Well posedness and stochastic derivation of a diffusion-growth-fragmentation equation in a chemostat

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Abstract

We study the existence and uniqueness of the solution of a non-linear coupled system constituted of a degenerate diffusion-growth-fragmentation equation and a differential equation, resulting from the modeling of bacterial growth in a chemostat. This system is derived, in a large population approximation, from a stochastic individual-based model where each individual is characterized by a non-negative real valued trait described by a diffusion. Two uniqueness results are highlighted. They differ in their hypotheses related to the influence of the resource on individual trait dynamics, the main difficulty being the non-linearity due to this dependence and the degeneracy of the diffusion coefficient. Further we show that the semi-group of the stochastic trait dynamics admits a density by probabilistic arguments, that allows the measure solution of the diffusion-growth-fragmentation equation to be a function with a certain Besov regularity.

Keywords Diffusion-growth-fragmentation equation coupled with resource · Stochastic Feller-type diffusion · Mild formulation · Large population approximation · Existence of density

Mathematics subject classification (2020) Primary 35K65 · 60K35 ; Secondary 35K61

1 Introduction

Developped by Novick and Szilard [25], and Monod [24], the chemostat is a laboratory device used by biologists to raise the microorganisms and study their interactions while at the same time regulating the population size and the experimental medium. It consists in a culture in a container of constant volume in which a substrate is continuously injected and extracted at the same rate. In the literature, it is extensively used to study the population dynamics, their interactions and the adaptative behavior of microorganisms [10, 12, 18]. In this vein, we are interested in the dynamics of a

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structured asexual population (typically bacteria) coupled with a resource in chemostat. Each individual is characterized by a quantitative trait $x \in \mathbb{R}_+$ which models a protein density and evolves according to a diffusion of which coefficients depend on the resource. The goal of this work is to show the existence and uniqueness of the solution of the large population dynamics described by the coupled system

$$\begin{cases} \partial_t u_t = \partial_x^2\big(xQ(x,R_t)u_t\big) - \partial_x\big(\zeta(x,R_t)u_t\big) + \mathcal{G}^\dagger\big[b(\cdot,R_t)u_t\big] - d(x)u_t\,, \text{ for } (x,t) \in \mathbb{R}_+^* \times (0,T] \\ \dot{R}_t = D(r_{in} - R_t) - \lambda \displaystyle\int_0^\infty b(x,R_t)u_t(x)dx, \forall t \in (0,T] \\ \zeta(0,R_t)u_t(0) - \partial_x\big(xQ(x,R_t)u_t\big)_{|x=0} = 0, \forall t \in (0,T] \end{cases} \tag{1.1}$$

with a given initial condition (u_0, R_0). The first two terms on the right side of the parabolic equation in this system are linked to the trait dynamics. The third one modelizes cellular division which is accompanied by a fragmentation of the trait according to the operator

$$\mathcal{G}^\dagger[f](x) = -f(x) + \int_0^1 \frac{2}{\alpha} f\left(\frac{x}{\alpha}\right) \bar{m}(d\alpha)$$

where $\bar{m}(d\alpha)$ is a symetric probability measure on $]0,1[$, and the last term represents death. The second equation describes the resource supply and its extraction at the same rate $D > 0$. The last term corresponds to its consumption by the population. The third equation is a no-flux boundary condition which ensures the quantitative variable to remain non negative.

This model extends the well known growth-fragmentation model that has been extensively studied in the literature (see Bertoin et *al*. [4, 5, 6, 7], or Doumic et *al*. [13, 14, 20] who investigated the asymptotic properties of the equation without resource, or Campillo and Fritsch [9] for its well posedness and stochastic derivation in a chemostat). Here, we also include the resource dynamics, and introduce a diffusion term that can be seen as an intrinsic noise resulting from very fast synthesis and degradations of the quantitative trait (see [3] for a similar framework). The main difficulty to get a uniqueness result lies in the degeneracy and the dependence in resource of this term. In order to deal with it, we will develop an approach based on the regularity of the individual trait dynamics Markovian semi-group. As it will also depend on the solution, it will be necessary to show some regularity according to this dependence and the state variable. Such an approach has been developed in [15] for a model without resource with Lipschitz continuous diffusion coefficient in the stochastic trait dynamics. Although this coefficient is less regular in our case, we show that it is possible to adapt their method when it does not depend on the resource dynamics. The general situation is more complicated because we are unable to get the Lipschitz dependence of the semi-group according to the solution, that holds in the previous case. We will then require more regularity and bounds on the coefficients in order to deal with this dependence thanks to the regularity of its generator.

Further, another major point consists in showing that the weak solution of the diffusion-growth-fragmentation equation is a function, even if the initial condition is a measure. Classically, it is directly linked to the quantitative trait stochastic dynamics. Then we will first show that at any time, the distribution of the trait admits a density with respect to the Lebesgue measure on $\mathbb{R}_+$, and extend the existence of the density to the solution thanks to a mild formulation of the equation. Again, the main difficulty lies in the degeneracy of the diffusion coefficient, but also its weaker regularity (locally

Hölder continuous). We refer to Sato-Ueno [29] (Theorem 2.1 and Appendix) when the diffusion term is uniformly elliptic, and Bouleau-Hirsch [8] (Theorem 2.1.3, p162) when it is Lipschitz continuous. Here, we adapt the method developed by Romito [28] for time dependent and locally Hölder continuous diffusion coefficients, to show that the density exists outside the degeneracy point 0. It remains to justify that the law of the trait at any time does not charge this very point, to conclude that the density exists on the entire non-negative real half-line.

The paper is then organized as follows : in Section 2, we show that (1.1) can be derived from an individual based model that describes each intrinsic dynamics and the resource. For that purpose, we propose a Markovian stochastic process that models the trait distribution within the population coupled to the resource dynamics, given a parameter that scales the initial number of individuals. We show that when this scale parameter goes to infinity, the sequence of laws of the processes admits some limiting values which satisfies a weak formulation of (1.1). In Section 3, we show that the solution of this weak formulation is unique in a certain set of continuous measure valued processes and under suitable assumptions. We highlight two cases which only differs in the dependence or not of the diffusion term in the resource dynamics. As described here above, the method we use in those cases are based on the regularity of the semi-group of the stochastic process that describes the trait dynamics, according to the state variable and the way it depends on the solution. Finally, in Section 4, we show that this solution admits a density at any time with a Besov regularity. To summarize and including the resource dynamics, we prove that the only weak solution of (1) belongs to $\mathbb{L}^\infty((0,T],\mathbb{L}^1(\mathbb{R}_+,(1+x)dx)\times\mathbb{R})$.

Notations For later use, we introduce the following notations :

- $\mathcal{M}_F(\mathbb{R}_+)$ is the space of finite measures on $\mathbb{R}_+$. It can be endowed with its vague (respectively weak) topology and denoted $(\mathcal{M}_F(\mathbb{R}_+),v)$ (respectively $(\mathcal{M}_F(\mathbb{R}_+),w)$), that is the weakest topology making all maps $\mathcal{M}_F(\mathbb{R}_+)\ni\mu\mapsto\int_{\mathbb{R}_+}f(x)\mu(dx)$ continuous for any $f\in\mathcal{C}_c(\mathbb{R}_+^n,\mathbb{R})$ (respectively $f\in\mathcal{C}_b(\mathbb{R}_+,\mathbb{R})$).
- $\mathcal{M}$ is the subspace of $\mathcal{M}_F(\mathbb{R}_+)$ constituted of the finite sums of Dirac masses, thus
$$\mathcal{M}=\left\{\sum_{i=1}^n\delta_{x_i}:n\in\mathbb{N},x_1,...,x_n\in\mathbb{R}_+\right\}$$
- $\mathcal{P}(\mathcal{E})$ denotes the space of probability measures on a given measurable space $\mathcal{E}$.
- $\mathcal{LB}(\mathbb{R}_+,\mathbb{R})$ denotes the space of functions from $\mathbb{R}_+$ to $\mathbb{R}$ that are bounded and Lipschitz continuous. It is endowed with the norm $\|\phi\|_{\mathcal{LB}}=\|\phi\|_{\text{Lip}}+\|\phi\|_\infty$ where
$$\|\phi\|_{\text{Lip}}=\inf\left\{c>0:\frac{|\phi(x)-\phi(y)|}{|x-y|}\le c,\forall x,y\in\mathbb{R}_+,x\neq y\right\}.$$
- $\mathcal{J}$ denotes an abstract countable set.
- The constant $C>0$ (or C_p) can change one line to an other.

