for every $S \in \widetilde{m}(\Gamma \cup \Delta)$, there is $T \in \widetilde{m}(\Lambda)$ such that $T \subseteq S$ and $R(\widetilde{m}(\Lambda)) \subseteq R(\widetilde{m}(\Gamma)) \cup R(\widetilde{m}(\Delta))$.*

The notions of truth of a formula in a frame, validity of a formula in a class of frames and logic of a class of frames are defined in a usual way. We distinguish the following classes of frames :

- K_{L3} is the set of all frames,
- K^S_{L3} is the set of all standard frames,
- $K_{L3}(m, n, j, k)$ is the set of all $^{mn}_{jk}$-normal frames,
- $K^S_{L3}(m, n, j, k)$ is the set of all standard $^{mn}_{jk}$-normal frames,
- $S5^S_{L3}$ is the set of all standard frames with equivalence relations.

Observe that :

Theorem 19 *Let $\mathcal{F} = (PAR, OB, R)$ be a frame with equivalence relations. If $\mathcal{F}$ is standard then all the instances of the following schema are true in $\mathcal{F}$:*

- $[\Gamma]A \rightarrow [\Delta]A \vee [\Lambda]A$, *for every* $\Gamma, \Delta, \Lambda \in CPAR$ *such that* $\Gamma \sqsubseteq \Delta \cap \Lambda$.

Proof Suppose that $\mathcal{F}$ is standard and let m be an assignment on $\mathcal{F}$. Let $\Gamma, \Delta, \Lambda \in CPAR$ be such that : $\Gamma \sqsubseteq \Delta \cap \Lambda$ and let $A \in CFOR$. Let $x \in OB$. If $x \notin \widetilde{m}([\Delta]A \vee [\Lambda]A)$ then there exists $y \in OB$ such that $x\ R(\widetilde{m}(\Delta))\ y$ and $y \notin \widetilde{m}(A)$ and there exists $z \in OB$ such that $x\ R(\widetilde{m}(\Lambda))\ z$ and $z \notin \widetilde{m}(A)$. Then, $x\ R(\widetilde{m}(\Delta \cup \Lambda))\ y$ and $x\ R(\widetilde{m}(\Delta \cup \Lambda))\ z$. Since $R(\widetilde{m}(\Delta \cup \Lambda))$ is a relation of equivalence, then $y\ R(\widetilde{m}(\Delta \cup \Lambda))\ z$ and either $y\ R(\widetilde{m}(\Delta))\ z$ or $y\ R(\widetilde{m}(\Lambda))\ z$. If $y\ R(\widetilde{m}(\Delta))\ z$ then $x\ R(\widetilde{m}(\Delta))\ z$ and $x\ R(\widetilde{m}(\Lambda))\ z$. Since $\mathcal{F}$ is standard, then $x\ R(\widetilde{m}(\Delta \cap \Lambda))\ z$. Since $\Gamma \sqsubseteq \Delta \cap \Lambda$, then $x\ R(\widetilde{m}(\Gamma))\ z$ and $x \notin \widetilde{m}([\Gamma]A)$. Similarly, if $y\ R(\widetilde{m}(\Lambda))\ z$ then $x\ R(\widetilde{m}(\Delta))\ y$ and $x\ R(\widetilde{m}(\Lambda))\ y$. Since $\mathcal{F}$ is standard, then $x\ R(\widetilde{m}(\Delta \cap \Lambda))\ y$. Since $\Gamma \sqsubseteq \Delta \cap \Lambda$, then $x\ R(\widetilde{m}(\Gamma))\ y$ and $x \notin \widetilde{m}([\Gamma]A)$.
$\dashv$

3.3 Axiomatic presentation

Together with the classical tautologies, all the instances of the following schemata are axioms of K_{L3} :

- $[\Gamma](A \rightarrow B) \rightarrow ([\Gamma]A \rightarrow [\Gamma]B)$, for every $\Gamma \in CPAR$,
- $[\Gamma]A \wedge [\Delta]A \rightarrow [\Lambda]A$, for every $\Gamma, \Delta, \Lambda \in CPAR$ such that $\Gamma \cup \Delta \sqsubseteq \Lambda$.

Together with the modus ponens, all the instances of the following schema are rules of K_{L3} :

- From A infer $[\Gamma]A$, for every $\Gamma \in CPAR$.

