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# Recovering non-monotonicity problems of voting rules\*

Umut Keskin<sup>†</sup>    M. Remzi Sanver<sup>‡</sup>    H. Berkay Tosunlu<sup>‡</sup>

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## Abstract

A social choice rule (SCR) is monotonic if raising a single alternative in voters' preferences while leaving the rankings otherwise unchanged is never detrimental to the prospects for winning of the raised alternative. Monotonicity is rather weak but well-known to discriminate against scoring elimination rules, such as plurality with a run off and single transferable vote. We define the minimal monotonic extension of an SCR as its unique monotonic supercorrespondence that is minimal with respect to set inclusion. After showing the existence of the concept, we characterize, for every non-monotonic SCR, the alternatives that its minimal monotonic extension must contain. As minimal monotonic extensions can entail coarse SCRs, we address the possibility of refining them without violating monotonicity provided that this refinement does not diverge from the original SCR more than the divergence prescribed by the minimal monotonic extension itself. We call these refinements monotonic adjustments and identify conditions over SCRs that ensure unique monotonic adjustments that are minimal with respect to set inclusion. As an application of our general findings, we consider plurality with a runoff, characterize its minimal monotonic extension as well as its (unique) minimal monotonic adjustment. Interestingly, this adjustment is not coarser than plurality with a runoff itself, hence we suggest it as a monotonic substitute to plurality with a runoff.

JEL Classifications: D71, D79.

**Keywords:** monotonicity, minimal monotonic extension, minimal monotonic adjustment, plurality with a runoff, voting rules.

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<sup>†</sup>İstanbul Bilgi University. Kazim Karabekir Cad. No: 2/13 34060. İstanbul, Turkey.

<sup>‡</sup>Université Paris-Dauphine, Université PSL, CNRS, LAMSADE, 75016 PARIS, FRANCE. remzi.sanver@dauphine.fr

# 1 Introduction

Monotonicity conditions imposed over social choice rules (SCRs) elaborate the idea that whenever one or more voters change their preferences in a certain “direction”, the collective choice must also “move towards a similar direction”. Depending on the precise meaning attributed to the two concepts within quotation marks, the literature admits a variety of monotonicity conditions, most of which have a normative appeal that rests on strategic concerns.

These conditions are usually strong, the central example being the one identified by Maskin (1999) as a necessary (but not sufficient) condition for Nash implementation. For resolute SCRs, Maskin monotonicity coincides with strong positive association which Muller & Satterthwaite (1977) show to be equivalent to strategy-proofness. Hence, by Gibbard (1973) and Satterthwaite (1975), only dictatorial or imposed resolute SCRs are Maskin monotonic. When resoluteness is not required, the class of Maskin monotonic SCRs expands but, as Jackson (2001) discusses, still excludes most of the interesting SCRs, such as scoring rules and Condorcet extensions.

There are three other prominent monotonicity conditions that can be expressed with reference to Maskin monotonicity:

- Danilov (1992) monotonicity characterizes Nash implementable SCRs, hence is stronger than Maskin monotonicity.
- Condition alpha (Abreu & Sen (1990)) replaces Maskin monotonicity when the target is implementation via subgame perfect equilibria. Condition alpha is considerably weaker than Maskin monotonicity. As Nunez and Sanver (2018) show, it is satisfied by several Condorcet extensions while failed by scoring rules.
- One-way monotonicity is suggested by Sanver & Zwicker (2009) as a weaker form of strategy-proofness. It is also weaker than Maskin monotonicity, indeed satisfied by scoring rules while failed by Condorcet extensions.

A monotonicity condition whose normative appeal is independent of any strategic concern is *simple monotonicity*<sup>1</sup> which asserts that raising a single alternative in voters’ preferences while leaving the rankings otherwise unchanged is never detrimental to the prospects for winning of the raised alternative.<sup>2</sup> Throughout the paper we refer to simple monotonicity as monotonicity. Monotonicity is rather weak and satisfied by most voting rules of the literature. However, it discriminates against scoring elimination rules which contain well-known

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<sup>1</sup>Simple monotonicity is perhaps the oldest known monotonicity condition in the literature. It has been expressed under different names during its relatively long history that predates modern social choice theory. For a comprehensive account, see Black et al. (1958), Brams & Fishburn (2002) and comments on page 120 of Fishburn (1982).

<sup>2</sup>The condition has a slightly stronger version which additionally requires that no new alternative is added to the chosen set. We chose to analyze the weaker version for reasons we discuss in Footnote 11.

members such as plurality with a runoff and single transferable vote (Smith 1973).<sup>3</sup>

Under an SCR that fails monotonicity, it is ambiguous to the candidates whether they should try to convince voters on the rightness of the cause they defend. This is a perverse incentive which is incompatible with the idea that healthiness of democratic outcomes comes from virtues of deliberation. Thus, conducting elections under a non-monotonic SCR is susceptible to give unhealthy outcomes.<sup>4</sup> Given the perversity induced by monotonicity failures, a question of interest is to compare SCRs according to their likelihood of non-monotonicity. This is addressed by Lepelley et al. (1996) for the comparison of plurality with runoff and antiplurality with runoff when there are three alternatives only. Their analysis gives a complete description of instances where plurality with a runoff fails monotonicity.<sup>5</sup>

We suggest recovering failures of monotonicity by minimally extending SCRs to their monotonic supercorrespondences.<sup>6</sup> The SCR that picks every alternative at every preference profile is trivially monotonic. As a result, every non-monotonic SCR admits at least one monotonic supercorrespondence. A more interesting and less obvious point is whether a unique minimal supercorrespondence that ensures monotonicity exists. To illustrate our point, consider two preference profiles  $P$  and  $P'$  where an alternative  $x$  is raised at the latter in the way expressed by the definition of simple monotonicity. Suppose an SCR  $F$  picks  $x$  at  $P$  but not at  $P'$ . In order to transform  $F$  into a monotonic SCR without discarding any of the originally chosen alternatives,  $x$  must be added to the choice set at  $P'$ . Such an analysis must be made for all pairs of preference profiles. Moreover, the added alternatives must also satisfy the requirements of monotonicity. Thus, transforming  $F$  into a monotonic SCR through adding new alternatives may be a complicated process and it is not obvious whether there is a unique way of minimally making these additions. Our results on minimal monotonic extensions answer this question affirmatively and characterize, for every non-monotonic SCR, the alternatives that its minimal monotonic extension must contain.<sup>7</sup>

As minimal monotonic extensions can entail coarse SCRs, we address the possibility of refining them without violating monotonicity. To be sure, such a refinement must discard some of the alternatives picked by the original SCR, as otherwise the monotonic extension that is refined would not be minimal. Nevertheless, we ask these refinements to coincide with the original SCR at profiles where the minimal monotonic extension does so, thus ensuring that they do not diverge from the original SCR more than the divergence prescribed

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<sup>3</sup>An early discussion of other monotonicity failures is given by Fishburn (1982) while a more recent and comprehensive account can be found in Felsenthal and Nurmi (2017).

<sup>4</sup>See Doron and Kronick (1977) for arguments against using non-monotonic SCRs.

<sup>5</sup>These instances, expressed by their Propositions 1 and 3, are special cases of the general characterization we give in our Theorem 5.1.

<sup>6</sup>Our approach is similar to the approach in Sen (1995) for Maskin monotonicity.

<sup>7</sup>Caragiannis et al. (2014) characterize what they call “the approximation with the least approximation ratio” for the non-monotonic Dodgson’s voting rule and this corresponds to the minimal monotonic extension as we define here.

by the minimal monotonic extension itself. We call these refinements *monotonic adjustments* and identify conditions over SCRs that ensure unique monotonic adjustments that are minimal with respect to set inclusion. As an illustration of the concept, consider the instance exemplified in the previous paragraph. The collective choice rendered monotonic must contain at least  $x$  and some other  $y$  at  $P'$ . Now, suppose we need to make a singleton choice at  $P'$  while  $x$  is kept at  $P$  and monotonicity is preserved. Inevitably,  $x$  will be the unique choice, hence discarding  $y$ , at  $P'$ . In case the choice of  $x$  at  $P$  is admissible and monotonicity is adopted, the choice of  $x$  at  $P'$  is well-justified.<sup>8</sup> On the other hand, it is not obvious that such distortions of  $F$  will yield a monotonic SCR. Our analysis establishes the conditions under which the minimal monotonic extension of  $F$  can be refined while monotonicity is preserved and  $F$  is minimally distorted.

