HAL open science

An arbitrary order and pointwise divergence-free finite element scheme for the incompressible 3D Navier-Stokes equations

Marien-Lorenzo Hanot

To cite this version:

Marien-Lorenzo Hanot. An arbitrary order and pointwise divergence-free finite element scheme for the incompressible 3D Navier-Stokes equations. 2021. hal-03250657v1

HAL Id: hal-03250657

https://hal.science/hal-03250657v1

Preprint submitted on 4 Jun 2021 (v1), last revised 11 May 2022 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An arbitrary order and pointwise divergence-free finite element scheme for the incompressible 3D Navier-Stokes equations.

Marien-Lorenzo HANOT.*

IMAG, UMR 5149, Université de Montpellier, 34090 Montpellier, FRANCE.

Abstract

In this paper we introduce a new discretization of the incompressible Navier-Stokes equations. We use the Lamb identity for the advection term $(u \cdot \nabla)u$ and the general idea allows a lot of freedom in the treatment of the non-linear term. The main advantage of this scheme is that the divergence of the fluid velocity is pointwise zero at the discrete level. This exactness allows for exactly conserved quantities and pressure robustness. Discrete spaces consist of piecewise polynomials, they may be taken of arbitrary order and are already implemented in most libraries. Although the nonlinear term may be implemented as is, most proofs here are done for the linearized equation. The whole problem is expressed in the finite element exterior calculus framework (developed in [15]). We also present numerical simulations, our codes are written with the FEniCS computing platform, version 2019.1.0. See `fenicsproject.org` and [6].

Keywords. Finite Element, exterior calculus, incompressible Navier-Stokes, Hodge decomposition, de Rham complex, mixed element.

MSC. 35Q30, 65N30, 76D07, 76M10.

Acknowledgements. I would like to thank Pascal Azerad for his comments and helpful advice.

1 Overview.

The purpose of this paper is to give a discrete formulation of the Navier-Stokes equations with pointwise divergence-free solutions in the space of piecewise polynomials of arbitrary degree.

*marien-lorenzo.hanot@umontpellier.fr

There are three main parts, first we give the formulation and state a few results in section 2, then sections 3 to 5 are dedicated to the proofs of the main results. Section 3 and 4 are technical. Lastly we present some numerical simulations done with our scheme in section 6. In the second part of the paper, we work in the finite element exterior calculus setting, and we will assume that the reader has some familiarity with the main results of the theory. No prior knowedge of exterior calculus are expected in the two other parts. In addition to the convergence rate, we show in section 6.3 that our scheme gives a good qualitative description of rotating fluids (specifically we are interested in critical speeds for Taylor-Couette flows).

1.1 State of art.

Recently much work has been done to design structure preserving methods, but while the construction of such methods was found early on in two dimensions, the three dimensional case remained difficult and the introduction of the finite element exterior calculus brought a significant breakthrough. An excellent review is given in [14]. The general idea taken from the finite element exterior calculus is to use a subcomplex of the De Rham complex. There are well-known discretizations of this complex with minimal regularity, however the discretization of smoother variants is still an active topic usually leading to shape functions of high degree, see for exemple the progress made in [10]. We chose to use the complex with minimal regularity, as this is often done for electromagnetism or recently for magnetohydrodynamics (see [19]).

The main difference from usual schemes lies in the regularity of the velocity field since we only require it to be in $H(div)$ and in the discrete adjoint of $H(curl)$. Whereas the continuous space regularity is the same than the usual one, since the adjoint of $(curl, H(curl))$ is $(curl, H_0(curl))$, and the velocity is sought in $H(div) \cap H_0(curl) \subset H^1$ (for a smooth enough domain, it is discussed in part 3.2 of [1]) this does not hold (in general) in the discrete case since for $V_h \subset H(curl)$ and $(curl^*, V_h^*)$ the adjoint of $(curl, V_h)$ we no longer have $V_h^* \subset H(curl)$. This has a fondamental impact both from the philosophical and practical point of view. In practice since $v \in H(div)$ (resp. $v \in H(curl)$) does not impose continuity of the tangential (resp. normal) components on faces, we will not have any degree of freedom corresponding to these and lack any way to set them in a Dirichlet boundary condition. This means that the normal and the tangential part of the boundary condition must be treated in two different ways. This may be viewed as an advantage since it allows to easily enforce a net zero outflow on a moving wall even when the mesh is only an approximation of the expected geometry. From the philosophical point of view it makes more sense in the exterior algebra and means that the fluid velocity is really sought as a "2-form" (mostly defined by its flux accross cell boundaries) which happens to be regular enough to also be in

the domain of the exterior derivative adjoint.

Let us summarize the main idea of the algorithm. In order to preserve the free divergence constraint, we have to consider u as a 2-form, which can be discretized by face elements. Then it is not straightforward to discretize the Laplacian in the usual way $\langle \nabla u, \nabla v \rangle$ because ∇u is not a natural quantity for a 2-form. Our simple trick is to use the fact

$$\begin{aligned} \Delta u =& \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) \\ =& - \nabla \times (\nabla \times u). \end{aligned}$$

Now $(\nabla \times u)$ is exactly the codifferential of u and is therefore a natural quantity. It is precisely a 1-form and can be discretized with edge elements.

2 Main results.

We recall the Navier-Stokes equations:

$$\begin{aligned} u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f \text{ on } \Omega \times (0,T), \\ \nabla \cdot u = 0 \text{ on } \Omega \times (0,T) \end{aligned} \tag{2.1}$$

together with some boundary and initial conditions, where Ω is a bounded domain of $\mathbb{R}^3$, u is the velocity of the fluid, p the pressure, ν the kinematic viscosity and f an external force.
Using the Lamb identity $(u \cdot \nabla)u = (\nabla \times u) \times u + \frac{1}{2}\nabla(u \cdot u)$, we get the following formulation

$$\begin{aligned} u_t + (\nabla \times u) \times u - \nu \Delta u + \nabla P = f \text{ on } \Omega \times (0,T), \\ \nabla \cdot u = 0 \text{ on } \Omega \times (0,T) \end{aligned} \tag{2.2}$$

where $P = p + \frac{1}{2} u \cdot u$.

Since $\nabla \cdot u = 0$ (in the continuous setting but also in the discrete one as we shall see below) we may rewrite the Laplacian as

$$\Delta u = -\nabla \times (\nabla \times u)$$

and our formulation becomes

$$\begin{aligned} u_t + (\nabla \times u) \times u + \nu \nabla \times (\nabla \times u) + \nabla P = f \text{ on } \Omega \times (0,T), \\ \nabla \cdot u = 0 \text{ on } \Omega \times (0,T). \end{aligned} \tag{2.3}$$

2.1 Function spaces and weak formulations.

Our scheme does not rely on a particular choice of discrete spaces, instead we make some assumptions on them and any space fulfilling these assumptions may be used. Our assumptions are given in the section 2.4, indeed adequate spaces are readily available on simplicial and cubic meshes (they are given e.g. in the periodic table of finite elements [8]). We shall need four spaces, namely $V^1 = H(curl, \Omega)$, $V^2 = H(div, \Omega)$, $V^3 = L^2(\Omega)$ and $\mathfrak{H} = div(V^2)^\perp \subset L^2$. Let $\mathbf{T}_h$ be a simplicial triangulation of Ω, we choose the following discretization:

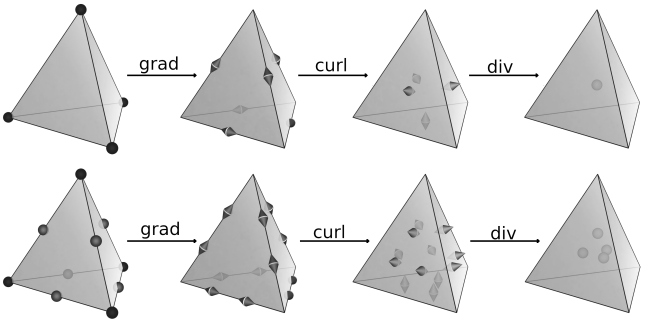

Figure 1: Degrees of freedom on reference elements for the two lowest polynomial degree $r = 1$ and $r = 2$.

- To define the curl space V_h^1 we use Nedelec's edge elements of the first kind of degree r (or $P_r^-\Lambda^1$ in the periodic table), $V_h^1 = \{\omega \in H(curl, \Omega); \omega_{|T} \in P_r^-\Lambda^1(T), \forall T \in \mathbf{T}_h\}$.
- To define the velocity space V_h^2 we use Nedelec's face elements of the first kind of degree r (or $P_r^-\Lambda^2$), $V_h^2 = \{\omega \in H(div, \Omega); \omega_{|T} \in P_r^-\Lambda^2(T), \forall T \in \mathbf{T}_h\}$.
- To define the pressure space V_h^3 we use discontinuous Galerkin elements of degree $r - 1$ (or $P_r^-\Lambda^3$), $V_h^3 = \{\omega \in L^2(\Omega); \omega_{|T} \in P_r^-\Lambda^3(T), \forall T \in \mathbf{T}_h\}$.
- $\mathfrak{H}_h$ is the orthogonal complement of $div(V_h^2)$ in V_h^3 (the kernel of the discrete gradient).

We may substitute the first kind by the second, at the expense of increasing the polynomial degrees. Nedelec elements of first and second kind are respectively the 3-dimensional equivalent of Raviart-Thomas and of Brezzi-Douglas-Marini elements. The space $\mathfrak{H}_h$ is just the natural way of fixing the pressure (usually only defined up to an arbitrary constant when it does not appear in boundary conditions). Figure 1 shows the degrees of freedom of this choice of finite elements for $r = 1$ and $r = 2$, here we only use the last three spaces on the sequence.

In full generality $\mathfrak{H}_h$ is simply the space of the discrete harmonic 3-forms. When there is no boundary condition on V_h^2, $\mathfrak{H}_h = 0$. When Dirichlet boundary conditions are prescribed everywhere $\mathfrak{H}_h$ is 1-dimensional. In the latter case, $\mathfrak{H}_h$ is equal to the space of constant functions. Thus the pressure is known up to a constant.

Remark 2.0.1. *In this case the space of discrete harmonic* 3*-forms is the same as the space of continuous harmonic* 3*-forms.*

The weak formulation is:

find $(\omega, u, p, \phi) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}$ such that $\forall(\tau, v, q, \chi) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}$:

$$\langle \omega, \tau \rangle - \langle u, \nabla \times \tau \rangle = 0, \tag{2.4}$$

$$\langle u_t, v \rangle + \nu \langle \nabla \times \omega, v \rangle + \langle \omega \times u, v \rangle - \langle p, \nabla \cdot v \rangle = \langle f, v \rangle, \tag{2.5}$$

$$\langle \nabla \cdot u, q \rangle + \langle \phi, q \rangle = 0 \tag{2.6}$$

$$\langle p, \chi \rangle = 0 \tag{2.7}$$

The the equation (2.4) is equivalent to $\omega = \nabla \times u$ and $u \in H_0(curl)$, (2.5) is equivalent to $\nu \nabla \times u + \omega \times u + u_t + \nabla p = f$ and $p \in H_0^1$, (2.6) is equivalent to $\nabla \cdot u = 0$, and (2.7) is equivalent to $p \in div(H(div))$.

Remark 2.0.2. *Equation* (2.6) *might seem odd but since ϕ is the harmonic projection of* 0 *so $\phi = 0$. So the equation is indeed equivalent to $\nabla \cdot u = 0$.*

The general semi-discrete (continuous in time) linearized formulation is:
find $(\omega, u, p, \phi) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h$ such that $\forall(\tau, v, q, \chi) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h$:

$$\begin{aligned} \langle \omega, \tau \rangle - \langle u, \nabla \times \tau \rangle &= 0, \\ \langle u_t, v \rangle + \nu \langle \nabla \times \omega, v \rangle + \langle l_3 \omega, v \rangle + \langle l_5 u, v \rangle - \langle p, \nabla \cdot v \rangle &= \langle f, v \rangle, \\ \langle \nabla \cdot u, q \rangle + \langle \phi, q \rangle &= 0 \\ \langle p, \chi \rangle &= 0 \end{aligned} \tag{2.8}$$

where l_3 and l_5 are arbitrary linear functions. We use l_5 for the linearization of $\omega \times u$ and for the discretization of u_t, whereas we solely use l_3 for the linearization of $\omega \times u$. Proof of the well-posedness and error estimates are given for this formulation, the main result being given in (4.13.1). For the above-mentioned choice of spaces we get:

$$\|u - u_h\| + \|\omega - \omega_h\| + \|\nabla \times (\omega - \omega_h)\| + \|p - p_h\| \lesssim O(h^r)$$

where (ω, u, p, ϕ) is the solution to the continuous problem and $(\omega_h, u_h, p_h, \phi_h)$ the solution to the discrete problem. Note that the hidden constant does depend on the operator norm of l_3 and of l_5, and that we assumed to be able to change l_5 by $l_5 + \epsilon I$ for an arbitrary small ϵ (in particular we do not have the well-posedness for $l_5 = 0$ on some domains).