It can be easily observed that, for every $\Gamma, \Delta \in CPAR$, all the instances of the following schemata are theorems of K_{L3} : $[\Gamma]A \wedge [\Delta]A \leftrightarrow [\Gamma \cup \Delta]A$ and $[\Gamma]A \vee [\Delta]A \rightarrow [\Gamma \cap \Delta]A$. Moreover, if $\Gamma \sqsubseteq \Delta$ then all the instances of the following schema are theorems of K_{L3} : $[\Gamma]A \rightarrow [\Delta]A$.

3.4 Completeness

It is easy to see that the following soundness theorem holds for logic K_{L3} :

Theorem 20 *For every $A \in CFOR$, if $\vdash_{K_{L3}} A$ then $\models_{K_{L3}} A$.*

Let OB be the set of the maximal consistent sets of formulas. Let R be the mapping of $2_f^{2_f^{APAR}}$ into the set of the binary relations on OB defined in the following way :

- For every $P \subseteq_f 2_f^{APAR}$ and for every $x, y \in OB$, $x\ R(P)\ y$ iff for every $\Gamma \in CPAR$ and for every $A \in CFOR$, if for every $S \in set(\Gamma)$ there is $T \in P$ such that $T \subseteq S$, then if $[\Gamma]A \in x$ then $A \in y$.

Let it be proved that the structure of the form $\mathcal{F} = (APAR, OB, R)$ is a frame :

$\mathcal{F}$ is a frame Let $P, Q \subseteq_f 2_f^{APAR}$, let $x, y \in OB$ be such that $x\ R(P \cup Q)\ y$ and let it be proved that $x\ R(P)\ y$ and $x\ R(Q)\ y$. Therefore, for every $\Gamma \in CPAR$ and for every $A \in CFOR$, if, for every $S \in set(\Gamma)$, there is $T \in P \cup Q$ such that $T \subseteq S$ and $[\Gamma]A \in x$ then $A \in y$. Since $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$, then, for every $\Gamma \in CPAR$ and for every $A \in CFOR$, if, for every $S \in set(\Gamma)$, there is $T \in P$ such that $T \subseteq S$ and $[\Gamma]A \in x$ then, for every $S \in set(\Gamma)$, there is $T \in P \cup Q$ such that $T \subseteq S$ and $A \in y$ and if, for every $S \in set(\Gamma)$, there is $T \in Q$ such that $T \subseteq S$ and $[\Gamma]A \in x$ then, similarly, for every $S \in set(\Gamma)$, there is $T \in P \cup Q$ such that $T \subseteq S$ and $A \in y$. Consequently, $x\ R(P)\ y$ and $x\ R(Q)\ y$. Let $P, Q \subseteq_f 2_f^{APAR}$, let $x, y \in OB$ be such that $x\ R(\{S \cup T : S \in P \text{ and } T \in Q\})\ y$ and let it be proved that $x\ R(P)\ y$ or $x\ R(Q)\ y$. Assume that $x\ \overline{R(P)}\ y$ and $x\ \overline{R(Q)}\ y$. Then there exists $\Gamma \in CPAR$ and there exists $A \in CFOR$ such that :

- for every $U \in set(\Gamma)$ there is $S \in P$ such that $S \subseteq U$,
- $[\Gamma]A \in x$ and
- $A \notin y$,

and there exists $\Delta \in CPAR$ and there exists $B \in CFOR$ such that :

- for every $V \in set(\Delta)$ there is $T \in Q$ such that $T \subseteq V$,
- $[\Delta]B \in x$ and
- $B \notin y$

Therefore, $[\Gamma](A \vee B) \in x$, $[\Delta](A \vee B) \in x$ and $[\Gamma \cup \Delta](A \vee B) \in x$. Moreover, for every $W \in set(\Gamma \cup \Delta)$, there is $U \in set(\Gamma)$ and there is $V \in set(\Delta)$ such that $W = U \cup V$ and, furthermore, there is $S \in P$ and there is $T \in Q$ such that $S \cup T \subseteq U \cup V$. Since $x\ R(\{S \cup T : S \in P \text{ and } T \in Q\})\ y$ and $[\Gamma \cup \Delta](A \vee B) \in x$, then $A \vee B \in y$, a contradiction. Therefore, either $x\ R(P)\ y$ or $x\ R(Q)\ y$. Reciprocally, let $P, Q \subseteq_f 2_f^{APAR}$, let $x, y \in OB$ be such that either $x\ R(P)\ y$ or $x\ R(Q)\ y$ and let it be proved that $x\ R(\{S \cup T : S \in P \text{ and } T \in Q\})\ y$. Suppose that $x\ R(P)\ y$. Then for every $\Gamma \in CPAR$ and for every $A \in CFOR$, if for every $S \in set(\Gamma)$ there is $T \in P$ such that $T \subseteq S$, then if $[\Gamma]A \in x$ then $A \in y$. Now, let $\Gamma \in CPAR$ be such that, for every $S \in set(\Gamma)$ there is $T \cup U \in \{T \cup U : T \in P \text{ and } U \in Q\}$ such that $T \cup U \subseteq S$. Therefore, for every $S \in set(\Gamma)$ there is $T \in P$ such that $T \subseteq S$. Now, for every $A \in CFOR$, if $[\Gamma]A \in x$ then $A \in y$. Then $x\ R(\{S \cup T : S \in P \text{ and } T \in Q\})\ y$.