As an application of our general findings, we consider plurality with a runoff, an SCR that is well-known to fail monotonicity. We characterize the minimal monotonic extension of plurality with a runoff, as well as its unique minimal monotonic adjustment. Interestingly, this adjustment is not coarser than plurality with a runoff itself, hence we suggest it as a monotonic substitute to plurality with a runoff.

Section 2 presents the basic notions and notation. Section 3 introduces the concept of a minimal monotonic extension, establishes its uniqueness and gives for every SCR, a characterization of the alternatives that its minimal monotonic extension must contain. Section 4 introduces the concept of monotonic adjustment and provide conditions an SCRs must satisfy to possess a unique minimal monotonic adjustment. Section 5 gives an application of these general findings to plurality with a runoff. Here, we compute the minimal monotonic extension and the minimal monotonic adjustment of this SCR and explain why the latter can be suggested as a monotonic substitute to plurality with a runoff. Section 6 concludes.

## 2 Basic notions and notation

Throughout the paper, for a given set  $X$ ,  $2^X$  denotes the power set of  $X$  and  $\#X$  denotes the cardinality of  $X$ . Consider a *society*  $N$  with  $\#N = n \geq 2$  confronting a set of *alternatives*  $A$  with  $\#A \geq 3$ . Each *voter*  $i \in N$  has a *preference*  $P_i \in L(A)$  where  $L(A)$  is the set of linear orders over  $A$ .<sup>9</sup> We write  $P \in L(A)^n$  for a (*preference*) *profile*. A *social choice rule* (SCR) is a mapping  $F : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$ . We may occasionally define an SCR over some restricted but non-empty domain  $D \subseteq L(A)^n$ , hence as a mapping  $f : D \rightarrow 2^A \setminus \{\emptyset\}$ .

Given any distinct  $P, P' \in L(A)^n$ , we say that  $P'$  is an *improvement* for  $x \in A$  with respect to  $P$  iff  $x P_i y \Rightarrow x P'_i y \forall y \in A \setminus \{x\}, \forall i \in N$  and  $y P_i z \iff y P'_i z \forall y, z \in A \setminus \{x\}, \forall i \in N$ . Note that the distinctness of  $P$  and  $P'$  implies

<sup>8</sup>We further address this point in Section 4. We thank an anonymous referee for raising the issue.

<sup>9</sup>So precisely one of  $x P_i y$  and  $y P_i x$  holds for any distinct  $x, y \in A$  while  $x P_i x$  fails for all  $x \in A$ . Moreover,  $x P_i y$  and  $y P_i z$  implies  $x P_i z$  for all  $x, y, z \in A$ .

the existence of some  $i \in N$  for whom  $y P_i x$  and  $x P'_i y$  for some  $y \in A \setminus \{x\}$ . We write  $\text{IMP}_x(P) \subsetneq L(A)^n$  for the set of profiles which are improvements for  $x$  with respect to  $P$ . When  $P'$  is an improvement for  $x$  with respect to  $P$ , we equivalently say that  $P$  is a *worsening* for  $x$  with respect to  $P'$  and let  $\text{WOR}_x(P') \subsetneq L(A)^n$  be the set of profiles which are worsenings for  $x$  with respect to  $P'$ .

An SCR  $F : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$  is called *monotonic* if and only if given any  $P \in L(A)^n$ , any  $x \in F(P)$  and any  $P' \in \text{IMP}_x(P)$ , we have  $x \in F(P')$ .<sup>10</sup> In case  $F$  is defined over a restricted domain  $D$ , the condition would require both  $P$  and  $P'$  belong to  $D$ .

Although quite desirable, not every SCR is monotonic. The next section suggests a solution to monotonicity violations.

### 3 Monotonic extensions: definition and characterization

Given two SCRs  $F, G : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$ , we say that  $G$  is an *extension* of  $F$  if and only if  $F(P) \subseteq G(P) \forall P \in L(A)^n$ , which we write as  $F \subseteq G$ . When  $G$  is also monotonic, we call it a *monotonic extension* of  $F$ . We write  $\mu(F)$  to denote the set of all monotonic extensions of  $F$ . Note that  $\mu(F)$  is non-empty for any SCR  $F$ , as the SCR  $K(P) = A \forall P \in L(A)^n$  is a monotonic extension of every  $F$ .

Given any two SCRs  $F$  and  $G$  with  $F(P) \cap G(P) \neq \emptyset$  for all  $P \in L(A)^n$ , we define the SCR  $F \cap G$  as  $F \cap G(P) = F(P) \cap G(P) \forall P \in L(A)^n$ . We first show that the intersection of any two monotonic SCRs is also monotonic.

**Proposition 3.1.** *Let  $F, G : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$  be two SCRs with  $F \cap G(P) \neq \emptyset$  for all  $P \in L(A)^n$ . If  $F$  and  $G$  are both monotonic, then  $F \cap G$  is also monotonic.*

*Proof:* Let  $F$  and  $G$  be as in the statement of the proposition. Take any  $P \in L(A)^n$ , any  $x \in F \cap G(P)$  and any  $P' \in \text{IMP}_x(P)$ . As  $x \in F \cap G(P)$  implies  $x \in F(P)$  and  $x \in G(P)$  while  $F$  and  $G$  are both monotonic, we have  $x \in F(P')$  and  $x \in G(P')$ , implying  $x \in F \cap G(P')$ , establishing that  $F \cap G$  is monotonic. Q.E.D.

Among all the monotonic extensions of a given SCR  $F$ , we are mainly interested in the smallest with respect to set inclusion. We define the *minimal monotonic extension* of  $F$  as the SCR  $\bar{F} = \bigcap_{G \in \mu(F)} G$ . As  $A$  and  $N$  are both finite, so is  $\mu(F)$ . Therefore, since  $H \cap G$  is well-defined for any  $H, G \in \mu(F)$ , so is  $\bar{F}$ . Also, monotonicity of  $\bar{F}$  follows from finitely many applications of Proposition 3.1 and, by construction, within  $\mu(F)$ ,  $\bar{F}$  is minimal with respect to set inclusion.<sup>11</sup>

<sup>10</sup>The stronger version which we mention in Footnote 2 would additionally impose  $F(P') \subseteq F(P)$ .

<sup>11</sup>Our conclusion on the existence of a unique minimal monotonic extension would not

Remark that while  $F \subseteq \bar{F}$  in general, we have  $F = \bar{F}$  if and only if  $F$  is monotonic. Let  $\delta_F(P) = \bar{F}(P) \setminus F(P) \forall P \in L(A)^n$  give the divergence between  $\bar{F}$  and  $F$ . Note that  $\delta_F$  is an SCR defined over the domain  $\Delta_F = \{P \in L(A)^n : \delta_F(P) \neq \emptyset\}$ .

We now present a characterization result which specifies the alternatives that minimal monotonic extensions contain.

**Theorem 3.1.** *Given any SCR  $F : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$ , any  $P \in L(A)^n$ , and any  $x \in A \setminus F(P)$ , we have  $x \in \delta_F(P)$  if and only if there exists  $P' \in \text{WOR}_x(P)$  such that  $x \in F(P')$ .*

Proof: To show the “if” part, take any SCR  $F$ , any  $P \in L(A)^n$ , any  $x \in A \setminus F(P)$  and any  $P' \in \text{WOR}_x(P)$  with  $x \in F(P')$ . Consider any  $G \in \mu(F)$ . Since  $G$  is an extension of  $F$ , we have  $x \in G(P') \supseteq F(P')$ . Moreover,  $x \in G(P)$ , as  $G$  is monotonic. Thus,  $x \in G(P)$  for any  $G \in \mu(F)$ , implying  $x \in \bar{F}(P) = \bigcap_{G \in \mu(F)} G(P)$ .

