An arbitrary order and pointwise divergence-free finite element scheme for the incompressible 3D Navier-Stokes equations

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AN ARBITRARY ORDER AND POINTWISE DIVERGENCE-FREE FINITE ELEMENT SCHEME FOR THE INCOMPRESSIBLE 3D NAVIER-STOKES EQUATIONS.

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Abstract. In this paper we discretize the incompressible Navier-Stokes equations in the framework of finite element exterior calculus. We make use of the Lamb identity to rewrite the equations into a vorticity-velocity-pressure form which fits into the de Rham complex of minimal regularity. We propose a discretization on a large class of finite elements, including arbitrary order polynomial spaces readily available in many libraries. The main advantage of this discretization is that the divergence of the fluid velocity is pointwise zero at the discrete level. This exactness ensures pressure robustness. We focus the analysis on a class of linearized equations for which we prove well-posedness and provide a priori error estimates. The results are validated with numerical simulations.

Key words. Finite Element, exterior calculus, incompressible Navier-Stokes, Hodge decomposition, de Rham complex, mixed element.

AMS subject classifications. 35Q30, 65N30, 76D07, 76M10.

1. Introduction. Recently much work has been done to design structure preserving methods, but while the construction of such methods was found early on in two dimensions, the three-dimensional case remained difficult and the introduction of the finite element exterior calculus brought a significant breakthrough. An excellent review is given by V. John et al. [17]. The general idea taken from the finite element exterior calculus is to use a subcomplex of the De Rham complex. There are well-known discrete counterparts of this complex with minimal regularity, however the discretization of smoother variants is still an active topic usually leading to shape functions of high degree, for example see [24]. We chose to use the complex with minimal regularity, as this is often done for electromagnetism or recently for magnetohydrodynamics (see [16]).

The main difference from usual schemes lies in the regularity of the velocity field since we only require it to be in $H(\mathrm{div})$ and in the discrete adjoint of $H(\mathrm{curl})$. Although the continuous space regularity is the same as the usual one, since the adjoint of $(\mathrm{curl}, H(\mathrm{curl}))$ is $(\mathrm{curl}, H_0(\mathrm{curl}))$, and the velocity is sought in $H(\mathrm{div}) \cap H_0(\mathrm{curl}) \subset H^1$ (for a smooth enough domain, it is discussed in part 3.2 of [14]). This does not hold (in general) in the discrete case since for $V_h \subset H(\mathrm{curl})$ and (curl^*, V_h^*) the adjoint of (curl, V_h) we no longer have $V_h^* \subset H(\mathrm{curl})$. This has a fundamental impact both from the philosophical and practical point of view. In practice $v \in H(\mathrm{div})$ (resp. $v \in H(\mathrm{curl})$) does not impose continuity of the tangential (resp. normal) components on faces. This suggests that we will not have any degree of freedom corresponding to these and lack any way to set them in a Dirichlet boundary condition. This means that the normal and the tangential part of the boundary condition must be treated in two different ways. It makes more sense in the exterior algebra and means that the fluid velocity is really sought as a 2-form (mostly defined by its flux across cell boundaries) which happens to be regular enough to also be in the domain of the exterior derivative adjoint.

Let us summarize the main idea of the algorithm. In order to preserve the free divergence constraint, we have to consider u as a 2-form, which can be discretized by

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face elements. Then it is not straightforward to discretize the Laplacian in the usual way $\langle \nabla u, \nabla v\rangle$ because ∇u is not a natural quantity for a 2-form. Our simple trick is to use $\nabla \cdot u = 0$ to rewrite the Laplacian:

$$\begin{aligned} \Delta u =& \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) \\ =& -\nabla \times (\nabla \times u). \end{aligned} \tag{1.1}$$

Let Ω be a bounded domain of $\mathbb{R}^3$ and $T > 0$, we recall the Navier-Stokes equations:

$$\begin{aligned} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p = f \text{ on } \Omega \times (0,T), \\ \nabla \cdot u = 0 \text{ on } \Omega \times (0,T) \end{aligned} \tag{1.2}$$

together with some boundary and initial conditions, where u is the velocity of the fluid, p the pressure, ν the kinematic viscosity and f an external force.
Using the Lamb identity $(u \cdot \nabla)u = (\nabla \times u) \times u + \frac{1}{2}\nabla(u\cdot u)$ and Equation (1.1), we get the following formulation

$$\begin{aligned} u_t + (\nabla \times u) \times u + \nu \nabla \times (\nabla \times u) + \nabla P = f \text{ on } \Omega \times (0,T), \\ \nabla \cdot u = 0 \text{ on } \Omega \times (0,T). \end{aligned} \tag{1.3}$$

where $P = p + \frac{1}{2} u \cdot u$ is the Bernoulli pressure.

Since u is a 2-form it is not natural to take $\nabla \times u$ (and it is unadvisable for reasons detailed in Remark 3.3). Therefore, we introduce an auxiliary variable $\omega = \nabla \times u$ (namely the vorticity) and work with a mixed problem. This is known as the vorticity-velocity-pressure formulation and was considered by many others (see [5, 2, 18, 10, 3]). The finite element exterior calculus framework is very flexible because it allows us to work in abstract spaces which can be discretized easily provided that some exactness properties are fulfilled. Therefore we shall use a generic name for our spaces, here $V^1 \times V^2 \times V^3$ for the continuous spaces and $V_h^1 \times V_h^2 \times V_h^3$ for the discrete spaces (indexed by the mesh size h). The exact requirements for these spaces require introducing some concepts and notations, hence for the sake of readability we postpone the definition to Section 2.1. Typically a valid choice is to take $V^1 = H(\mathrm{curl}, \Omega)$, $V^2 = H(\mathrm{div}, \Omega)$ and $V^3 = L^2(\Omega)$. We also need another space $\mathfrak{H}^3 \subset V^3$. This is a vector space that does not only depend on V^3 but on the couple $V^2 \times V^3$ and which is typically of small dimension, i.e. of dimension 0 or 1 in the case of interest to us. From the point of view of exterior calculus $\mathfrak{H}^3$ is the set of harmonic 3-forms, for a more practical point of view $\mathfrak{H}^3$ can be understood as a set of Lagrange multipliers.

Remark 1.1. The choice of boundary conditions is encoded in the choice of $V^1 \times V^2 \times V^3$. More details are given later in Section 2.2.

An example of discrete in time, mixed and linearized weak formulation is:
Given $f^n \in L^2(\Omega)$, find $(\omega^n, u^n, p^n, \phi^n) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$ such that $\forall(\tau, v, q, \chi) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$,

$$\langle \omega^n, \tau\rangle - \langle u^n, \nabla \times \tau\rangle = 0, \tag{1.4a}$$

$$\begin{aligned} \langle \frac{1}{\delta t} u^n, v\rangle + \langle \nu\nabla \times \omega^n + \theta\omega^n \times u^{n-1} + (1-\theta)\omega^{n-1} \times u^n, v\rangle \\ -\langle p^n, \nabla \cdot v\rangle = \langle \frac{1}{\delta t} u^{n-1}, v\rangle + \langle f^n, v\rangle, \end{aligned} \tag{1.4b}$$

$$\langle \nabla \cdot u^n + \phi^n, q\rangle = 0 \tag{1.4c}$$

$$\langle p^n, \chi\rangle = 0. \tag{1.4d}$$

Here we used an implicit Euler discretization in time and consider a sequence of solutions at each time step indexed by an integer $n \geq 1$. The real number $\theta \in [0,1]$ is an arbitrary parameter. In the following we take $\theta = \frac{1}{2}$. Let us look at a specific example in order to give some explanation on Equation (1.4): If we take $V^1 = H(\mathrm{curl}, \Omega)$, $V^2 = H(\mathrm{div}, \Omega)$ and $V^3 = L^2(\Omega)$ then we must have $\mathfrak{H}^3 = \{0\}$. Equation (1.4a) is equivalent to $\omega^n = \nabla \times u^n$ and $u^n \in H_0(\mathrm{curl}, \Omega)$, (1.4b) is equivalent to $\frac{u^n - u^{n-1}}{\delta t} + \nu \nabla \times (\nabla \times u^n) + \frac{1}{2}(\omega^n \times u^{n-1} + \omega^{n-1} \times u^n) + \nabla p^n = f^n$ and $p^n \in H_0^1$, (1.4c) is equivalent to $\nabla \cdot u^n = 0$, and (1.4d) is trivial here (since $\mathfrak{H}^3 = \{0\}$). This formulation is similar to the one studied by Anaya et al. [3]. The main difference is that our formulation is studied in the framework of finite element exterior calculus and for arbitrary low order perturbations (see Equation (1.5) below). In particular the abstraction made on discrete spaces allows using any discrete subcomplex (as defined in Section 2.4). Two families are given as example in Section 2.1 but more exist on different kind of meshes (e.g. the cubical elements [4, Chapter 7.7]). The construction of such families is still an active topic and the independence over the choice of discrete subcomplex is a great feature of finite element exterior calculus allowing to choose any family without modification to the proofs.

Remark 1.2. Equations (1.4c) and (1.4d) might seem odd but in fact are quite natural. From the exterior calculus point of view ϕ and χ are harmonic forms of V^3, in practice they can be viewed as Lagrange multipliers. For the given example $\mathfrak{H}^3$ is trivial but if instead we consider $V^1 = H_0(\mathrm{curl}, \Omega)$, $V^2 = H_0(\mathrm{div}, \Omega)$ and $V^3 = L^2(\Omega)$ then we must have $\mathfrak{H}^3 \approx \mathbb{R}$ (the 1-dimensional vector space of constant functions). Equation (1.4c) ensure that the system is onto, although here the right-hand side is null so ϕ^n will always be zero. Equation (1.4d) ensure that p^n is orthogonal to the space of harmonics 3-forms (i.e. here that $\int_\Omega p^n = 0$).

Abstracting the linearization and time discretization scheme we simply consider two linear maps: l_3 and l_5 defined on $L^2(\Omega) \to L^2(\Omega)$ (the name of which follows the convention of Arnold [7]). And we define the problem:
Given $f_2, f_3 \in L^2(\Omega)$, find $(\omega, u, p, \phi) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$ such that $\forall (\tau, v, q, \chi) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$:

$$
\begin{aligned}
\langle \omega, \tau \rangle - \langle u, \nabla \times \tau \rangle &= 0, \\
\langle \nu \nabla \times \omega + l_3 \omega + l_5 u, v \rangle - \langle p, \nabla \cdot v \rangle &= \langle f_2, v \rangle, \\
\langle \nabla \cdot u, q \rangle + \langle \phi, q \rangle &= \langle f_3, q \rangle, \\
\langle p, \chi \rangle &= 0.
\end{aligned} \tag{1.5}
$$

We easily see that a suitable choice of l_3 and l_5 (namely $l_3 = (v \to \frac{1}{2} v \times u^{n-1})$ and $l_5 = (v \to \frac{1}{\delta t} v + \frac{1}{2}\omega^{n-1} \times v)$ allows recovering (1.4). We redefine the problem in the framework of exterior calculus in (4.8). We also define a discrete counterpart to this problem in (4.14).

Under mild assumptions on l_3, l_5 and Ω detailed in (4.5) we prove the well-posedness of the problem (4.8) (or equivalently (1.5)) and of its discrete counterpart (4.14). If we write (ω, u, p, ϕ) (resp. $(\omega_h, u_h, p_h, \phi_h)$) the solution of (4.8) (resp. (4.14)) then we derive an optimal a priori error estimate on the energy norm proportional to the approximation properties of the discrete spaces used. The result is stated in Corollary 4.17. When $f_3 = 0$ (which is the case for (1.4)) we show that the velocity is exactly divergence free even at the discrete level, i.e. that $\nabla \cdot u = \nabla \cdot u_h = 0$ holds pointwise. We also show that the scheme is pressure-robust (see Section 5.3). This allows to derive an error estimate for the vorticity and velocity independent of the pressure in Theorem 5.5.

The remaining of this paper is divided as follows: We define the notations used in the paper and discuss some applications of the scheme in Section 2. We introduce an intermediary problem akin to a Stokes problem in Section 3. We show well-posedness and derive improved error estimates for this intermediary problem that will be useful in Section 4. Section 4 is dedicated to the analysis of problem (1.5). This is the most technical section. We derive some additional results in Section 5, and finally present a variety of numerical simulations done with our scheme in Section 6 that validate our results and give some perspectives. The exterior calculus formalism is introduced in Section 2 and heavily used in Section 3 and 4. We assume that the reader has some familiarity with exterior calculus in those two sections (3 and 4). However, no prior knowledge of exterior calculus are expected in Section 5 and 6.

2. Setting. The exterior calculus framework allows getting a uniform vision on many objects. As such it may appear abstract and confusing at first. Therefore, we will start by giving some explicit examples of spaces to fix the ideas. We will also discuss how to deal with boundary conditions and how to use the scheme in a 2-dimensional space. Only then we will introduce the full notations and specifications of exterior calculus which will be used in the remaining of the paper.