2 The individual-based model and stochastic derivation

In this section, we propose a stochastic individual-based model describing the population dynamics in the chemostat at the individual level with a carrying capacity $K > 1$. Our goal is to derive a weak solution of (1.1) from this stochastic model when K goes to infinity.

2.1 Description

Let us first introduce a scaling parameter $K > 1$ that represents the carrying capacity of the environment. We assume that it is roughtly proportional to the initial number of individuals. The bacterial population is described by a stochastic measure valued process $(\nu_t^K)_{t\geq 0}$ with values in $\frac{1}{K}\mathcal{M}$ given at any time by the sum of the Dirac masses at the trait of each living cell so that each individual has the same mass $1/K$, and the resource is described by a positive continuous process $(R_t^K)_{t\geq 0}$ that corresponds to its mass concentration. Let us denote by $V_t^K \subset \mathcal{J}$ the collection of the living cells at any time $t \geq 0$. Each individual is characterized by a positive trait that evolves during its life-time, and may give birth or die depending on its trait and the availability of the resource as described as follows :

Between jumps, the trait dynamics of an individual $i \in V_{t^-}^K$ is described by a non negative diffusion process $X^{i,K}$ that modelizes its random evolution with a drift coefficient $\zeta(x,r)$ and a degenerate diffusion coefficient $\sqrt{2xQ(x,r)}$ during its life-time. We assume that for all $x \geq 0$ and locally in $r \geq 0$,

$$0 \leq Q(x,r) \leq \bar{Q}\,;\, \zeta(0,r) \geq 0. \tag{2.1}$$

Furthermore, an individual $i \in V_{t^-}^K$ with trait $x \geq 0$ divides at rate $b(x, R_{t^-}^K)$ and its trait is split at the same time so that the mother cell stays with a random proportion $\alpha \in]0,1[$ chosen according to the probability distribution $m(d\alpha)$, and the daughter starts its life with the remaining proportion $1-\alpha$. We assume that the birth rate is bounded and there is no division without resource, thus for all $x \geq 0$ and locally in $r \geq 0$,

$$b(x,0) = 0\,;\, 0 \leq b(x,r) \leq \bar{b}. \tag{2.2}$$

That individual dies at rate $d(x)$ either by extraction at rate $D > 0$ or by natural death. Let us assume that this rate is bounded so that for all $x \geq 0$,

$$D \leq d(x) \leq \bar{d}. \tag{2.3}$$

Finally, the mechanisms of the resource dynamics are well known for a culture in chemostat and consist in a continuous supply of concentration r_{in} and extraction of the resource at rate $D \geq 0$, and its consumption by the micro-organisms during their life-time with a small conversion rate λ/K from each individual to resource. Then,

$$R_t^K = R_0 + \int_0^t \left\{ D(r_{in} - R_s^K) - \frac{\lambda}{K} \sum_{i\in V_s^K} b(X_s^{i,K}, R_s^K) \right\} ds\,,\ \forall t \geq 0. \tag{2.4}$$

Remark 2.1. *The condition $\zeta(0,r) \geq 0$ in (2.1) ensures that the trait is positive, since its diffusion coefficient vanishes and its drift is non-negative when it reaches*

0. *Similarly, the condition* $b(x,0) = 0$ *in (2.2) ensures that the process* $(R_t^K)_{t\geq 0}$ *is positive, since its drift is positive when it reaches* 0. *Moreover, it follows from the comparison principle that*

$$R_t^K \leq r_{in} + (R_0 - r_{in})e^{-Dt} \leq r_{in} \vee R_0 \text{ for all } t \geq 0. \tag{2.5}$$

Then, one has a uniform bound on the resource dynamics, that is

$$\bar{R} = r_{in} \vee R_0. \tag{2.6}$$

In order to describe the coupled dynamics of the population and the resource, we introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ so that the initial condition (ν_0^K, R_0) is almost surely $\frac{1}{K}\mathcal{M} \times \mathbb{R}_+$-valued for each $K > 1$. Let us also introduce a family $\mathcal{W} = \left\{W^i, i \in \mathcal{J}\right\}$ of independant Brownian motions that is independent of the sequence of initial conditions $\{(\nu_0^K, R_0), K > 1\}$, and independent Poisson point measures $\mathcal{N}_1(d\alpha, dz, di, ds), \mathcal{N}_2(dz, di, ds)$ on $[0,1] \times [0,1] \times \mathcal{J} \times \mathbb{R}_+$ and $[0,1] \times \mathcal{J} \times \mathbb{R}_+$ with intensities $\bar{b}\, m(d\alpha)\eta(dz, di, ds)$ and $\bar{d}\,\eta(dz, di, ds)$ respectively, where

$$\eta(dz, di, ds) = dz \left(\sum_{j \in \mathcal{J}} \delta_j(di) \right) ds. \tag{2.7}$$

Those Poisson point measures are independent of the families $\mathcal{W}$ and $\{\nu_0^K, K > 1\}$. We finally introduce the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the sequence of initial conditions $\{\nu_0^K, K > 1\}$, the familly of Brownian motions $\mathcal{W}$ and the Poisson point measures $\mathcal{N}_1, \mathcal{N}_2$. This filtration is assumed to be complete and right continuous.

We are interested in the dynamics of a coupled stochastic process composed of the $\frac{1}{K}\mathcal{M}$-valued process defined by

$$\nu_t^K = \frac{1}{K} \sum_{i \in V_t^K} \delta_{X_t^{i,K}}, \forall t \geq 0 \tag{2.8}$$

where $X_t^{i,K}$ corresponds to the trait of the individual $i \in V_t^K$ at time $t \geq 0$, and the continuous real valued process $(R_t^K)_{t\geq 0}$. The trait of the i-th individual satisfies the stochastic differential equation

$$dX_t^{i,K} = \zeta(X_t^{i,K}, R_t^K)dt + \sqrt{2X_t^{i,K} Q(X_t^{i,K}, R_t^K)}\, dW_t^i \tag{2.9}$$

during its life-time. The trajectories of this coupled stochastic process can be repre-

sented by the following system

$$\begin{cases} \text{for all } f \in \mathcal{C}_b^2(\mathbb{R}_+, \mathbb{R}), \\ \langle \nu_t^K, f \rangle = \langle \nu_0^K, f \rangle + \dfrac{1}{K} \displaystyle\int_0^t \sum_{i \in V_s^K} \sqrt{2X_s^{i,K} Q(X_s^{i,K}, R_s^K)}\, f'(X_s^{i,K}) dW_s^i \\ \qquad + \displaystyle\int_0^t \int_0^\infty \left\{ \zeta(x, R_s^K) f'(x) + x Q(x, R_s^K) f''(x) \right\} \nu_s^K(dx) ds \\ \qquad + \dfrac{1}{K} \displaystyle\iiiint_{[0,t]\times\mathcal{J}\times[0,1]^2} 1_{\left\{i \in V_{s-}^K\right\} \cap \left\{z \le b(X_{s-}^{i,K}, R_{s-}^K)/\bar{b}\right\}} \\ \qquad\qquad \left[-f(X_{s-}^{i,K}) + f(\alpha X_{s-}^{i,K}) + f((1-\alpha) X_{s-}^{i,K}) \right] \mathcal{N}_1(d\alpha, dz, di, ds) \\ \qquad - \dfrac{1}{K} \displaystyle\iiint_{[0,t]\times\mathcal{J}\times[0,1]} 1_{\left\{i \in V_{s-}^K\right\} \cap \left\{z \le d(X_{s-}^{i,K})/\bar{d}\right\}} f(X_{s-}^{i,K}) \mathcal{N}_2(dz, di, ds), \\ R_t^K = R_0 + \displaystyle\int_0^t \left\{ D(r_{in} - R_s^K) - \lambda \int_0^\infty b(x, R_s^K) \nu_s^K(dx) \right\} ds. \end{cases} \tag{2.10}$$