Lastly we shall give the fully discrete scheme we used in our numerical simulations, the time is discretized with an implicit Euler method and we chose a symmetric linearization of the nonlinear term $\omega \times u$. We take $(\omega^0, u^0, p^0, \phi^0) = (\nabla \times u_0, u_0, 0, 0)$ if an initial condition u_0 is given (else we take $u_0 = 0$) and take δt a timestep. We may now construct iteratively a sequence (for $n \geq 1$) by the problem:
find $(\omega^n, u^n, p^n, \phi^n) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h$ such that $\forall(\tau, v, q, \chi) \in V_h^1 \times$

$V_h^2 \times V_h^3 \times \mathfrak{H}_h$:

$$\begin{aligned}
\langle \omega^n, \tau \rangle - \langle u^n, \nabla \times \tau \rangle &= 0, \\
\langle u^n, v \rangle + \delta t(\nu \langle \nabla \times \omega^n, v \rangle + \langle \frac{1}{2}(\omega^n \times u^{n-1} + \omega^{n-1} \times u^n), v \rangle & \\
- \langle p^n, \nabla \cdot v \rangle) &= \langle u^{n-1}, v \rangle + \delta t \langle f, v \rangle, \\
\langle \nabla \cdot u^n, q \rangle + \langle \phi^n, q \rangle &= 0 \\
\langle p^n, \chi \rangle &= 0.
\end{aligned} \tag{2.9}$$

Remark 2.0.3. *Steady problems are sought iteratively with this scheme. Although one may think that this is a simple trick to get what can be computed in a single step of a nonlinear problem, we like to think of them as asymptoticaly steady problems. Indeed they are typicaly driven by some non-zero boundary conditions physicaly corresponding to some moving parts. Since we start our iteration from a fluid at rest, we may simply view it as a sharp increase of the speed of the boundary (in less than a timestep). Numerical studies ([7]) show that we obtain a different steady state whether we start from a fluid at rest and increase sharply the speed, increase gradualy (at each step) the speed or start from a faster moving state and gradualy decrease the speed. A single, non-linear step may not differentiate between them.*

2.2 Boundary conditions.

We give a quick review of the boundary conditions readily available. Let n be the unit outward normal to the boundary and g and h arbitrary functions in a suitable space. Equalities below are always understood on $\partial\Omega$ and any combination of the following conditions may be used:

- $\omega \times n = g, \ u \cdot n = h.$
- $\omega \times n = g, \ p = h.$
- $u = g.$
- $u \times n = g, \ p = h.$

The first two conditions are uncommon since they ask for the tangential trace of the vorticity. The third is the most commonly used since it allows to enforce the noslip condition with $u \cdot n = 0$, $u \times n = 0$.

In order to implement them, we use essential Dirichlet boundary conditions for:

- $\omega \times n = g$ and take test functions in $H_0(curl)$.
- $u \cdot n = g$ and take test functions in $H_0(div)$.

And natural conditions for the other two:

- $u \times n = g$ by adding $-\int_{\partial\Omega}(g \times \tau) \cdot n ds$ to the left-hand side of equations (2.9).
- $P = g$ by adding $\int_{\partial\Omega} g v \cdot n ds$ to the left-hand side of equations (2.9).

2.3 Two dimensional formulation.

While everything is expressed in $\mathbb{R}^3$ it is perfectly possible to apply this to an 2-dimensional problem. One shall take the Raviart-Thomas's face elements (and not edge since we wish to preserve a pointwise zero divergence) or Brezzi-Douglas-Marini's face elements for V_h^2 and substitute $\nabla \times \omega$ by its usual two dimensional expression:

$$\nabla \times \omega = \nabla \times \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \text{ and } \omega \times u = \omega u^{\perp} = \omega \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix}.$$

Everything said and proved below also works in two dimensions.

2.4 Notations.

We shall make extensive use of the following notations, which are explained in greater detail in [15].

- d is the exterior derivative, d^* its adjoint or codifferential,
- $W^0 \to W^1 \to W^2 \to W^3$ is the L^2-De Rham complex of a bounded domain of $\mathbb{R}^3$. We shall only use the last three of them,
- $V^0 \to V^1 \to V^2 \to V^3$ is a dense subcomplex on which the exterior derivative is defined (and not just densely defined). Here we have $H^1(\Omega) \to H(curl, \Omega) \to H(div, \Omega) \to L^2(\Omega)$.
- V_k^* is the domain of the adjoint of (d, V^k),
- $\|\cdot\|$ is the L^2-norm, scalar or vectorial,
- $\|\cdot\|_V$ is the V norm, defined by $\|\cdot\|_V = \|\cdot\| + \|d\cdot\|$,
- Sometimes we take the norm on $V \cap V^*$, it is given by $\|\cdot\|_V + \|d^*\cdot\|$,
- $V_h^0 \to V_h^1 \to V_h^2 \to V_h^3$ is a discrete subcomplex parametrized by h,
- P_h is the orthogonal projection on the discrete subcomplex and in general P_A is the orthogonal projection on A.

We assume that our complexes have the compactness property, which means that the inclusion $V^k \cap V_k^* \subset W^k$ is compact for each k.
We also assume that there exists a cochain projection $\pi_h^k : V^k \to V_h^k$, bounded for the W-norm.
Each space has the Hodge-decomposition $W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}_k^*$, where $\mathfrak{B}^k = d(W^{k-1})$, $\mathfrak{B}_k^* = d^*(W^{k+1})$ and $\mathfrak{H}^k = \mathfrak{Z}^k/\mathfrak{B}^k$, $\mathfrak{Z}^k$ being the kernel of $d : W^k \to W^{k+1}$.

In order to measure the approximation properties of the discrete subspaces we introduce the notation $E = E^k$, defined by:

$$\forall k, \forall \sigma \in V^k, E^k(\sigma) := \inf_{\tau \in V_h^k} \|\sigma - \tau\| . \tag{2.10}$$

The choice of spaces introduced in section 2.1 does indeed verify all these properties with $E = \mathcal{O}(h^r)$ on a dense subset; this is proved in [5].

When it is obvious from the context we shall drop the exponent. When we apply an operator (such as π_h, d or d^*) to a product space, we mean to apply it to each component. When we add a suffix $_h$ such as d_h instead of d we refer to the discrete counterpart of the object.
We will often add a numerical suffix such as z_2 for $z \in V^2 \times V^3$, this means we take the V^2 component of z and will be clear from the context.

Usually when dealing with a primal formulation we will use the variable (u, p), and they can indeed be seen as the velocity and pressure. However when we deal with mixed formulations, we will often write (u_1, u_2, u_3), 1,2 and 3 here refer to 1-form, 2-form and 3-form and have nothing to do with components in a frame of the velocity field. Specifically we will have the identification $u_1 = \omega$, $u_2 = u$ and $u_3 = p$.

The symbol $A \lesssim B$ means that there exists hidden constant C independent of A and B (usualy depending solely on few specified parameters) such that $A \leq CB$.

3 Linear steady problem.

We first study a simpler problem, analogous to a Stokes problem, which is also closely related to the Hodge-Laplace problem (see [15]). The problem is given $f = (f_2, f_3)$:
find (u, p) such that

$$\begin{aligned} \nu \nabla \times (\nabla \times u) + \nabla p &= f_2 \text{ on } \Omega, \\ \nabla \cdot u &= f_3 \text{ on } \Omega \end{aligned}$$

3.1 Primal formulation.

We give the primal formulation in order to explain more clearly what we seek in the mixed formulation, and because we shall need this operator in section 4. Since we have no way to reach the harmonic part of f_2, we must include a second harmonic space (this time of 2-forms).

Definition 3.1. *Let* $D_0 := \{(u, p) \in (V^2 \cap V_2^*) \times V_3^* |\ d^* u \in V^1\}$, $f \in W^2 \times W^3$ *and*

$$L_0 := \begin{bmatrix} \nu dd^* & -d^* \\ d & 0 \end{bmatrix} = \begin{bmatrix} \nu \nabla \times \nabla \times & -\nabla \\ \nabla \cdot & 0 \end{bmatrix}. \tag{3.1}$$

The problem is to find $(u, p) \in P_{\mathfrak{H}^\perp} D_0$, such that

$$\forall (v, q) \in P_{\mathfrak{H}^\perp} D_0, \langle L_0(u, p), (v, q) \rangle = \langle f, (v, q) \rangle.$$

Remark 3.1.1. *Since* $d : V^3 \to 0$*, the Hodge decomposition on* W^3 *reads* $W^3 = \mathfrak{B}^3 \oplus \mathfrak{H}^3$*. We also have the equality between the* V*-norm and the* W*-norm.*

3.2 Continuous Well-posedness.

Now we introduce the mixed formulation, it is characterized by the introduction of the auxiliary variable u_1 corresponding to $\nabla \times u_2$ or $d^* u_2$. The problem reads: given $(f_2, f_3) \in W^2 \times W^3$, find $(u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ such that $\forall (v_1, v_2, v_3, \chi_2, \chi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$,

$$\begin{aligned} \langle u_1, v_1 \rangle - \langle u_2, dv_1 \rangle &= 0, \\ \nu \langle du_1, v_2 \rangle - \langle u_3, dv_2 \rangle + \langle \phi_2, v_2 \rangle &= \langle f_2, v_2 \rangle, \\ \langle du_2, v_3 \rangle + \langle \phi_3, v_3 \rangle &= \langle f_3, v_3 \rangle, \\ \langle u_2, \chi_2 \rangle + \langle u_3, \chi_3 \rangle &= 0. \end{aligned} \tag{3.2}$$

The associated bilinear form is noted B_0. We took the space $\mathfrak{H}^2$ to deals with the general case, however it is null on most domains.

Lemma 3.2. *There is $\alpha > 0$, such that $\forall (u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ we have:*

$$\sup_{(v_1, v_2, v_3, \chi_2, \chi_3)} \frac{B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3))}{\|(v_1, v_2, v_3, \chi_2, \chi_3)\|_V} \geq \alpha \|(u_1, u_2, u_3, \phi_2, \phi_3)\|_V.$$

Proof. For any $(u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$, let $\rho_1 \in \mathfrak{B}_1^* \cap V^1$ be such that $d\rho_1 = P_{\mathfrak{B}} u_2$, $\rho_2 \in \mathfrak{B}_2^* \cap V^2$ be such that $d\rho_2 = P_{\mathfrak{B}} u_3 + du_2$. And take $v_1 = \nu u_1 - \frac{\nu}{c_p^2} \rho_1$, $v_2 = P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2$, $v_3 = du_2 + \phi_3$, $\chi_2 = P_{\mathfrak{H}} u_2$, $\chi_3 = P_{\mathfrak{H}} u_3$. The Poincaré inequality gives:

$$\|\rho_1\|_V \leq c_p \|P_{\mathfrak{B}} u_2\|, \tag{3.3}$$

$$\|\rho_2\|_V \leq c_p \|P_{\mathfrak{B}} u_3\| + c_p \|du_2\|. \tag{3.4}$$

And we easily see that:

$$\|v_1\|_V + \|v_2\|_V + \|v_3\|_V + \|\chi_2\| + \|\chi_3\| \lesssim \|u_1\|_V + \|u_2\|_V + \|u_3\|_V + \|\phi_2\| + \|\phi_3\| \tag{3.5}$$

where the hidden constant only depends on ν and c_p. Then we have $B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) =$

$$\begin{aligned} &\langle u_1, \nu u_1 - \frac{\nu}{c_p^2} \rho_1 \rangle - \langle u_2, d(\nu u_1 - \frac{\nu}{c_p^2} \rho_1) \rangle \\ &+ \nu \langle du_1, P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2 \rangle - \langle u_3, d(P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2) \rangle \\ &+ \langle du_2, du_2 + \phi_3 \rangle + \langle u_2, P_{\mathfrak{H}} u_2 \rangle + \langle u_3, P_{\mathfrak{H}} u_3 \rangle \\ &+ \langle \phi_2, P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2 \rangle + \langle \phi_3, du_2 + \phi_3 \rangle. \end{aligned}$$