2.1. Function spaces. Our scheme does not rely on a particular choice of discrete spaces, instead we make some assumptions (given in Section 2.4) on them and any space fulfilling these assumptions can be used. Adequate spaces are readily available on simplicial and cubic meshes (they are given e.g. in the periodic table of finite elements [8]). We illustrate here an example of sensible choice. Since the discrete spaces depend on the continuous one (through boundary conditions) we begin by setting the continuous spaces for this example. Let $V^1 = H_0(\mathrm{curl}, \Omega)$, $V^2 = H_0(\mathrm{div}, \Omega)$, $V^3 = L^2(\Omega)$ and $\mathfrak{H}^3 = \mathbb{R} \subset L^2(\Omega)$. The discrete spaces depend on a parameter $r \in \mathbb{N}$ which is a polynomial degree. Let $\mathbf{T}_h$ be a simplicial triangulation of Ω, we choose the following discretization:

- To define the curl space V_h^1 we use Nedelec's edge elements of the first kind of degree r (or $P_r^- \Lambda^1$ in the periodic table), $V_h^1 = \{\omega \in H_0(\mathrm{curl}, \Omega);\ \omega_{|T} \in P_r^- \Lambda^1(T), \forall T \in \mathbf{T}_h\}$.
- To define the velocity space V_h^2 we use Nedelec's face elements of the first kind of degree r (or $P_r^- \Lambda^2$), $V_h^2 = \{u \in H_0(\mathrm{div}, \Omega);\ u_{|T} \in P_r^- \Lambda^2(T), \forall T \in \mathbf{T}_h\}$.
- To define the pressure space V_h^3 we use discontinuous Galerkin elements of degree $r-1$ (or $P_r^- \Lambda^3$), $V_h^3 = \{p \in L^2(\Omega);\ p_{|T} \in P_r^- \Lambda^3(T), \forall T \in \mathbf{T}_h\}$.
- The space of discrete harmonic forms $\mathfrak{H}_h^3 = \mathbb{R} \subset L^2(\Omega)$ is the L^2-orthogonal complement of $\mathrm{div}(V_h^2)$ in V_h^3.

We can substitute the first kind by the second, at the expense of increasing the polynomial degrees. Nedelec elements of first and second kind are respectively the 3-dimensional equivalent of Raviart-Thomas and of Brezzi-Douglas-Marini elements. The space $\mathfrak{H}_h^3$ is just the natural way of fixing the pressure (usually only defined up to an arbitrary constant when it does not appear in boundary conditions). Figure 2.1 shows the degrees of freedom of this choice of finite elements for $r = 1$ and $r = 2$. The first element shown on the left corresponds to the space V_h^0 that plays no role and will not appear in this paper.

Remark 2.1. In full generality $\mathfrak{H}_h^3$ is simply the space of the discrete harmonic 3-forms. In this case the space of discrete harmonic 3-forms is the same as the space of continuous harmonic 3-forms. If we were to take $V^1 = H_0(\mathrm{curl}, \Omega)$, $V^2 = H_0(\mathrm{div}, \Omega)$

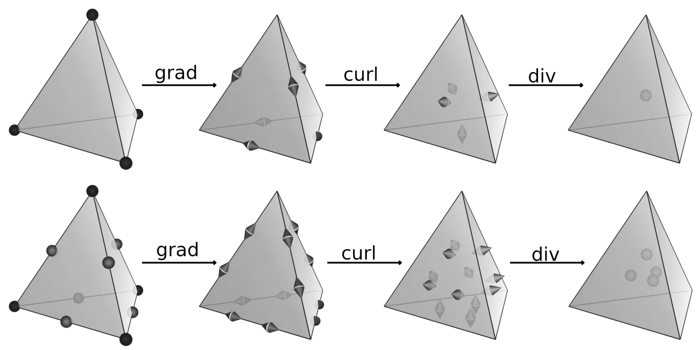

FIG. 2.1. *Degrees of freedom on reference elements for the two lowest polynomial degrees $r = 1$ and $r = 2$.*

and $V^3 = L^2(\Omega)$ instead then we would have $\mathfrak{H}^3 = \mathfrak{H}^3_h = \{0\}$.

For this choice we can make explicit the approximation properties in terms of the size of the mesh h and of the polynomial degree r. For (ω, u, p, ϕ) (resp. $(\omega_h, u_h, p_h, \phi_h)$) the solution of (4.8) (resp. (4.14)) the estimate of Corollary 4.17 reads:

$$\|\omega - \omega_h\|_{H(\mathrm{curl})} + \|u - u_h\|_{H(\mathrm{div})} + \|p - p_h\|_{L^2} + \|\phi - \phi_h\|_{L^2} = \mathcal{O}(h^r).$$

2.2. Boundary conditions. We give a quick review of the boundary conditions readily available. Let n be the unit outward normal to the boundary and g and h arbitrary functions in a suitable space. Equalities below are always understood on $\partial\Omega$ and any combination of the following conditions can be used:

$$\omega \times n = g, \; u \cdot n = h. \tag{2.1}$$
$$\omega \times n = g, \; p = h. \tag{2.2}$$
$$u = g. \tag{2.3}$$
$$u \times n = g, \; p = h. \tag{2.4}$$

The first two conditions are uncommon since they prescribe the tangential trace of the vorticity. The third is the most commonly used since it allows enforcing the no slip condition with $u \cdot n = 0$, $u \times n = 0$.

In order to implement them, we use essential Dirichlet boundary conditions for:
- $\omega \times n = g$ and take test functions in $H_0(\mathrm{curl})$.
- $u \cdot n = g$ and take test functions in $H_0(\mathrm{div})$.

And natural conditions for the other two:
- $u \times n = g$ by adding $-\int_{\partial\Omega} (g \times \tau) \cdot n ds$ to the left-hand side of equations (1.4).
- $p = h$ by adding $\int_{\partial\Omega} hv \cdot n ds$ to the left-hand side of equations (1.4).

Remark 2.2. These are conditions that can be easily implemented. They have the particularity of treating independently the tangential and normal components of the functions. Although they can all be imposed, only condition (2.1) and (2.4) gives a complex. The following proofs are based on the use of complexes, so we assume that one of these two conditions is chosen. When we do not use a complex we still have the well-posedness, however the rate of convergence can be impacted. The case of condition (2.3) which is used to impose the no slip condition is studied by Arnold et al. [5, 9]. He shows that the rate of convergence on the vorticity (for the norm $H(\mathrm{curl})$)

and on the pressure (for the norm L^2) is slightly degraded. However, for a polynomial degree of at least 2 the rate of convergence of the velocity (for the norm $H(\mathrm{div})$) is not impacted.

2.3. Two-dimensional formulation. While everything is expressed in $\mathbb{R}^3$ it is perfectly possible to apply this to a 2-dimensional problem. One shall take the Raviart-Thomas's face elements (and not edge since we wish to preserve a pointwise zero divergence) or Brezzi-Douglas-Marini's face elements for V_h^2. The meaning of ω in (1.4a) is then $\omega = \nabla \times u = \partial_1 u_2 - \partial_2 u_1$ and the expressions appearing in (1.4b) become:

$$\nabla \times \omega = \begin{bmatrix} \partial_2 \omega \\ -\partial_1 \omega \end{bmatrix} \text{ and } \omega \times u = \omega u^\perp = \omega \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix}.$$

Everything said and proved below also works in two dimensions. The independence w.r.t. dimension is a major advantage of exterior calculus formalism.

2.4. Notations. Finally we state the full framework of exterior calculus which will be used. We make extensive use of the following notations introduced by D. Arnold [4]. They are explained in greater detail in the original reference.

- d is the exterior derivative, d^* its adjoint or codifferential,
- $W^0 \to W^1 \to W^2 \to W^3$ is the L^2 de Rham complex of a bounded domain of $\mathbb{R}^3$. We shall only use the last three of them,
- $V^0 \to V^1 \to V^2 \to V^3$ is a dense subcomplex on which the exterior derivative is defined (and not just densely defined). Here we have $H^1(\Omega) \to H(\mathrm{curl}, \Omega) \to H(\mathrm{div}, \Omega) \to L^2(\Omega)$.
- V_k^* is the domain of the adjoint of (d, V^k),
- $\|\cdot\|$ is the L^2-norm for scalar or vector valued functions,
- $\|\cdot\|_V$ is the V norm, defined by $\|\cdot\|_V = \|\cdot\| + \|d \cdot\|$,
- Sometimes we take the norm on $V \cap V^*$, it is given by $\|\cdot\|_V + \|d^* \cdot\|$,
- $V_h^0 \to V_h^1 \to V_h^2 \to V_h^3$ is a discrete subcomplex parametrized by h,
- P_h is the L^2-orthogonal projection on the discrete subcomplex and in general P_A is the L^2-orthogonal projection on A.

We assume that our complexes have the compactness property, which means that the inclusion $V^k \cap V_k^* \subset W^k$ is compact for each k.
We also assume that there exists a cochain projection $\pi_h^k : V^k \to V_h^k$, bounded for the W-norm, uniformly in h.
Each space has the Hodge-decomposition $W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}_k^*$, where $\mathfrak{B}^k = d(V^{k-1})$, $\mathfrak{B}_k^* = d^*(V_{k+1}^*)$ and $\mathfrak{H}^k = \mathfrak{Z}^k \cap (\mathfrak{B}^k)^\perp$, $\mathfrak{Z}^k$ being the kernel of $d : V^k \to V^{k+1}$.

In order to measure the approximation properties of the discrete subspaces we introduce the notation $E = E^k$, defined by:

$$\forall k, \forall \sigma \in V^k, E^k(\sigma) := \inf_{\tau \in V_h^k} \|\sigma - \tau\| \,. \tag{2.5}$$

The choice of spaces introduced in Section 2.1 does indeed verify all these properties with $E = \mathcal{O}(h^r)$ on a dense subset (see [6]). Each space has a Poincaré inequality (see [4, Theorem 4.6]). That is to say, for each k, $\exists c_p^k > 0$ such that $\forall z \in \mathfrak{Z}^{k\perp_{V^k}}$, $\|z\|_{V^k} \leq c_p^k \|dz\|_{L^2}$. Moreover this also holds for the discrete subcomplex with constants bounded by $c_p^k \, \|\pi_h^k\|_{W^k}$ (see [4, Theorem 5.3]). We set $c_p = \max_{k,h} c_p^k \, \|\pi_h^k\|_{W^k}$, so that the Poincaré inequality holds for c_p regardless of the space.

When it is obvious from the context we shall drop the exponent. Hence, when applied to a 1-form $v_1 \in V^1$ we mean $dv_1 := \mathrm{curl}(v_1)$, and when applied to a 2-form

$v_2 \in V^2$ we mean $dv_2 := \mathrm{div}(v_2)$. When we apply an operator (such as π_h, d or d^*) to a product space, we mean to apply it to each component (i.e. $d(v_1, v_2, v_3) := (dv_1, dv_2, 0)$). When we add a suffix $_h$ such as d_h instead of d we refer to the discrete counterpart of the object.
We will often add a numerical suffix such as z_2 for $z \in V^2 \times V^3$, this means we take the V^2 component of z and will be clear from the context.

Usually when dealing with a primal formulation we will use the variable (u, p), and they can indeed be seen as the velocity and pressure. However, when we deal with mixed formulations we will frequently write (u_1, u_2, u_3). Here 1,2 and 3 refer to 1-form, 2-form and 3-form and have nothing to do with components in a frame of the velocity field. Specifically we will have the identification $u_1 = \omega$, $u_2 = u$ and $u_3 = p$.

The symbol $A \lesssim B$ means that there exists a constant $C \in \mathbb{R}_+^*$ independent of A and B (depending solely on few specified parameters) such that $A \leq CB$.

3. Linear steady problem. We first study a simpler problem, analogous to a Stokes problem, which is also closely related to the Hodge-Laplace problem (see [4]). Given $f = (f_2, f_3) \in W^2 \times W^3$, the problem is:
Find (u, p) such that

$$\begin{aligned} \nu \nabla \times (\nabla \times u) + \nabla p &= f_2 \text{ on } \Omega, \\ \nabla \cdot u &= f_3 \text{ on } \Omega. \end{aligned} \tag{3.1}$$

3.1. Primal formulation. We give the primal formulation in order to explain more clearly what we seek in the mixed formulation, and because we shall need this operator in Section 4. Since we have no way to reach the harmonic part of f_2, we must include a second harmonic space (this time of 2-forms).