The first and the second integrals in the first equations correspond to the trait dynamics and follows from the Itô formula applied on (2.9), and the jumps correspond to births and deaths respectively. We refer to [11, 16] for the construction of a such process. Let us first make the following assumptions.

Assumption 2.1.

(A.1) *The coefficients $\zeta(x,r), Q(x,r), b(x,r), d(x)$ are Lipschitz continuous for $x \ge 0$ and $r \in [0, \bar{R}]$.*

(A.2) *Hypotheses (2.1), (2.2), (2.3) hold for $r \in [0, \bar{R}]$.*

Then we have the following result.

Lemma 2.1. *If there exist $p, q \ge 1$ such that $\sup_K \mathbb{E}\left(\langle \nu_0^K, x^p \rangle + \langle \nu_0^K, 1 \rangle^q\right) < \infty$, then under Assumption 2.1 the following bound holds*

$$\sup_K \mathbb{E}\left\{ \sup_{0 \le t \le T} \left[\langle \nu_t^K, x^p \rangle + \langle \nu_t^K, 1 \rangle^q \right] \right\} < \infty \text{ for all } T > 0. \tag{2.11}$$

In addition, the process $(\nu_t^K)_{t \ge 0}$ admits the Doob-Meyer decomposition

$$\langle \nu_t^K, f \rangle = \langle \nu_0^K, f \rangle + V_t^{K,f} + M_t^{K,f}, \forall t \ge 0, \forall f \in \mathcal{C}_b^2(\mathbb{R}_+, \mathbb{R}). \tag{2.12}$$

where $V^{K,f}$ is the process with finite variation defined by

$$\begin{aligned} V_t^{K,f} &= \int_0^t \int_0^\infty \left\{ \zeta(x, R_s^K) f'(x) + x Q(x, R_s^K) f''(x) \right\} \nu_s^K(dx) ds \\ &\quad + \int_0^t \int_0^\infty \left\{ b(x, R_s^K) \left[-f(x) + \int_0^1 \left[f(\alpha x) + f((1-\alpha)x) \right] m(d\alpha) \right] - d(x) f(x) \right\} \nu_s^K(dx) ds \end{aligned} \tag{2.13}$$

and $M^{K,f}$ *a square integrable martingale of quadratic variation*

$$\left\langle M^{K,f}\right\rangle_t = \frac{2}{K}\int_0^t\int_0^\infty xQ(x,R_s^K)|f_s'(x)|^2\nu_s^K(dx)ds + \frac{1}{K}\int_0^t\int_0^\infty d(x)f(x)^2\nu_s^K(dx)ds$$
$$+ \frac{1}{K}\int_0^t\int_0^\infty b(x,R_s^K)\int_0^1\left\{-f(x)+f(\alpha x)+f((1-\alpha)x)\right\}^2 m(d\alpha)\nu_s^K(dx)ds. \tag{2.14}$$

The proof is relatively classical and similar to the one obtained in [11, 15, 21, 30]. It is left to the reader.

Remark 2.2. *It follows from (2.14) that the quadratic variation of the martingale part of the above Doob-Meyer decomposition satisfies the property*

$$\left\langle M^{K,f}\right\rangle_t \leq C_T\,\frac{\|f\|_\infty^2+\|f'\|_\infty^2}{K}\sup_{0\leq u\leq T}\left\langle \nu_u^K, 1+x\right\rangle, \forall t\in[0,T]. \tag{2.15}$$

2.2 Large population approximation and existence theorem

In this section, we are interested in a large population approximation of the above stochastic model. The main result consists in showing that under suitable assumptions, the sequence of laws of the coupled stochastic dynamics of the population and resource is tight in the set of probability distributions $\mathcal{P}\left(\mathbb{D}([0,T],(\mathcal{M}_F(\mathbb{R}_+)\times\mathbb{R},w\otimes|.|)\right)$, and the processes in its limiting values are the solutions of (1.1) in the weak sense. Let us make the following assumptions.

Assumption 2.2.

(A.1) *There exists* $\varrho>0$ *such that the initial conditions* $\nu_0^K\in\frac{1}{K}\mathcal{M}$ *satisfy*

$$\sup_K \mathbb{E}\left[\left\langle\nu_0^K,1\right\rangle^{1+\varrho}+\left\langle\nu_0^K,x\right\rangle\right]<+\infty.$$

(A.2) *The sequence* $\left\{\nu_0^K, K>1\right\}$ *converges in law in* $(\mathcal{M}_F(\mathbb{R}_+),w)$ *towards a mesure valued initial condition* $\nu_0\in\mathcal{M}_F(\mathbb{R}_+)$.

We introduce the symetric probability distribution

$$\bar{m}(d\alpha)=\frac{1}{2}m(d\alpha)+\frac{1}{2}m(1-d\alpha) \tag{2.16}$$

where $m(1-d\alpha)$ is the image measure of $m(d\alpha)$ by the function $\alpha\to1-\alpha$, and consider the operator

$$\mathcal{G}[f](x)=-f(x)+2\int_0^1 f(\alpha x)\bar{m}(d\alpha), \forall f\in\mathcal{C}_b(\mathbb{R}_+,\mathbb{R}), \forall x\geq0. \tag{2.17}$$

The main result of this section follows.

Theorem 2.1. *Under Assumptions 2.1 and 2.2, the sequence of laws* $\left\{\mathcal{Z}^K=\mathcal{L}(\nu^K,R^K), K>1\right\}$ *is tight in the space of probability measures* $\mathcal{P}(\mathbb{D}([0,T],(\mathcal{M}_F(\mathbb{R}_+)\times\mathbb{R},w\otimes|.|)))$ *for*

all $T > 0$. In addition, each process (ν, R) in the support of its limiting values is continuous, satisfies the bound $\sup_{0\le t\le T}\langle \nu_t, 1+x\rangle < \infty$ and the system

$$\begin{cases} \forall t\in[0,T], \forall f\in \mathcal{C}_b^2(\mathbb{R}_+,\mathbb{R}), \\ \langle \nu_t, f\rangle = \langle \nu_0, f\rangle + \displaystyle\int_0^t \int_0^\infty \big\{\zeta(x,R_s)f'(x) + xQ(x,R_s)f''(x) \\ \qquad\qquad + b(x,R_s)\mathcal{G}[f](x) - d(x)f(x)\big\}\nu_s(dx)ds \\ R_t = R_0 + \displaystyle\int_0^t \left\{D(r_{in}-R_s) - \lambda\int_0^\infty b(x,R_s)\nu_s(dx)\right\}ds \end{cases} \tag{2.18}$$

that is a weak formulation of (1.1) with the initial condition (ν_0, R_0).