Let m be the assignment on $\mathcal{F}$ defined in the following way :

- For every atomic parameter α, $m(\alpha) = \{\{\alpha\}\}$,
- For every atomic formula A, $m(A) = \{x \in OB : A \in x\}$.

Let $\mathcal{M} = (\mathcal{F}, m)$. The proof is done by induction on Γ that, for every $\Gamma \in CPAR$, $\widetilde{m}(\Gamma) = set(\Gamma)$.

Basis For every atomic parameter α, $\widetilde{m}(\alpha) = m(\alpha) = \{\{\alpha\}\} = set(\Gamma)$.

Hypothesis There is $\Gamma, \Delta \in CPAR$ such that $\widetilde{m}(\Gamma) = set(\Gamma)$ and $\widetilde{m}(\Delta) = set(\Delta)$.

Step $\widetilde{m}(\Gamma \cap \Delta) = \widetilde{m}(\Gamma) \cup \widetilde{m}(\Delta) =$, by the hypothesis, $set(\Gamma) \cup set(\Delta) = set(\Gamma \cap \Delta)$. $\widetilde{m}(\Gamma \cup \Delta) = \{S \cup T : S \in \widetilde{m}(\Gamma)$ and

$T \in \widetilde{m}(\Delta)\} =$, by the hypothesis, $\{S \cup T : S \in set(\Gamma)$ and $T \in set(\Delta)\} = set(\Gamma \cup \Delta)$.

The proof is done by induction on A that, for every $A \in CFOR$, $\widetilde{m}(A) = \{x \in OB : A \in x\}$.

Basis For every atomic formula A, $\widetilde{m}(A) = m(A) = \{x \in OB : A \in x\}$.

Hypothesis There is $A \in CFOR$ such that $\widetilde{m}(A) = \{x \in OB : A \in x\}$.

Step For every $\Gamma \in CPAR$, let $x \in OB$ be such that $[\Gamma]A \in x$. Consequently, for every $y \in OB$, if $x\ R(\widetilde{m}(\Gamma))\ y$ then $A \in y$ and, by the hypothesis, $y \in \widetilde{m}(A)$. Therefore, $x \in \widetilde{m}([\Gamma]A)$. For every $\Gamma \in CPAR$, let $x \in OB$ be such that $[\Gamma]A \notin x$. Let y be a maximal consistent set of formulas containing $\{\neg A\} \cup \{B \in CFOR : [\Gamma]B \in x\}$. Direct calculations would lead to the conclusion that $x\ R(\widetilde{m}(\Gamma))\ y$. Therefore, $x \notin \widetilde{m}([\Gamma]A)$.

Therefore :

Theorem 21 *For every $A \in CFOR$, if $\models_{K_{L3}} A$ then $\vdash_{K_{L3}} A$.*

3.5 Copying

Let $\mathcal{F} = (PAR, OB, R)$ and $\mathcal{F}' = (PAR, OB', R')$ be frames. Let I be a set of mappings of OB into OB'. I is a *copying of $\mathcal{F}$ into $\mathcal{F}'$* whenever the following conditions are satisfied :

- For every $x' \in OB'$, there is $f \in I$ and there is $x \in OB$ such that $f(x) = x'$,
- For every $f, g \in I$ and for every $x, y \in OB$, if $f(x) = g(y)$ then $x = y$,
- For every $P \subseteq_f 2_f^{PAR}$, for every $f \in I$ and for every $x, y \in OB$, $x\ R(P)\ y$ iff there is $g \in I$ such that $f(x)\ R'(P)\ g(y)$.