Definition 3.1. *Let* $D_0 := \{(u,p) \in (V^2 \cap V_2^*) \times V_3^* |\ d^* u \in V^1\}$, $f \in W^2 \times W^3$ *and*

$$L_0 := \begin{bmatrix} \nu dd^* & -d^* \\ d & 0 \end{bmatrix} = \begin{bmatrix} \nu \nabla \times \nabla \times & \nabla \\ \nabla \cdot & 0 \end{bmatrix}. \tag{3.2}$$

The problem is to find $(u, p) \in P_{\mathfrak{H}^\perp} D_0$, such that

$$\forall (v,q) \in P_{\mathfrak{H}^\perp} D_0, \langle L_0(u,p), (v,q) \rangle = \langle f, (v,q) \rangle. \tag{3.3}$$

Remark 3.2. Since $d : V^3 \to 0$, the Hodge decomposition on W^3 reads $W^3 = \mathfrak{B}^3 \oplus \mathfrak{H}^3$. We also have the equality between the V-norm and the W-norm.

3.2. Continuous Well-posedness. Now we introduce the mixed formulation and switch to the exterior calculus notation. Recall that u_2 (resp. u_3) defined below corresponds to u (resp. p) in (3.1). The mixed formulation is characterized by the introduction of the auxiliary variable u_1 corresponding to $\nabla \times u_2$ or $d^* u_2$. The problem reads:
Given $(f_2, f_3) \in W^2 \times W^3$, find $(u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ such that $\forall (v_1, v_2, v_3, \chi_2, \chi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$,

$$\begin{aligned} \langle u_1, v_1 \rangle - \langle u_2, dv_1 \rangle &= 0, \\ \nu \langle du_1, v_2 \rangle - \langle u_3, dv_2 \rangle + \langle \phi_2, v_2 \rangle &= \langle f_2, v_2 \rangle, \\ \langle du_2, v_3 \rangle + \langle \phi_3, v_3 \rangle &= \langle f_3, v_3 \rangle, \\ \langle u_2, \chi_2 \rangle + \langle u_3, \chi_3 \rangle &= 0. \end{aligned} \tag{3.4}$$

The associated bilinear form is noted B_0.

Remark 3.3. There are two important reasons to remove d^* from the formulation. They become clear when we will consider the discrete counterpart of the formulation. The first one is that d^* and d_h^* are barely related, and it is much harder to analyze the difference between them than between d and d_h. The second one is because while d_h is a local operator, d_h^* is in general a global operator. This greatly deteriorates the sparsity pattern of the system.

LEMMA 3.4. *There is $\alpha > 0$, such that $\forall (u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ we have:*

$$\sup_{(v_1,v_2,v_3,\chi_2,\chi_3)} \frac{B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3))}{\|(v_1, v_2, v_3, \chi_2, \chi_3)\|_V} \geq \alpha \|(u_1, u_2, u_3, \phi_2, \phi_3)\|_V.$$

Proof. For any $(u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$, let $\rho_1 \in \mathfrak{B}_1^* \cap V^1$ be such that $d\rho_1 = P_{\mathfrak{B}} u_2$, $\rho_2 \in \mathfrak{B}_2^* \cap V^2$ be such that $d\rho_2 = P_{\mathfrak{B}} u_3 + du_2$. And take $v_1 = \nu u_1 - \frac{\nu}{c_p^2}\rho_1$, $v_2 = P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2$, $v_3 = du_2 - P_{\mathfrak{B}} u_3 + \phi_3$, $\chi_2 = P_{\mathfrak{H}} u_2$, $\chi_3 = P_{\mathfrak{H}} u_3$. The Poincaré inequality gives:

$$\|\rho_1\|_{V^1} \leq c_p \|P_{\mathfrak{B}} u_2\|, \tag{3.5}$$

$$\|\rho_2\|_{V^2} \leq c_p \|P_{\mathfrak{B}} u_3\| + c_p \|du_2\|. \tag{3.6}$$

And we easily see that:

$$\|v_1\|_{V^1} + \|v_2\|_{V^2} + \|v_3\|_{V^3} + \|\chi_2\| + \|\chi_3\| \lesssim \|u_1\|_{V^1} + \|u_2\|_{V^2} + \|u_3\|_{V^3} + \|\phi_2\| + \|\phi_3\|,$$

where the hidden constant only depends on ν and c_p. Using orthogonality of the Hodge decomposition we get:

$$\begin{aligned}
&B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) \\
&\quad = \langle u_1, \nu u_1 - \frac{\nu}{c_p^2}\rho_1 \rangle - \langle u_2, d(\nu u_1 - \frac{\nu}{c_p^2}\rho_1) \rangle \\
&\quad + \nu \langle du_1, P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2 \rangle - \langle u_3, d(P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2) \rangle \\
&\quad + \langle du_2, du_2 - P_{\mathfrak{B}} u_3 + \phi_3 \rangle + \langle u_2, P_{\mathfrak{H}} u_2 \rangle + \langle u_3, P_{\mathfrak{H}} u_3 \rangle \\
&\quad + \langle \phi_2, P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2 \rangle + \langle \phi_3, du_2 - P_{\mathfrak{B}} u_3 + \phi_3 \rangle \\
&\quad = \nu \langle u_1, u_1 \rangle - \frac{\nu}{c_p^2} \langle u_1, \rho_1 \rangle - \nu \langle u_2, du_1 \rangle + \frac{\nu}{c_p^2} \langle u_2, P_{\mathfrak{B}} u_2 \rangle \\
&\quad + \nu \langle du_1, u_2 \rangle + \nu \langle du_1, du_1 \rangle + \langle u_3, P_{\mathfrak{B}} u_3 \rangle + \langle u_3, du_2 \rangle \\
&\quad + \langle du_2, du_2 \rangle - \langle du_2, u_3 \rangle + \langle u_2, P_{\mathfrak{H}} u_2 \rangle + \langle u_3, P_{\mathfrak{H}} u_3 \rangle \\
&\quad + \langle \phi_2, \phi_2 \rangle + \langle \phi_3, \phi_3 \rangle \\
&\quad = \nu \|u_1\|^2 - \frac{\nu}{c_p^2} \langle u_1, \rho_1 \rangle + \frac{\nu}{c_p^2} \|P_{\mathfrak{B}} u_2\|^2 + \nu \|du_1\|^2 + \|P_{\mathfrak{B}} u_3\|^2 \\
&\quad + \|du_2\|^2 + \|P_{\mathfrak{H}} u_2\|^2 + \|P_{\mathfrak{H}} u_3\|^2 + \|\phi_2\|^2 + \|\phi_3\|^2.
\end{aligned} \tag{3.7}$$

Now since $\langle u_1, \rho_1 \rangle \leq \|u_1\| \|\rho_1\| \leq \frac{c_p^2}{2} \|u_1\|^2 + \frac{1}{2c_p^2} \|\rho_1\|^2$, using (3.5) and injecting in the expression (3.7) we get:

$$\begin{aligned}
&B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) \geq \\
&\quad \frac{\nu}{2} \|u_1\|^2 + \frac{\nu}{2c_p^2} \|P_{\mathfrak{B}} u_2\|^2 + \nu \|du_1\|^2 + \|u_3\|^2 + \|du_2\|^2 \\
&\quad + \|P_{\mathfrak{H}} u_2\|^2 + \|P_{\mathfrak{H}} u_3\|^2 + \|\phi_2\|^2 + \|\phi_3\|^2.
\end{aligned}$$

Finally, since $du_3 = 0$, $d\phi_2 = d\phi_3 = 0$ and $\|P_{\mathfrak{B}^*}u_2\| \leq c_p\|dP_{\mathfrak{B}^*}u_2\| = \|du_2\|$ we have:

$$B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) \gtrsim \|(u_1, u_2, u_3, \phi_2, \phi_3)\|_V^2,$$

where the hidden constant depends only on c_p and ν. ∎

LEMMA 3.5. *Given any $(v_1, v_2, v_3, \chi_2, \chi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ there is $(u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ such that*

$$B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) > 0\ .$$

Proof. If $P_{\mathfrak{B}}v_2 \neq 0$ take $u_1 \in \mathfrak{B}_1^*$ such that $du_1 = P_{\mathfrak{B}}v_2$, $\phi_2 = 0$, $u_3 = \phi_3 = 0$. Then if $\langle u_1, v_1\rangle = 0$ take $u_2 = 0$ else take $u_2 = dv_1\frac{\langle u_1, v_1\rangle}{\langle dv_1, dv_1\rangle}$ ($dv_1 \neq 0$ since $P_{\mathfrak{B}^*}v_1 \neq 0$ as $\langle u_1, v_1\rangle \neq 0$). If $P_{\mathfrak{B}}v_2 = 0$, simply take $u_1 = v_1$, u_2 such that $du_2 = P_{\mathfrak{B}}v_3, P_{\mathfrak{B}}u_2 = -dv_1, P_{\mathfrak{H}}u_2 = \chi_2$ (this is possible by the Hodge decomposition), u_3 such that $P_{\mathfrak{B}}u_3 = -dv_2, P_{\mathfrak{H}}u_3 = \chi_3$, $\phi_2 = P_{\mathfrak{H}}v_2$ and $\phi_3 = P_{\mathfrak{H}}v_3$. ∎

Lemma 3.4 together with Lemma 3.5 give the conditions to apply the Babuška–Lax–Milgram theorem. This proves the continuous well-posedness. Moreover, we have:

$$\|u_1\|_{V^1} + \|u_2\|_{V^2} + \|u_3\|_{V^3} + \|\phi_2\|_{V^2} + \|\phi_3\|_{V^3} \leq c\|f\| \tag{3.8}$$

where c depends only on c_p and ν.

3.3. Discrete well-posedness. The discrete problem reads:
Given $(f_2, f_3) \in W^2 \times W^3$, find $(u_{1h}, u_{2h}, u_{3h}, \phi_{2h}, \phi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$ such that $\forall(v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$,

$$B_0((u_{1h}, u_{2h}, u_{3h}, \phi_{2h}, \phi_{3h}), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) = \langle f_2, v_{2h}\rangle + \langle f_3, v_{3h}\rangle. \tag{3.9}$$

Since we have a discrete Poincaré inequality (see [4]) and use a subcomplex we can apply exactly the same proof as in the continuous case, substituting continuous spaces for their discrete counterparts.

Remark 3.6. The only differential operator used in the mixed formulation is d and its discrete counterpart is merely its restriction to the discrete space. However since the discrete space of harmonic forms $\mathfrak{H}_h$ is not necessarily a subspace of $\mathfrak{H}$. When $\mathfrak{H}^2$ or $\mathfrak{H}^3$ is not zero, the method may be non-conforming.

3.4. Error estimate. We will only derive a basic error estimate for a global norm here. First we define:

$$\mu := \max_{k\in\{2,3\}} \sup_{r\in\mathfrak{H}^k, \|r\|=1} \|(I - \pi_h^k)r\|. \tag{3.10}$$

THEOREM 3.7. *Given $(f_2, f_3) \in W^2 \times W^3$, let $(u_1, u_2, u_3, \phi_2, \phi_3)$ be the solution of the continuous problem (3.4) and $(u_{1h}, u_{2h}, u_{3h}, \phi_{2h}, \phi_{3h})$ the solution of the discrete problem (3.9), then for E given by (2.5) it holds:*

$$\begin{aligned}
&\|(u_1 - u_{1h}, u_2 - u_{2h}, u_3 - u_{3h})\|_V + \|(\phi_2 - \phi_{2h}, \phi_3 - \phi_{3h})\| \\
&\quad\lesssim \inf_{v_1\ \in V_h^1} \|u_1 - v_1\|_{V^1} + \inf_{v_2\ \in V_h^2} \|u_2 - v_2\|_{V^2} + E(u_3) + E(\phi_2) + E(\phi_3) \\
&\qquad + \mu(E(P_{\mathfrak{B}}u_2) + E(P_{\mathfrak{B}}u_3)).
\end{aligned}$$

Proof. For all $(v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$ we have:

$$\begin{aligned} B_0((u_1, u_2, u_3, \phi_2, \phi_3), &(v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\ &= \langle f_2, v_{2h} \rangle + \langle f_3, v_{3h} \rangle + \langle u_2, \chi_{2h} \rangle + \langle u_3, \chi_{3h} \rangle \ . \end{aligned}$$

Let $(v_1, v_2, v_3, \chi_2, \chi_3)$ be the V-orthogonal projection of $(u_1, u_2, u_3, \phi_2, \phi_3)$ into their respective discrete spaces. By the continuity of B_0 it holds,
$\forall (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$

$$\begin{aligned} &B_0((u_{1h} - v_1, u_{2h} - v_2, u_{3h} - v_3, \phi_{2h} - \chi_2, \phi_{3h} - \chi_3), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\ &\quad = B_0((u_1 - v_1, u_2 - v_2, u_3 - v_3, \phi_2 - \chi_2, \phi_3 - \chi_3), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\ &\quad - \langle u_2, \chi_{2h} \rangle - \langle u_3, \chi_{3h} \rangle \\ &\quad = B_0((u_1 - v_1, u_2 - v_2, u_3 - v_3, \phi_2 - \chi_2, \phi_3 - \chi_3), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\ &\quad - \langle P_{\mathfrak{H}_h} u_2, \chi_{2h} \rangle - \langle P_{\mathfrak{H}_h} u_3, \chi_{3h} \rangle \\ &\quad \lesssim (\|u_1 - v_1\|_{V^1} + \|u_2 - v_2\|_{V^2} + \|u_3 - v_3\| + \|\phi_2 - \chi_2\| + \|\phi_3 - \chi_3\| \\ &\quad + \|P_{\mathfrak{H}_h} u_2\|_{V^2} + \|P_{\mathfrak{H}_h} u_3\|_{V^3})(\|v_{1h}\|_{V^1} + \|v_{2h}\|_{V^2} + \|v_{3h}\| + \|\chi_{2h}\| + \|\chi_{3h}\|), \end{aligned}$$

where the hidden constant only depends on ν.