Remark 2.3. *Equation (1.1) has therefore at least one weak solution.*

The proof of Theorem 2.1 is relatively classical and uses the techniques developed in [1, 11, 15, 16, 21, 27] or [2], Theorem 7.4. It consists in three main steps : a uniform control of trait distribution tails in the population, the tightness of the sequence of laws $\{\mathcal{Z}^K, K>1\}$ and identification of the limit. It is detailed in Appendix A.

3 Uniqueness of the solution

In this section, we show that there exists a unique solution of the system (2.18). The main difficulties are the degeneracy of the diffusion coefficient and the non linearity of the drift and diffusion coefficients. Then we will first of all proceed by a decoupling of this system in order to get only one non-linear equation on the measure-valued process and establish some important properties of that decoupling. Secondly, we will write a mild formulation of this equation with continuous test functions thanks to the semigroup of the individual trait. Finally, we will use the properties of this semigroup to show in two different cases that the system (2.18) admits a unique solution.

3.1 Decoupling and mild formulation of the limit system

Let us start with the following results that ensure us that the decoupling makes sense, so that the limit system (2.18) could be resumed in only one equation that describes the process $(\nu_t)_{t\in[0,T]}$ for the given initial condition (ν_0, R_0).

Proposition 3.1. *If $(\nu, R)\in \mathcal{C}([0,T],(\mathcal{M}_F(\mathbb{R}_+)\times\mathbb{R}, w\otimes|.|))$ is a solution of (2.18), then the process $(R_t)_{t\in[0,T]}$ is completely defined by the measure valued process $(\nu_t)_{t\in[0,T]}$ in the sense that if $(\nu_t)_{t\in[0,T]}$ is given, then $(R_t)_{t\in[0,T]}$ exists and is unique. More precisely, at any time $t\ge 0$ the quantity R_t is characterized by the history of the measure valued process ν until the time t.*

Proof. Given the process $(\nu_t)_{t\in[0,T]}$, the resource dynamics is described by the equation

$$R_t = R_0 + \int_0^t F^\nu(s,R_s)ds\,,\ \forall t\in[0,T] \tag{3.1}$$

with

$$F^\nu(t,r) = D(r_{in}-r) - \lambda\int_0^\infty b(x,r)\nu_t(dx)\,,\ (t,r)\in[0,T]\times\mathbb{R}. \tag{3.2}$$

Since $F^\nu(t,0) = Dr_{in} > 0$, each solution of (3.1) is non-negative. In addition,

$$b(x,r) = |b(x,r) - b(x,0)| \leq r\|b\|_{\text{Lip}}.$$

Then if we set

$$\rho_T^\nu = \lambda\|b\|_{\text{Lip}} \sup_{0\leq t\leq T} \langle \nu_t, 1\rangle, \tag{3.3}$$

we obtain the inequality

$$(D+\rho_T^\nu)\left[\frac{Dr_{in}}{D+\rho_T^\nu} - r\right] = D(r_{in}-r) - \rho_T^\nu r \leq F^\nu(t,r) \leq D(r_{in}-r).$$

Thanks to the comparison theorem for ordinary differential equations, that implies that each solution of (3.1) satisfies the bound

$$\frac{Dr_{in}}{D+\rho_T^\nu} + \left(R_0 - \frac{Dr_{in}}{D+\rho_T^\nu}\right) e^{-(D+\rho_T^\nu)t} \leq R_t \leq r_{in} + (R_0 - r_{in}e^{-Dt}).$$

We deduce the bound property

$$R_0 e^{-(D+\rho_T^\nu)T} \leq R_t \leq r_{in} \vee R_0 \leq \bar{R}, \forall t \in [0,T]. \tag{3.4}$$

Moreover, for all $t \in [0,T]$ and $r_1, r_2 \in [0,\bar{R}]$,

$$\begin{aligned}
|F^\nu(t,r_1) - F^\nu(t,r_2)| &= |D(r_2-r_1) + \lambda \int_0^\infty [b(x,r_2) - b(x,r_1)]\, \nu_t(dx)| \\
&\leq D|r_1 - r_2| + \lambda \int_0^\infty |b(x,r_1) - b(x,r_2)| \nu_t(dx) \\
&\leq (D+\rho_T^\nu)\, |r_1 - r_2|
\end{aligned}$$

It then follows from the Cauchy-Lipschitz Theorem that the equation (3.1) admits a unique solution. □

The limit system (2.18) is then decoupled by introducing the new notation

$$g[\nu](x,t) = g(x,R_t), \forall (x,t) \in \mathbb{R}_+ \times [0,T] \tag{3.5}$$

for each continuous function $g(x,r)$. This notation is consistent with the regularity according to the first variable. Indeed, if $g(x,r)$ is of class $\mathcal{C}^k$ in its first variable with continuous derivatives, then the function $g[\nu](x,t)$ will be as well. Moreover, the function F^ν given in (3.2) being continuous, the trajectory $t \to R_t$ is differentiable and then for each $g(x,r)$ of class $\mathcal{C}^1$ in its second variable, the function $g[\nu](x,t)$ will be as well. We then introduce the following lemma whose proof is left to the reader.

Lemma 3.1. *Let $(x,r) \mapsto g(x,r)$ be a function of class $\mathcal{C}^1$ on $\mathbb{R}_+ \times [0,\bar{R}]$, then for each $M > 0$, the local Lipschitz property holds*

$$\left|g[\nu](x,s) - g[\nu](y,t)\right| \leq C_{M,T}\left(|x-y| + |t-s|\right), \forall s,t \leq T, \forall x,y \in [0,M]. \tag{3.6}$$

We can now rewrite equation (2.18) in the equivalent form

$$\left\{\begin{array}{l}
\forall t \in [0,T], \forall f \in \mathcal{C}_b^2(\mathbb{R}_+,\mathbb{R}), \\
\\
\langle \nu_t, f\rangle = \langle \nu_0, f\rangle + \displaystyle\int_0^t \int_0^\infty \Big(\zeta[\nu](x,s) f'(x) + xQ[\nu](x,s) f''(x)\Big)\nu_s(dx)ds \\
\qquad\qquad + \displaystyle\int_0^t \int_0^\infty \Big(b[\nu](x,s)\mathcal{G}[f](x) - d(x)f(x)\Big)\nu_s(dx)ds.
\end{array}\right. \tag{3.7}$$

In order to get the mild formulation of this system, let us consider for a given solution $(\nu_t)_{t\in[0,T]}$ of (3.7) the stochastic differential equation

$$dX_t = \zeta[\nu](X_t,t)dt + \sqrt{2X_tQ[\nu](X_t,t)}\,dW_t \tag{3.8}$$

that admits for every $x \geq 0$ and $s \in [0,T]$ a unique weak solution that starts at x at time $s \geq 0$ (see Lemma B.1 in Appendix B). This solution is almost surely non negative at any time and satisfies the control of moments

$$\forall p > 0,\ \mathbb{E}_{x,s}\left\{\sup_{s\leq u\leq T}(X_u)^p\right\} \leq C_{p,T}(1+x^p) \tag{3.9}$$

(see Lemma B.1). Let us denote by $(X^{\nu,x}_{s,t})_{t\in[s,T]}$ its trajectory, $(P^\nu_{s,t})_{s,t\geq 0}$ its semi-group defined by

$$P^\nu_{s,t}\phi(x) = \mathbb{E}\left[\phi(X^{\nu,x}_{s,s+t})\right],\ \forall \phi \in \mathcal{C}_b(\mathbb{R}_+,\mathbb{R}) \tag{3.10}$$

and $(L^\nu_s)_{s\in[0,T]}$ its infinitesimal generator that satisfies

$$L^\nu_s f(x) = \zeta[\nu](x,s)f'(x) + xQ(x)f''(x),\ \forall f \in \mathcal{C}^2_b(\mathbb{R}_+,\mathbb{R}). \tag{3.11}$$

We then have the following result.