To show the “only if” part, take any SCR  $F$  and suppose there exists  $P^* \in L(A)^n$  with some  $x^* \in \delta_F(P^*)$  while  $x^* \in F(P)$  for no  $P \in \text{WOR}_{x^*}(P^*)$ . Construct the SCR  $F^*$  defined at each  $P \in L(A)^n$  as

$$F^*(P) = \begin{cases} \bar{F}(P) & \text{if } P \notin \text{WOR}_{x^*}(P^*) \\ \bar{F}(P) \setminus \{x^*\} & \text{if } P \in \text{WOR}_{x^*}(P^*) \cup \{P^*\} \end{cases}$$

Note that  $F^*(P)$  is well-defined because  $x^* \in F(P)$  for no  $P \in \text{WOR}_{x^*}(P^*) \cup \{P^*\}$ . We have  $F^* \subseteq \bar{F}$  by construction while  $x^* \in \delta_F(P^*)$ . So,  $F^* \subsetneq \bar{F}$ . We now show  $F \subseteq F^*$ . Take any  $P \in L(A)^n$  and any  $x \in F(P) \subseteq \bar{F}(P)$ . If  $P \notin \text{WOR}_{x^*}(P^*)$ , then  $\bar{F}(P) = F^*(P)$ , hence  $x \in F^*(P)$ . If  $P \in \text{WOR}_{x^*}(P^*) \cup \{P^*\}$  and  $x \neq x^*$ , then  $x \in \bar{F}(P) \setminus \{x^*\} = F^*(P)$ . The case  $P \in \text{WOR}_{x^*}(P^*)$  and  $x = x^*$  contradicts our supposition that  $x^* \notin F(P) \forall P \in \text{WOR}_{x^*}(P^*)$ . Finally, the case  $x = x^*$  and  $P = P^*$  is null because  $x^* \in \delta_F(P^*)$ . Therefore  $F \subseteq F^*$ .

Thus,  $F^*$  is both an extension of  $F$  and a proper subcorrespondence of  $\bar{F}$ . We complete the proof by showing that  $F^*$  is monotonic which contradicts the minimality of  $\bar{F}$ . Take any  $P \in L(A)^n$ , any  $x \in F^*(P)$  and any  $P' \in \text{IMP}_x(P)$ . Consider first the case where  $P \in \text{WOR}_{x^*}(P^*) \cup \{P^*\}$ . Since  $F^*(P) = \bar{F}(P) \setminus \{x^*\}$ , we have  $x \neq x^*$  implying  $x \in \bar{F}(P)$  which implies  $x \in \bar{F}(P')$  by the monotonicity of  $\bar{F}$  while  $x \neq x^*$  ensures  $x \in F^*(P')$ . Now consider the case where  $P \notin \text{WOR}_{x^*}(P^*)$ . So  $x \in \bar{F}(P) = F^*(P)$  and  $x \in \bar{F}(P')$  by the monotonicity of  $\bar{F}$ . If  $P' \notin \text{WOR}_{x^*}(P^*)$ , then  $\bar{F}(P') = F^*(P')$ , thus  $x \in F^*(P')$ . If  $P' \in \text{WOR}_{x^*}(P^*)$ , then  $P^* \in \text{IMP}_x(P)$  which implies  $x \neq x^*$ , as otherwise  $P \notin \text{WOR}_{x^*}(P^*)$  is contradicted. When  $x \neq x^*$ , we have  $x \in \bar{F}(P) = F^*(P)$  and  $x \in \bar{F}(P')$  by the monotonicity of  $\bar{F}$ . Again,  $x \neq x^*$  ensures  $x \in F^*(P') = \bar{F}(P')$ , establishing the monotonicity of  $F^*$ . Q.E.D.

So, coming back to the illustration at the introduction, when  $F$  picks  $x$  at  $P$  but not at  $P'$  where  $x$  is raised, it is straightforward to see that  $x$  must

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be valid under the stronger version of monotonicity expressed in Footnote 10, as the non-emptiness of  $\mu(F)$  could not be ensured. We thank David Pennock for raising this issue.

be added to the choice set at  $P'$ . Theorem 3.1 formally shows this but more importantly establishes that this addition does not create further complications at other profiles. In other words, when  $x$  is chosen at  $P$  but not at  $P'$  where  $x$  is raised, it is necessary but also sufficient to add  $x$  to the collective choice at  $P'$ , hence the minimal monotonic extension.

## 4 Refining the extension: monotonic adjustments

Minimal monotonic extensions may be coarse. By definition they cannot be refined by preserving both their monotonicity and their property of being an extension of the considered SCR. However, for any  $F$ , we may aim to consider a monotonic refinement  $G$  of  $\bar{F}$  which satisfies  $G(P) = F(P) \forall P \in L(A)^n \setminus \Delta_F$ . So  $G$  coincides with  $F$  over the domain  $L(A)^n \setminus \Delta_F$  and selects a subset of  $\bar{F}$  over  $\Delta_F$ . As  $\bar{F}$  is the minimal monotonic extension of  $F$ ,  $G$  cannot be an extension of  $F$ . Thus,  $G$  is a monotonic refinement of  $\bar{F}$ , obtained at the expense of discarding some outcomes prescribed by  $F$ , i.e.,  $F(P) \not\subseteq G(P)$  at some  $P \in \Delta_F$ . We call  $G$  a *monotonic adjustment* of  $F$ .

Given any  $F$ , we define the SCR

$$\tilde{F}(P) = \begin{cases} F(P) & \text{if } P \in L(A)^n \setminus \Delta_F \\ \delta_F(P) & \text{if } P \in \Delta_F \end{cases}$$

So  $\tilde{F}$  coincides with  $F$  over the domain  $L(A)^n \setminus \Delta_F$  while over  $\Delta_F$  it discards all alternatives prescribed by  $F$  and picks those in  $\delta_F$ . As a matter of fact,  $\tilde{F}$  is constructed with the aim of obtaining from  $F$  a monotonic and maximally resolute SCR that distorts the original prescriptions of  $F$  as little as possible. As it will occur in Section 5, there may be instances where  $\tilde{F}$  can be proposed as a monotonic substitute for  $F$ . At a first glance, it may seem counterintuitive to refine  $\bar{F}$  by picking the alternatives  $\delta_F(P)$  that are rejected by  $F$ . To justify this choice, two remarks are in order:

- $\delta_F(P)$  consists of alternatives that are found plausible by  $F$  at some profile  $P'$  which is a worsening for those alternatives with respect to  $P$ . This observation, combined with adopting monotonicity as a reasonable condition makes the choice of  $\delta_F(P)$  well-justified.<sup>12</sup> In other words, if  $F$  is “interesting” (in spite of its non-monotonicity) and we wish to render it monotonic, then  $\tilde{F}$  is also “interesting”.

- The outcomes of  $\bar{F}$  represent alternatives from which a final singleton choice will be made. Thus, if  $\bar{F}$  itself is an acceptable SCR, then so are its refinements. Moreover,  $\tilde{F}$  distorts  $F$  only at profiles where  $\bar{F}$  distorts  $F$ . Thus, in case  $\bar{F}$  is seen as a reasonable solution to the non-monotonicity problem of  $F$  (which is the case when the notion of a minimal monotonic extension is adopted), then so can be seen  $\tilde{F}$ .

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<sup>12</sup>This argument would fail in case we were refining a monotonic extension of  $F$  that is not minimal. We thank an anonymous referee for raising this issue.

To be sure, the monotonicity of  $\tilde{F}$  is not granted, as we can see from Example 4.1 below. In the sequel we write  $r(x, P_i) = \#\{y \in A \mid y P_i x\} + 1$  for the rank of  $x \in A$  at  $P_i \in L(A)$ .

**Example 4.1.** Fix some  $x^*, y^* \in A$ . The SCR  $F_1$  chooses  $A \setminus \{y^*\}$  at every profile  $P$  where  $r(x^*, P_i) = r(y^*, P_j) = 1$  for some  $i, j \in N$  and  $F_1$  chooses  $A$  otherwise.

To see that  $\tilde{F}_1$  is not monotonic, let  $A = \{x^*, y^*, z\}$ ,  $n = 2$  and consider the two profiles  $P, R$  with

$P_1$	$P_2$	$R_1$	$R_2$
$z$	$y^*$	$x^*$	$y^*$
$y^*$	$z$	$z$	$x^*$
$x^*$	$x^*$	$y^*$	$z$

where  $\tilde{F}_1(P) = F_1(P) = \{x^*, y^*, z\}$  and  $\tilde{F}_1(R) = \delta_{F_1}(R) = \{y^*\}$ . So  $x^* \in \tilde{F}_1(P) \setminus \tilde{F}_1(R)$  while  $R \in \text{IMP}_{x^*}(P)$ , showing the non-monotonicity of  $\tilde{F}_1$ .

We introduce two conditions over  $F$  that render  $\tilde{F}$  monotonic. Given any  $x \in A$  and any  $P, Q, R \in L(A)^n$ , we say that  $Q$  is between  $P$  and  $R$  for  $x$  whenever  $R \in \text{IMP}_x(P)$  and  $Q \in \text{WOR}_x(R) \cap \text{IMP}_x(P)$ .

**Condition  $\gamma_1$ :** Take any  $x \in A$  and any  $P, Q, R \in L(A)^n$  such that  $Q$  is between  $P$  and  $R$  for  $x$ . If  $x \in F(P) \cap F(R)$  and  $x \notin F(Q)$ , then  $\delta_F(R) = \emptyset$ .