From the discrete inf-sup condition we have

$$\begin{aligned} (\|u_{1h} - v_1\|_{V^1} + \|u_{2h} - v_2\|_{V^2} &+ \|u_{3h} - v_3\| + \|\phi_{2h} - \chi_2\| + \|\phi_{3h} - \chi_3\|) \\ &\lesssim (\|u_1 - v_1\|_{V^1} + \|u_2 - v_2\|_{V^2} + \|u_3 - v_3\| + \|\phi_2 - \chi_2\| \\ &\quad + \|\phi_3 - \chi_3\| + \|P_{\mathfrak{H}_h} u_2\|_{V^2} + \|P_{\mathfrak{H}_h} u_3\|), \end{aligned}$$

where the hidden constant only depends on ν and the discrete constant of Poincaré.

The theorem follows from

$$\|P_{\mathfrak{H}_h} u_i\|_V \lesssim \mu E(P_{\mathfrak{B}} u_i), \forall i \in \{2, 3\}.$$

This is proved in Theorem 5.2 of [4]. ∎

Remark 3.8. $E(\phi_i)$ is understood as viewing ϕ_i as an element of $V^i \supset \mathfrak{H}^i$,

$$E(\phi_i) = \inf_{q \in V_h^i} \|\phi_i - q\|.$$

4. Linearized problem. Starting from the linear steady problem (3.4), we add some terms of lower order since this will allow us to construct our scheme.
Again we recall the correspondence with the names used for variables in the introduction: $(u_1, u_2, u_3, u_p) \equiv (\omega, u, p, \phi)$ in (1.5). To keep the notations bearable we shall also write $u_{\mathfrak{B}} = P_{\mathfrak{B}} u$, $u_{\mathfrak{H}} = P_{\mathfrak{H}} u$ and so on.

We consider two linear maps $l_3 : W^1 \to W^2$ and $l_5 : W^2 \to W^2$ (we chose these names to match those used by D. Arnold [7]). Define $D = \{(u, p) \in (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp}) | \ d^* u \in V^1\}$, $W = W^2 \times (W^3 \cap \mathfrak{H}^{3\perp})$,

$$L := \begin{bmatrix} (\nu d + l_3) d^* + l_5 & -d^* \\ d & 0 \end{bmatrix}. \tag{4.1}$$

We consider the primal problem:
Given $f \in W$, to find $(u, p) \in D$ such that

$$L(u, p) = f. \tag{4.2}$$

We also define the dual operator L' on $D' = \{(u,p) \in (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp}) |\ (\nu d^* + l_3^*)u \in V^1\}$ by:

$$L' := \begin{bmatrix} d(\nu d^* + l_3^*) + l_5^* & d^* \\ -d & 0 \end{bmatrix} \tag{4.3}$$

As an intermediary step, we wish to extend L on a larger domain and introduce the following notations:

$$\begin{aligned} \overline{L_\lambda} &: (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp}) \to (V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))', \\ \overline{L_\lambda}(u,p)(v,q) &:= \langle \nu d^* u, d^* v\rangle + \langle l_3 d^* u + l_5 u - \lambda d^* p, v\rangle + \langle \lambda du, q\rangle, \\ \overline{L'_\lambda} &: (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp}) \to (V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))', \\ \overline{L'_\lambda}(u,p)(v,q) &:= \langle \nu d^* u + l_3^* u, d^* v\rangle + \langle l_5^* u + \lambda d^* p, v\rangle + \langle -\lambda du, q\rangle. \end{aligned} \tag{4.4}$$

Here λ is a positive parameter introduced to simplify the proof of Theorem 4.1. In the Theorem 4.1 we shall see that they are almost always isomorphisms. We define the solution operator $K = (\overline{L'_1})^{-1}$, and we assume that

$$d^*(\overline{L_1}^{-1})_2(W) \subset V^1, (\nu d^* + l_3^*)(K)_2(W) \subset V^1 \tag{4.5}$$

where $(\overline{L_1}^{-1})_2$ and $(K)_2$ are the projections on the first component of the product space taken after the operators. Moreover, we assume that $\|dd^*(K)_2\|_{W\to W^2}$ and $\|dl_3^*(K)_2\|_{W\to W^2}$ are bounded. We show in Section 5.1 that these assumptions are satisfied when l_3 and l_5 are those used in our scheme.

The proof follows the same outline as [7]. First we prove that the continuous primal formulation gives an isomorphism, then we prove that the continuous mixed formulation is well-posed. Lastly we prove the well-posedness of the discrete mixed formulation and give an estimation of the error in energy norm.

4.1. Continuous primal formulation.

THEOREM 4.1. *$\overline{L_\lambda} + \mu\langle\cdot,\cdot\rangle$ is a bounded isomorphism for all $\mu \in \mathbb{C}$ except for a countable subset.*

Proof. Let $c = max(\|l_3\|, \|l_5\|)$, for $(u,p) \in (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp})$ take $v_{\mathfrak{B}} = u_{\mathfrak{B}}$, $v_{\mathfrak{H}} = u_{\mathfrak{H}}$, $v_{\mathfrak{B}^*} = -d^*p$, $q = du$. We have:

$$\begin{aligned} (\mu\langle\cdot,\cdot\rangle &+ \overline{L_\lambda})(u,p)(v,q) \\ &= \nu\langle d^*u, d^*u\rangle + \lambda\langle d^*p, d^*p\rangle + \lambda\langle du, du\rangle \\ &\quad + \langle l_3 d^* u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle + \langle l_5 u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle \\ &\quad + \mu\langle u, u_{\mathfrak{B}} + u_{\mathfrak{H}}\rangle - \mu\langle u, d^*p\rangle + \mu\langle p, du\rangle \\ &= \nu\|d^*u\|^2 + \lambda\|d^*p\|^2 + \lambda\|du\|^2 + \mu\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 \\ &\quad + \langle l_3 d^* u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle + \langle l_5 u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle. \end{aligned} \tag{4.6}$$

With the Cauchy–Schwarz inequality and the identity $ab \le 2^{-1}(a^2+b^2)$ we bound the last line from (4.6):

$$\begin{aligned} |\langle l_3 d^* u, &u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle| + |\langle l_5 u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle| \\ &\le \frac{c^2}{2\nu}(\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 + \|d^*p\|^2) + \frac{\nu}{2}\|d^*u\|^2 \\ &\quad + \frac{c^2}{2}(\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 + \|d^*p\|^2) + \frac{1}{2}\|u\|^2. \end{aligned}$$

Since $\|u_{\mathfrak{B}^*}\|^2 \leq c_p^2\|du\|^2$ and $\|u\|^2 = \|u_{\mathfrak{B}^*}\|^2 + \|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2$ we have:

$$(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) \geq (\mu - \frac{c^2}{2\nu} - \frac{c^2}{2} - \frac{1}{2})\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 + \frac{\nu}{2}\|d^*u\|^2$$
$$+ (\lambda - \frac{c^2}{2\nu} - \frac{c^2}{2})\|d^*p\|^2 + (\lambda - \frac{c_p^2}{2})\|du\|^2.$$

Using once again the Poincaré inequality to bound $\|u_{\mathfrak{B}^*}\|$ by $\|du\|$ and $\|p_{\mathfrak{B}}\|$ by $\|d^*p\|$ (on the dual complex), and since $\|p_{\mathfrak{B}}\| = \|p\|$ (as $p \in V^3 \cap \mathfrak{H}^{3\perp}$) we have for λ and μ large enough (solely depending on c, ν and c_p):

$$\begin{aligned}(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) &\gtrsim \|u\|^2 + \|du\|^2 + \|d^*u\|^2 + \|p\|^2 + \|d^*p\|^2\\ &\gtrsim \|(u,p)\|^2_{V^2\cap V_2^*\times V_3^*}.\end{aligned}$$

Clearly $\|(v,q)\|_{V_2^*\times W^3} \leq \|(u,p)\|_{V^2\cap V_2^*\times V_3^*}$ as $\|d^*v\| = \|d^*u_{\mathfrak{B}}\| \leq \|d^*u\|$ and $(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})$ is continuous as a bilinear form from $(V^2\cap V_2^*\times V_3^*)\times(V_2^*\times(W^3\cap\mathfrak{H}^{3\perp}))$. The only thing left to show in order to use the Babuška–Lax–Milgram theorem is the second condition. For any $(v,q)\neq 0$ we must find (u,p) such that $(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) > 0$. We can take λ and μ such that $\lambda^2 = \mu\nu$. We consider two cases: When $v \neq 0$, we can find $u \in \{w \in V^2\cap V_2^* | d^*w \in V^1, dw \in V_3^*\}$ such that $((\nu d + l_3)d^* + l_5 + \mu + \nu d^*d)u = v$ (see [7]). Take $p = -\nu/\lambda du$, then since $\lambda = \mu\nu/\lambda$ it holds:

$$\begin{aligned}(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) &= \langle\nu dd^*u, v\rangle + \langle(l_3d^* + l_5 + \mu)u, v\rangle - \lambda\langle d^*p, v\rangle\\ &\quad + \lambda\langle du, q\rangle + \mu\langle p, q\rangle\\ &= \langle((\nu d + l_3)d^* + l_5 + \mu + \nu d^*d)u, v\rangle\\ &\quad + \lambda\langle du, q\rangle - \frac{\mu\nu}{\lambda}\langle du, q\rangle\\ &= \langle v, v\rangle.\end{aligned}$$

When $v = 0$ take u solution of the Hodge-Dirac problem (see [20]): $du = q$, $d^*u = 0$. Then

$$(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,0)(0,q) = \lambda\langle du, q\rangle = \lambda\langle q, q\rangle > 0.$$

This shows that $(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})$ is a bounded isomorphism from $(V^2\cap V_2^*)\times(V_3^*\cap\mathfrak{H}^{3\perp})$ to $(V_2^*\times(W^3\cap\mathfrak{H}^{3\perp}))'$. Since $I : (V^2\cap V_2^*)\times(V_3^*\cap\mathfrak{H}_3^{\perp}) \to (V_2^*\times(W^3\cap\mathfrak{H}^{3\perp}))'$ is compact by the compactness property, $I(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})^{-1}$ is also compact. Since the spectrum of a compact operator is at most countable, we have that $Id + \eta I(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})^{-1}$ has a bounded inverse for all $\eta \in \mathbb{C}$ except for a countable subset. Therefore, by composing to the right with $\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda}$ we get that $\overline{L_\lambda} + (\mu + \eta)I$ has almost always a bounded inverse. ∎

Hence, up to an arbitrary small perturbation, $\overline{L_\lambda}$ is a bounded isomorphism from $(V^2\cap V_2^*)\times(V_3^*\cap\mathfrak{H}_3^{\perp})$ to $(V_2^*\times(W^3\cap\mathfrak{H}^{3\perp}))'$.

Remark 4.2. We could have left $\mathfrak{H}^3$ in the domain and the proof above would still work. However, in this case, $\overline{L_\lambda}$ would never have been an isomorphism since its image cannot reach $\mathfrak{H}^3$.

We have the same result for the dual problem.

LEMMA 4.3. *For all $\mu \in \mathbb{C}$ except for a countable subset, $\overline{L'_\lambda} + \mu\langle\cdot,\cdot\rangle$ is a bounded isomorphism.*

Proof. The same proof as the one of Theorem 4.1 works. The only differences will be a sign in the chosen (v, q), l_5 and l_3 substituted by l_5^* and l_3^* and $(l_3 d^* u, v)$ changed to $(l_3^* u, d^* v)$, which does not add any difficulty in the proof. ∎

Remark 4.4. The proof Theorem 4.1 requires taking λ to be sufficiently large, however for any $f_2 \in {V_2^*}'$, $f_3 \in (W^3 \cap \mathfrak{H}^{3\perp})'$, $\lambda_0 > 0$, $\lambda_1 > 0$ we have the following equivalence:

$$\overline{L_{\lambda_0}}(u, p) = (f_2, f_3) \Leftrightarrow \overline{L_{\lambda_1}}(u, \frac{\lambda_0}{\lambda_1} p) = (f_2, \frac{\lambda_1}{\lambda_0} f_3) .$$

Therefore, if $\overline{L_{\lambda_0}}$ is a bounded isomorphism for a given λ_0, it easily follows that $\overline{L_{\lambda_1}}$ is a bounded isomorphism for any $\lambda_1 > 0$, in particular for $\lambda_1 = 1$. The same argument works for $\overline{L'_\lambda}$.