Direct calculations would lead to the conclusion that :

Theorem 22 *Let $\mathcal{F} = (PAR, OB, R)$ and $\mathcal{F}' = (PAR, OB', R')$ be frames. Let I be a copying of $\mathcal{F}$ into $\mathcal{F}'$. Let m be an assignment on $\mathcal{F}$. Let m' be the assignment on $\mathcal{F}'$ defined in the following way :*

- *For every atomic parameter α, $m'(\alpha) = m(\alpha)$,*

- *For every atomic formula A, $m'(A) = \{f(x) : f \in I$ and $x \in m(A)\}$.*

Then for every $A \in CFOR$, $\widetilde{m'}(A) = \{f(x) : f \in I$ and $x \in \tilde{m}(A)\}$.

3.6 Standard completeness

Let $\mathcal{F} = (PAR, OB, R)$ be a frame. For every $P \subseteq_f 2_o^{PAR}$, let $\sigma(P)$ be the mapping of $OB \times OB$ into 2^{OB} such that, for every $x, y \in OB$, $\sigma(P)(x, y) = \emptyset$ if $x\ R(P)\ y$, otherwise $\sigma(P)(x, y) = OB$. Let I be the set of the mappings of $2_f^{2_o^{PAR}} \times 2_o^{PAR}$ — 2_o^{PAR} is the set of the singletons of PAR and $2_f^{2_o^{PAR}}$ is the set of the finite sets of singletons of PAR — into 2^{OB}. Let $OB' = OB \times I$. For every $P \subseteq_f 2_o^{PAR}$, let $R'(P)$ be the binary relation on OB' defined in the following way :

- For every $f, g \in I$ and for every $x, y \in OB$, $(x, f)\ R'(P)\ (y, g)$ iff :
  - For every $O \subseteq_f 2_o^{PAR}$ and for every $\alpha \in 2_o^{PAR}$, if $\alpha \in O$ and $\alpha \in P$ then $g(O, \alpha) = f(O, \alpha)$, and
  - For every $O \subseteq_f 2_o^{PAR}$, $\Sigma_{\alpha \in O} f(O, \alpha) + g(O, \alpha) = \sigma(O)(x, y)$.

For every $P \subseteq_f 2_f^{PAR}$, let $R'(P)$ be the binary relation on OB' defined in the following way :

- For every $f, g \in I$ and for every $x, y \in OB$, $(x, f)\ R'(P)\ (y, g)$ iff there is a mapping ϕ of P into 2_o^{PAR} such that :
  - For every $S \in P$, $\phi(S) \subseteq S$,
  - $(x, f)\ R'(\{\phi(S) : S \in P\})\ (y, g)$.

The proofs of the three following lemmas are similar to the proofs of the lemmas 1, 2 and 3 :

Lemma 10 *Let $P \subseteq_f 2_f^{PAR}$, let $f, g \in I$ and let $x, y \in OB$ be such that $(x, f)\ R'(P)\ (y, g)$. Then $x\ R(P)\ y$.*

Lemma 11 *Let $P \subseteq_f 2_f^{PAR}$, let $f \in I$ and let $x, y \in OB$ be such that $x\ R(P)\ y$. There is $g \in I$ such that $(x, f)\ R'(P)\ (y, g)$.*

Lemma 12 *The structure of the form $\mathcal{F}' = (PAR, OB', R')$ is a standard frame and I — considered as a set of mappings of OB into OB' — is a copying of $\mathcal{F}$ into $\mathcal{F}'$.*

It follows that we have the following completeness theorem with respect to the standard frames :

Theorem 23 *For every $A \in CFOR$, if $\models_{K^S_{L3}} A$ then $\vdash_{K_{L3}} A$.*

3.7 Extensions

This subsection presents two extensions — $S5_{L3}$ and $K_{L3}(m,n,j,k)$ — of the modal logic with strong and pseudo-weak accessibility relations.

3.7.1 $S5_{L3}$

Together with the axioms and the rules of K_{L3}, all the instances of the following schemata are axioms of $S5_{L3}$:

- $[\Gamma]A \rightarrow A$, for every $\Gamma \in CPAR$,
- $\langle\Gamma\rangle A \rightarrow [\Gamma]\langle\Gamma\rangle A$, for every $\Gamma \in CPAR$,
- $[\Gamma]A \rightarrow [\Delta]A \vee [\Lambda]A$, for every $\Gamma, \Delta, \Lambda \in CPAR$ such that $\Gamma \sqsubseteq \Delta \cap \Lambda$.