$\gamma_1$  says that if  $x$  is chosen at  $P$  but creates a monotonicity problem at  $Q$  where it is raised while this problem is resolved at  $R$  where it is further raised, then there should be no other alternative that creates a monotonicity problem at  $R$ .

**Condition  $\gamma_2$ :** Take any  $x \in A$  and any  $P, R \in L(A)^n$  with  $R \in \text{IMP}_x(P)$  and  $x \in F(P) \cap F(R)$ . If  $x \in F(Q)$  for any  $Q$  that is between  $P$  and  $R$  for  $x$  and  $\delta_F(P) = \emptyset$  then  $\delta_F(R) = \emptyset$ .

$\gamma_2$  says that if  $x$  is chosen in all profiles between  $P$  and  $R$  while  $P$  exhibits no monotonicity problem, then  $R$  exhibits no monotonicity problems as well.

The two conditions are rather technical but they characterize the monotonicity of  $\tilde{F}$ . Before showing this result, we establish the logical independence of the two conditions. In fact,  $F_1$  in Example 4.1 exemplifies an SCR that satisfies  $\gamma_1$  but fails  $\gamma_2$ . To see that  $F_1$  satisfies  $\gamma_1$ , take  $x, P, Q, R$  as in the statement of the condition. As  $x \notin F(Q)$ , we have  $x = y^*$  and  $y^* \in F(R)$  implies  $F(R) = A$  which in turn implies  $\delta_{F_1}(R) = \emptyset$ . The violation of  $\gamma_2$  can be observed through Example 4.1 by noting that  $x^* \in F(P) \cap F(R)$  and  $x^* \in F(Q)$  for any  $Q$  that is between  $P$  and  $R$  for  $x^*$ , while  $\delta_{F_1}(P) = \emptyset$  but  $\delta_{F_1}(R) \neq \emptyset$ .

We now give an example of an SCR that fails  $\gamma_1$  but satisfies  $\gamma_2$ .

**Example 4.2.** The SCR  $F_2$  picks the unique plurality winner (ties broken by an exogenous linear order) at every profile  $P$ . Moreover, if  $P \in L(A)^n$  is such that  $r(x, P_i) = 2$  for all  $i \in N$ , for some  $x \in A$ , then  $x$  is also picked at  $P$ .

To see the failure of  $\gamma_1$ , let  $A = \{x, y, z\}$ ,  $n = 3$  and consider the three profiles  $P, Q, R$  with

$P_1$	$P_2$	$P_3$	$Q_1$	$Q_2$	$Q_3$	$R_1$	$R_2$	$R_3$
$y$	$y$	$y$	$x$	$y$	$y$	$x$	$x$	$y$
$x$	$x$	$x$	$y$	$x$	$x$	$y$	$y$	$x$
$z$	$z$	$z$	$z$	$z$	$z$	$z$	$z$	$z$

Here,  $Q$  is between  $P$  and  $R$  for  $x$ , and  $x \in F_2(P) \cap F_2(R)$  while  $x \notin F_2(Q)$ . However,  $\delta_{F_2}(R) = \{y\} \neq \emptyset$ , showing the failure of  $\gamma_1$ . To see the satisfaction of  $\gamma_2$ , take  $x, P, R$  as in the statement of the condition. As  $x \in F(P) \cap F(R)$  and  $x \in F(Q)$  for any  $Q$  that is between  $P$  and  $R$  for  $x$ , it must be that  $x$  is the plurality winner at  $P$ . As  $\delta_{F_1}(P) = \emptyset$ , there exists no  $y \in A \setminus \{x\}$  with  $r(y, P_i) \leq 1$  for all  $i \in N$ .  $I_1$  and  $r(y, P_j) = 2$  for all  $j \in I_2$ . As  $R \in \text{IMP}_x(P)$ ,  $x$  is the plurality winner at  $R$  while there exists no  $y \in A \setminus \{x\}$  with  $r(y, R_i) \leq 1$  for all  $i \in N$ , implying  $\delta_{F_2}(R) = \emptyset$ , hence the satisfaction of  $\gamma_2$ .

We first show that  $\gamma_1$  is equivalent to the monotonicity of  $\delta_F$ :

**Lemma 4.1.** *Given any SCR  $F : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$ ,  $\delta_F : \Delta_F \rightarrow 2^A \setminus \{\emptyset\}$  is monotonic if and only if  $F$  satisfies  $\gamma_1$ .*

Proof: To show the “if” part, let  $F$  satisfy  $\gamma_1$ . Take any  $Q \in \Delta_F$ , any  $x \in \delta_F(Q)$  and any  $R \in \text{IMP}_x(Q) \cap \Delta_F$ . As  $x \in \delta_F(Q)$ ,  $\exists P$  such that  $Q \in \text{IMP}_x(P)$  and  $x \in F(P)$ . Note that  $Q \in \text{WOR}_x(R)$ . Therefore,  $Q$  is between  $P$  and  $R$  for  $x$ . Since  $\bar{F}$  is monotonic,  $x \in F(R) \cup \delta_F(R)$ . As  $x \in F(P)$ ,  $x \notin F(Q)$  and  $R \in \Delta_F$ , by  $\gamma_1$ ,  $x \notin F(R)$ . Therefore  $x \in \delta_F(R)$ , establishing the monotonicity of  $\delta_F$ .

To show the “only if” part, let  $\delta_F$  be monotonic. Take any  $x \in A$ , any  $P, Q, R \in L(A)^n$  such that  $Q$  is between  $P$  and  $R$  for  $x$  while  $x \in F(P) \cap F(R)$  but  $x \notin F(Q)$ . Since  $P \in \text{WOR}_x(Q)$ ,  $x \in F(P)$  and  $x \notin F(Q)$ , by Theorem 3.1,  $x \in \delta_F(Q)$ . Suppose, for a contradiction,  $R \in \Delta_F$ . Since  $R \in \text{IMP}_x(Q)$  and  $\delta_F$  is monotonic, we have  $x \in \delta_F(R)$ , contradicting  $x \in F(R)$ . Therefore  $R \notin \Delta_F$  i.e.,  $\delta_F(R) = \emptyset$ , showing that  $\gamma_1$  holds. Q.E.D.

So, for example,  $F_1$  in Example 4.1 satisfies  $\gamma_1$ , which implies the monotonicity of  $\delta_{F_1}$  while  $\delta_{F_2}$  is not monotonic for  $F_2$  in Example 4.2, which fails  $\gamma_1$ .<sup>13</sup>

**Theorem 4.1.** *Given any SCR  $F : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$ ,  $\tilde{F}$  is a monotonic adjustment of  $F$  if and only if  $F$  satisfies  $\gamma_1$  and  $\gamma_2$ .*

Proof: We first prove the “if” part. By construction,  $\tilde{F}$  is a refinement of  $\bar{F}$ . We now show that  $\tilde{F}$  is monotonic. Take any  $P \in L(A)^n$ , any  $x \in \tilde{F}(P)$  and any  $P' \in \text{IMP}_x(P)$ . We show that  $x \in \tilde{F}(P')$  for the four exhaustive cases below, thus establishing the monotonicity of  $\tilde{F}$ .

**Case 1.**  $P, P' \in \Delta_F$ . Thus  $\tilde{F}(P) = \delta_F(P)$  and  $\tilde{F}(P') = \delta_F(P')$ . By Lemma 4.1, monotonicity of  $\delta_F$  implies  $x \in \tilde{F}(P')$ .

<sup>13</sup>One can see this through the profiles in Example 4.2 where  $\delta_{F_2}(Q) = \{x\}$  and  $R \in \text{IMP}_x(Q)$  but  $x \notin \delta_{F_2}(R) = \{y\}$ .

**Case 2.**  $P \in \Delta_F, P' \in L(A)^n \setminus \Delta_F$ . Since  $\bar{F}$  is monotonic,  $x \in \bar{F}(P') = \tilde{F}(P')$ .

**Case 3.**  $P, P' \in L(A)^n \setminus \Delta_F$ . Thus  $x \in F(P)$ , implying  $x \in F(P') = \tilde{F}(P')$ , as otherwise we would have  $P' \in \Delta_F$ .