From here onward, we assume that $\overline{L_1}$ and $\overline{L'_1}$ are bounded isomorphisms.

4.2. Well-posedness of the continuous mixed formulation. As we did in the unperturbed case, we introduce an auxiliary variable in the problem. In the following, $\mathbf{u}$ is a shortcut for (u_1, u_2, u_3, u_p).

We define B by:

$$\begin{aligned} B(\mathbf{u}, \mathbf{v}) := {} & \langle u_1, v_1\rangle - \langle u_2, dv_1\rangle \\ & + \nu\langle du_1, v_2\rangle - \langle u_3, dv_2\rangle + \langle du_2, v_3\rangle \\ & + \langle l_3 u_1, v_2\rangle + \langle l_5 u_2, v_2\rangle + \langle u_p, v_3\rangle + \langle u_3, v_p\rangle. \end{aligned} \tag{4.7}$$

The mixed formulation is:
Given $(f_2, f_3) \in W$, find $\mathbf{u} = (u_1, u_2, u_3, u_p) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$ such that $\forall \mathbf{v} = (v_1, v_2, v_3, v_p) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$,

$$B(\mathbf{u}, \mathbf{v}) = \langle f_2, v_2\rangle + \langle f_3, v_3\rangle. \tag{4.8}$$

For (u, p) solution of (4.2), it immediately appears that $(d^* u, u, p, 0)$ solves (4.8). Now for (u_1, u_2, u_3, u_p) solution of 4.8, the first line implies that $u_2 \in V_2^*$, $d^* u_2 = u_1$ and therefore $d^* u_2 \in V^1$. The second line implies that $u_3 \in V_3^*$. And the last implies that $u_3 \perp \mathfrak{H}^3$. Therefore $(u_2, u_3) \in D$, and it obviously solves (4.2).

THEOREM 4.5. *Under the regularity assumption (4.5), Problem (4.8) is well-posed.*

The proof follows from the Babuška–Lax–Milgram theorem along with Lemma 4.8 and inf-sup condition (4.13). In the following hidden constants only depend on $\|l_3\|$, $\|l_5\|$, $\|K\|$, $\|d(d^* + l_3^*)(K)_2\|$, ν and on constants of Poincaré. We shall write $V = V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$.

LEMMA 4.6. *For all $(0, u_2, u_3, 0) = \mathbf{u} \in V$, there exists $\mathbf{z} \in V$ such that $\|\mathbf{z}\|_V \lesssim \|\mathbf{u}\|_V$ and $\forall \omega \in V, B(\omega, \mathbf{z}) = \langle \omega, \mathbf{u}\rangle$.*

Proof. Take $(z_2, z_3) = K(u_2, P_{\mathfrak{H}^\perp} u_3)$ and $\xi = -(\nu d^* + l_3^*) z_2$. Then $\xi \in V^1$ by assumption (4.5) and by definition of K we have $\forall \omega \in V$:

$$\begin{aligned} -\langle dz_2, \omega_3\rangle &= \langle P_{\mathfrak{H}^\perp} u_3, \omega_3\rangle, \\ \langle d(\nu d^* + l_3^*) z_2 + l_5^* z_2 + d^* z_3, \omega_2\rangle &= \langle u_2, \omega_2\rangle. \end{aligned} \tag{4.9}$$

We set $\mathbf{z} := (\xi, z_2, z_3, u_{3\mathfrak{H}})$. Applying (4.9) we have:

$$\begin{aligned}
B(\omega, \mathbf{z}) =& \langle \omega_1, \xi\rangle - \langle \omega_2, d\xi\rangle \\
&+ \langle (\nu d + l_3)\omega_1, z_2\rangle - \langle \omega_3, dz_2\rangle + \langle l_5\omega_2, z_2\rangle \\
&+ \langle d\omega_2, z_3\rangle + \langle \omega_3, u_{3\mathfrak{H}}\rangle + \langle \omega_p, z_3\rangle \\
=& \langle \omega_1, \xi\rangle - \langle \omega_2, d\xi\rangle \\
&+ \langle \omega_1, (\nu d^* + l_3^*)z_2\rangle + \langle \omega_3, P_{\mathfrak{H}^\perp} u_3\rangle + \langle \omega_2, l_5^* z_2\rangle \\
&+ \langle \omega_2, d^* z_3\rangle + \langle \omega_3, u_{3\mathfrak{H}}\rangle \\
=& \langle \omega_1, \xi\rangle - \langle \omega_2, d\xi\rangle \\
&+ \langle \omega_1, (\nu d^* + l_3^*)z_2\rangle - \langle \omega_2, d(\nu d^* + l_3^*)z_2\rangle \\
&+ \langle \omega_2, u_2\rangle + \langle \omega_3, P_{\mathfrak{H}^\perp} u_3\rangle + \langle \omega_3, u_{3\mathfrak{H}}\rangle \\
=& -\langle \omega_1, (\nu d^* + l_3^*)z_2\rangle + \langle \omega_2, d(\nu d^* + l_3^*)z_2\rangle \\
&+ \langle \omega_1, (\nu d^* + l_3^*)z_2\rangle - \langle \omega_2, d(\nu d^* + l_3^*)z_2\rangle \\
&+ \langle \omega_3, u_3\rangle + \langle \omega_2, u_2\rangle \\
=& \langle \omega_2, u_2\rangle + \langle \omega_3, u_3\rangle.
\end{aligned}$$

Moreover since $\overline{L_1'}$ is a bounded isomorphism, so is K thus

$$\begin{aligned}
\|\mathbf{z}\|_V &\leq \|\nu d^* z_2\| + \|l_3^* z_2\| + \|z_2\|_{V^2} + \|z_3\| + \|d\xi\| + \|u_{3\mathfrak{H}}\| \\
&\lesssim \|z_2\|_{V^2 \cap V_2^*} + \|z_3\|_{V_3^*} + \|d\xi\| + \|u_{3\mathfrak{H}}\| \\
&\lesssim (\|K\| + \|d(d^* + l_3^*)K\| + 1)\|\mathbf{u}\| \lesssim \|\mathbf{u}\|_V.
\end{aligned}$$

∎

LEMMA 4.7. *For all $\mathbf{u} \in V$, there exists $\mathbf{z} \in V$ such that $\|\mathbf{z}\|_V \lesssim \|\mathbf{u}\|_V$ and $B(\mathbf{u}, \mathbf{z}) \gtrsim \|du_1\|^2 + \|du_2\|^2 + \|u_1\|^2 + \|u_p\|^2 - \|u_2\|^2$.*

Proof. Let $c = \max(\|l_3\|, \|l_5\|)$. We begin with some preliminary computations:

$$\begin{aligned}
B(\mathbf{u}, (\nu u_1, u_{2\mathfrak{B}}, du_2, 0)) =& \nu\langle u_1, u_1\rangle - \nu\langle u_2, du_1\rangle + \nu\langle du_1, u_{2\mathfrak{B}}\rangle \\
&- \langle u_3, du_{2\mathfrak{B}}\rangle + \langle du_2, du_2\rangle + \langle l_3 u_1, u_{2\mathfrak{B}}\rangle \\
&+ \langle l_5 u_2, u_{2\mathfrak{B}}\rangle + \langle u_p, du_2\rangle + \langle u_3, 0\rangle \\
\geq& \frac{\nu}{2}\|u_1\|^2 + \|du_2\|^2 - \left(\frac{c^2}{2\nu} + c\right)\|u_2\|^2,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
B(\mathbf{u}, (0, du_1, 0, 0)) =& \nu\langle du_1, du_1\rangle + \langle l_3 u_1, du_1\rangle + \langle l_5 u_2, du_1\rangle \\
\geq& \frac{1}{2}\nu\|du_1\|^2 - \frac{c^2}{\nu}(\|u_1\|^2 + \|u_2\|^2),
\end{aligned} \tag{4.11}$$

$$B(\mathbf{u}, (0, 0, u_p, 0)) = \langle u_p, u_p\rangle. \tag{4.12}$$

Clearly it is possible to construct a suitable $\mathbf{z}$ from a linear combination of (4.10), (4.11) and (4.12). Bounds on norms are easily checked, for example:

$$\|(0, du_1, 0, 0)\|_V = \|du_1\| + \|ddu_1\| = \|du_1\| \lesssim \|\mathbf{u}\|_V.$$

∎

Combining the two preceding lemmas gives

$$\forall \mathbf{u} \in V, \sup_{\|\mathbf{v}\|_V = 1} |B(\mathbf{u}, \mathbf{v})| \gtrsim \|\mathbf{u}\|_V\ . \tag{4.13}$$

LEMMA 4.8. *For any* $\mathbf{v} \in V$*, there is* $\mathbf{u} \in V$ *such that* $B(\mathbf{u}, \mathbf{v}) > 0$.

Proof. Given $\mathbf{v} \neq 0 \in V$, if $v_2 = 0$, $v_3 = 0$ and $v_p = 0$ take $\mathbf{u} = (v_1, 0, 0, 0)$ and

$$B(\mathbf{u}, \mathbf{v}) = \langle v_1, v_1 \rangle > 0.$$

Else take $(u_2, u_3) = \overline{L_1}^{-1}(v_2, P_{\mathfrak{H}^\perp} v_3) + (0, v_p)$, $u_p = P_{\mathfrak{H}} v_3$ and $u_1 = d^* u_2$ ($u_1 \in V^1$ by assumption (4.5)) then

$$B(\mathbf{u}, \mathbf{v}) = \langle v_2, v_2 \rangle + \langle v_3, v_3 \rangle + \langle q, q \rangle > 0. \qquad \square$$

4.3. Discrete well-posedness. We introduce the notation $V_h = V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^3$. The discrete variational problem is the same as the continuous, substituting V by V_h. Hence we shall still use the notation B, this time as a function from $V_h \times V_h$ to $\mathbb{R}$. So that the discrete problem is:
Given $(f_2, f_3) \in W$, find $\mathbf{u_h} = (u_{1h}, u_{2h}, u_{3h}, u_{ph}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^3$ such that $\forall \mathbf{v_h} = (v_{1h}, v_{2h}, v_{3h}, v_{ph}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^3$,

$$B(\mathbf{u_h}, \mathbf{v_h}) = \langle f_2, v_{2h} \rangle + \langle f_3, v_{3h} \rangle. \tag{4.14}$$

Considering the dual problem to the unperturbed problem with $\nu = 1$. We have: $D_0 = \{(u, p) \in (V^2 \cap V_2^*) \times V_*^3 |\ d^* u \in V^1\}$ and $L_0'(u, p) = (dd^* u + d^* p, -du)$. Let K_0 be the solution operator of the dual problem L_0'. We have $K_0 = (L_0')^{-1}$ when L_0' is viewed as an isomorphism from $P_{\mathfrak{H}^\perp} D_0$ to $P_{\mathfrak{H}^\perp}(W^2 \times W^3)$ and K_0 is extended by 0 on $\mathfrak{H}$. Explicitly we have the decomposition: $\forall (f_2, f_3) \in W^2 \times W^3$,

$$(f_2, f_3) = (dd^*(K_0)_2(f_2, f_3) + d^*(K_0)_3(f_2, f_3), -d(K_0)_2(f_2, f_3)) + (P_{\mathfrak{H}} f_2, P_{\mathfrak{H}} f_3) \tag{4.15}$$

and a similar expression for their discrete counterparts.
Therefore, $\forall (z_2, z_3) \in D_0, (P_{\mathfrak{H}^\perp} z_2, P_{\mathfrak{H}^\perp} z_3) = L_0' K_0(z_2, z_3) = K_0 L_0'(z_2, z_3)$.

As mentioned before this problem is closely related to the one studied by D. Arnold [7]. Since the mixed variable part is almost unchanged we shall use the generalized canonical projection Π_h (see [7]) and we state its properties below.

LEMMA 4.9. *Under the condition of [7, Theorem 5.1]:*
- Π_h *is a projection uniformly bounded in the V-norm.*
- $d\Pi_h = P_{\mathfrak{B}_h} d$.
- $\forall w \in V^k, \|(I - \Pi_h) w\| \lesssim \|(I - \pi_h) w\| + \eta_0' \|dw\|$.
- $\forall w, v \in V^k, |\langle (I - \Pi_h) w, v \rangle| \lesssim (\|(I - \pi_h) w\| + \eta_0' \|dw\|)(\|(I - \pi_h) v\| + \eta_0' \|dv\|) + \alpha_0' \|dw\| \|dv\|$.

where $\eta_0', \alpha_0' \to 0$ when $h \to 0$, they are given, along the proof in the reference [7].