Lemma 3.2. *Equation (3.7) admits the mild formulation*

$$\begin{cases} \forall t \geq 0, \forall \phi \in \mathcal{C}_b(\mathbb{R}_+,\mathbb{R}) \\ \langle \nu_t,\phi\rangle = \langle \nu_0, P^\nu_{0,t}\phi\rangle + \displaystyle\int_0^t\int_0^\infty \left\{b[\nu](x,s)\mathcal{G}[P^\nu_{s,t-s}\phi](x) - d(x)P^\nu_{s,t-s}\phi(x)\right\}\nu_s(dx)ds. \end{cases} \tag{3.12}$$

Proof. It classically follows from equation (3.7), that for each test function $f_s(x) = f(s,x) \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}_+,\mathbb{R})$ and $t \in [0,T]$,

$$\langle \nu_t, f_t\rangle = \langle \nu_0, f_0\rangle + \int_0^t\int_0^\infty \left\{\partial_s f_s(x) + L^\nu_s f_s(x) + b[\nu](x,s)\mathcal{G}[f_s](x) - d(x)f_s(x)\right\}\nu_s(dx)ds. \tag{3.13}$$

In addition, for each test function $\phi \in \mathcal{C}^2_b(\mathbb{R}_+,\mathbb{R})$, the function

$$f_s(x) = P^\nu_{s,t-s}\phi(x), \forall (s,x) \in [0,t]\times\mathbb{R}_+$$

is a solution of the problem

$$\begin{cases} \partial_s f_s(x) + L^\nu_s f_s(x) = 0, (s,x) \in [0,t]\times\mathbb{R}_+ \\ f_{\big|_{s=t}} = \phi \text{ sur } \mathbb{R}_+. \end{cases}$$

We then use this function in (3.13) and obtain (3.12) for $\phi \in \mathcal{C}^2_b(\mathbb{R}_+,\mathbb{R})$. It is easily extended by a density argument to continuous and bounded test functions. □

We also have the following property on the semi-group, which means that it is consistent with the linear space $\mathcal{LB}(\mathbb{R}_+,\mathbb{R})$:

Lemma 3.3. *The operator $P^\nu_{s,t}$ satisfies the uniform property*

$$\sup_{t+s\leq T}\left|P^\nu_{s,t}\phi(x) - P^\nu_{s,t}\phi(y)\right| \leq C\|\phi\|_{Lip}|x-y|, \forall \phi \in \mathcal{LB}(\mathbb{R}_+,\mathbb{R}). \tag{3.14}$$

Proof. For all $s,t \geq 0$ such that $s+t \leq T$ and $x,y \geq 0$, we have

$$\begin{aligned}\left|P^{\nu}_{s,t}\phi(x) - P^{\nu}_{s,t}\phi(y)\right| &= \left|\mathbb{E}\left[\phi(X^{\nu,x}_{s,t+s}) - \phi(X^{\nu,y}_{s,t+s})\right]\right| \\ &\leq \mathbb{E}\left(\left|\phi(X^{\nu,x}_{s,t+s}) - \phi(X^{\nu,y}_{s,t+s})\right|\right) \\ &\leq \|\phi\|_{\mathrm{Lip}}\mathbb{E}\left(\left|X^{\nu,x}_{s,t+s} - X^{\nu,y}_{s,t+s}\right|\right)\end{aligned}$$

where $(X^{\nu,x}_{s,u})_{u\in[s,T]}$ and $(X^{\nu,y}_{s,u})_{u\in[s,T]}$ satisfy the stochastic differential equation (3.8) with the same Brownian motion. It then follows from Lemma B.1 in Appendix B that

$$\left|P^{\nu}_{s,t}\phi(x) - P^{\nu}_{s,t}\phi(y)\right| \leq C\|\phi\|_{\mathrm{Lip}}|x-y|, \forall s+t \leq T.$$

□

3.2 A simpler case of uniqueness

In this section, we show a uniqueness result when the diffusion coefficient does not depend on the resource dynamics. The method is adapted from that of [15] with a local Hölder continuous diffusion coefficient for the stochastic differential equation that describes the individual trait dynamics. Let us introduce the distance on $\mathcal{M}_F(\mathbb{R}_+)$ defined by

$$|||\mu_1 - \mu_2|||_{\mathcal{LB}} = \sup_{\phi\in\mathcal{LB}(\mathbb{R}_+,\mathbb{R});\|\phi\|_{\mathcal{LB}}\leq 1} \langle \mu_1 - \mu_2, \phi\rangle, \forall \mu_1,\mu_2 \in \mathcal{M}_F(\mathbb{R}_+). \tag{3.15}$$

We start with the following results that highlight the consistence of the decoupling and the semigroup according to the solution considered.

Lemma 3.4. *Let $(x,r) \mapsto g(x,r)$ be a Lipschitz continuous fonction from $\mathbb{R}_+ \times [0,\bar{R}]$ to $\mathbb{R}$ and $\nu^1,\nu^2 \in \mathcal{C}([0,T],(\mathcal{M}_F(\mathbb{R}_+),w))$ be two solutions of (3.7), then*

$$\sup_{(x,s)\in\mathbb{R}_+\times[0,t]} \left|g[\nu^1](x,s) - g[\nu^2](x,s)\right| \leq C\|g\|_{Lip}\int_0^t |||\nu^1_u - \nu^2_u|||_{\mathcal{LB}}\, du, \forall t \in [0,T]. \tag{3.16}$$

Proof. Let us denote by R^{ν^1} and R^{ν^2} the processes that are solutions of (3.1) for ν^1 and ν^2 respectively, for the same initial condition R_0. Then thanks to the decoupling formula (3.5) we have for all $t \in [0,T]$ and $(x,s) \in \mathbb{R}_+ \times [0,t]$,

$$\left|g[\nu^1](x,s) - g[\nu^2](x,s)\right| = \left|g(x,R^{\nu^1}_s) - g(x,R^{\nu^2}_s)\right| \leq \|g\|_{\mathrm{Lip}}\left|R^{\nu^1}_s - R^{\nu^2}_s\right|. \tag{3.17}$$

In addition, it follows from (3.1) that

$$\begin{aligned}\left|R^{\nu^1}_s - R^{\nu^2}_s\right| &= \left|\int_0^s \left\{D(R^{\nu^2}_u - R^{\nu^1}_u) - \lambda\int_0^\infty b(x,R^{\nu^1}_u)\nu^1_u(dx) + \lambda\int_0^\infty b(x,R^{\nu^2}_u)\nu^2_u(dx)\right\}du\right| \\ &\leq \int_0^s \left\{D\left|R^{\nu^2}_u - R^{\nu^1}_u\right| + \lambda\int_0^\infty \left|b(x,R^{\nu^2}_u) - b(x,R^{\nu^1}_u)\right|\nu^1_u(dx)\right. \\ &\qquad \left. + \lambda\left|\int_0^\infty b(x,R^{\nu^2}_u)(\nu^2_u - \nu^1_u)(dx)\right|\right\}du \\ &\leq \left(D + \|b\|_{\mathrm{Lip}}\lambda \sup_{0\leq z\leq T}\langle \nu^1_z, 1\rangle\right)\int_0^s \left|R^1_u - R^2_u\right|du + \lambda(\bar{b} + \|b\|_{\mathrm{Lip}})\int_0^t |||\nu^1_u - \nu^2_u|||_{\mathcal{LB}}\, du\end{aligned}$$

and by the Gronwall lemma, for all $s \le t$,

$$\left|R_s^{\nu^1} - R_s^{\nu^2}\right| \le \lambda(\bar{b}+\|b\|_{\mathrm{Lip}}) \exp\left\{\left(D + \|b\|_{\mathrm{Lip}}\lambda \sup_{0\le z\le T}\langle \nu_z^1, 1\rangle\right) T\right\} \int_0^t |||\nu_u^1 - \nu_u^2|||_{\mathcal{LB}}\, du.$$