**Case 4.**  $P \in L(A)^n \setminus \Delta_F, P' \in \Delta_F$ . Suppose  $x \notin \tilde{F}(P')$ . As  $\tilde{F}(P) = F(P)$  and  $P' \in \text{IMP}_x(P)$ , we have  $x \in \bar{F}(P')$ , thus  $x \in F(P')$ . By monotonicity of  $\bar{F}$ , for any  $Q$  between  $P$  and  $P'$  for  $x$ , either  $x \in F(Q)$  or  $x \in \delta_F(Q)$ . If  $x \in \delta_F(Q^*)$  for some  $Q^*$  between  $P$  and  $P'$  for  $x$ , then by  $\gamma_1, \delta_F(P') = \emptyset$  which contradicts  $P' \in \Delta_F$ . If  $x \in F(Q)$  for all  $Q$  between  $P$  and  $P'$  for  $x$ , then by  $\gamma_2, \delta_F(P') = \emptyset$  which again contradicts  $P' \in \Delta_F$ . Therefore  $x \in \tilde{F}(P')$ .

We now prove the “only if” part. Let  $\tilde{F}$  be a monotonic adjustment of  $F$ . Thus  $\delta_F : \Delta_F \rightarrow 2^A \setminus \{\emptyset\}$  is monotonic and by Lemma 4.1,  $F$  satisfies  $\gamma_1$ . Take any  $x \in A$ , any  $P, R \in L(A)^n$  with  $R \in \text{IMP}_x(P), x \in F(P) \cap F(R)$  and  $x \in F(Q)$  for any  $Q$  that is between  $P$  and  $R$  for  $x$ . Let also  $\delta_F(P) = \emptyset$ , which ensures  $\tilde{F}(P) = F(P)$ , implying  $x \in \tilde{F}(P)$ . As  $R \in \text{IMP}_x(P)$ , monotonicity of  $\tilde{F}(P)$  implies  $x \in \tilde{F}(R)$ . If  $\delta_F(R) \neq \emptyset$  then  $\tilde{F}(R) = \delta_F(R)$  which means  $x \in \delta_F(R)$ , contradicting  $x \in F(R)$ . Therefore  $\delta_F(R) = \emptyset$  i.e.,  $\gamma_2$  holds. Q.E.D.

Several remarks are in order.

$\gamma_1$  and  $\gamma_2$  are logically compatible, as they are jointly satisfied by  $F_3$ , as defined below:

**Example 4.3.** The SCR  $F_3$  picks at every profile the unique plurality loser (ties being broken by an exogenous linear order).

Note that  $\Delta_{F_3} = L(A)^n$ . Therefore,  $\gamma_2$  holds trivially. Also, for any  $P \in L(A)^n$  and  $Q \in \text{IMP}_x(P)$  if  $x \in F_3(P)$  and  $x \notin F_3(Q)$ , then  $x$  is the plurality loser at  $P$  and not the plurality loser at  $Q$ . Thus, for any  $R \in \text{IMP}_x(Q)$ ,  $x$  is still not the plurality loser, hence  $x \notin F_3(R)$ . Therefore,  $\gamma_1$  holds trivially as well.

To complete the picture on logical independence, we note that the SCR  $F_4$  defined below fails both conditions:

**Example 4.4.** The SCR  $F_4$  picks some fixed  $x^* \in A$  at every profile; moreover the unique plurality loser (ties broken by an exogenous linear order) at  $P$  if and only if  $r(x^*, P_i) = 1$  for all  $i \in N$ ; and also  $y \in A$  with  $r(y, P_i) = 2$  for all  $i \in N$ .

When  $\tilde{F}$  is ensured to be a monotonic adjustment, it need not be unique nor minimal (with respect to set inclusion). This can be seen through  $F_3$  in Example 4.3 where  $\bar{F}_3$  picks  $A$  at every profile while  $\Delta_{F_3} = L(A)^n$  which implies that any monotonic SCR is indeed a monotonic adjustment of  $F_3$ . Nevertheless, if  $\tilde{F}$  is a monotonic adjustment of  $F$  while  $F$  satisfies the following Condition  $\gamma_3$ , then  $\tilde{F}$  turns out to be the unique monotonic adjustment of  $F$  that is minimal with respect to set inclusion:

**Condition  $\gamma_3$ :** Given any  $P \in \Delta_F$  and any  $x \in \delta_F(P)$ , we have  $x \in F(R)$  for some  $R \in \text{WOR}_x(P) \setminus \Delta_F$ .

$\gamma_3$  says that if  $x$  creates a monotonicity problem at  $P$ , then among the profiles where  $x$  is lowered but chosen (such profiles exist by Theorem 3.1) at least one must exhibit no monotonicity problem.

Before stating our result, we note the logical independence of  $\gamma_1$ ,  $\gamma_2$  and  $\gamma_3$ . In fact,  $F_1$  in Example 4.1 satisfies  $\gamma_1$  and  $\gamma_3$  but fails  $\gamma_2$ ;  $F_2$  in Example 4.2 satisfies  $\gamma_2$  and  $\gamma_3$  but fails  $\gamma_1$ ;  $F_3$  in Example 4.3 satisfies  $\gamma_1$  and  $\gamma_2$  but fails  $\gamma_3$ .

**Theorem 4.2.** *Given any SCR  $F : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$ ,  $\tilde{F}$  is the unique minimal monotonic adjustment of  $F$  if and only if  $F$  satisfies  $\gamma_1$ ,  $\gamma_2$  and  $\gamma_3$ .*

Proof: We first prove the “if” part. By Theorem 4.1,  $\tilde{F}$  is a monotonic adjustment of  $F$ . We now show that  $\tilde{F}$  is unique and minimal with respect to set inclusion. Take any monotonic  $G$  which is a refinement of  $\tilde{F}$  and coincides with  $F$  on  $L(A)^n \setminus \Delta_F$ . So  $G(P) = \tilde{F}(P)$  for all  $P \in L(A)^n \setminus \Delta_F$ . Now take any  $P \in \Delta_F$  and any  $x \in \tilde{F}(P) = \delta_F(P)$ . By  $\gamma_3$ , there exists  $P' \in \text{WOR}_x(P) \setminus \Delta_F$  such that  $x \in F(P') = G(P')$ . Since  $G$  is monotonic and  $P \in \text{IMP}_x(P')$  we have  $x \in G(P)$ . Therefore,  $G(P) \supseteq \tilde{F}(P) = \delta_F(P)$  for all  $P \in \Delta_F$ , hence  $\tilde{F}$  is the unique minimal monotonic adjustment of  $F$ .

We now prove the “only if” part. Let  $\tilde{F}$  be the unique minimal monotonic adjustment of  $F$ . By Theorem 4.1,  $F$  satisfies  $\gamma_1$  and  $\gamma_2$ . For  $\gamma_3$ , take any  $P \in \Delta_F$  and any  $x \in \delta_F(P)$ . Suppose, for a contradiction, that  $x \notin F(R) \forall R \in \text{WOR}_x(P) \setminus \Delta_F$ . Let  $D = \{R \in \text{WOR}_x(P) \mid x \in \delta_F(R)\}$ . Consider the following SCR  $\hat{F} : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$  defined as

$$\hat{F}(Q) = \begin{cases} \delta_F(Q) \setminus \{x\} & \text{if } Q \in D \cup \{P\} \\ \tilde{F}(Q) & \text{otherwise} \end{cases}$$

Note that  $\hat{F}$  is a refinement of  $\tilde{F}$  and  $\hat{F} = F$  on  $L(A)^n \setminus \Delta_F$ , i.e.,  $\hat{F}$  is an adjustment of  $\tilde{F}$ . Next we argue that  $\hat{F}$  is monotonic. To see this, note that for any  $y \neq x$ , any  $R \in L(A)^n$  and any  $R' \in \text{IMP}_y(R)$ ,  $y \in \hat{F}(R)$  yields  $y \in \tilde{F}(R)$  which implies  $y \in \tilde{F}(R')$  due to monotonicity of  $\tilde{F}$ , so  $y \in \hat{F}(R')$  as  $y \neq x$ . Now consider  $x \in A$ . For any  $Q \in D \cup \{P\}$ ,  $x \notin \hat{F}(Q)$ , so checking monotonicity condition would be void in this case. When  $Q \notin D \cup \{P\}$  and  $x \in \hat{F}(R) = \tilde{F}(R)$ , due to our contrary assumption there is no  $Q' \in D \cup \{P\}$  which is an improvement for  $x$  with respect to  $Q$ . Therefore monotonicity check would be void in this case too. In all other profiles  $\hat{F}$  coincides with  $\tilde{F}$  whose monotonicity gives us the monotonicity of  $\hat{F}$ . Therefore,  $\hat{F}$  is a monotonic adjustment of  $\tilde{F}$ . Also by construction, we have  $\hat{F} \subsetneq \tilde{F}$  which contradicts minimality of  $\tilde{F}$ . Hence,  $F$  satisfies  $\gamma_3$  as well. Q.E.D.