DEFINITION 4.10. *We shall use the following notations in this section:*

$$\delta_0 = \|(I - \pi_h) K_0\|, \mu_0 = \|(I - \pi_h) P_{\mathfrak{H}}\|,$$
$$\eta_0 = \max\{\|(I - \pi_h) dK_0\|, \|(I - \pi_h) d^*(K_0)_2\|\},$$
$$\alpha_0 = \eta_0(1 + \eta_0) + \mu_0 + \delta_0 + \mu_0 \delta_0 + \eta_0',$$
$$\eta = \max\{\delta_0, \mu_0, \eta_0, \|(I - \pi_h) l_3^*(K)_2\|, \|(I - \pi_h) dl_3^*(K)_2\|\}.$$

LEMMA 4.11. *We have:*

$$\|K_0 - K_{0h} P_h\| \lesssim \alpha_0,$$
$$\|dK_0 - dK_{0h} P_h\| + \|d^*(K_0)_2 - d_h^*(K_{0h})_2 P_h\| \lesssim \eta_0.$$

Proof. The idea is for $(f_2, f_3) \in (W^2 \times W^3)$ to apply the error estimate of Theorem 3.7 for $(u_2, u_3) = K_0(f_2, f_3)$, $(\phi_2, \phi_3) = P_{\mathfrak{H}}(f_2, f_3)$, $u_1 = d^* u_2$, $(u_{2h}, u_{3h}) = K_{0h} P_h(f_2, f_3)$, $(\phi_{2h}, \phi_{3h}) = P_{\mathfrak{H}_h} P_h(f_2, f_3)$, $u_{1h} = d_h^* u_{2h}$. Unfortunately we cannot conclude with the crude estimate of Theorem 3.7 because of the error on u_1. We need improved estimates that give

$$\begin{aligned} \|u_2 - u_{2h}\| + \|u_3 - u_{3h}\| &\lesssim (1+\mu_0)E(u_2) + E(u_3) + \eta_0 E(u_1) \\ &\qquad (\eta_0^2 + \delta_0 + \eta_0')E(du_1) + \eta_0' E(\phi_2) + E(du_2), \\ \|du_2 - du_{2h}\| + \|u_1 - u_{1h}\| &\lesssim E(du_2) + E(u_1) + \eta_0 E(du_1). \end{aligned}$$

We conclude since $E(u_2) + E(u_3) \leq \delta_0 \|(f_2, f_3)\|$, $E(u_1) + E(du_2) \leq \eta_0 \|(f_2, f_3)\|$ and $E(du_1) + E(\phi_2) \leq \|(f_2, f_3)\|$, the last coming from $du_1 = P_{\mathfrak{B}} f_2$. These proofs are lengthy, technical and mostly follow those in theorem 3.11 of [6]. ∎

Lemma 4.12. *For $f \in \mathfrak{H}^\perp$ we have $\|P_{\mathfrak{H}_h} f\| \lesssim \mu_0 \|f\|$*

Proof. We recall the mixed formulation for the Hodge-Laplacian problem. The bilinear form is given by:

$$B((\sigma, u, p), (\tau, v, q)) = \langle \sigma, \tau \rangle - \langle u, d\tau \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle + \langle u, q \rangle.$$

In the continuous case the bilinear form acts on $(V^1 \times V^2 \times \mathfrak{H}^2)^2$ and on $(V_h^1 \times V_h^2 \times \mathfrak{H}_h^2)^2$ in the discrete case. Let $(\sigma, u, p) \in V^1 \times V^2 \times \mathfrak{H}^2$ be such that $\forall (\tau, v, q) \in V^1 \times V^2 \times \mathfrak{H}^2$, $B((\sigma, u, p), (\tau, v, q)) = (f, v)$, and let $(\sigma_h, u_h, p_h) \in V_h^1 \times V_h^2 \times \mathfrak{H}_h^2$ be such that $\forall (\tau, v, q) \in V_h^1 \times V_h^2 \times \mathfrak{H}_h^2$, $B((\sigma_h, u_h, p_h), (\tau, v, q)) = (f, v)$. Then [4, Theorem 5.6 p. 62] gives the error estimate $\|p - p_h\| \lesssim E(p) + \mu_0 E(d\sigma)$. We have $P_{\mathfrak{H}} f = p = 0$ and $P_{\mathfrak{H}_h} f = p_h$ thus

$$\|P_{\mathfrak{H}_h} f\| = \|p - p_h\| \lesssim 0 + \mu_0 E(d\sigma) \lesssim \mu_0 \|f\|$$

since $E(p) = 0$ and by the well-posedness of the Hodge-Laplacian problem for the last inequality. ∎

Theorem 4.13. *For $z = (z_2, z_3) \in D_0$, let $z_h = (z_{2h}, z_{3h}) = K_{0h} P_h L_0' z + P_{\mathfrak{H}_h} P_{\mathfrak{H}} z$, we have:*

$$\|z - z_h\| \lesssim \alpha_0 \|L_0' z\|, \|d(z - z_h)\| + \|d^* z_2 - d_h^* z_{2h}\| \lesssim \eta_0 \|L_0' z\|,$$

$$\|P_h(dd^* z_2 + d^* z_3) - (dd_h^* z_{2h} + d_h^* z_{3h})\| \leq \mu_0 \|L_0' z\|.$$

Proof. The same proof as [7, Theorem 5.2] works. It is duplicated here since the proof is short and demonstrates that our change in the definition of L_0' does not interfere.

$$\begin{aligned} z - z_h &= (P_{\mathfrak{H}^\perp} z - P_{\mathfrak{H}_h^\perp} z_h) + (P_{\mathfrak{H}} z - P_{\mathfrak{H}_h} z_h) \\ &= (K_0 - K_{0h} P_h) L_0' z + (I - P_{\mathfrak{H}_h}) P_{\mathfrak{H}} z. \end{aligned}$$

As $P_{\mathfrak{H}_h} P_{\mathfrak{H}} = P_{\mathfrak{Z}_h} P_{\mathfrak{H}}$ and $\pi_h \mathfrak{Z} \subset \mathfrak{Z}_h$ we have:

$$\|(I - P_{\mathfrak{H}_h}) P_{\mathfrak{H}} z\| \leq \|(I - \pi_h) P_{\mathfrak{H}} z\| \leq \mu_0 \|z\| \lesssim \alpha_0 \|L_0' z\|.$$

We then get the expected result by Lemma 4.11. The second part follows directly from Lemma 4.11. Finally for the last estimate, (4.15) gives

$$dd_h^* z_{2h} + d_h^* z_{3h} = (L_{0h}')_2 z_h = (L_{0h}' K_{0h} P_h)_2 L_0' z = (P_{\mathfrak{B}_h} + P_{\mathfrak{B}_h^*}) P_h (L_0')_2 z$$

so:

$$\begin{aligned}\|P_h(dd^*z_2 + d^*z_3) - (dd_h^*z_{2h} + d_h^*z_{3h})\| &= \|(I - (P_{\mathfrak{B}_h} + P_{\mathfrak{B}_h^*}))P_h(L_0')_2 z\| \\ &= \|P_{\mathfrak{H}_h}(L_0')_2 z\|.\end{aligned}$$

And we conclude with Lemma 4.12. ∎

Given $\mathbf{u} \in V_h$, define $g = (u_2, P_{\mathfrak{H}^\perp} u_3)$, $z = Kg$, $\xi = -(d^* + l_3^*)z_2$ and $\mathbf{z} = (\xi, z, P_{\mathfrak{H}} u_3)$.

THEOREM 4.14. *There is $\mathbf{z_h} \in V_h$ such that $\forall \omega \in V_h$, $\|\mathbf{z_h}\|_V \lesssim \|\mathbf{z}\|_V$ uniformly in h and $|B(\omega, \mathbf{z} - \mathbf{z_h})| \le \epsilon_h \|\omega\|_V \|\mathbf{u}\|$, where $\epsilon_h \to 0$ as $h \to 0$.*

Proof. Take $z_h = K_{0h} P_h L_0' z + P_{\mathfrak{H}_h} P_{\mathfrak{H}} z$, $\xi_h = -d_h^* z_{2h} - \Pi_h l_3^* z_2$. By Theorem 4.13 we have:

$$\begin{aligned}\|z - z_h\| \lesssim \alpha_0 \|g\|,\ \|d(z - z_h)\| \lesssim \eta_0 \|g\|,\ \|d^*z_2 - d_h^*z_{2h}\| \lesssim \eta_0 \|g\|, \\ \|\xi - \xi_h\| \le \|d^*z_2 - d_h^*z_{2h}\| + \|(I - \Pi_h)l_3^*z_2\|.\end{aligned} \tag{4.16}$$

Using Lemma 4.9 and the boundedness of $d\,l_3^*\,K$ we get

$$\|(I - \Pi_h)l_3^*z_2\| \lesssim \|(I - \pi_h)l_3^*z_2\| + \eta_0' \|dl_3^*z_2\| \lesssim (\eta + \eta_0')\|g\|.$$

Finally, since $\forall \omega_2 \in V_h^2 \subset V^2$

$$\langle d\omega_2, z_3\rangle = \langle \omega_2, d^*z_3\rangle, \langle d\omega_2, z_{3h}\rangle = \langle \omega_2, d_h^*z_{3h}\rangle\ ,$$

we have $\forall \omega \in V_h$:

$$\begin{aligned}|B(\omega,&(\xi - \xi_h, z_2 - z_{2h}, z_3 - z_{3h}, P_{\mathfrak{H}}u_3 - P_{\mathfrak{H}_h}P_{\mathfrak{H}}u_3))| = \ |\langle \omega_1, \xi - \xi_h\rangle \\ &- \langle \omega_2, d(\xi - \xi_h)\rangle + \langle (\nu d + l_3)\omega_1, z_2 - z_{2h}\rangle - \langle \omega_3, d(z_2 - z_{2h})\rangle \\ &+ \langle l_5\omega_2, z_2 - z_{2h}\rangle + \langle d\omega_2, z_3 - z_{3h}\rangle \\ &+ \langle \omega_p, z_3 - z_{3h}\rangle + \langle \omega_3, P_{\mathfrak{H}}u_3 - P_{\mathfrak{H}_h}P_{\mathfrak{H}}u_3\rangle| \\ &\lesssim\ \|\omega\|_V (\|\xi - \xi_h\| + 2\|z_2 - z_{2h}\| + \|d(z_2 - z_{2h})\| \\ &+ \|z_3 - z_{3h}\| + \|(I - P_{\mathfrak{H}_h})P_{\mathfrak{H}}u_3\|) + |\langle \omega_2, P_h d^*z_3 - d_h^*z_{3h} - P_h d(\xi - \xi_h)\rangle| \\ &\lesssim\ \|\omega\|_V\,[(\eta + 2\eta_0 + 2\alpha_0 + \eta_0' + \mu_0)\|\mathbf{u}\| + \|P_h d^*z_3 - d_h^*z_{3h} - P_h d(\xi - \xi_h)\|]\,.\end{aligned}$$

Since η, η_0, η_0', μ_0 and α_0 all converge toward 0 when $h \to 0$, the only thing left to prove is that $\|P_h d^*z_3 - d_h^*z_{3h} - P_h d(\xi - \xi_h)\| \lesssim \epsilon \|\mathbf{u}\|$ where $\epsilon \to 0$ when $h \to 0$. We have

$$-d(\xi - \xi_h) = d(d^* + l_3^*)z_2 - dd_h^*z_{2h} - d\Pi_h l_3^*z_2 = dd^*z_2 - dd_h^*z_{2h} + d(I - \Pi_h)l_3^*z_2.$$

Theorem 4.13 gives:

$$\|P_h d^*z_3 - d_h^*z_{3h} + P_h dd^*z_2 - dd_h^*z_{2h}\| \lesssim \mu_0 \|g\|.$$

And we conclude with Lemma 4.9 since we can find a bounded cochain projection π_h such that $\pi_h d = P_{\mathfrak{B}_h} d$ (see [6, Theorem 3.7]) so

$$\|d(I - \Pi_h)l_3^*z_2\| = \|(I - P_{\mathfrak{B}_h})dl_3^*z_2\| \lesssim \|(I - \pi_h)dl_3^*z_2\| \le \eta \|g\|. \qquad ∎$$

LEMMA 4.15. *For all $\mathbf{u} \in V_h$ and $\mathbf{z} \in V$ defined in Theorem 4.14, there exists a constant c independent of h and $\sigma \in V_h$ such that $\|\sigma\|_V \lesssim \|\mathbf{u}\|_V$ and $B(\mathbf{u}, \mathbf{z} + \sigma) \geq c\|\mathbf{u}\|_V^2$.*