We introduce this inequality in (3.17) to obtain (3.16). □

Lemma 3.5. *For each test function $\phi \in \mathcal{LB}(\mathbb{R}_+, \mathbb{R})$ and $\nu^1, \nu^2 \in \mathcal{C}([0,T], (\mathcal{M}_F(\mathbb{R}_+), w))$ two solutions of (3.7), one has*

$$\sup_{s\le t} \left\| \left(P_{s,t-s}^{\nu^1} - P_{s,t-s}^{\nu^2}\right)\phi \right\|_\infty \le C\|\phi\|_{Lip} \int_0^t |||\nu_u^1 - \nu_u^2|||_{\mathcal{LB}}\, du. \tag{3.18}$$

Proof. For all $0 \le s \le t \ge T$ and $x \ge 0$, we have

$$\begin{aligned} \left|P_{s,t-s}^{\nu^1}\phi(x) - P_{s,t-s}^{\nu^2}\phi(x)\right| &= \left|\mathbb{E}\left[\phi(X_{s,t}^{\nu^1,x}) - \phi(X_{s,t}^{\nu^2,x})\right]\right| \\ &\le \mathbb{E}\left(\left|\phi(X_{s,t}^{\nu^1,x}) - \phi(X_{s,t}^{\nu^2,x})\right|\right) \\ &\le \|\phi\|_{\mathrm{Lip}}\mathbb{E}\left(\left|X_{s,t}^{\nu^1,x} - X_{s,t}^{\nu^2,x}\right|\right) \end{aligned} \tag{3.19}$$

where $(X_{s,u}^{\nu^1,x})_{u\in[s,T]}$ and $(X_{s,u}^{\nu^2,x})_{u\in[s,T]}$ satisfy (3.8) with the same Brownian motion. In addition for all $s \le t' \le t$ and $\varepsilon > 0$,

$$\begin{aligned} &\int_s^{t'} 1_{\{0<X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x} \le \varepsilon\}} \frac{d\left\langle X_{s,\cdot}^{\nu^1,x} - X_{s,\cdot}^{\nu^2,x}\right\rangle_u}{X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x}} \\ &= \int_s^{t'} 1_{\{0<X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x} \le \varepsilon\}} \frac{2\left|\sqrt{X_{s,u}^{\nu^1,x}Q(X_{s,u}^{\nu^1,x})} - \sqrt{X_{s,u}^{\nu^2,x}Q(Z_{s,u}^{\nu^2,x})}\right|^2}{X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x}} du \\ &\le 2\int_s^{t'} 1_{\{0<X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x} \le \varepsilon\}} \frac{\left|X_{s,u}^{\nu^1,x}Q(X_{s,u}^{\nu^1,x}) - X_{s,u}^{\nu^2,x}Q(X_{s,u}^{\nu^2,x})\right|}{X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x}} du \\ &\le 2\int_s^{t'} 1_{\{0<X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x} \le \varepsilon\}} \frac{\left|(X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x})Q(X_{s,u}^{\nu^1,x}) + X_{s,u}^{\nu^2,x}\left(Q(X_{s,u}^{\nu^1,x}) - Q(X_{s,u}^{\nu^2,x})\right)\right|}{X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x}} du \\ &\le 2\int_s^{t'} 1_{\{0<X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x} \le \varepsilon\}} (\overline{Q} + CX_{s,u}^{\nu^2,x}) du < +\infty. \end{aligned}$$

We deduce thanks to [26], Ch. IX, lemma 3.3 that the process $X_{s,\cdot}^{\nu^1,x} - X_{s,\cdot}^{\nu^2,x}$ has a zero local time at 0 and the Tanaka formula gives for all $s \le t' \le t$

$$\left|X_{s,t'}^{\nu^1,x} - X_{s,t'}^{\nu^2,x}\right| = \int_s^{t'} \mathrm{sign}(X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x}) d(X_{s,u}^{\nu^1,x} - X_{s,u}^{\nu^2,x}).$$

Thanks to the control of moments (B.2) in Lemma B.1, that implies

$$\mathbb{E}\left(\left|X_{s,t'}^{\nu^1,x}-X_{s,t'}^{\nu^2,x}\right|\right)=\mathbb{E}\left(\int_s^{t'}\text{sign}(X_{s,u}^{\nu^1,x}-X_{s,u}^{\nu^2,x})\left[\zeta[\nu^1](X_{s,u}^{\nu^1,x},u)-\zeta[\nu^2](X_{s,u}^{\nu^2,x},u)\right]du\right)$$
$$\leq\mathbb{E}\left(\int_s^{t'}\left|\zeta[\nu^1](X_{s,u}^{\nu^1,x},u)-\zeta[\nu^2](X_{s,u}^{\nu^2,x},u)\right|du\right)$$
$$\leq\int_s^{t'}\mathbb{E}\left(\left|\zeta[\nu^1]-\zeta[\nu^2]\right|(X_{s,u}^{\nu^1,x},u)\right)du+\|\zeta[\nu^2]\|_{\text{Lip}}\int_s^{t'}\mathbb{E}\left(\left|X_{s,u}^{\nu^1,x}-X_{s,u}^{\nu^2,x}\right|\right)du.$$

Thanks to Lemma 3.4 and Gronwall's Lemma, it follows that

$$\mathbb{E}\left(\left|X_{s,t'}^{\nu^1,x}-X_{s,t'}^{\nu^2,x}\right|\right)\leq C(t-s)e^{\|\zeta\|(t'-s)}\int_s^t|||\nu_u^1-\nu_u^2|||_{\mathcal{LB}}\,du.$$

We inject this inequality for $t'=t$ in (3.19) to obtain (3.18). □

The main result for this part follows :

Theorem 3.1. *Under Assumptions 2.1 and 2.2, the system (2.18) admits a unique solution in $\mathcal{C}([0,T],(\mathcal{M}_F(\mathbb{R}_+)\times\mathbb{R},w\otimes|.|))$. In addition, it satisfies the bound*

$$\sup_{0\leq t\leq T}\langle\nu_t,1+x\rangle<\infty.$$

Proof. The existence of the solution and the bound comes from Theorem 2.1 here above. It then suffices to show that the solution is unique. For that purpose, let us assume that there exist two solutions denoted (ν^1,R^1) and (ν^2,R^2) in $\mathcal{C}([0,T],(\mathcal{M}_F(\mathbb{R}_+)\times\mathbb{R},w\otimes|.|))$. Then it follows from (3.12) that for all test function $\phi\in\mathcal{LB}(\mathbb{R}_+,\mathbb{R})$,

$$\begin{aligned}\langle\nu_t^1-\nu_t^2,\phi\rangle&=\left\langle\nu_0,\left(P_{0,t}^{\nu^1}-P_{0,t}^{\nu^2}\right)\phi\right\rangle\\&+\int_0^t\int_0^\infty\left\{b[\nu^1](x,s)\mathcal{G}[P_{s,t-s}^{\nu^1}\phi](x)-d(x)P_{s,t-s}^{\nu^1}\phi(x)\right\}(\nu_s^1-\nu_s^2)(dx)ds\\&+\int_0^t\int_0^\infty\left\{b[\nu^1](x,s)\mathcal{G}[\left(P_{s,t-s}^{\nu^1}-P_{s,t-s}^{\nu^2}\right)\phi](x)-d(x)\left(P_{s,t-s}^{\nu^1}-P_{s,t-s}^{\nu^2}\right)\phi(x)\right\}\nu_s^2(dx)ds\\&+\int_0^t\int_0^\infty\mathcal{G}[P_{s,t-s}^{\nu^2}\phi](x)\left\{b[\nu^1](x,s)-b[\nu^2](x,s)\right\}\nu_s^2(dx)ds.\end{aligned}\tag{3.20}$$