SCRs that satisfy all three conditions and, thus, are covered by Theorem 4.2 exist, as we discuss in the next section.

We close the section by discussing an issue raised by an anonymous referee: How “close” is  $\tilde{F}$  to  $F$  compared to other monotonic refinements of  $\tilde{F}$ ? Such an analysis needs to measure the distance between two SCRs. Our framework allows several such measures. For example, one can define the distance between

two SCRs  $F$  and  $G$  as the number of preference profiles where  $F$  and  $G$  disagree<sup>14</sup>. This is a measure which overlooks whether the disagreement occurs at a profile where monotonicity is violated or not. As a result, it allows monotonic refinements of  $\bar{F}$  that are closer to  $F$  than  $\tilde{F}$  is.<sup>15</sup> However, when  $\tilde{F}$  is unique, these refinements inevitably distort  $F$  at profiles where monotonicity is not violated. So, under a measure that honors monotonicity and reflects the spirit of  $\bar{F}$  by giving a sufficiently low (but non-zero) weight to disagreements at profiles where monotonicity is violated,  $\tilde{F}$  would turn out to be closer to  $F$  than any other other monotonic refinement of  $\bar{F}$ .

## 5 An application: plurality with a runoff

For any  $x \in A$  and  $P \in L(A)^n$ , define  $\tau(x, P) = \#\{i \in N \mid r(x, P_i) = 1\}$  as the number of voters who rank  $x$  at the top. Let  $\pi_1(P) = \{x \in A \mid \tau(x, P) \geq \tau(y, P) \forall y \in A\}$  be the set of plurality winners, i.e., the alternatives ranked at the top by the highest number of voters. Similarly,  $\pi_2(P) = \{x \in A \setminus \pi_1(P) \mid \tau(x, P) \geq \tau(y, P) \forall y \in A \setminus \pi_1(P)\}$ .

Let

$$RO(P) = \begin{cases} \{\{x, y\}\}_{x \in \pi_1(P), y \in \pi_1(P) \setminus \{x\}} & \text{if } \#\pi_1(P) \geq 2 \\ \{\{x, y\}\}_{\{x\} = \pi_1(P), y \in \pi_2(P)} & \text{if } \#\pi_1(P) = 1 \end{cases}$$

be defined as the set of unordered pairs of alternatives that go for a runoff at  $P \in L(A)^n$ . Note that  $RO$  is not an SCR and due to ties,  $RO(P)$  may contain more than one unordered pair. We now define the SCR  $\rho : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$  which picks at each  $P \in L(A)^n$  the set of candidates that go for a runoff, i.e.,

$$\rho(P) = \bigcup_{\{x, y\} \in RO(P)} \{x, y\}.$$

Plurality with a runoff is the SCR  $F_{PR} : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$  defined for each  $P \in L(A)^n$  as

$$F_{PR}(P) = \{x \in A \mid \exists \{x, y\} \in RO(P) \text{ and } \#\{i \in N \mid x P_i y\} \geq \#\{i \in N \mid y P_i x\}\}.$$

It is well-known that  $F_{PR}$  is not monotonic.<sup>16</sup>

<sup>14</sup>We could define a more sophisticated measure that also takes into account the amount of disagreement at a given profile but we don't wish to deal with details that are unnecessary for the argument we are about to make.

<sup>15</sup>To see this, let  $A = \{x, y\}$  and  $N = \{1, 2\}$ . Define  $F(P) = \{x, y\}$  when  $x P_i y \forall i \in N$ ;  $F(P) = \{y\}$  when  $y P_i x \forall i \in N$ ; and  $F(P) = \{x\}$  when  $x P_i y$  and  $y P_j x$  for  $i \neq j$ . Here, monotonicity is violated at the two profiles where the outcome is  $\{x\}$ , hence  $\bar{F}$  disagrees with  $F$  at those two profiles. On the other hand, the SCR  $G$  that agrees with  $F$  at every profile except the one where  $x$  is ranked first by both voters ( $G$  picks  $\{x\}$  rather than  $\{x, y\}$  at this profile) is monotonic. So, under the measure we described,  $G$  is closer to  $F$  than  $\bar{F}$ .

<sup>16</sup>For example, let  $n = 93$ ,  $A = \{a, b, c\}$  and  $P$  be a profile where 42 voters have the preference  $aP_i bP_i c$ ; 27 voters have the preference  $bP_i cP_i a$ ; and 24 voters have the preference  $cP_i aP_i b$ . So  $RO(P) = \{\{a, b\}\}$  and  $F_{PR}(P) = \{a\}$ . Let  $P'$  be a profile where 46 voters have the preference  $aP'_i bP'_i c$ ; 23 voters have the preference  $bP'_i cP'_i a$ ; and 24 voters have the preference  $cP'_i aP'_i b$ . Now  $RO(P') = \{\{a, c\}\}$  and  $F_{PR}(P') = \{c\}$  while  $P' \in \text{IMP}_a(P)$ .

We first show that the minimal monotonic extension of plurality with a runoff can only contain elements who went for a runoff.

**Proposition 5.1.**  $\bar{F}_{PR} \subseteq \rho$ .

Proof: We have  $F_{PR} \subseteq \rho$  by definition. So, if  $\rho$  is monotonic, then  $\rho \in \mu(F_{PR})$  which proves the proposition, as  $\bar{F}_{PR} = \bigcap_{G \in \mu(F_{PR})} G$ . Thus, we complete the proof by establishing the monotonicity of  $\rho$ .

Take any  $P \in L(A)^n$ , any  $x \in \rho(P)$  and any  $P' \in \text{IMP}_x(P)$ . As  $x \in \rho(P)$ , we have  $\{x, y\} \in RO(P)$  for some  $y \in A \setminus \{x\}$ . If  $\#\pi_1(P) \geq 2$ , then  $x \in \pi_1(P)$ . By  $P' \in \text{IMP}_x(P)$ , we have  $x \in \pi_1(P')$ , which implies  $x \in \rho(P')$ . If  $\#\pi_1(P) = 1$  and  $\pi_1(P) = \{x\}$ , then  $\pi_1(P') = \{x\}$  as well, implying  $x \in \rho(P')$ . Finally, if  $\#\pi_1(P) = 1$  and  $\pi_1(P) \neq \{x\}$ , then  $x \in \pi_1(P') \cup \pi_2(P')$ . Moreover,  $x \in \pi_1(P')$  in case  $\#\pi_1(P') \geq 2$ , implying  $x \in \rho(P')$ . Q.E.D.

The next proposition identifies the elements of  $\rho$  excluded from  $\bar{F}_{PR}$ . As a matter of fact, an alternative who does not go to a runoff as an untied plurality winner is never included to the minimal monotonic extension.

**Proposition 5.2.** For any  $P \in L(A)^n$  and any  $x \in \rho(P) \setminus F_{PR}(P)$ , if  $\tau(x, P) \leq \tau(y, P)$  for all  $\{x, y\} \in RO(P)$ , then  $x \notin \bar{F}_{PR}(P)$ .

Proof: Take any  $P \in L(A)^n$  and any  $x \in \rho(P) \setminus F_{PR}(P)$  with  $\tau(x, P) \leq \tau(y, P) \forall \{x, y\} \in RO(P)$ . Suppose  $x \in \bar{F}_{PR}(P)$ . By Theorem 3.1,  $\exists P' \in \text{WOR}_x(P)$  with  $x \in F_{PR}(P')$ .

First observe that  $\pi_1(P') = \{x\}$  does not hold because otherwise  $P \in \text{IMP}_x(P')$  would imply  $\pi_1(P) = \{x\}$  as well, which would contradict  $\tau(x, P) \leq \tau(y, P) \forall \{x, y\} \in RO(P)$ . As a result, the following two cases are exhaustive:

**Case 1:**  $x \in \pi_1(P')$  and  $\#\pi_1(P') \geq 2$ .

So  $\tau(y, P') = \tau(z, P') \forall \{y, z\} \in RO(P')$ . Note that  $P \in \text{IMP}_x(P')$  and  $\tau(x, P) \leq \tau(y, P) \forall \{x, y\} \in RO(P)$  imply  $RO(P) = RO(P')$ . As  $x \in F_{PR}(P')$ ,  $\exists \{x, y\} \in RO(P')$  with  $\#\{i \in N \mid x P'_i y\} \geq \#\{i \in N \mid y P'_i x\}$ . By  $RO(P) = RO(P')$  we note  $\{x, y\} \in RO(P)$  and by  $P \in \text{IMP}_x(P')$  we have  $\#\{i \in N \mid x P_i y\} \geq \#\{i \in N \mid y P_i x\}$ , implying  $x \in F_{PR}(P)$ , contradicting  $x \in \rho(P) \setminus F_{PR}(P)$ .