Proof. Starting from Lemma 4.6, we construct σ in the same way as we did in Lemma 4.7 in the continuous case. We must simply add

$$B(\omega, (0, 0, -P_{\mathfrak{H}_h} z_3, 0)) = -\langle \omega_p, z_3 \rangle$$

to correct the harmonic part. ∎

THEOREM 4.16. *There are two positive constants h_0 and C_0 such that for all $h \in (0, h_0]$, there exists a unique $\mathbf{u} \in V_h$ such that $\forall \mathbf{v} \in V_h$, $B(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$. Moreover we have $\|\mathbf{u}\|_V \leq C_0\|\mathbf{f}\|$.*

Proof. For $\mathbf{u} \in V_h$ and $\mathbf{z}$, $\mathbf{z_h}$ defined in Theorem 4.14, Lemma 4.15 gives $\sigma \in V_h$ with $\|\sigma\|_V \lesssim \|\mathbf{u}\|_V$ and a constant c independent of h such that:

$$|B(\mathbf{u}, \mathbf{z} + \sigma)| \geq c\|\mathbf{u}\|_V^2.$$

By Theorem 4.14 we have for a constant b independent of h.

$$B(\mathbf{u}, \mathbf{z_h} - \mathbf{z}) \leq \epsilon_h b\|\mathbf{u}\|_V^2.$$

Therefore:

$$\begin{aligned} |B(\mathbf{u}, \sigma + \mathbf{z_h})| &= |B(\mathbf{u}, \mathbf{z} + \sigma) + B(\mathbf{u}, \mathbf{z_h} - \mathbf{z})| \\ &\geq |B(\mathbf{u}, \mathbf{z} + \sigma)| - |B(\mathbf{u}, \mathbf{z_h} - \mathbf{z})| \\ &\geq c\|\mathbf{u}\|_V^2 - \epsilon_h b\|\mathbf{u}\|_V^2 \geq (c - \epsilon_h b)\|\mathbf{u}\|_V^2. \end{aligned}$$

Since $\epsilon_h \to 0$ as $h \to 0$ we can find h_0 such that $\forall h \in (0, h_0]$,
$c - \epsilon_h b > c - \epsilon_{h_0} b > 0$.

By Theorem 4.14 and by the expression of σ we find:

$$\|\sigma + \mathbf{z_h}\|_V \lesssim \|\mathbf{u}\|_V + \|\mathbf{z}\|_V \lesssim \|\mathbf{u}\|_V.$$

This ends the proof since V_h has finite dimension. ∎

COROLLARY 4.17. *If assumption (4.5) holds then for $h \leq h_0$ given by Theorem 4.16, and for $\mathbf{u}$ (resp. $\mathbf{u_h}$) the solution of the continuous problem (4.8) (resp. of the discrete problem (4.14)) it holds:*

$$\|\mathbf{u} - \mathbf{u_h}\|_V \lesssim \inf_{v_1 \in V_h^1} \|u_1 - v_1\|_{V^1} + \inf_{v_2 \in V_h^2} \|u_2 - v_2\|_{V^2} + E(u_3) + E(u_p) + \mu_0 E(P_{\mathfrak{B}} u_3).$$

Proof. The proof is the same as in Theorem 3.7 ∎

Remark 4.18. The hidden constant of Corollary 4.17 depends on $\|l_5\|$ which, in the case of problem (1.4), blows up when $\delta t \to 0$. A more subtle analysis is required to explicit the dependency of the error on δt. For a single time step in the setting of Corollary 4.17 the error will actually decrease when $\delta t \to 0$.

5. Conserved quantities. Lastly we prove that our scheme does indeed verify properties mentioned in the introduction as well as the regularity assumption (4.5).

5.1. Regularity assumptions. Problem (1.4) is a special case of Problem (4.8) taking suitable l_3 and l_5. We prove below that assumptions (4.5) are valid if $u^{n-1} \in H^2(\Omega)$ and if the domain is smooth enough to have $H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega) \subset H^1(\Omega)$, as discussed in [14, Chapter 3.2]. We use the notation $H^1(\Omega)$ both for scalar and for vector fields. First we need the following lemma:

LEMMA 5.1. *If $B \in H^1(\Omega)$ and $A \in H^2(\Omega)$ then $A \times B \in H^1(\Omega)$ for a smooth enough domain Ω of $\mathbb{R}^3$ (or $\mathbb{R}^2$).*

Proof. We have $H^1(\Omega) \subset L^4(\Omega)$ and $H^2(\Omega) \subset C^0(\overline{\Omega})$ by Sobolev Embedding theorems thus $\forall i, j, k \in \{x, y, z\}$, $\partial_i A_j \in L^4(\Omega)$, $\partial_i B_j \in L^2(\Omega)$ and terms of the form $A_i\, \partial_j B_k$ are the product of a bounded function with a function in $L^2(\Omega)$ and those of the form $B_i\, \partial_j A_k$ are the product of two functions in $L^4(\Omega)$. ∎

Going back to problem (1.4), we take $V^1 = H(\mathrm{curl}, \Omega)$, $V^2 = H(\mathrm{div}, \Omega)$, $V^3 = L^2(\Omega)$ and $\mathfrak{H}^3 = 0$. Regarding assumption (4.5), K is a bounded isomorphism from $L^2 \times L^2$ to $(H(\mathrm{div}) \cap H_0(\mathrm{curl})) \times H_0^1 \subset H^1 \times H^1$. If we assume $u \in H^2$ and $v \in H^1$ then using the scalar triple product we have $\forall \sigma \in H(\mathrm{curl})$:

$$2\langle l_3\sigma, v\rangle = \int (\sigma \times u^{n-1}) \cdot v = \int (u^{n-1} \times v) \cdot \sigma = 2\langle \sigma, l_3^* v\rangle.$$

Thus $l_3^* v = \frac{1}{2} u^{n-1} \times v$ on H^1, l_3^* maps H^1 into itself by Lemma 5.1, and

$$l_3^*(K)_2(W) \subset H^1 \subset H(\mathrm{curl}) = V^1.$$

Thus we have $\|(\nabla\times) l_3^*(K)_2\|_{W\to L^2} \lesssim \|(K)_2\|_{W\to H^1}$ and by the boundedness of $\|K\|_{L^2\times L^2 \to H^1 \times H^1}$ we get the boundedness of $\|dl_3^*(K)_2\|_{W\to L^2}$. Finally we have $\nu dd^*(K)_2 = I - dl_3^*(K)_2 + l_5^*(K)_2 + d^*(K)_3$ as distributions. From the $L^2 \to L^2$ boundedness of the right-hand side we get both $\nu d^*(K)_2(W) \subset H(\mathrm{curl}) = V^1$ and the boundedness of $\|dd^*(K)_2\|_{W\to L^2}$.

The same argument applied to $\overline{L}^{-1}$ shows that $d^*(\overline{L}^{-1})_2(W) \subset V^1$.
Hence (4.5) is fulfilled.

Remark 5.2. Assuming $u^{n-1} \in H^2(\Omega)$ is very mild as any solution u of (4.2) must have $\nabla \times u \in H(\mathrm{curl}, \Omega)$, $\nabla \cdot u = 0$ thus $\Delta u \in L^2(\Omega)$. Hence by elliptic regularity for Ω smooth enough and if u satisfies appropriate boundary conditions then $u \in H^2$.

5.2. Pointwise vanishing divergence. This is a simple fact that follows from the use of a discrete subcomplex. The operator $d_h = \mathrm{div}$ used in the formulation is simply the restriction of the continuous operator to the discrete space V_h^2 and its image is contained in the discrete space V_h^3. This holds even for condition (2.2) and (2.3) which does not give a complex structure. From (1.4) we have $\forall q_h \in V_h^3$, $\langle \nabla \cdot u_h + \phi_h, q_h\rangle = 0$ with $\nabla \cdot u_h \perp \phi_h$ and $\nabla \cdot u_h \in V_h^3$ by construction. Therefore, taking $q_h = \nabla \cdot u_h$ we have $\langle \nabla \cdot u_h + \phi_h, \nabla \cdot u_h\rangle = \|\nabla \cdot u_h\|^2 = 0$.

5.3. Pressure-robustness. A scheme is called pressure-robust ([21, 22, 19]) if only the pressure (and not the velocity) changes when the external forces acting on the system are modified by a gradient. This property is only valid if there are no harmonic 2-forms, i.e. if the domain is simply connected.

Every vector field $f \in L^2(\Omega)$ can be written as $\nabla \times g + \nabla p$ for some fields g and p. In a bounded domain we only have uniqueness with correct boundary conditions on g and p. As long as these boundary conditions match the ones given in the complex, we have by viewing f as a 2-form, $\nabla \times g = P_{\mathfrak{B}} f$ and $\nabla p = P_{\mathfrak{B}^*} f$. Let $\bar{f} \in L^2(\Omega)$ be another source term and write $(\omega_h, u_h, p_h, \phi_h)$ (resp. $(\bar{\omega}_h, \bar{u}_h, \bar{p}_h, \bar{\phi}_h)$) the solution of (4.14) (resp. (4.14)) for the external force $(f, 0)$ (resp. $(\bar{f}, 0)$).

THEOREM 5.3. *If f and $\bar{f}$ are such that $\exists p \in V^3$, $\bar{f} = f + \nabla p$ then $\omega_h = \bar{\omega}_h$ and $u_h = \bar{u}_h$.*

Proof. If there is $p \in V^3$ such that $f + \nabla p = \bar{f}$ then $P_{\mathfrak{B}}(f - \bar{f}) = 0$ so $\forall g \in V^1, \langle f - \bar{f}, \nabla \times g\rangle = 0$ and in particular

$$\forall g_h \in V_h^1 \subset V^1, \langle f - \bar{f}, \nabla \times g_h\rangle = 0 \ .$$

Therefore $P_{\mathfrak{B}_h}(f - \bar{f}) = 0$ and since we assumed that there were no harmonic 2-forms, $P_h(f - \bar{f}) = P_{\mathfrak{B}_h^*}(f - \bar{f})$. Thus we can find $\xi_h \in V_h^3$ such that $\xi_h \perp \mathfrak{H}_h$ and $d_h^* \xi_h = -P_h(f - \bar{f})$. Moreover $(0, 0, \xi_h, 0)$ verifies: $\forall (\tau_h, v_h, q_h, \chi_h) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h$:

$$\begin{aligned}
\langle 0, \tau_h\rangle - \langle 0, \nabla \times \tau_h\rangle &= 0,\\
\langle \nu d0 + l_3 0 + l_5 0, v_h\rangle - \langle \xi_h, \nabla \cdot v_h\rangle) &= \langle f - \bar{f}, v_h\rangle,\\
\langle \nabla \cdot 0, q_h\rangle + \langle 0, q_h\rangle &= 0,\\
\langle \xi_h, \chi_h\rangle &= 0.
\end{aligned}$$

Therefore by linearity and uniqueness of the solution we have
$(\omega_h, u_h, p_h, \phi_h) = (\bar{\omega}_h, \bar{u}_h, \bar{p}_h, \bar{\phi}_h) + (0, 0, \xi_h, 0)$ and $\omega_h = \bar{\omega}_h, u_h = \bar{u}_h$. ∎

Remark 5.4. Consider a time iterating scheme. At any time step n, the linear maps l_3^n, l_5^n involved in the determination of $(\omega_h^n, u_h^n, p_h^n, \phi_h^n)$ are function of u_h^{n-1} and ω_h^{n-1} (more precisely $l_3^n = l_3(u_h^{n-1})$ and $l_5^n = l_5(\omega_h^{n-1})$). However since both ω_h and u_h remain unchanged when the external force is modified by a gradient, as long as $u_h^0 = \bar{u}_h^0$ and $\omega_h^0 = \bar{\omega}_h^0$ an immediate recursion shows that for all step n, $u_h^n = \bar{u}_h^n$ and $\omega_h^n = \bar{\omega}_h^n$.