Theorem 24 *For every $A \in CFOR$, if $\vdash_{S5_{L3}} A$ then $\models_{S5^S_{L3}} A$.*

Moreover :

Lemma 13 *The structure of the form $\mathcal{F} = (APAR, OB, R)$ defined in the subsection 3.4 is a standard frame with equivalence relations.*

Proof : Suppose that $\mathcal{F}$ is not standard. Then there exists $P, Q \subseteq_f 2_f^{APAR}$ and there exists $x, y \in OB$ such that $x\ R(P)\ y$, $x\ R(Q)\ y$ and $x\ \overline{R(P \cup Q)}\ y$. Let $\Delta, \Lambda \in CPAR$ be such that $set(\Delta) = P$ and $set(\Lambda) = Q$. Since $x\ \overline{R(P \cup Q)}\ y$, then there exists $\Gamma \in CPAR$ and there exists $A \in CFOR$ such that :

- For every $S \in set(\Gamma)$, there exists $T \in P \cup Q$ such that $T \subseteq S$.
- $[\Gamma]A \in x$.
- $A \notin y$.

Since $P \cup Q = set(\Delta \cap \Lambda)$, then $\Gamma \sqsubseteq \Delta \cap \Lambda$ and either $[\Delta]A \in x$ or $[\Lambda]A \in x$. Consequently, $A \in y$: a contradiction.
$\dashv$
Therefore :

Theorem 25 *For every $A \in CFOR$, if $\models_{S5^S_{L3}} A$ then $\vdash_{S5_{L3}} A$.*

3.7.2 $K_{L3}(m, n, j, k)$

Let $m, n, j, k \geq 0$. Together with the axioms and the rules of K_{L3}, all the instances of the following schema are axioms of $K_{L3}(m, n, j, k)$:

- $\langle \Gamma \rangle^m [\Gamma]^n A \rightarrow [\Gamma]^j \langle \Gamma \rangle^k A$, for every $\Gamma \in CPAR$.

We have the following soundness theorem :

Theorem 26 *For every $A \in CFOR$, if $\vdash_{K_{L3}(m,n,j,k)} A$ then $\models_{K_{L3}(m,n,j,k)} A$.*

Observe that the structure of the form $\mathcal{F} = (APAR, OB, R)$ defined in the subsection 3.4 is an $^{mn}_{jk}$-normal frame. Therefore :

Theorem 27 *For every $A \in CFOR$, if $\models_{K_{L3}(m,n,j,k)} A$ then $\vdash_{K_{L3}(m,n,j,k)} A$.*

The results analogous to theorems 13, 14, 15 can be obtained for the logics of the class $L3$. The proofs of the three following theorems are similar to the proofs of the theorems 13, 14 and 15 :

Theorem 28 *For every $A \in CFOR$, if $\models_{K^S_{L3}(0,1,0,0)} A$ then $\vdash_{K_{L3}(0,1,0,0)} A$.*

Theorem 29 *For every $A \in CFOR$, if $\models_{K^S_{L3}(1,1,0,0)} A$ then $\vdash_{K_{L3}(1,1,0,0)} A$.*

Theorem 30 *If $n+k \geq 2$ then, for every $A \in CFOR$, if $\models_{K^S_{L3}(m,n,j,k)} A$ then $\vdash_{K_{L3}(m,n,j,k)} A$.*

3.8 Examples

The language of the relative modal logics of the class $L3$ is defined by the operations $\cap$ and $\cup$. Therefore, it is a sublanguage of the language of the Boolean modal logic introduced by Gargov, Passy, Tinchev [11] [12] and K_{L3} is nothing but a fragment of BML. As well, the extensions $K_{L3}(m, n, j, k)$ are axiomatizable extensions of this fragment. In other respects, the relative modal logic $S5_{L3}$ is exactly the data analysis logic with local agreement $DALLA$ introduced by Gargov [10] and further developed by Demri [5].

4 Conclusion

In this paper, we developed a general framework for presenting and studying modal logics based on frames with relative accessibility relations. We defined two major hierarchies of the classes of these logics : logics of class $L2$ and logics of class $L3$. For each class of logics we defined semantic structures of the two kinds : general structures and standard structures. We presented an axiomatization of several classes of logics from the given hierarchies and we studied their completeness with respect to both general and standard semantic structures. We showed that several modal logics for information systems are members of the classes $L2$ and $L3$.

For the modal logics of class $L2$, the accessibility relations R of the models (PAR, OB, R, m) are parametrized by the elements of 2_f^{PAR}, the set of the finite subsets of PAR. For the modal logics of class $L3$, the accessibility relations R are parametrized by the elements of $2_f^{2_f^{PAR}}$, the set of the finite subsets of the set of the finite subsets of PAR. What would be the modal logic the models of which are relational structures of the form (PAR, OB, R, m) where the accessibility relations R are parametrized by the elements of the set of the finite subsets of the set of the finite subsets of the set of the finite subsets of PAR Γ

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