Using orthogonality of the Hodge decomposition this gives:

$$\begin{aligned} &\nu \langle u_1, u_1 \rangle - \frac{\nu}{c_p^2} \langle u_1, \rho_1 \rangle - \nu \langle u_2, du_1 \rangle + \frac{\nu}{c_p^2} \langle u_2, P_{\mathfrak{B}} u_2 \rangle \\ &+ \nu \langle du_1, u_2 \rangle + \nu \langle du_1, du_1 \rangle + \langle u_3, P_{\mathfrak{B}} u_3 \rangle \\ &+ \langle du_2, du_2 \rangle + \langle u_2, P_{\mathfrak{H}} u_2 \rangle + \langle u_3, P_{\mathfrak{H}} u_3 \rangle \\ &+ \langle \phi_2, \phi_2 \rangle + \langle \phi_3, \phi_3 \rangle. \end{aligned}$$

Therefore:

$$
\begin{aligned}
B_0((u_1,u_2,u_3,\phi_2,\phi_3),(v_1,v_2,v_3,\chi_2,\chi_3)) = \\
\nu\|u_1\|^2 - \frac{\nu}{c_p^2}\langle u_1,\rho_1\rangle + \frac{\nu}{c_p^2}\|P_{\mathfrak{B}}u_2\|^2 + \nu\|du_1\|^2 + \|u_3\|^2 \\
+ \|du_2\|^2 + \|P_{\mathfrak{H}}u_2\|^2 + \|P_{\mathfrak{H}}u_3\|^2 + \|\phi_2\|^2 + \|\phi_3\|^2.
\end{aligned} \tag{3.6}
$$

Now $\langle u_1,\rho_1\rangle \le \|u_1\|\|\rho_1\| \le \frac{c_p^2}{2}\|u_1\|^2 + \frac{1}{2c_p^2}\|\rho_1\|^2$, using (3.3) and injecting in the expression (3.6) we get:

$$
\begin{aligned}
B_0((u_1,u_2,u_3,\phi_2,\phi_3),(v_1,v_2,v_3,\chi_2,\chi_3)) \ge \\
\frac{\nu}{2}\|u_1\|^2 + \frac{\nu}{2c_p^2}\|P_{\mathfrak{B}}u_2\|^2 + \nu\|du_1\|^2 + \|u_3\|^2 + \|du_2\|^2 \\
+ \|P_{\mathfrak{H}}u_2\|^2 + \|P_{\mathfrak{H}}u_3\|^2 + \|\phi_2\|^2 + \|\phi_3\|^2.
\end{aligned}
$$

Finally, since $du_3 = 0$, $d\phi_2 = d\phi_3 = 0$ and $\|P_{\mathfrak{B}^*}u_2\| \le c_p\|dP_{\mathfrak{B}^*}u_2\| = \|du_2\|$ we have:

$$
B_0((u_1,u_2,u_3,\phi_2,\phi_3),(v_1,v_2,v_3,\chi_2,\chi_3)) \gtrsim \|(u_1,u_2,u_3,\phi_2,\phi_3)\|_V^2
$$

where the hidden constant depends only on c_p and ν. □

Lemma 3.3. *Given any* $(v_1,v_2,v_3,\chi_2,\chi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ *there is* $(u_1,u_2,u_3,\phi_2,\phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ *such that*

$$
B_0((u_1,u_2,u_3,\phi_2,\phi_3),(v_1,v_2,v_3,\chi_2,\chi_3)) > 0 \ .
$$

Proof. If $P_{\mathfrak{B}}v_2 \neq 0$ take $u_1 \in \mathfrak{B}_1^*$ such that $du_1 = P_{\mathfrak{B}}v_2$, $\phi_2 = 0$, $u_3 = \phi_3 = 0$, then if $\langle u_1,v_1\rangle = 0$ take $u_2 = 0$ else take $u_2 = dv_1\frac{\langle u_1,v_1\rangle}{\langle dv_1,dv_1\rangle}$ ($dv_1 \neq 0$ since $P_{\mathfrak{B}^*}v_1 \neq 0$ as $\langle u_1,v_1\rangle \neq 0$). If $P_{\mathfrak{B}}v_2 = 0$, simply take $u_1 = v_1$, u_2 such that $du_2 = P_{\mathfrak{B}}v_3, P_{\mathfrak{B}}u_2 = -dv_1, P_{\mathfrak{H}}u_2 = \chi_2$ (this is possible by the Hodge decomposition), u_3 such that $P_{\mathfrak{B}}u_3 = -dv_2, P_{\mathfrak{H}}u_3 = \chi_3$, $\phi_2 = P_{\mathfrak{H}}v_2$ and $\phi_3 = P_{\mathfrak{H}}v_3$. □

Lemma 3.2 together with 3.3 give the conditions to apply the Babuška–Lax–Milgram theorem. This proves the continuous well-posedness. Moreover we have:

$$
\|u_1\|_V + \|u_2\|_V + \|u_3\|_V + \|\phi_2\|_V + \|\phi_3\|_V \le c\|f\| \tag{3.7}
$$

where c depends only on c_p and ν.

3.3 Discrete well-posedness.

The discrete problem reads:
Given $(f_2,f_3) \in W^2 \times W^3$, find $(u_{1h},u_{2h},u_{3h},\phi_{2h},\phi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$ such that $\forall (v_{1h},v_{2h},v_{3h},\chi_{2h},\chi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$,

$$B_0((u_{1h}, u_{2h}, u_{3h}, \phi_{2h}, \phi_{3h}), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) = \langle f_2, v_{2h}\rangle + \langle f_3, v_{3h}\rangle.$$

Since we have a discrete Poincaré inequality (from [15]) and use a subcomplex we may apply exactly the same proof as in the continuous case, substituting continuous spaces for their discrete counterparts.

Remark 3.3.1. *The only differential operator used in the mixed formulation is d, its discrete counterpart is then merely its restriction on the discrete space. However since the discrete space of harmonic forms $\mathfrak{H}_h$ is not necessarily a subspace of $\mathfrak{H}$. When $\mathfrak{H}^2$ or $\mathfrak{H}^3$ is not zero, the method may be non-conforming.*

3.4 Error estimate.

We will only derive a basic error estimate. First we define:

$$\mu := \max_{k\in\{2,3\}} \sup_{r\in\mathfrak{H}^k, \|r\|=1} \|(I - \pi_h^k)r\|. \tag{3.8}$$

Theorem 3.4. *Given $(f_2, f_3) \in W^2 \times W^3$, let $(u_1, u_2, u_3, \phi_2, \phi_3)$ be the solution of the continuous problem and $(u_{1h}, u_{2h}, u_{3h}, \phi_{2h}, \phi_{3h})$ the solution of the discrete problem, then:*

$$\begin{aligned}
&\|(u_1 - u_{1h}, u_2 - u_{2h}, u_3 - u_{3h})\|_V + \|(\phi_2 - \phi_{2h}, \phi_3 - \phi_{3h})\| \\
&\quad \lesssim E(u_1) + E(du_1) + E(u_2) + E(du_2) + E(u_3) + E(\phi_2) + E(\phi_3) \\
&\quad + \mu(E(P_{\mathfrak{B}} u_2) + E(P_{\mathfrak{B}} u_3))
\end{aligned}$$

where E is given by (2.10).

Proof. For all $(v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$ we have:

$$\begin{aligned}
&B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\
&\qquad = \langle f_2, v_{2h}\rangle + \langle f_3, v_{3h}\rangle + \langle u_2, \chi_{2h}\rangle + \langle u_3, \chi_{2h}\rangle \ .
\end{aligned}$$

Let $(v_1, v_2, v_3, \chi_2, \chi_3)$ be the orthogonal projection of $(u_1, u_2, u_3, \phi_2, \phi_3)$ into their respective discrete spaces, then by the continuity of B_0, $\forall (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$

$$\begin{aligned}
&B_0((u_{1h} - v_1, u_{2h} - v_2, u_{3h} - v_3, \phi_{2h} - \chi_2, \phi_{3h} - \chi_3), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\
&= B_0((u_1 - v_1, u_2 - v_2, u_3 - v_3, \phi_2 - \chi_2, \phi_3 - \chi_3), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\
&\ - \langle u_2, \chi_{2h}\rangle - \langle u_3, \chi_{2h}\rangle \\
&= B_0((u_1 - v_1, u_2 - v_2, u_3 - v_3, \phi_2 - \chi_2, \phi_3 - \chi_3), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\
&\ - \langle P_{\mathfrak{H}_h} u_2, \chi_{2h}\rangle - \langle P_{\mathfrak{H}_h} u_3, \chi_{2h}\rangle \\
&\lesssim (\|u_1 - v_1\|_V + \|u_2 - v_2\|_V + \|u_3 - v_3\| + \|\phi_2 - \chi_2\| + \|\phi_3 - \chi_3\| \\
&\ + \|P_{\mathfrak{H}_h} u_2\|_V + \|P_{\mathfrak{H}_h} u_3\|_V)(\|v_{1h}\|_V + \|v_{2h}\|_V + \|v_{3h}\| + \|\chi_{2h}\| + \|\chi_{3h}\|)
\end{aligned}$$

where the hidden constant only depends on ν.

From the discrete inf-sup condition we have

$$\begin{aligned}(\|u_{1h}-v_1\|_V+\|u_{2h}-v_2\|_V+\|u_{3h}-v_3\|+\|\phi_{2h}-\chi_2\|+\|\phi_{3h}-\chi_3\|)\\ \lesssim(\|u_1-v_1\|_V+\|u_2-v_2\|_V+\|u_3-v_3\|_V+\|\phi_2-\chi_2\|\\ +\|\phi_3-\chi_3\|+\|P_{\mathfrak{H}_h}u_2\|_V+\|P_{\mathfrak{H}_h}u_3\|_V).\end{aligned}$$

where the hidden constant only depends on ν and the discrete constant of Poincaré.

The theorem follows from

$$\|\phi_i-\chi_i\|\lesssim E(\phi_i),\|P_{\mathfrak{H}_h}u_i\|_V\lesssim\mu E(P_{\mathfrak{B}}u_i),\forall i\in\{2,3\}.$$

This is proved in the theorem 5.2 of [15]. □

Remark 3.4.1. *$E(\phi_i)$ is understood as viewing ϕ_i as an element of $V^i\supset\mathfrak{H}^i$,*

$$E(\phi_i)=\inf_{q\in V_h^i}\|\phi_i-q\|.$$

4 Linearized problem.

Starting from the linear steady problem (3.2), we add some terms of lower order since this will allow us to construct our scheme.
In the following, $\mathbf{u}$ is a shortcut for (u_1,u_2,u_3,u_p).

Now we consider:

$$\begin{aligned}B(\mathbf{u},\mathbf{v}):=&\langle u_1,v_1\rangle-\langle u_2,dv_1\rangle+\\ &\nu\langle du_1,v_2\rangle-\langle u_3,dv_2\rangle+\langle l_3u_1,v_2\rangle+\langle l_5u_2,v_2\rangle+\\ &\langle du_2,v_3\rangle+\langle u_3,v_p\rangle+\langle u_p,v_3\rangle\end{aligned}$$

where $l_3:W^1\to W^2$ and $l_5:W^2\to W^2$ are linear functionals (names are taken from [11]).