Furthermore, the operator $\mathcal{G}$ is consistent with the linear space $\mathcal{LB}(\mathbb{R}_+,\mathbb{R})$. Indeed, for all test function $f\in\mathcal{LB}(\mathbb{R}_+,\mathbb{R})$ we have

$$|\mathcal{G}[f](x)|\leq|f(x)|+2\int_0^1|f(\alpha x)|\bar{m}(d\alpha)\leq3\|f\|_\infty,\forall x\geq0$$

and for all $x,y\geq0$,

$$\begin{aligned}\left|\mathcal{G}[f](x)-\mathcal{G}[f](y)\right|&=\left|f(x)-f(y)+2\int_0^1\left[f(\alpha x)-f(\alpha y)\right]\bar{m}(d\alpha)\right|\\&\leq|f(x)-f(y)|+2\int_0^1\left|f(\alpha x)-f(\alpha y)\right|\bar{m}(d\alpha)\\&\leq2\|f\|_{\text{Lip}}|x-y|.\end{aligned}$$

Hence, we can deduce from Lemma 3.5 that for each $s \leq t$ the function

$$\mathbb{R}_+ \ni x \mapsto b[\nu^1](x,s)\mathcal{G}[P^{\nu^1}_{s,t-s}\phi](x) - d(x)P^{\nu^1}_{s,t-s}\phi(x)$$

is in $\mathcal{LB}(\mathbb{R}_+,\mathbb{R})$ so that

$$\begin{aligned}\left\|b[\nu^1](\cdot,s)\mathcal{G}[P^{\nu^1}_{s,t-s}\phi] - d(\cdot)P^{\nu^1}_{s,t-s}\phi\right\|_{\mathcal{LB}} &= \left\|b[\nu^1](\cdot,s)\mathcal{G}[P^{\nu^1}_{s,t-s}\phi] - d(\cdot)P^{\nu^1}_{s,t-s}\phi\right\|_{\infty} \\ &\quad + \left\|b[\nu^1](\cdot,s)\mathcal{G}[P^{\nu^1}_{s,t-s}\phi] - d(\cdot)P^{\nu^1}_{s,t-s}\phi\right\|_{\mathrm{Lip}} \\ &\leq (3\bar{b}+\bar{d})\|\phi\|_\infty + 2\bar{b}\|P^{\nu^1}_{s,t-s}\phi\|_{\mathrm{Lip}} + 3\|\phi\|_\infty\|b\|_{\mathrm{Lip}} \\ &\leq (3\bar{b}+\bar{d}+3\|b\|_{\mathrm{Lip}})\|\phi\|_\infty + 2\bar{b}C\|\phi\|_{\mathrm{Lip}}\end{aligned}$$

and then

$$\left\|b[\nu^1](\cdot,s)\mathcal{G}[P^{\nu^1}_{s,t-s}\phi] - d(\cdot)P^{\nu^1}_{s,t-s}\phi\right\|_{\mathcal{LB}} \leq C\|\phi\|_{\mathcal{LB}}, \forall s \leq t.$$

With equation (3.20) and Lemmas 3.5, 3.4, that implies that

$$\left\langle \nu^1_t - \nu^2_t, \phi\right\rangle \leq C\|\phi\|_{\mathcal{LB}}\left(1 + \sup_{0\leq s\leq T}\langle \nu^2_u, 1\rangle\right)\int_0^t |||\nu^1_s - \nu^2_s|||_{\mathcal{LB}}\, ds.$$

It follows that for all $t \in [0,T]$

$$|||\nu^1_t - \nu^2_t|||_{\mathcal{LB}} = \sup_{\phi\in\mathcal{LB}(\mathbb{R}_+,\mathbb{R});\|\phi\|_{\mathcal{LB}}\leq 1}\left\langle \nu^1_t - \nu^2_t, \phi\right\rangle \leq C\left(1 + \sup_{0\leq s\leq T}\langle \nu^2_u, 1\rangle\right)\int_0^t |||\nu^1_s - \nu^2_s|||_{\mathcal{LB}}\, ds$$

and by Gronwall's lemma, we deduce that $\nu^1_t = \nu^2_t$ for all $t \in [0,T]$. We conclude that $(\nu^1, R^1) = (\nu^2, R^2)$. □

We saw that for this method, the control on the difference $P^{\nu^1}_{s,t-s} - P^{\nu^2}_{s,t-s}$ given in Lemma 3.5 is very important. However, in the general situation where the diffusion coefficient of the individual trait dynamics depends on the resource, we are not able to get such an inequality. In the following section, we suggest another method for this general case that requires more regularity on the coefficients.

3.3 The general case of uniqueness

In this section, we show a more complicated uniqueness result for (2.18) in the general case where the diffusion coefficient depends on the resource dynamics. As in the previous section, the method consists in building a solution of an underlying parabolic partial differential equation in order to remove the derivatives of the test function. We need this function to be regular enough. Let us then introduce the new distance on $\mathcal{M}_F(\mathbb{R}_+)$ defined by

$$|||\mu^1 - \mu^2||| = \sup_{\phi\in\mathcal{C}^2_b(\mathbb{R}_+,\mathbb{R});\|\phi\|\leq 1}\left\langle \mu^1 - \mu^2, \phi\right\rangle, \forall \mu^1, \mu^2 \in \mathcal{M}_F(\mathbb{R}_+) \tag{3.21}$$

where

$$\|\phi\| = \|\phi\|_\infty + \|\phi'\|_\infty + \|\phi''\|_\infty, \forall \phi \in \mathcal{C}^2_b(\mathbb{R}_+,\mathbb{R}). \tag{3.22}$$

We consider the additional assumptions.

Assumption 3.1.

(A.1) $\zeta \in \mathcal{C}^2(\mathbb{R}_+ \times [0, \bar{R}], \mathbb{R})$ *is such that* $\|\partial_x \zeta\|_\infty + \|\partial_r \zeta\|_\infty + \|\partial_x^2 \zeta\|_\infty < +\infty$,

(A.2) $Q \in \mathcal{C}^2(\mathbb{R}_+ \times [0, \bar{R}], \mathbb{R})$ *is such that* $\|\partial_x Q\|_\infty + \|\partial_r Q\|_\infty + \|\partial_x^2 (xQ)\|_\infty < +\infty$,

(A.3) $b \in \mathcal{C}^{2,1}(\mathbb{R}_+ \times [0, \bar{R}], \mathbb{R})$ *is such that* $\|\partial_x b\|_\infty + \|\partial_r b\|_\infty + \|\partial_x^2 b\|_\infty < +\infty$,

(A.4) $d \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R})$ *is such that* $\|d'\|_\infty + \|d''\|_\infty < +\infty$,

(A.5) $Q(x, r) > 0$ *for all* $(x, r) \in (0, \infty) \times (0, \bar{R}]$.

Let us start with an important result that states the regularity and some bounds of the semigroup of the stochastic individual trait dynamics.