**Case 2:**  $x \in \pi_2(P')$  and  $\pi_1(P') = \{y\}$  for some  $y \in A \setminus \{x\}$ .

So  $\{x, y\} \in RO(P')$  and  $\{x, z\} \notin RO(P') \forall z \in A \setminus \{x, y\}$ . As  $x \in F_{PR}(P')$ , we have  $\#\{i \in N \mid x P'_i y\} \geq \#\{i \in N \mid y P'_i x\}$ . Note that  $P \in \text{IMP}_x(P')$  implies  $\{x, y\} \in RO(P)$  and  $\#\{i \in N \mid x P_i y\} \geq \#\{i \in N \mid y P_i x\}$ , implying  $x \in F_{PR}(P)$ , contradicting  $x \in \rho(P) \setminus F_{PR}(P)$ .

Therefore, we conclude  $x \notin \bar{F}_{PR}(P)$ . Q.E.D.

Thus, an alternative which is included to the minimal monotonic extension at profile  $P$  must be the unique untied plurality winner at  $P$ . On the other hand, untied plurality winners are not always included to the extension. Our next result identifies the precise conditions under which this inclusion happens. Before stating the formal result, let us denote the number of voters who ranks an alternative  $x$  on top and  $z$  as second in profile  $P$ , by  $N_{x,z}(P) = \#\{i \in N \mid x P_i z P_i y \forall y \in A \setminus \{x, z\}\}$ .

**Proposition 5.3.** *For any  $P \in L(A)^n$  and any  $x \in A \setminus F_{PR}(P)$  with  $\pi_1(P) = \{x\}$ , we have  $x \in \overline{F}_{PR}(P)$  if and only if for all  $y \in \pi_2(P)$ , there exists  $z \in A \setminus \{x, y\}$  such that*

- (i)  $\tau(x, P) - \tau(y, P) \geq \tau(y, P) - \tau(z, P)$
- (ii)  $N_{x,z}(P) \geq \tau(y, P) - \tau(z, P)$
- (iii)  $\#(i \in N \mid x P_i z) - \#(i \in N \mid z P_i x) \geq 2(\tau(y, P) - \tau(z, P))$ .

Proof: Take any  $P \in L(A)^n$  and  $x \in A \setminus F_{PR}(P)$  where  $\pi_1(P) = \{x\}$  and consider any  $y \in \pi_2(P)$ .

To see the “only if” part, let  $x \in \overline{F}_{PR}(P)$ . By Proposition 3.1,  $\exists P' \in WOR_x(P)$  such that  $x \in F_{PR}(P')$ . As  $x \notin F_{PR}(P)$ ,  $\#(i \in N \mid x P_i y) < \#(i \in N \mid y P_i x)$ . As  $P' \in WOR_x(P)$ ,  $x \notin F_{PR}(P)$ ,  $\#(i \in N \mid x P'_i y) < \#(i \in N \mid y P'_i x)$ . So  $\exists z \in A \setminus \{x, y\}$  such that  $\{x, z\} \in RO(P')$  while  $\#(i \in N \mid x P'_i z) \geq \#(i \in N \mid z P'_i x)$  because otherwise  $x \in F_{PR}(P')$  would not be possible. Moreover  $\{x, z\} \notin RO(P)$  because otherwise we would have  $x \in F_{PR}(P)$ . Thus, noting  $\{x, z\} \in RO(P') \setminus RO(P)$ , we observe the existence of some  $K \subseteq (i \in N \mid P_i \neq P'_i)$ .  $r(x, P_i) = 1$  and  $r(z, P_i) = 2 \forall i \in K$ .

Moreover,  $\#K \geq \tau(y, P) - \tau(z, P)$  and  $\#K \geq \tau(x, P) - \tau(y, P)$  which establishes (i) and (ii).

As  $\#(i \in N \mid x P'_i z) \geq \#(i \in N \mid z P'_i x)$ , we have  $\#(i \in N \mid x P'_i z) - \#(i \in N \mid z P'_i x) \geq 2\#K$  which establishes (iii).

To see “if” part, suppose  $\exists z \in A \setminus \{x, y\}$  that satisfies conditions (i), (ii) and (iii) in the statement of the lemma. Take some  $K \subseteq N_{x,z}(P)$  with  $\#K = \tau(y, P) - \tau(z, P)$ . Condition (i) ensures the existence of  $K$ . Now, take  $P' \in L(A)^n$  with  $P'_i = P_i \forall i \in N \setminus K$  and  $s P'_i t \iff s P_i t \forall s, t \in A \setminus \{x, z\} \forall i \in K$  and  $z P'_i x P_i s \forall s \in A \setminus \{x, z\} \forall i \in K$ . Note that  $P' \in WOR_x(P)$ . Moreover, as  $\#K = \tau(y, P) - \tau(z, P)$ , we have  $\tau(y, P') = \tau(z, P')$ . Also, by condition (ii),  $\tau(x, P') \geq \tau(z, P')$ . Hence  $\{x, z\} \in RO(P')$ . By condition (iii),  $\#(i \in N \mid x P'_i z) \geq \#(i \in N \mid z P'_i x)$ . Thus,  $x \in F_{PR}(P')$  with, by Proposition 3.1,  $x \in \overline{F}_{PR}(P)$ . Q.E.D.

Note that replacing in the statement of the proposition the universal quantification of  $y$  with the existential one makes an equivalent statement, as this change has no impact on the satisfaction of conditions (i), (ii), (iii). Moreover, having a unique plurality winner which is a plurality runoff loser implies  $\pi_2(P) = F_{PR}(P)$ .

Propositions 5.1, 5.2 and 5.3 lead to the following theorem as a corollary:

**Theorem 5.1.** *For each  $P \in L(A)^n$ , for each  $x \in A \setminus F_{PR}(P)$ , we have  $x \in \delta_{F_{PR}}(P)$  if and only if  $\pi_1(P) = \{x\}$  and given any  $y \in \pi_2(P) = F_{PR}(P)$ , there exists  $z \in A \setminus \{x, y\}$  such that*

- (i)  $\tau(x, P) - \tau(y, P) \geq \tau(y, P) - \tau(z, P)$
- (ii)  $N_{x,z}(P) \geq \tau(y, P) - \tau(z, P)$
- (iii)  $\#(i \in N \mid x P_i z) - \#(i \in N \mid z P_i x) \geq 2(\tau(y, P) - \tau(z, P))$ .

So at each  $P$ ,  $\delta_{F_{PR}}(P)$  is either empty or contains a single alternative which is the unique plurality winner (but plurality with a runoff loser) satisfying the conditions identified by Proposition 5.3.

Plurality with a runoff also satisfies the conditions that endows it with a unique minimal monotonic adjustment.

**Theorem 5.2.**  $F_{PR} : L(A)^n \rightarrow 2^A \setminus \{\emptyset\}$  admits a unique minimal monotonic adjustment  $\tilde{F}_{PR}$  defined as  $\tilde{F}_{PR}(P) = F_{PR}(P)$  for all  $P \in L(A)^n \setminus \Delta_{F_{PR}}$  and  $\tilde{F}_{PR}(P) = \delta_{F_{PR}}(P)$  for all  $P \in \Delta_{F_{PR}}$ .