Define $D=\{(u,p)\in(V^2\cap V_2^*)\times(V_3^*\cap\mathfrak{H}^{3\perp})|\ d^*u\in V^1\}$, $W=W^2\times(W^3\cap\mathfrak{H}^{3\perp})$,

$$L:=\begin{bmatrix}(\nu d+l_3)d^*+l_5 & -d^*\\ d & 0\end{bmatrix}\tag{4.1}$$

We consider the primal problem:
Given $f\in W$, to find $(u,p)\in D$ such that

$$L(u,p)=f.\tag{4.2}$$

We also define the dual operator L' on $D'=\{(u,p)\in(V^2\cap V_2^*)\times(V_3^*\cap\mathfrak{H}^{3\perp})|\ (\nu d^*+l_3^*)u\in V^1\}$ by:

$$L':=\begin{bmatrix}d(\nu d^*+l_3^*)+l_5^* & d^*\\ -d & 0\end{bmatrix}\tag{4.3}$$

As an intermediary step, we wish to extend L on a larger domain and introduce the following notations:

$$\begin{aligned}
\overline{L_\lambda} &: (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp}) \to (V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))', \\
\overline{L_\lambda}(u,p)(v,q) &:= \langle \nu d^* u, d^* v\rangle + \langle l_3 d^* u + l_5 u - \lambda d^* p, v\rangle + \langle \lambda du, q\rangle, \\
\overline{L'_\lambda} &: (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp}) \to (V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))', \\
\overline{L'_\lambda}(u,p)(v,q) &:= \langle \nu d^* u + l_3^* u, d^* v\rangle + \langle l_5^* u + \lambda d^* p, v\rangle + \langle -\lambda du, q\rangle.
\end{aligned} \tag{4.4}$$

Where λ is a positive parameter introduced to simplify the proof of theorem 4.1. In the theorem 4.1 we shall see that they are almost always isomorphims. We define the solution operator $K = (\overline{L'_1})^{-1}$ and we assume that

$$d^*(\overline{L_1}^{-1})_2(W) \subset V^1, (\nu d^* + l_3^*)(K)_2(W) \subset V^1 \tag{4.5}$$

where $(\overline{L_1}^{-1})_2$ and $(K)_2$ are the projections on the first component of the product space taken after the operators. Moreover we assume that $\|dd^*(K)_2\|_{W\to W^2}$ and $\|dl_3^*(K)_2\|_{W\to W^2}$ are bounded. We show in 5.1 that these assumptions are satisfied when l_3 and l_5 are those used in our scheme.

The proof follows the same outline as [11]. First we prove that the continuous primal formulation gives an isomorphism, then we prove that the continuous mixed formulation is well-posed. Lastly we prove the well-posedness of the discrete mixed formulation and give an estimation of the error in energy norm.

4.1 Continuous primal formulation.

Theorem 4.1. *$\overline{L_\lambda} + \mu\langle\cdot,\cdot\rangle$ is a bounded isomorphism for all $\mu \in \mathbb{C}$ except for a countable subset.*

Proof. Let $c = max(\|l_3\|, \|l_5\|)$, for $(u,p) \in (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp})$ take $v_{\mathfrak{B}} = u_{\mathfrak{B}}$, $v_{\mathfrak{H}} = u_{\mathfrak{H}}$, $v_{\mathfrak{B}^*} = -d^* p$, $q = du$. We have:

$$\begin{aligned}
&(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) \\
&\qquad = \nu\langle d^* u, d^* u\rangle + \lambda\langle d^* p, d^* p\rangle + \lambda\langle du, du\rangle \\
&\qquad\quad + \langle l_3 d^* u, u_{\mathfrak{B}} - d^* p + u_{\mathfrak{H}}\rangle + \langle l_5 u, u_{\mathfrak{B}} - d^* p + u_h\rangle \\
&\qquad\quad + \mu\langle u, u_{\mathfrak{B}} + u_{\mathfrak{H}}\rangle - \mu\langle u, d^* p\rangle + \mu\langle p, du\rangle \\
&\qquad = \nu\|d^* u\|^2 + \lambda\|d^* p\|^2 + \lambda\|du\|^2 + \mu\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 \\
&\qquad\quad + \langle l_3 d^* u, u_{\mathfrak{B}} - d^* p + u_{\mathfrak{H}}\rangle + \langle l_5 u, u_{\mathfrak{B}} - d^* p + u_{\mathfrak{H}}\rangle.
\end{aligned} \tag{4.6}$$

With the Cauchy–Schwarz inequality and the identity $ab \le 2^{-1}(a^2 + b^2)$ we bound the last line from (4.6):

$$\begin{aligned}
&\langle l_3 d^* u, u_{\mathfrak{B}} - d^* p + u_{\mathfrak{H}}\rangle + \langle l_5 u, u_{\mathfrak{B}} - d^* p + u_{\mathfrak{H}}\rangle \\
&\qquad \le \frac{c^2}{2\nu}(\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 + \|d^* p\|^2) + \frac{\nu}{2}\|d^* u\|^2 \\
&\qquad + \frac{c^2}{2}(\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 + \|d^* p\|^2) + \frac{1}{2}\|u\|^2.
\end{aligned}$$

Since $\|u_{\mathfrak{B}^*}\|^2 \leq c_p\|du\|^2$ and $\|u\|^2 = \|u_{\mathfrak{B}^*}\|^2 + \|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2$ we have:

$$\begin{aligned}(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) \geq (\mu - \frac{c^2}{2\nu} - \frac{c^2}{2} - \frac{1}{2})\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 + \frac{\nu}{2}\|d^*u\|^2 \\ + (\lambda - \frac{c^2}{2\nu} - \frac{c^2}{2})\|d^*p\|^2 + (\lambda - \frac{c_p}{2})\|du\|^2.\end{aligned}$$

Using once again the Poincaré inequality to bound $\|u_{\mathfrak{B}^*}\|$ by $\|du\|$ and $\|p_{\mathfrak{B}}\|$ by $\|d^*p\|$ (on the dual complex), and since $\|p_{\mathfrak{B}}\| = \|p\|$ (as $p \in V^3 \cap \mathfrak{H}^{3\perp}$) we have for λ and μ large enough (solely depending on c and c_p):

$$\begin{aligned}(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) &\gtrsim \|u\|^2 + \|du\|^2 + \|d^*u\|^2 + \|p\|^2 + \|d^*p\|^2 \\ &\gtrsim \|(u,p)\|^2_{V^2\cap V_2^*\times V_3^*}.\end{aligned}$$

Clearly $\|(v,q)\|_{V_2^*\times W^3} \leq \|(u,p)\|_{V^2\cap V_2^*\times V_3^*}$ as $\|d^*v\| = \|d^*u_{\mathfrak{B}}\| \leq \|d^*u\|$ and $(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})$ is continuous as a bilinear form from $(V^2 \cap V_2^* \times V_3^*) \times (V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))$. The only thing left to show in order to use the Babuška–Lax–Milgram theorem is the second condition: for any $(v,q) \neq 0$ we must find (u,p) such that $(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) > 0$. We might take λ and μ such that $\lambda^2 = \mu\nu$. We consider two cases:
When $v \neq 0$, we might find $u \in \{w \in V^2 \cap V_2^* | d^*w \in V^1, dw \in V_3^*\}$ such that $((\nu d + l_3)d^* + l_5 + \mu + \nu d^*d)u = v$ by [11]. Take $p = -\nu/\lambda du$, then since $\lambda = \mu\nu/\lambda$:

$$\begin{aligned}(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) &= \langle\nu dd^*u, v\rangle + \langle(l_3d^* + l_5 + \mu)u, v\rangle - \lambda\langle d^*p, v\rangle \\ &\quad + \lambda\langle du, q\rangle + \mu\langle p, q\rangle \\ &= \langle((\nu d + l_3)d^* + l_5 + \nu d^*d)u, v\rangle \\ &\quad + \lambda\langle du, q\rangle - \frac{\mu\nu}{\lambda}\langle du, q\rangle \\ &= \langle v, v\rangle.\end{aligned}$$

When $v = 0$ take u solution of the Hodge-Dirac problem (see [12]): $du = q$, $d^*u = 0$. Then

$$(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,0)(0,q) = \lambda\langle du, q\rangle = \lambda\langle q, q\rangle > 0.$$

This shows that $(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})$ is a bounded isomorphism from $(V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp})$ to $(V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp})'$. Since $I : (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}_3^\perp) \to (V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))'$ is compact by the compactness property, $I(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})^{-1}$ is also compact. Since the spectrum of a compact operator is at most countable, we have that $Id + \eta I(\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda})^{-1}$ has a bounded inverse for all $\eta \in \mathbb{C}$ except for a countable subset. Therefore by composing to the right with $\mu\langle\cdot,\cdot\rangle + \overline{L_\lambda}$ we get that $\overline{L_\lambda} + (\mu + \eta)I$ has almost always a bounded inverse. □

Hence, up to an arbitrary small pertubation, $\overline{L_\lambda}$ is a bounded isomorphism from $(V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}_3^\perp)$ to $(V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))' = W^2 \times (W^3 \cap \mathfrak{H}^{3\perp})$.

Remark 4.1.1. *We could have left $\mathfrak{H}^3$ in the domain and the proof above would still work. However in this case, $\overline{L_\lambda}$ would never have been an isomorphism since its image cannot reach $\mathfrak{H}^3$.*

We have the same result for the dual problem.

Lemma 4.2. *$\overline{L'_\lambda}$ is (almost always) a bounded isomorphism.*

Proof. The same proof than the one of the theorem 4.1 works, the only differences will be a sign in the chosen (v, q), l_5 and l_3 substituted by l_5^* and l_3^* and $(l_3 d^* u, v)$ changed to $(l_3^* u, d^* v)$, which does not add any difficulty in the proof. □

Remark 4.2.1. *The proof of theorem 4.1 requires taking λ to be sufficiently large, however for any $f_2 \in V_2^{*'}$, $f_3 \in (W^3 \cap \mathfrak{H}^{3\perp})'$, $\lambda_0 > 0$, $\lambda_1 > 0$ we have the following equivalence:*

$$\overline{L_{\lambda_0}}(u, p) = (f_2, f_3) \Leftrightarrow \overline{L_{\lambda_1}}(u, \frac{\lambda_0}{\lambda_1} p) = (f_2, \frac{\lambda_1}{\lambda_0} f_3) \ .$$

Therefore, if $\overline{L_{\lambda_0}}$ is a bounded isomorphism for a given λ_0, it easily follows that $\overline{L_{\lambda_1}}$ is a bounded isomorphism for any $\lambda_1 > 0$, in particular for $\lambda_1 = 1$. The same argument works for $\overline{L'_\lambda}$.

4.2 Well-posedness of the continuous mixed formulation.

As we did in the unperturbed case, we introduce an auxiliary variable in the problem. The goal is to remove any d^* from the formulation, since they will not translate well in the discrete setting.

We define B by:

$$\begin{aligned} B(\mathbf{u}, \mathbf{v}) := \langle u_1, v_1 \rangle - \langle u_2, dv_1 \rangle \\ + \nu \langle du_1, v_2 \rangle - \langle u_3, dv_2 \rangle + \langle du_2, v_3 \rangle \\ + \langle l_3 u_1, v_2 \rangle + \langle l_5 u_2, v_2 \rangle + \langle u_p, v_3 \rangle + \langle u_3, v_p \rangle. \end{aligned} \tag{4.7}$$

The mixed formulation is:
Given $(f_2, f_3) \in W$, find $(u_1, u_2, u_3, u_p) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$ such that $\forall (v_1, v_2, v_3, v_p) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$,

$$B(\mathbf{u}, \mathbf{v}) = \langle f_2, v_2 \rangle + \langle f_3, v_3 \rangle. \tag{4.8}$$

For (u, p) solution of 4.2, it immediately appears that $(d^* u, u, p, 0)$ solves 4.8. Now for (u_1, u_2, u_3, u_p) solution of 4.8, the first line implies that $u_2 \in V_2^*$, $d^* u_2 = u_1$ and therefore $d^* u_2 \in V^1$. The second line implies that $u_3 \in V_3^*$. And the last implies that $u_3 \perp \mathfrak{H}^3$. Therefore $(u_2, u_3) \in D$ and it obviously solves 4.2.

We may now proceed to the proof of the well-posedness of the mixed problem. We require the regularity assumption (4.5). In the following hidden constants only depend on $\|l_3\|$, $\|l_5\|$, $\|K\|$, $\|d(d^* + l_3^*)(K)_2\|$ and on constants of Poincaré. We shall write $V = V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$.

Lemma 4.3. *For all* $(0, u_2, u_3, 0) = \mathbf{u} \in V$*, there exists* $\mathbf{z} \in V$ *such that* $\|\mathbf{z}\|_V \lesssim \|\mathbf{u}\|_V$ *and* $\forall \omega \in V, B(\omega, \mathbf{z}) = \langle \omega, \mathbf{u} \rangle$.