5.4. Pressure independent estimate. The pressure-robustness allows us to remove the pressure from the error estimate.

THEOREM 5.5 (Pressure-robust estimate). *Let $(\omega, u, p, \phi) = (u_1, u_2, u_3, u_p)$ be the solution of the continuous problem (4.8) and $(\omega_h, u_h, p_h, \phi_h) = (u_{1h}, u_{2h}, u_{3h}, u_{ph})$ be the solution of the discrete problem (4.14). Then it holds:*

$$\|\omega - \omega_h\| + \|\nabla \times (\omega - \omega_h)\| + \|u - u_h\| \lesssim \inf_{\chi \in V_h^1} \|\omega - \chi\|_{V^1} + \inf_{v \in V_h^2} \|u - v\|_{V^2}.$$

Proof. Consider an alternative problem with the same l_3 and l_5 as before but with the source term replaced by $\tilde{f}_2 := f_2 - \nabla p$. Let $(\tilde{\omega}, \tilde{u}, \tilde{p}, \tilde{\phi})$ and $(\tilde{\omega}_h, \tilde{u}_h, \tilde{p}_h, \tilde{\phi}_h)$ be respectively the continuous and discrete solution to this alternative problem. By construction, we have $\tilde{p} = 0$, $\tilde{\phi} = 0$. Hence the estimate of Corollary 4.17 gives:

$$\|\tilde{\omega} - \tilde{\omega}_h\| + \|\nabla \times (\tilde{\omega} - \tilde{\omega_h})\| + \|\tilde{u} - \tilde{u_h}\| \lesssim \inf_{\chi \in V_h^1} \|\tilde{\omega} - \chi\|_{V^1} + \inf_{v \in V_h^2} \|\tilde{u} - v\|_{V^2}.$$

We conclude since by the pressure-robustness we must have:

$$\tilde{\omega} = \omega,\ \tilde{u} = u,\ \tilde{\omega}_h = \omega_h,\ \tilde{u}_h = u_h \ .$$

∎

6. Numerical simulations. We give the results of three numerical simulations done to demonstrate the validity of our scheme. We checked the norm of the divergence of the fluid velocity at every time step, and we found a value vanishing up to machine precision. The first simulation aims to verify the pressure-robustness property, the second is based on an exact and fully 3D solution of the Navier-Stokes equation constructed by Ethier [11]. We use it to check the convergence rate in space, first on a steady problem then on an unsteady problem. The last simulation focuses on a system of two rotating cylinders and shows the good agreement with the theory on the value of the critical speed of the inner cylinder at which Taylor vortices appear. It is based upon [1, 13, 15]. In any case, we took a unit kinematic viscosity and polynomials of degree 2. Our codes are written with the FEniCS computing platform, version 2019.1.0 (See fenicsproject.org and [23]) and are available at https://github.com/mlhanot/Navier-Stokes-feec.

6.1. Pressure robustness. We wish to verify that if the external forces acting on two flows differ only by a gradient, then only the pressure differs between the flows. We took the Stokes no-flow problem in a (3-dimensional) glass (see [12]). The setup is rather simple, the mesh is a cylinder along the z axis of height 2.0, base radius 1.0 and top radius 1.5 and the force f derives from a potential:

$$f = \frac{\nabla \Phi}{\int_\Omega \Phi}, \Phi = z^\gamma$$

for $\gamma = 1, 2, 4, 7$. We start from a fluid at rest and enforce a no slip condition on the whole boundary. In every case we found a velocity equal to zero at the order of the machine accuracy. This is not trivial as the same test conducted with Taylor-Hood elements (P_2/P_1) gave a velocity of norm up to 3.3×10^{-4} for $\gamma = 7$.

6.2. Convergence rate to an exact solution. We have conducted a convergence analysis with an exact solution. The expression for the solution is given by Ethier [11] and depends on two real parameters a and b. It is given by:

$$u = \begin{bmatrix} -a(\exp(ax)\sin(ay+dz) + \exp(az)\cos(ax+dy))\exp(-d^2t) \\ -a(\exp(ay)\sin(az+dx) + \exp(ax)\cos(ay+dz))\exp(-d^2t) \\ -a(\exp(az)\sin(ax+dy) + \exp(ay)\cos(az+dx))\exp(-d^2t) \end{bmatrix}.$$

We have performed two sets of experiments: the first with $a = 2$ and $d = 0$ and the second with $a = 2$ and $d = 1$. The domain consists of a cylinder of height 2 and radius 1. In the latter case computations were done for t between 0 and 1 with a time step of 1.0×10^{-3}. We set the velocity to be equal to the velocity of the exact solution on the boundary (without enforcing any boundary condition on the vorticity). For the stationary case ($d = 0$) we start from a fluid at rest, otherwise we start from the exact solution at $t = 0$. We found a rate of convergence in space for the velocity of order 2.0 in both cases which is in agreement with the theory. Figure 6.1 shows the convergence of the velocity in the relative $H(\mathrm{div})$-norm $\frac{\|u_h - u\|_{H(\mathrm{div})}}{\|u\|_{H(\mathrm{div})}}$ (which is the same as the L^2-norm since the solution is exactly divergence free) with a log-log scale.

Remark 6.1. Despite the fact that the time discretization is Euler implicit and only first order by taking the time step small enough we can observe the second order convergence rate in space.

6.3. Taylor-Couette flow. This test focuses on Taylor-Couette flow. We follow the work of Gebhardt et al. [1, 13, 15]. The geometry consists in two concentric cylinders of constant radius R_i for the inner and R_o for the outer, rotating at an

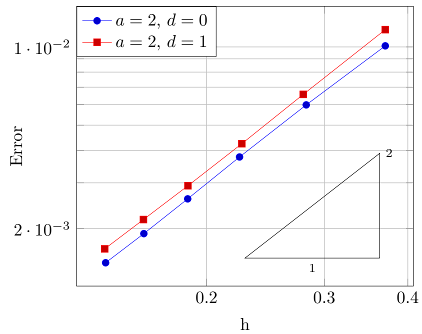

FIG. 6.1. *Convergence rate of the velocity with relative $H(div)$ error on a log log scale.*

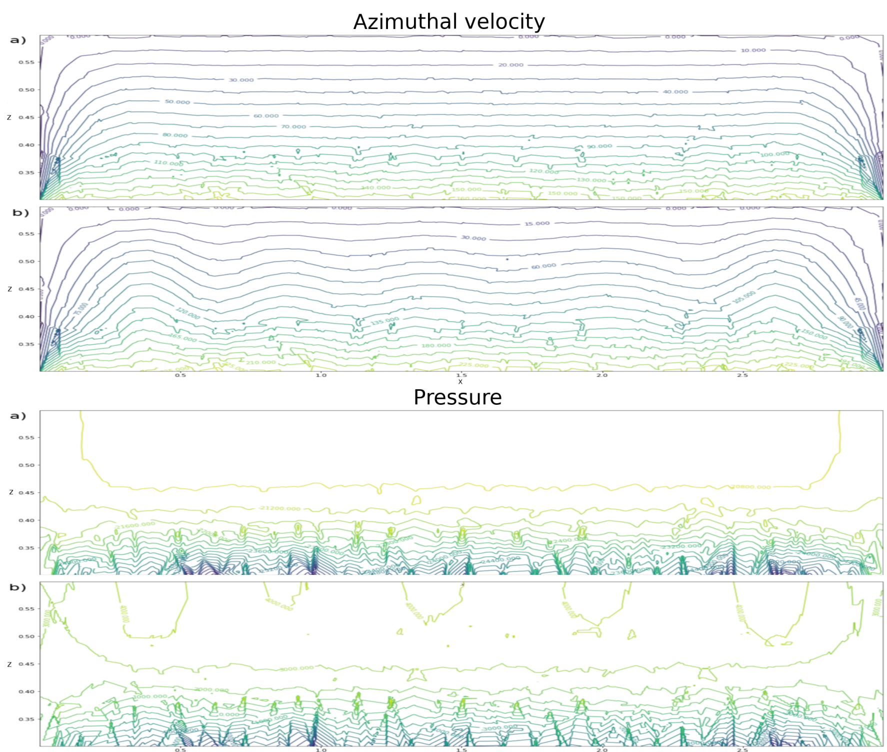

FIG. 6.2. *Isolines for azimuthal velocity and the pressure for $Re_o = 0$ at $Re_i = 50$ (a) and $Re_i = 72$ (b).*

angular velocities Ω_i and Ω_o respectively, and both of height a. The system is closed by two fixed lids at the bottom and top ends. We characterized the system by two geometric parameters: $\eta = R_i/R_o$ and $\Lambda = a/d$ with the gap $d = R_o - R_i$. We also need to define two quantities: the inner Reynolds number $Re_i = \Omega_i R_i d/\nu$ and the outer Reynolds number $Re_0 = \Omega_o R_o d/\nu$ where ν is the kinematic viscosity. It is a well known fact that (for an infinite height a) at low speed the flow is steady and fully azimuthal and that vortices start to form at a critical speed. Since a is finite we expect to see vortices near the lids for speeds way under the critical value (they are however fundamentally different from the Taylor vortices, see [15]).

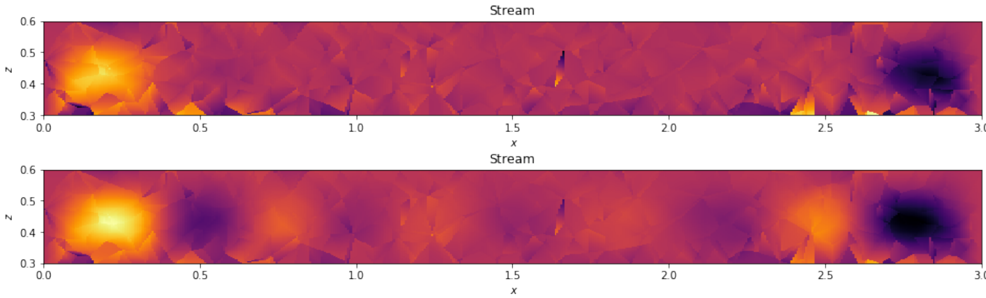

FIG. 6.3. *Stream function on cross section $y = 0$, $x > 0$ for $\eta = 0.5$, $Re_o = 0$ at $Re_i = 50$ (top) and $Re_i = 72$ (bottom).*

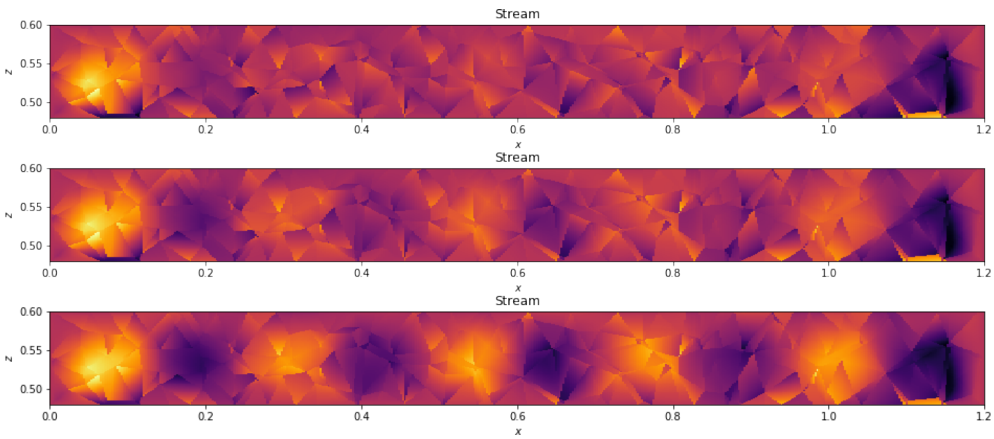

FIG. 6.4. *Stream function on cross section $y = 0$, $x > 0$ for $\eta = 0.8$, $Re_o = 0$ at $Re_i = 90$ (top), $Re_i = 95$ and $Re_i = 100$ (bottom).*

We compare results obtained from our code with the reference [13, 15]. The simulations are done starting from a fluid at rest with no slip boundary condition and studied at $t = 0.1$. The time step depends on the angular velocities but is most often at $\delta t = 0.001$ and only decreased for the highest velocities. We check the value of Re_i at which the transition occurs for various values of Re_o and η. We see very good agreement, though we used a much coarser mesh and a smaller aspect ratio Λ of 10 instead of 20, for computational cost reasons. We display some values taken on the half plane $y = 0, x > 0$. Figure 6.2 shows a comparison between the azimuthal velocity and the pressure for two values of Re_i at $\eta = 0.5$ and $Re_o = 0$. In Figure 6.3 we see the stream function (the azimuthal component of the vector potential) on the same system. Stream functions for three values of Re_i at $\eta = 0.8$ are shown in Figure 6.4. Lastly Figures 6.5 and 6.6 give the aforementioned comparison, our results being shown in red and the reference curve in black.

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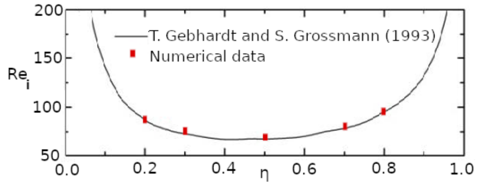

Fig. 6.5. *Comparison of the critical value of Re_i at $Re_o = 0$ for various η.*

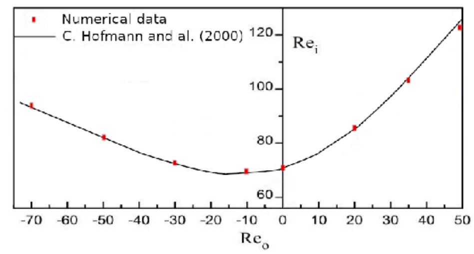

Fig. 6.6. *Comparison of the critical value of Re_i at $\eta = 0.5$ for various Re_o.*

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