Proof: We prove the result through Theorem 4.2, by showing that  $F_{PR}$  satisfies  $\gamma_1$ ,  $\gamma_2$  and  $\gamma_3$ . For  $\gamma_1$ , take any  $x \in A$ , and any  $P, Q, R$  such that  $R \in \text{IMP}_x(P)$  and  $Q$  is between  $P$  and  $R$  for  $x$  while  $x \in F_{PR}(P) \cap F_{PR}(R)$  and  $x \notin F_{PR}(Q)$ . Since  $x \in F_{PR}(P)$  and  $Q \in \text{IMP}_x(P)$ ,  $x \notin F_{PR}(Q)$  implies  $x \in \delta_{F_{PR}}(Q)$  by Theorem 3.1. Thus, by Theorem 5.1,  $\pi_1(Q) = \{x\}$ . Since  $R \in \text{IMP}_x(Q)$ , we have  $\pi_1(R) = \{x\}$  as well. As  $x \in F_{PR}(R)$ , we have  $\delta_{F_{PR}}(R) = \emptyset$  by Proposition 5.3. Therefore,  $\gamma_1$  is satisfied. For  $\gamma_2$ , let  $x \in A$ ,  $P, R \in L(A)^n$  be as in  $\gamma_2$ . Suppose  $\delta_{F_{PR}}(R) \neq \emptyset$ , which implies  $\{y\} = \pi_1(R)$  for some  $y \neq x$  and  $\{x\} \in \pi_2(R)$  by Theorem 5.1. Note that, as  $R \in \text{IMP}_x(P)$ ,  $\{y\} = \pi_1(P)$  and  $\{x\} \in \pi_2(P)$ . Since  $\delta_{F_{PR}}(P) = \emptyset$ , by Proposition 5.3

$$\tau(y, P) - \tau(x, P) > \tau(x, P) - \tau(z, P)$$

$\forall z \in A \setminus \{x, y\}$ . Since  $R \in \text{IMP}_x(P)$ , we have  $\tau(x, R) \geq \tau(x, P)$ ,  $\tau(y, R) \leq \tau(y, P)$  and  $\tau(z, R) \leq \tau(z, P) \forall z \in A \setminus \{x, y\}$ . Therefore,

$$\tau(y, R) - \tau(x, R) > \tau(x, R) - \tau(z, R)$$

which contradicts Proposition 5.3. Therefore,  $\delta_{F_{PR}}(R) = \emptyset$ . So  $\gamma_2$  is satisfied.

As for  $\gamma_3$ , take any  $P \in \Delta_{F_{PR}}$ . By Proposition 5.3,  $\delta_{F_{PR}}(P) = \pi_1(P) = \{x\}$ . If  $\delta_{F_{PR}}(Q) \neq \emptyset$  for all  $Q \in \text{WOR}_x(P)$  then it is not possible to find  $y, z$  that satisfies conditions (i), (ii) and (iii) of Proposition 5.3 which contradicts the proposition. Therefore,  $\exists P' \in \text{WOR}_x(P)$  with  $x \in F(P')$  i.e.,  $\gamma_3$  is satisfied. So all three conditions are met, making  $\tilde{F}_{PR}$  the unique minimal monotonic adjustment of  $F$ . Q.E.D.

$\tilde{F}_{PR}$  is an SCR that suggests to replace the plurality with a runoff winner with the plurality winner at profiles where plurality with a runoff exhibits a monotonicity problem. More precisely,  $\tilde{F}_{PR}$  coincides with plurality with a runoff unless there is a different unique plurality winner that satisfies the three conditions of Theorem 5.1, in which case this plurality winner is the outcome. Thus,  $\tilde{F}_{PR}$  is a combination of plurality with a runoff with plurality that overcomes the non-monotonicity problem of plurality with a runoff. Four properties of  $\tilde{F}_{PR}$  are worth to be mentioned:

- As there can be at most one alternative who goes to a runoff as an untied plurality winner, by Theorems 5.1 and 5.2,  $\tilde{F}_{PR}$  is singleton-valued at every profile where it diverges from plurality with a runoff, thus being sufficiently refined to be of practical use.
- The plurality rule is capable to pick a Condorcet loser (in fact, even an alternative ranked last by a majority), a well-known defect that is recov-

ered by plurality with a runoff.<sup>17</sup>  $\tilde{F}_{PR}$  is also free from this defect and never picks a Condorcet loser, a matter that is ensured by condition (iii) of Theorem 5.1.

- Although plurality with a runoff needs the full ranking of every voter, at the cost of making a two round election, it can be implemented by simply asking voters their best alternative.  $\tilde{F}_{PR}$  also needs the full ranking of every voter while it does not admit an informationally simpler two round version.
- Saari (1990) presents *weak consistency* as a central unifying theme for a wide class of social choice paradoxes. Under weak consistency, if the choice of an SCR  $F$  agrees at two profiles  $P$  and  $Q$  belonging respectively to two disjoint societies  $N$  and  $M$ , then  $F$  must make the same choice at the profile  $P \cup Q$  belonging to the combined society  $N \cup M$ .<sup>18</sup> The failure of weak consistency by plurality with a runoff is not recovered by its minimal monotonic extension nor by its minimal monotonic adjustment. To see this take  $A = \{w, x, y, z, t\}$ , a society  $N$  with 9 voters with the profile  $P$  where 4 voters' preference is  $x \succ w \succ y \succ z \succ t$ , 3 voters' preference is  $y \succ t \succ w \succ x \succ z$ , 3 voters' preference is  $t \succ w \succ x \succ y \succ z$ , and a society  $M$  with 9 voters with the profile  $Q$  where 4 voters' preference is  $x \succ w \succ y \succ z \succ t$ , 3 voters' preference is  $z \succ t \succ w \succ x \succ y$ , 3 voters' preference is  $t \succ w \succ x \succ z \succ y$ . Note that  $F_{PR}(P) = \bar{F}_{PR}(P) = \tilde{F}_{PR}(P) = F_{PR}(Q) = \bar{F}_{PR}(Q) = \tilde{F}_{PR}(Q) = \{x\}$  while  $F_{PR}(P \cup Q) = \bar{F}_{PR}(P \cup Q) = \tilde{F}_{PR}(P \cup Q) = \{t\}$ . Therefore, none of  $F_{PR}$ ,  $\bar{F}_{PR}$  and  $\tilde{F}_{PR}$  are weakly consistent.

## 6 Conclusion

We introduce two concepts of interest in the treatment of SCRs that fail monotonicity. We start with the minimal monotonic extension of an SCR  $F$  which is the monotonic supercorrespondence  $\bar{F}$  of  $F$  that is minimal with respect to set inclusion. Every non-monotonic  $F$  admits a unique minimal monotonic extension  $\bar{F}$ . However,  $\bar{F}$  may be too coarse to be used as an SCR. As a result we suggest to refine it while preserving its monotonicity, a concept that we call a monotonic adjustment of  $F$ . We introduce an SCR  $\tilde{F}$  which agrees with  $F$  at profiles where  $\bar{F}$  agrees with  $F$ . In other words, in case  $\bar{F}$  leaves  $F$  intact,  $\tilde{F}$  does not distort  $F$ . On the other hand, when  $\bar{F}$  adds alternatives to  $F$ ,  $\tilde{F}$  refines  $\bar{F}$  by discarding all outcomes prescribed by  $\bar{F}$  and preserving all outcomes added by  $F$ . We provide conditions under which  $\tilde{F}$  turns out to be the unique minimal monotonic adjustment of  $F$ . To illustrate the use of all this analysis, we consider a well-known non-monotonic SCR, namely plurality with a runoff

<sup>17</sup>Sertel and Kalaycıoğlu (1995) comprehensively discussed this issue while campaigning against the usage of the plurality rule in Turkish political elections.

<sup>18</sup>A more formal and complete description of these concepts is beyond the scope of this paper but can be found in Saari (1990).

and identify its minimal monotonic extension as well as its minimal monotonic adjustment.

As already argued in Section 4,  $\bar{F}$  is an irresolute SCR, thus prescribing at each profile the list of acceptable alternatives from which a final singleton choice will be made. Therefore, in case  $\bar{F}$  is proposed as a solution to the monotonicity failure of  $F$ , subcorrespondences of  $\bar{F}$  could also present a solution in this direction. Moreover, the outcomes added by  $\bar{F}$  at some profile  $P$  are alternatives that are prescribed by  $F$  at some other profile  $P'$  which is a worsening for those alternatives with respect to  $P$ . This observation, combined with adopting monotonicity as a reasonable condition allows to conclude that if  $F$  is “interesting” (in spite of its non-monotonicity) and we wish to render it monotonic, then  $\bar{F}$  is also “interesting”. Thus, substituting a non-monotonic  $F$  with its minimal monotonic adjustment arises as a possibly plausible recovery of the non-monotonicity problem. In fact, plurality with a runoff presents an instance where its minimal monotonic adjustment turns out to be an SCR of interest.

We think several point run off procedures, including the single transferable vote, would be covered by our Theorem 4.2. Moreover, our analysis inspires the derivation a formal index of non-monotonicity based on the shape of  $\delta_F : \Delta_F \rightarrow 2^A \setminus \{\emptyset\}$ . This would allow to compare the degree of non-monotonicity of SCRs, such as plurality with a runoff and the single transferable vote, by computing their minimal monotonic extensions. We leave all these as open questions raised by our analysis.

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