Proof. Take $(z_2, z_3) = K(u_2, P_{\mathfrak{H}^\perp} u_3)$, $\xi = -(\nu d^* + l_3^*) z_2$ then $\xi \in V^1$ by assumption (4.5) and by definition of K, we have $\forall \omega \in V$,

$$-\langle dz_2, \omega_3 \rangle = \langle P_{\mathfrak{H}^\perp} u_3, \omega_3 \rangle,$$

$$\langle d(\nu d^* + l_3^*) z_2 + l_5^* z_2 + d^* z_3, \omega_2 \rangle = \langle u_2, \omega_2 \rangle,$$

thus we have:

$$\begin{aligned}
B(\omega, (\xi, z_2, z_3, u_{3\mathfrak{H}})) =& \langle \omega_1, \xi \rangle - \langle \omega_2, d\xi \rangle \\
&+ \langle (\nu d + l_3)\omega_1, z_2 \rangle - \langle \omega_3, dz_2 \rangle + \langle l_5 \omega_2, z_2 \rangle \\
&+ \langle d\omega_2, z_3 \rangle + \langle \omega_3, u_{3\mathfrak{H}} \rangle + \langle \omega_p, z_3 \rangle \\
=& \langle \omega_1, \xi \rangle - \langle \omega_2, d\xi \rangle \\
&+ \langle \omega_1, (\nu d^* + l_3^*) z_2 \rangle + \langle \omega_3, P_{\mathfrak{H}^\perp} u_3 \rangle + \langle \omega_2, l_5^* z_2 \rangle \\
&+ \langle \omega_2, d^* z_3 \rangle + \langle \omega_3, u_{3\mathfrak{H}} \rangle \\
=& \langle \omega_1, \xi \rangle - \langle \omega_2, d\xi \rangle \\
&+ \langle \omega_1, (\nu d^* + l_3^*) z_2 \rangle - \langle \omega_2, d(\nu d^* + l_3^*) z_2 \rangle \\
&+ \langle \omega_2, u_2 \rangle + \langle \omega_3, P_{\mathfrak{H}^\perp} u_3 \rangle + \langle \omega_3, u_{3\mathfrak{H}} \rangle \\
=& - \langle \omega_1, (\nu d^* + l_3^*) z_2 \rangle + \langle \omega_2, d(\nu d^* + l_3^*) z_2 \rangle \\
&+ \langle \omega_1, (\nu d^* + l_3^*) z_2 \rangle - \langle \omega_2, d(\nu d^* + l_3^*) z_2 \rangle \\
&+ \langle \omega_3, u_3 \rangle + \langle \omega_2, u_2 \rangle \\
=& \langle \omega_2, u_2 \rangle + \langle \omega_3, u_3 \rangle.
\end{aligned}$$

Moreover since $\overline{L_1'}$ is a bounded isomorphism, so is K thus

$$\begin{aligned}
\|\mathbf{z}\|_V &= \|\nu d^* z_2\| + \|l_3^* z_2\| + \|z_2\|_V + \|z_3\| + \|d\xi\| + \|u_{3\mathfrak{H}}\| \\
&\lesssim \|z_2\|_{V^2 \cap V_2^*} + \|z_3\|_{V_3^*} + \|d\xi\| + \|u_{3\mathfrak{H}}\| \\
&\lesssim (\|K\| + \|d(d^* + l_3^*)K\| + 1)\|u\| \lesssim \|u\|_V.
\end{aligned}$$

□

Lemma 4.4. *For all* $\mathbf{u} \in V$*, there exists* $\mathbf{z} \in V$ *such that* $\|\mathbf{z}\|_V \lesssim \|\mathbf{u}\|_V$ *and* $B(\mathbf{u}, \mathbf{z}) \gtrsim \|du_1\|^2 + \|du_2\|^2 + \|u_1\|^2 + \|u_p\|^2 - \|u_2\|^2 - \|u_3\|^2$.

Proof. Let $c = max(\|l_3\|, \|l_3\|)$,

$$\begin{aligned}
B(\mathbf{u}, (\nu u_1, u_{2\mathfrak{B}}, du_2, 0)) =& \nu \langle u_1, u_1 \rangle - \nu \langle u_2, du_1 \rangle + \nu \langle du_1, u_{2\mathfrak{B}} \rangle \\
&- \langle u_3, du_2 \rangle + \langle du_2, du_2 \rangle + \langle l_3 u_1, u_{2\mathfrak{B}} \rangle \\
&+ \langle l_5 u_2, u_{2\mathfrak{B}} \rangle + \langle u_p, du_2 \rangle + \langle u_3, 0 \rangle \\
\geq& \frac{1}{2}(\nu \|u_1\|^2 + \|du_2\|^2) - \left(\frac{c^2}{2\nu} + c \right) \|u_2\|^2 \\
&- \frac{1}{2} \|P_{\mathfrak{H}^\perp} u_3\|^2,
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
B(\mathbf{u},(0,du_1,0,0)) &= \nu\langle du_1,du_1\rangle + \langle l_3u_1,du_1\rangle + \langle l_5u_2,du_1\rangle \\
&\geq \frac{1}{2}\nu\|du_1\|^2 - \frac{c^2}{\nu}(\|u_1\|^2+\|u_2\|^2),
\end{aligned} \tag{4.10}
$$

$$
B(\mathbf{u},(0,0,u_p,0)) = \langle u_p,u_p\rangle. \tag{4.11}
$$

We get the expected result by combining (4.9), (4.10) and (4.11). Bounds on norms are easily checked, for exemple:

$$
\|(0,du_1,0,0)\|_V = \nu\|du_1\| + \nu\|ddu_1\| = \nu\|du_1\| \lesssim \|\mathbf{u}\|_V.
$$

□

Combining the two preceding lemmas gives

$$
\forall \mathbf{u}\in V,\ \sup_{\|\mathbf{v}\|_V=1} |B(\mathbf{u},\mathbf{v})| \gtrsim \|\mathbf{u}\|_V\ . \tag{4.12}
$$

Lemma 4.5. *For any* $\mathbf{v}\in V$*, there is* $\mathbf{u}\in V$ *such that* $B(\mathbf{u},\mathbf{v})>0$.

Proof. Given $\mathbf{v}\neq 0\in V$, if $v_2=0$,$v_3=0$ and $v_p=0$ take $\mathbf{u}=(v_1,0,0,0)$ and

$$
B(\mathbf{u},\mathbf{v}) = \langle v_1,v_1\rangle > 0.
$$

Else take $(u_2,u_3)=\overline{L_1}^{-1}(v_2,P_{\mathfrak{H}^\perp}v_3)+(0,v_p)$, $u_p=P_{\mathfrak{H}}v_3$ and $u_1=d^*u_2$ ($u_1\in V^1$ by assumption (4.5)) then

$$
B(\mathbf{u},\mathbf{v}) = \langle v_2,v_2\rangle + \langle v_3,v_3\rangle + \langle q,q\rangle > 0.
$$

□

Since $B: V\times V\to\mathbb{R}$ is cleary continous we may apply the Babuška–Lax–Milgram theorem with lemma 4.5 and (4.12).

4.3 Discrete well-posedness.

We introduce the notation $V_h = V_h^1\times V_h^2\times V_h^3\times\mathfrak{H}_h^3$. The discrete variational problem is the same as the continuous, substituting V by V_h. Hence we shall still use the notation B, this time as a function from $V_h\times V_h$ to $\mathbb{R}$.

Considering the dual problem to the unperturbed problem with $\nu=1$, we have:
$D_0=\{(u,p)\in(V^2\cap V_2^*)\times V_*^3 |\ d^*u\in V^1\}$ and $L_0'(u,p)=(dd^*u+d^*p,-du)$. Let K_0 the solution operator of the dual problem L_0', we have $K_0=(L_0')^{-1}$ when L_0' is viewed as an isomorphism from $P_{\mathfrak{H}^\perp}D_0$ to $P_{\mathfrak{H}^\perp}(W^2\times W^3)$ and K_0 is extended by 0 on $\mathfrak{H}$. Explicitly we have the decomposition: $\forall(f_2,f_3)\in W^2\times W^3$,

$$
(f_2,f_3) = (dd^*(K_0)_2(f_2,f_3)+d^*(K_0)_3(f_2,f_3), -d(K_0)_2(f_2,f_3))+(P_{\mathfrak{H}}f_2,P_{\mathfrak{H}}f_3) \tag{4.13}
$$

and a similar expression for their discrete counterparts.
Therefore $\forall (z_2, z_3) \in D_0, (P_{\mathfrak{H}^\perp} z_2, P_{\mathfrak{H}^\perp} z_3) = L_0' K_0(z_2, z_3) = K_0 L_0'(z_2, z_3)$.

As mentioned before this problem is closely related to [11]. Since the mixed variable part is almost unchanged we shall use the generalized canonical projection Π_h from [11] and we state its properties below.

Lemma 4.6. *Under the condition of the Theorem 5.1 in [11]:*

- Π_h *is a projection uniformly bounded in the V-norm.*
- $d\Pi_h = P_{\mathfrak{B}_h} d$.
- $\forall w \in V^k, \|(I - \Pi_h)w\| \lesssim \|(I - \pi_h)w\| + \eta_0' \|dw\|$.
- $\forall w, v \in V^k, |\langle (I - \Pi_h)w, v\rangle| \lesssim (\|(I - \pi_h)w\| + \eta_0' \|dw\|)(\|(I - \pi_h)w\| + \eta_0' \|dw\|) + \alpha_0' \|dv\| \|dw\|$.

where $\eta_0', \alpha_0' \to 0$ as h does, they are given, along the proof in [11].

Definition 4.7. *We shall use the following notations in this section:*

$$\delta_0 = \|(I - \pi_h)K_0\|, \mu_0 = \|(I - \pi_h)P_{\mathfrak{H}}\|,$$

$$\eta_0 = \max\{\|(I - \pi_h)dK_0\|, \|(I - \pi_h)d^*(K_0)_2\|\},$$
$$\alpha_0 = \eta_0(1 + \eta_0) + \mu_0 + \delta_0 + \mu_0\delta_0 + \eta_0',$$
$$\eta = \max\{\delta_0, \mu_0, \eta_0, \|(I - \pi_h)l_3^*(K)_2\|, \|(I - \pi_h)dl_3^*(K)_2\|\}.$$

Lemma 4.8. *We have:*

$$\|K_0 - K_{0h}P_h\| \lesssim \alpha_0,$$

$$\|dK_0 - dK_{0h}P_h\| + \|d^*(K_0)_2 - d_h^*(K_{0h})_2 P_h\| \lesssim \eta_0.$$

Proof. The idea is for $(f_2, f_3) \in (W^2 \times W^3)$ to apply the error estimate 3.4 for $(u_2, u_3) = K_0(f_2, f_3)$, $(\phi_2, \phi_3) = P_{\mathfrak{H}}(f_2, f_3)$, $u_1 = d^* u_2$, $(u_{2h}, u_{3h}) = K_{0h}P_h(f_2, f_3)$, $(\phi_{2h}, \phi_{3h}) = P_{\mathfrak{H}_h} P_h(f_2, f_3)$, $u_{1h} = d_h^* u_{2h}$. Unfortunately the crude estimate 3.4 does not allow to conclude because of the term $E(du_1)$. We need improved estimates that gives

$$\|u_2 - u_{2h}\| + \|u_3 - u_{3h}\| \lesssim (1 + \mu_0)E(u_2) + E(u_3) + \eta_0 E(u_1)$$
$$(\eta_0^2 + \delta_0 + \eta_0')E(du_1) + \eta_0' E(\phi_2) + E(du_2),$$
$$\|du_2 - du_{2h}\| + \|u_1 - u_{1h}\| \lesssim E(du_2) + E(u_1) + \eta_0 E(du_1)$$

and may conclude since $E(u_2) + E(u_3) \leq \delta_0 \|(f_2, f_3)\|$, $E(u_1) + E(du_2) \leq \eta_0 \|(f_2, f_3)\|$ and $E(du_1) + E(\phi_2) \leq \|(f_2, f_3)\|$, the last coming from $du_1 = P_{\mathfrak{B}} f_2$. Theses proofs are lengthy, technical and mostly follow those in theorem 3.11 of [5] with some adaptations such as $P_{\mathfrak{B}_2^*} = (d^*)_2 K_0$ instead of $P_{\mathfrak{B}_2^*} = d^* K d$, therefore we shall not duplicate them here. □

Lemma 4.9. *For* $f \in \mathfrak{H}^\perp$ *we have* $\|P_{\mathfrak{H}_h} f\| \lesssim \mu_0 \|f\|$

Proof. We recall the mixed formulation for the Hodge-Laplacian problem: the bilinear form is given by:

$$B((\sigma,u,p),(\tau,v,q)) = \langle\sigma,\tau\rangle - \langle u,d\tau\rangle + \langle d\sigma,v\rangle + \langle du,dv\rangle + \langle p,v\rangle + \langle u,q\rangle.$$

In the continuous case the bilinear form acts on $(V^1\times V^2\times\mathfrak{H}^2)^2$ and on $(V_h^1\times V_h^2\times\mathfrak{H}_h^2)^2$ in the discrete case. Let $(\sigma,u,p)\in V^1\times V^2\times\mathfrak{H}^2$ be such that $\forall(\tau,v,q)\in V^1\times V^2\times\mathfrak{H}^2$, $B((\sigma,u,p),(\tau,v,q)=(f,v)$, and $(\sigma_h,u_h,p_h)\in V_h^1\times V_h^2\times\mathfrak{H}_h^2$ be such that $\forall(\tau,v,q)\in V_h^1\times V_h^2\times\mathfrak{H}_h^2$, $B((\sigma_h,u_h,p_h),(\tau,v,q)=(f,v)$. Then [15] gives the error estimate $\|p-p_h\|\lesssim E(p)+\mu_0E(d\sigma)$. We have $P_{\mathfrak{H}}f=p=0$ and $P_{\mathfrak{H}_h}f=p_h$ thus

$$\|P_{\mathfrak{H}_h}f\| = \|p-p_h\| \lesssim 0+\mu_0E(d\sigma)\lesssim\mu_0\|f\|$$

since $E(p)=0$ and by the well-posedness of the Hodge-Laplacian problem for the last inequality. □

Theorem 4.10. *For $z=(z_2,z_3)\in D_0$, let $z_h=(z_{2h},z_{3h})=K_{0h}P_hL_0'z+P_{\mathfrak{H}_h}P_{\mathfrak{H}}z$, we have:*

$$\|z-z_h\|\lesssim\alpha_0\|L_0'z\|, \|d(z-z_h)\|+\|d^*z_2-d_h^*z_{2h}\|\lesssim\eta_0\|L_0'z\|,$$

$$\|P_h(dd^*z_2+d^*z_3)-(dd_h^*z_{2h}+d_h^*z_{3h})\|\le\mu_0\|L_0'z\|.$$

Proof. The same proof as in theorem 5.2 of [11] works, it is duplicated here since the proof is short and demonstrates that our change in the definition of L_0' does not interfere.

$$\begin{aligned} z-z_h &= (P_{\mathfrak{H}^\perp}z-P_{\mathfrak{H}_h^\perp}z_h)+(P_{\mathfrak{H}}z-P_{\mathfrak{H}_h}z_h)\\ &= (K_0-K_{0h}P_h)L_0'z+(I-P_{\mathfrak{H}_h})P_{\mathfrak{H}}z. \end{aligned}$$

As $P_{\mathfrak{H}_h}P_{\mathfrak{H}}=P_{\mathfrak{Z}_h}P_{\mathfrak{H}}$ and $\pi_h\mathfrak{Z}\subset\mathfrak{Z}_h$ we have:

$$\|(I-P_{\mathfrak{H}_h})P_{\mathfrak{H}}z\|\le\|(I-\pi_h)P_{\mathfrak{H}}z\|\le\mu_0\|z\|.$$

We then get the expected result by 4.8. The second part follows directly from 4.8. Finally for the last estimate, (4.13) gives

$$dd_h^*z_{2h}+d_h^*z_{3h}=(L_{0h}')_2z_h=(L_{0h}'K_{0h}P_h)_2L_0'z=(P_{\mathfrak{B}_h}+P_{\mathfrak{B}_h^*})P_h(L_0')_2z$$

so:

$$\begin{aligned} \|P_h(dd^*z_2+d^*z_3)-(dd_h^*z_{2h}+d_h^*z_{3h})\| &= \|(I-(P_{\mathfrak{B}_h}+P_{\mathfrak{B}_h^*}))P_h(L_0')_2z\|\\ &= \|P_{\mathfrak{H}_h}(L_0')_2z\|. \end{aligned}$$

And we conclude with lemma 4.9. □

Given $\mathbf{u}\in V_h$, define $g=(u_2,P_{\mathfrak{H}^\perp}u_3)$, $z=Kg$, $\xi=-(d^*+l_3^*)z_2$ and $\mathbf{z}=(\xi,z,P_{\mathfrak{H}}u_3)$.

Theorem 4.11. *There is $\mathbf{z_h}\in V_h$ such that $\forall\omega\in V_h$, $\|\mathbf{z_h}\|_V\lesssim\|\mathbf{z}\|_V$ uniformly in h and $|B(\omega,\mathbf{z}-\mathbf{z_h})|\le\epsilon_h\|\omega\|_V\|\mathbf{u}\|$, where $\epsilon_h\to0$ as $h\to0$.*

Proof. Take $z_h = K_{0h} P_h L_0' z + P_{\mathfrak{H}_h} P_{\mathfrak{H}} z$, $\xi_h = -d_h^* z_{h2} - \Pi_h l_3^* z_2$. By 4.10 we have:

$$\|z - z_h\| \lesssim \alpha_0 \|g\|, \ \|d(z - z_h)\| \lesssim \eta_0 \|g\|$$

and also:

$$\|\xi - \xi_h\| \leq \|d^* z_2 - d_h^* z_{h2}\| + \|(I - \Pi_h) l_3^* z_2\|$$

and $\|d^* z_2 - d_h^* z_{h2}\| \lesssim \eta_0 \|g\|$ by 4.10.
Using 4.6 and the boundedness of $d\, l_3^*\, K$ we get

$$\|(I - \Pi_h) l_3^* z_2\| \lesssim \|(I - \pi_h) l_3^* z_2\| + \eta_0' \|d l_3^* z_2\| \lesssim (\eta + \eta_0')\|g\|.$$

Finally, since $\forall \omega_2 \in V_h^2 \subset V^2$

$$\langle d\omega_2, z_3 \rangle = \langle \omega_2, d^* z_3 \rangle, \langle d\omega_2, z_{3h} \rangle = \langle \omega_2, d_h^* z_{3h} \rangle \ ,$$

we have $\forall \omega \in V_h$:

$$\begin{aligned}
|B(\omega,(\xi - \xi_h, z_2 - z_{h2}, z_3 - z_{h3}, P_{\mathfrak{H}} u_3 - P_{\mathfrak{H}_h} P_{\mathfrak{H}} u_3))| = \ & |\langle \omega_1, \xi - \xi_h \rangle \\
& - \langle \omega_2, d(\xi - \xi_h) \rangle + \langle (\nu d + l_3)\omega_1, z_2 - z_{h2} \rangle - \langle \omega_3, d(z_2 - z_{h2}) \rangle \\
& + \langle l_5 \omega_2, z_2 - z_{h2} \rangle + \langle d\omega_2, z_3 - z_{h3} \rangle \\
& + \langle \omega_p, z_3 - z_{h3} \rangle + \langle \omega_3, P_{\mathfrak{H}} u_3 - P_{\mathfrak{H}_h} P_{\mathfrak{H}} u_3 \rangle| \\
\lesssim \ & \|\omega\|_V (\|\xi - \xi_h\| + 2\|z_2 - z_{h2}\| + \|d(z_2 - z_{h2})\| \\
& + \|z_3 - z_{h3}\| + \|(I - P_{\mathfrak{H}_h}) P_{\mathfrak{H}} u_3\|) + |\langle \omega_2, d^* z_3 - d_h^* z_{3h} - d(\xi - \xi_h) \rangle| \\
\lesssim \ & \|\omega\|_V \left[(\eta + 2\eta_0 + 2\alpha_0 + \eta_0' + \mu_0)\|\mathbf{u}\| + \|d^* z_3 - d_h^* z_{3h} - d(\xi - \xi_h)\| \right].
\end{aligned}$$

Since η,η_0,η_0',μ_0 and α_0 all converge toward 0 when $h \to 0$, the only thing left to prove is that $\|d^* z_3 - d_h^* z_{3h} - d(\xi - \xi_h)\| \lesssim \epsilon \|\mathbf{u}\|$ where $\epsilon \to 0$ when $h \to 0$ with

$$\|d^* z_3 - d_h^* z_{3h} - d(\xi - \xi_h)\| \leq \|d^* z_3 - d_h^* z_{3h} + d(d^* z_2 - d_h^* z_{h2})\| + \|d(I - \Pi_h) l_3^* z_2\|.$$

Lemma 4.10 gives:

$$\|d_h^* z_{h2} + d_h^* z_{3h} - d(d^* z_2 + d^* z_3)\| \lesssim \mu_0 \|g\|.$$

And we conclude with lemma 4.6 since we may find a bounded cochain projection π_h such that $\pi_h d = P_{\mathfrak{B}_h} d$ (see [5] Theorem 3.7) so

$$\|d(I - \Pi_h) l_3^* z_2\| = \|(I - P_{\mathfrak{B}_h}) d l_3^* z_2\| \lesssim \|(I - \pi_h) d l_3^* z_2\| \leq \eta \|g\|.$$

□

Lemma 4.12. *For all $\mathbf{u} \in V_h$ and $\mathbf{z} \in V$ defined in theorem 4.11, there exists a constant c independent of h and $\sigma \in V_h$ such that $\|\sigma\|_V \lesssim \|\mathbf{u}\|_V$ and $B(\mathbf{u}, \mathbf{z} + \sigma) \geq c\|\mathbf{u}\|_V^2$.*

Proof. Starting from lemma 4.3, we construct σ in the same way as we did in lemma 4.4 in the continuous case, we must simply add

$$B(\omega, (0, 0, -P_{\mathfrak{H}_h} z_3, 0)) = -\langle \omega_p, z_3 \rangle$$

to correct the harmonic part. □

Theorem 4.13. *There are two positive constants h_0 and C_0 such that for all $h \in (0, h_0]$, there exists a unique $\mathbf{u} \in V_h$ such that $\forall \mathbf{v} \in V_h$, $B(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$. Moreover we have $\|\mathbf{u}\|_V \leq C_0 \|\mathbf{f}\|$.*

Proof. For $\mathbf{u} \in V_h$ and $\mathbf{z}$, $\mathbf{z_h}$ defined in theorem 4.11, lemma 4.12 gives $\sigma \in V_h$ with $\|\sigma\|_V \lesssim \|u\|_V$ and a constant c independent of h such that:

$$|B(\mathbf{u}, \mathbf{z} + \sigma) \geq c\|\mathbf{u}\|_V^2.$$

By 4.11 we have for a constant b independent of h.

$$B(\mathbf{u}, \mathbf{z_h} - \mathbf{z}) \leq \epsilon_h b \|\mathbf{u}\|_V^2.$$

Therefore:

$$\begin{aligned} |B(\mathbf{u}, \sigma + \mathbf{z_h})| &= |B(\mathbf{u}, \mathbf{z} + \sigma) + B(\mathbf{u}, \mathbf{z_h} - \mathbf{z})| \\ &\geq |B(\mathbf{u}, \mathbf{z} + \sigma)| - |B(\mathbf{u}, \mathbf{z_h} - \mathbf{z})| \\ &\geq c\|\mathbf{u}\|_V^2 - \epsilon_h b \|\mathbf{u}\|_V^2 \geq (c - \epsilon_h b)\|\mathbf{u}\|_V^2. \end{aligned}$$

Since $\epsilon_h \to 0$ as $h \to 0$ we might find h_0 such that $\forall h \in (0, h_0]$, $c - \epsilon_h b > c - \epsilon_{h_0} b > 0$.

By 4.11 and by the expression of σ we find:

$$\|\sigma + \mathbf{z_h}\|_V \lesssim \|\mathbf{u}\|_V + \|\mathbf{z}\|_V \lesssim \|\mathbf{u}\|_V.$$

This ends the proof since V_h has finite dimension. □

Corollary 4.13.1. *Under the same assumptions, if $h \leq h_0$, the discrete solution satisfies:*

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u_h}\|_V \lesssim \\ &\quad E(u_1) + E(du_1) + E(u_2) + E(du_2) + E(u_3) + E(u_p) + \mu_0 E(P_{\mathfrak{B}} u_3). \end{aligned}$$

Proof. The proof is the same as in section 3.4 □

5 Conserved quantities.

Lastly we prove that our scheme does indeed verify properties mentioned in the introduction as well as the regularity assumption (4.5). Before that we state the explicit construction of the scheme (2.9): we take $l_5 u^n = \frac{1}{\delta t} u^n + \frac{1}{2}\omega^{n-1} \times u^n$ and $l_3 \omega^n = \frac{1}{2}\omega^n \times u^{n-1}$ where ω^{n-1} and u^{n-1} are given by the previous step. In the following we keep the notations of (2.9).

5.1 Regularity assumptions.

We prove here that in our case assumptions (4.5) are valid if $u^{n-1} \in H^2(\Omega)$ and if the domain is smooth enough to have $H_0(curl, \Omega) \cap H(div, \Omega) \subset H^1(\Omega)$, as discussed in part 3.2 of [1]. First we need the following lemma:

Lemma 5.1. *If $B \in H^1(\Omega)$ and $A \in H^2(\Omega)$ then $A \times B \in H^1(\Omega)$ for a smooth enough domain Ω of $\mathbb{R}^3$ (or $\mathbb{R}^2$).*

Proof. We have $H^1(\Omega) \subset L^6(\Omega)$ and $H^2(\Omega) \subset C^{0,\frac{1}{2}}(\Omega)$ by Sobolev Embedding theorems thus $\forall i, j, k \in \{x, y, z\}$, $\partial_i A_j \in L^6$, $\partial_i B_j \in L^2$ and terms of the form $A_i \partial_j B_k$ are the product of a bounded function with a function in L^2 and those of the form $B_i \partial_j A_k$ are the product of two functions in L^6. □

Going back to our problem, we place ourself in the continuous setting with $V^1 = H(curl)$, $V^2 = H(div)$, $V^3 = L^2$ and $\mathfrak{H}^3 = 0$, thus K is a bounded isomorphism from $L^2 \times L^2$ to $(H(div) \cap H_0(curl)) \times H_0^1 \subset H^1 \times H^1$. If we assume $u \in H^2(\Omega)$ and $v \in H^1 \subset L^2$ then using the scalar triple product we have $\forall \sigma \in H(curl)$:

$$2\langle l_3 \sigma, v\rangle = \int (\sigma \times u^{n-1}) \cdot v = \int (u^{n-1} \times v) \cdot \sigma = 2\langle \sigma, l_3^* v\rangle$$

thus $l_3^* v = \frac{1}{2} u^{n-1} \times v$ on H^1, l_3^* maps H^1 in H^1 by lemma 5.1 and

$$l_3^*(K)_2(W) \subset H^1 \subset H(curl) = V^1.$$

Thus we have $\|(\nabla \times) l_3^*(K)_2\|_{W \to L^2} \lesssim \|(K)_2\|_{W \to H^1}$ and by the boundedness of $\|K\|_{L^2 \times L^2 \to H^1 \times H^1}$ we get the boundedness of $\|d l_3^*(K)_2\|_{W \to L^2}$. Finally we have $\nu d d^*(K)_2 = I - d l_3^*(K)_2 + l_5^*(K)_2 + d^*(K)_3$ as distributions and from the $L^2 \to L^2$ boundedness of the right-hand side we get both $\nu d^*(K)_2(W) \subset H(curl) = V^1$ and the boundedness of $\|d d^*(K)_2\|_{W \to L^2}$.

The same argument applied to $\overline{L}^{-1}$ shows that $d^*(\overline{L}^{-1})_2(W) \subset V^1$. Hence (4.5) is fulfilled.

Remark 5.1.1. *Assuming $u^{n-1} \in H^2(\Omega)$ is very mild as any solution u of (4.2) must have $\nabla \times u \in H(curl, \Omega)$, $\nabla \cdot u = 0$ thus $\Delta u \in L^2(\Omega)$. Hence by elliptic regularity for Ω smooth enough and if u satisfies appropriate boundary conditions then $u \in H^2$.*

5.2 Pointwise divergence free.

This a simple fact that follows from the use of a discrete subcomplex, we have

$$d_h : V_h^2 \to V_h^3 = d_{|V_h^2}.$$

Hence the operator $d_h = div$ used in the formulation is the restriction of the continuous operator to the discrete space and its image is contained in the discrete space V_h^3.

From (2.8) we have $\forall q \in V_h^3$, $\langle \nabla \cdot u^n + \phi^n, q\rangle = 0$ with $\nabla \cdot u^n \perp \phi^n$ and $\nabla \cdot u^n \in V_h^3$ by construction, therefore taking $q = \nabla \cdot u^n$ we have $\langle \nabla \cdot u^n + \phi^n, \nabla \cdot u^n\rangle = \|\nabla \cdot u^n\|^2 = 0$.
We will check this property in the numerical simulation in section 6.

5.3 Pressure-robustness.

A scheme is called pressure-robust ([13], [18], [17]) if only the pressure (and not the velocity) changes when the external forces acting on the system are modified by a gradient.

Every function $f \in L^2$ may be written as $\nabla \times g + \nabla p$ for some functions g and p (if there are no harmonic form, for example if the domain is simply connected). In a bounded domain we only have uniqueness with correct boundary conditions on g and p. As long as the same conditions are given in the complex, we have by viewing f as a 2-form, $\nabla \times g = P_{\mathfrak{B}} f$ and $\nabla p = P_{\mathfrak{B}^*} f$.

If f and $\bar{f}$ differ only by a gradient we must have $P_{\mathfrak{B}}(f - \bar{f}) = 0$ so $\forall g \in V^1, \langle f - \bar{f}, \nabla \times g\rangle = 0$ and in particular

$$\forall g_h \in V_h^1 \subset V^1, \langle f - \bar{f}, \nabla \times g_h\rangle = 0 \ .$$

Therefore $P_{\mathfrak{B}_h}(f - \bar{f}) = 0$ and since we assumed that there were no harmonic 2-forms, $P_h(f - \bar{f}) = P_{\mathfrak{B}_h^*}(f - \bar{f})$ and we can find $\xi \in V_h^3$ such that $\xi \perp \mathfrak{H}_h$ and $d_h^* \xi = -P_h(f - \bar{f})$.

We write $(\omega^n, u^n, p^n, \phi^n)$ the solution at step n for the external force f and $(\bar{\omega}^n, \bar{u}^n, \bar{p}^n, \bar{\phi}^n)$ the solution at the same step for the external force $\bar{f}$ both with the same initial condition and boundaries conditions. We wish to prove that $\forall n$, $\omega^n = \bar{\omega}^n$ and $u^n = \bar{u}^n$, by a simple induction on n. This is true for $n = 0$ since we used the same initial condition for both problem, and if $\omega^{n-1} = \bar{\omega}^{n-1}$ and $u^{n-1} = \bar{u}^{n-1}$ then $(0, 0, \xi, 0)$ for ξ defined above (for $f, \bar{f}$ taken at the correct time if they are time-dependent) verifies: $\forall (\tau, v, q, \chi) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h$:

$$\begin{aligned}
\langle 0, \tau\rangle - \langle 0, \nabla \times \tau\rangle &= 0, \\
\langle 0, v\rangle + \delta t(\mu\langle \nabla \times 0, v\rangle + \langle \frac{1}{2}(0 \times u^{n-1} + \omega^{n-1} \times 0), v\rangle & \\
-\langle \xi, \nabla \cdot v\rangle) &= \langle u^{n-1} - u^{n-1}, v\rangle + \delta t\langle f - \bar{f}, v\rangle, \\
\langle \nabla \cdot 0, q\rangle &= 0, \\
\langle 0, q\rangle + \langle \xi, \chi\rangle &= 0.
\end{aligned}$$

Therefore by linearity and uniqueness of the solution we have $(\omega^n, u^n, p^n, \phi^n) = (\bar{\omega}^n, \bar{u}^n, \bar{p}^n, \bar{\phi}^n) + (0, 0, \xi, 0)$ and $\omega^n = \bar{\omega}^n, u^n = \bar{u}^n$.

Remark 5.1.2. *We used the fact that no step depends on the pressure of any previous step and that there are no harmonic 2-forms. If it were not the case then the harmonic part of $f - \bar{f}$ would have to be reached by $l_5 u$ or $l_3 \omega$ which would influence the field u (in a non trivial way since neither l_3 nor l_5 respect the discrete Hodge-decomposition).*

6 Numerical simulations.

We finaly give the results of three numerical simulations done to demonstrate the validity of our scheme. The norm of the divergence of the fluid velocity is checked everytime and we found a value in the order of the machine accuracy. The first simulation aims to verify the pressure-robustness property, the second is based on an exact and fully 3D solution of the Navier-Stokes equation constructed in [3]. We use it to check the convergence rate in space, first on a steady problem then on a unsteady problem. The last simulation focusses on a system of two rotating cylinders and shows the good agreement with the theory on the value of the critital speed of the inner cylinder at which Taylor vortices appear and is based on [9], [2] and [4]. In any case, we took a unit kinematic viscosity and polynomials of degree 2.

6.1 Pressure Robustness.

We wish to verify that if the external forces acting on two flows differ only by a gradient, then only the pressure differs between the flows. We took the Stokes no-flow problem in a (3-dimensional) glass from [16]. The setup is rather simple, the mesh is a cylinder along the z axis of height 2.0, base radius 1.0 and top radius 1.5 and the force f derives from a potential:

$$f = \frac{\nabla \Phi}{\int_\Omega \Phi}, \Phi = z^\gamma$$

for $\gamma = 1, 2, 4, 7$. In any case we found a velocity equal to zero at the order of the machine accuracy. This is not trivial as the same test conducted with Taylor-Hood elements (P_2/P_1) gave a velocity of norm up to 3.3×10^{-4} for $\gamma = 7$.

6.2 Convergence rate to an exact solution.

We have conducted a convergence analysis with an exact solution. The expression for the solution is taken from [3] and depends on two real parameters a and b. It is given by:

$$u = \begin{bmatrix} -a(\exp(ax)\sin(ay+dz) + \exp(az)\cos(ax+dy))\exp(-d^2t) \\ -a(\exp(ay)\sin(az+dx) + \exp(ax)\cos(ay+dz))\exp(-d^2t) \\ -a(\exp(az)\sin(ax+dy) + \exp(ay)\cos(az+dx))\exp(-d^2t) \end{bmatrix}.$$

We have performed two sets of experiments: the first with $a = 2$ and $d = 0$ and the second with $a = 2$ and $d = 1$, the velocity field of the first is shown in figure 2. The mesh consists of a cylinder with maximum edge size ranging from 0.37 to 0.13. In the latter case, computations were done for t between 0 and 1 with a timestep of 1.0×10^{-3} and we found a rate of convergence in space of order 2.0 in both cases which is in agreement with the theory. Figure 3 shows the convergence with a log-log scale.

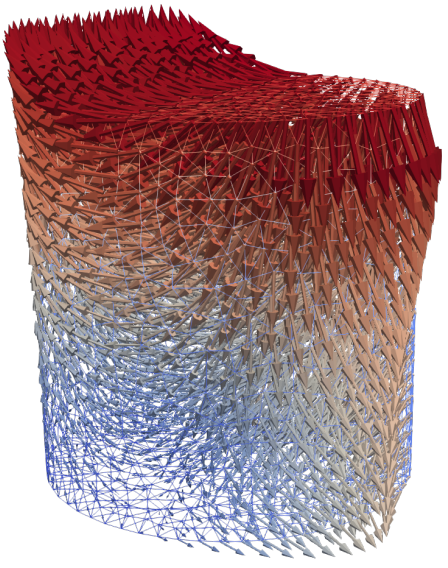

Figure 2: Velocity field for $a = 2$ and $d = 0$ with logarithmic scaling.

Remark 6.0.1. *Despite the fact that the time discretization is Euler implicit and only first order by taking the time step small enough we can observe the second order convergence rate in space.*

6.3 Taylor-Couette flow.

This test focusses on Taylor-Couette flow, we follow the work of [9], [2] and [4]. The geometry consists in two concentric cylinders of constant radius R_i for the inner and R_o for the outer, rotating at angular velocities of Ω_i and Ω_o respectively and both of height h. The system is closed by two nonmoving lids at the bottom and top ends. We characterized the system by two geometric parameters: $\eta = R_i/R_o$ and $\Lambda = h/d$ with the gap $d = R_o - R_i$. We also need to define two quantities: the inner Reynolds number $Re_i = \Omega_i R_i d/\nu$ and the outer Reynolds number $Re_0 = \Omega_o R_o d/\nu$ where ν is the kinematic viscosity. It is a well known fact that (for a infinite height h) at low speed the flow is steady and fully azimuthal and that vortices start to form at a critical speed. Since h is finite we expect to see vortices near the lids for speeds way under the critical value (they are however fundamentaly different from the Taylor vortices, see [4]).

The whole purpose is to compare results obtained from our code with those derived in [2] and [4]. We check the value of Re_i at which the transition occurs for various values of Re_o and η. We see very good

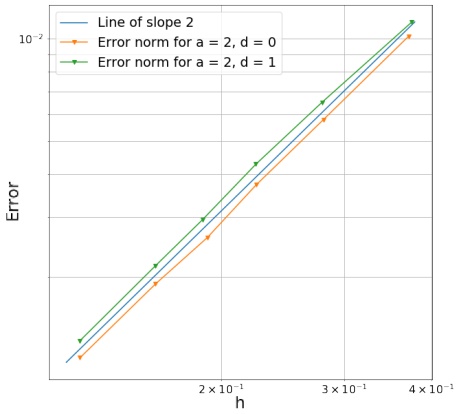

Figure 3: Convergence rate on a log log scale.

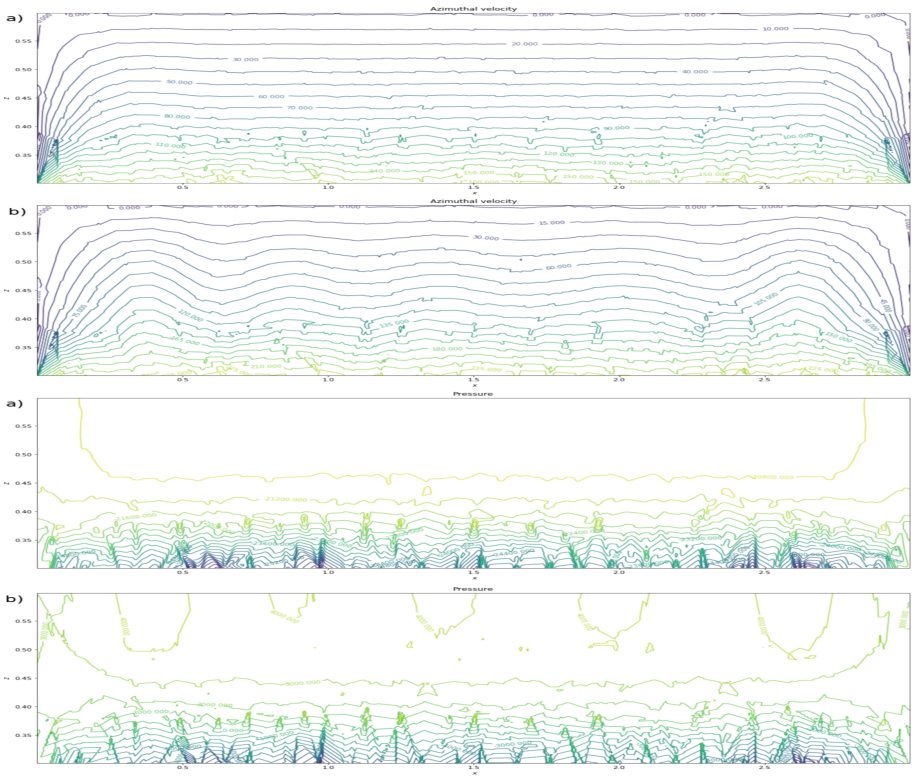

Figure 4: Isolines for azimuthal velocity and the pressure at $Re_i = 50$ and $Re_i = 72$.

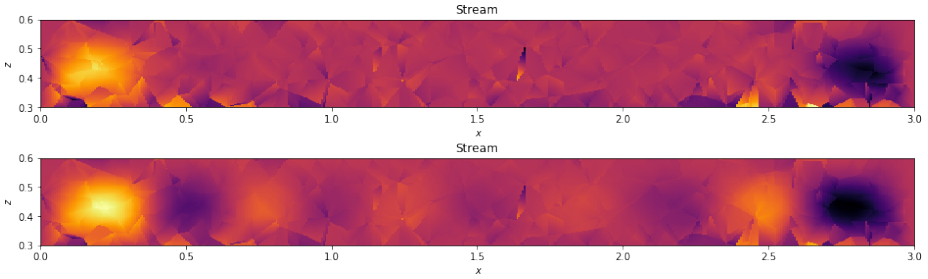

Figure 5: Stream function on the plane for $\eta = 0.5$ at $Re_i = 50$ and $Re_i = 72$.

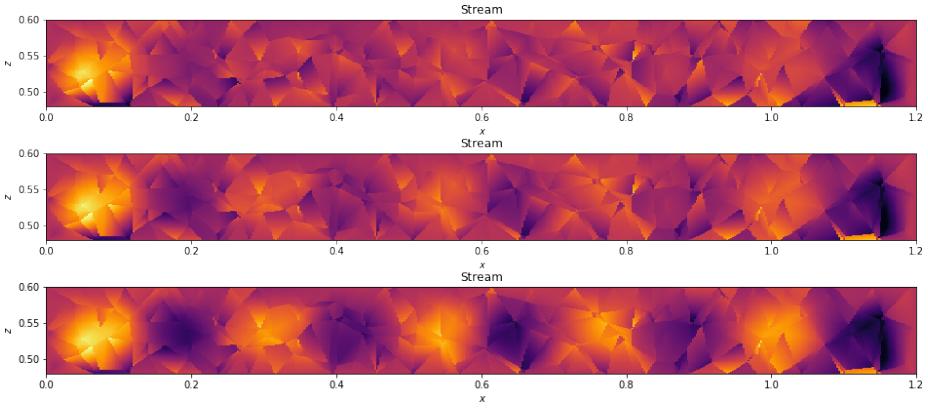

Figure 6: Stream function on the plane for $\eta = 0.8$ at $Re_i = 90$, $Re_i = 95$ and $Re_i = 100$.

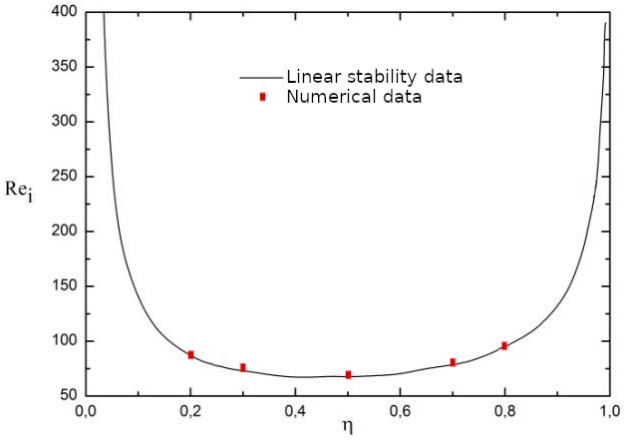

Figure 7: Comparison of the critical value of Re_i at $Re_o = 0$ for various η.

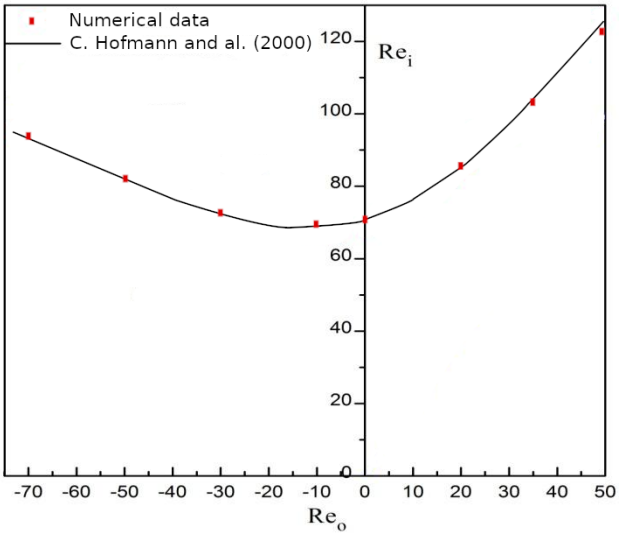

Figure 8: Comparison of the critical value of Re_i at $\eta = 0.5$ for various Re_o.

agreement, though we used a much coarser mesh and a smaller aspect ratio Λ of 10 instead of 20, for computational cost reasons.

We display some values taken on the half plane $y = 0, x > 0$. Figure 4 shows a comparison between the azimuthal velocity and the pressure for two values of Re_i at $\eta = 0.5$. In figure 5 we see the stream function (the azimuthal component of the vector potential) on the same system. Stream functions for three values of Re_i at $\eta = 0.8$ are shown in figure 6. Lastly figure 7 and 8 give the aforementioned comparison, our results being shown in red and the reference curve in black.

References

[1] Vivette Girault and Pierre-Arnaud Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer, Mar. 1986.

[2] Thomas Gebhardt and Siegfried Grossmann, *The Taylor-Couette eigenvalue problem with independently rotating cylinders*, Zeitschrift für Physik B Condensed Matter 90.4 (Dec. 1993), pp. 475–490, doi: 10.1007/bf01308827.

[3] C. Ross Ethier and D. A. Steinman, *Exact fully 3D Navier-Stokes solutions for benchmarking*, International Journal for Numerical Methods in Fluids 19.5 (Sept. 1994), pp. 369–375, doi: 10.1002/fld.1650190502.

[4] Ch. Hoffmann, M. Lücke, and A. Pinter, *Spiral vortices traveling between two rotating defects in the Taylor-Couette system*, Physical Review E 72.5 (Nov. 2005), p. 056311, doi: 10.1103/physreve.72.056311.

[5] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bulletin of the American Mathematical Society 47.2 (Jan. 2010), pp. 281–354, doi: 10.1090/s0273-0979-10-01278-4.

[6] Anders Logg, Kent-Andre Mardal, Garth N. Wells, et al., *Automated Solution of Differential Equations by the Finite Element Method*, Springer, 2012, doi: 10.1007/978-3-642-23099-8.

[7] Xiantao Zhou, Yan Shi, and Yinghuang Kong, *Study on Performance of Laminar Taylor-Couette Flow with Different Developed Procedures*, Modern Mechanical Engineering 02.01 (2012), pp. 14–23, doi: 10.4236/mme.2012.21003.

[8] D. N. Arnold and A. Logg, *Periodic table of the Finite Elements.* SIAM News 47.9 (Nov. 2014).

[9] Nabila Ait-Moussa, Sébastien Poncet, and Abdelrahmane Ghezal, *Numerical Simulations of Co- and Counter-Taylor-Couette Flows: Influence of the Cavity Radius Ratio on the Appearance of Taylor Vortices*, American Journal of Fluid Dynamics 5.1 (2015), pp. 17–22, doi: 10.5923/j.ajfd.20150501.02.

[10] Michael Neilan, *Discrete and conforming smooth de Rham complexes in three dimensions*, Mathematics of Computation 84.295 (Mar. 2015), pp. 2059–2081, doi: 10.1090/s0025-5718-2015-02958-5.

[11] Douglas N. Arnold and Lizao Li, *Finite element exterior calculus with lower-order terms*, Mathematics of Computation 86.307 (Oct. 2016), pp. 2193–2212, doi: 10.1090/mcom/3158.

[12] Paul Leopardi and Ari Stern, *The Abstract Hodge–Dirac Operator and Its Stable Discretization*, SIAM Journal on Numerical Analysis 54.6 (Jan. 2016), pp. 3258–3279, doi: 10.1137/15m1047684.

[13] A. Linke and C. Merdon, *Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier-Stokes equations*, English, Comput. Methods Appl. Mech. Eng. 311 (2016), pp. 304–326.

[14] Volker John et al., *On the Divergence Constraint in Mixed Finite Element Methods for Incompressible Flows*, SIAM Review 59.3 (Jan. 2017), pp. 492–544, doi: 10.1137/15m1047696.

[15] Douglas N. Arnold, *Finite Element Exterior Calculus*, vol. 93, CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2018, pp. xii+120.

[16] Nicolas R. Gauger, Alexander Linke, and Philipp W. Schroeder, *On high-order pressure-robust space discretisations, their advantages for incompressible high Reynolds number generalised Beltrami flows and beyond*, en, The SMAI journal of computational mathematics 5 (2019), pp. 89–129, doi: 10.5802/smai-jcm.44.

[17] Philip Lukas Lederer, Christian Merdon, and Joachim Schöberl, *Refined a posteriori error estimation for classical and pressure-robust Stokes finite element methods*, Numerische Mathematik 142.3 (May 2019), pp. 713–748, doi: 10.1007/s00211-019-01049-3.

[18] Alexander Linke, Christian Merdon, and Michael Neilan, *Pressure-robustness in quasi-optimal a priori estimates for the Stokes problem*, ETNA - Electronic Transactions on Numerical Analysis 52 (2020), pp. 281–294, doi: 10.1553/etna_vol52s281.

[19] Kaibo Hu, Young-Ju Lee, and Jinchao Xu, *Helicity-conservative finite element discretization for incompressible MHD systems*, Journal of Computational Physics 436 (July 2021), p. 110284, doi: 10.1016/j.jcp.2021.110284.