Theorem 3.2. *Under (A.1), (A.2), (A.5) in Assumption 3.1, for given* $t \in (0, T]$ *and a test function* $\phi \in \mathcal{C}_b^2(\mathbb{R}_+, \mathbb{R})$*, the parabolic problem*

$$\begin{cases} \partial_s f_s(x) + \zeta[\nu](x, s)\partial_x f_s(x) + xQ[\nu](x, s)\partial_x^2 f_s(x) = 0, x > 0, s \in [0, t) \\ f_{|s=t} = \phi \text{ on } \mathbb{R}_+ \end{cases} \tag{3.23}$$

admits a unique solution in the subspace of $\mathcal{C}([0, t] \times \mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}^{1,2}([0, t) \times (0, \infty), \mathbb{R})$ *constituted of functions* f *that satisfy*

$$\exists C, p > 0, |\partial_x f_s(x)| + |\partial_x^2 f_s(x)| \leq C\big(1 + x^p\big), \forall s < t, \forall x > 0. \tag{3.24}$$

In addition, this solution belongs to $\mathcal{C}^{1,1}([0, t) \times \mathbb{R}_+, \mathbb{R})$*, is given by*

$$f_s^\phi(x) = \mathbb{E}\left[\phi(X_{s,t}^{\nu,x})\right], \forall (s, x) \in [0, t] \times \mathbb{R}_+ \tag{3.25}$$

and satisfies the property

$$|f_s^\phi(x)| + |\partial_x f_s^\phi(x)| + |\partial_x^2 f_s^\phi(x)| \leq C_T \|\phi\|, \forall s < t, \forall x > 0. \tag{3.26}$$

Proof. We split this proof into the following steps :

Step 1 : Uniqueness and characterization of the solution. Let us consider that the system (3.23) admits a solution $(s, x) \mapsto f_s(x)$ in $\mathcal{C}([0, t] \times \mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}^{1,2}([0, t) \times (0, \infty), \mathbb{R})$ that satisfies the polynomial bound (3.24). Then its time derivative satisfies the property

$$|\partial_s f_s(x)| \leq C(1 + x^{1+p}), \forall s < t, \forall x > 0. \tag{3.27}$$

Since this solution is not differentiable on the straight line $x = 0$, we consider the regularising sequence defined for each integer $n \geq 1$ by

$$\rho_n(x) = n\rho(nx), \forall x \geq 0$$

such that

$$\rho \in \mathcal{C}_c^\infty(\mathbb{R}_+, \mathbb{R}), \rho \geq 0\,, \text{ Supp}(\rho) \subseteq [1, 2] \text{ et } \int_0^\infty \rho(y)dy = 1,$$

and we introduce the sequence of functions defined by

$$f_s^n(x) = \int_0^\infty f_s(x + y)\rho_n(y)dy\,, \forall s \in [0, t], \forall x \in \mathbb{R}_+. \tag{3.28}$$

Then for each $n \geq 1$ we have $f^n \in \mathcal{C}([0,t]\times\mathbb{R}_+,\mathbb{R})\cap\mathcal{C}^{1,\infty}([0,t)\times\mathbb{R}_+,\mathbb{R})$ that converges pointwise towards f on $[0,t]\times\mathbb{R}_+$ as $n\to\infty$. In addition, it satisfies on $[0,t)\times(0,\infty)$

$$\partial_s f_s^n(x) = \int_0^\infty \partial_s f_s(x+y)\rho_n(y)dy \xrightarrow[n\to\infty]{} \partial_s f_s(x),$$
$$\partial_x f_s^n(x) = \int_0^\infty \partial_x f_s(x+y)\rho_n(y)dy \xrightarrow[n\to\infty]{} \partial_x f_s(x),$$
$$\partial_x^2 f_s^n(x) = \int_0^\infty \partial_x^2 f_s(x+y)\rho_n(y)dy \xrightarrow[n\to\infty]{} \partial_x^2 f_s(x),$$

and the following uniform bound holds

$$|\partial_s f_s^n(x)| + |\partial_x f_s^n(x)| + |\partial_x^2 f_s^n(x)| \leq C(1+x^{1+p})\,,\ \forall s<t\,,\ \forall x>0. \tag{3.29}$$

It follows from the Itô formula that

$$f_t^n(X_{s,t}^x) = f_s^n(x) + \int_s^t \left\{\partial_u f_u^n(X_{s,u}^x) + X_{s,u}^x Q[\nu](X_{s,u}^x,u)\partial_x^2 f_u^n(X_{s,u}^x)\right\}du + \int_s^t \partial_x f_u^n(X_{s,u}^x)dX_{s,u}^x \tag{3.30}$$

of which the local martingale part in the last term is a true martingale thanks to (3.9) and (3.29). Then for all $0<\varepsilon\leq 1$,

$$\begin{aligned}
\mathbb{E}\left[f_t^n(X_{s,t}^x)\right] &= f_s^n(x) \\
&\quad + \mathbb{E}\int_s^t \left\{\partial_u f_u^n(X_{s,u}^x) + \zeta[\nu](X_{s,u}^x,u)\partial_x f_u^n(X_{s,u}^x) + X_{s,u}^x Q[\nu](X_{s,u}^x,u)\partial_x^2 f_u^n(X_{s,u}^x)\right\}du \\
&= f_s^n(x) \\
&\quad + \underbrace{\mathbb{E}\left(\int_s^t \left\{\partial_u f_u^n(X_{s,u}^x) + \zeta[\nu](X_{s,u}^x,u)\partial_x f_u^n(X_{s,u}^x) + X_{s,u}^x Q[\nu](X_{s,u}^x,u)\partial_x^2 f_u^n(X_{s,u}^x)\right\}1_{\{X_{s,u}^x\leq\varepsilon\}}du\right)}_{I_\varepsilon^n} \\
&\quad + \underbrace{\mathbb{E}\left(\int_s^t \left\{\partial_u f_u^n(X_{s,u}^x) + \zeta[\nu](X_{s,u}^x,u)\partial_x f_u^n(X_{s,u}^x) + X_{s,u}^x Q[\nu](X_{s,u}^x,u)\partial_x^2 f_u^n(X_{s,u}^x)\right\}1_{\{X_{s,u}^x>\varepsilon\}}du\right)}_{J_\varepsilon^n}.
\end{aligned}$$

It follows from the bound (3.29) that

$$|I_\varepsilon^n| \leq C\mathbb{E}\left[\int_s^t \left\{1+(X_{s,u}^x)^{1+p}\right\}1_{\{X_{s,u}^x\leq\varepsilon\}}du\right] \leq C\mathbb{E}\left\{\int_s^t 1_{\{X_{s,u}^x\leq\varepsilon\}}du\right\} \xrightarrow{\varepsilon\to 0} 0$$

and by the dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty}\lim_{\varepsilon\to 0} J_\varepsilon^n = \mathbb{E}\Bigg[\int_s^t \Big\{&\partial_u f_u(X_{s,u}^x) + \zeta[\nu](X_{s,u}^x,u)\partial_x f_u(X_{s,u}^x) \\
&+ X_{s,u}^x Q[\nu](X_{s,u}^x,u)\partial_x^2 f_u(X_{s,u}^x)\Big\}1_{\{X_{s,u}^x>0\}}du\Bigg] = 0.
\end{aligned}$$

Hence, by the dominated convergence theorem again, we have

$$\mathbb{E}\left[\phi(X_{s,t}^{\nu,x})\right] = \lim_{n\to\infty}\mathbb{E}\left[f_t^n(X_{s,t}^{\nu,x})\right] = \lim_{n\to\infty} f_s^n(x) = f_s(x), \forall s\leq t, \forall x\geq 0$$

that is the Feynman-Kac characterization (3.25). The uniqueness follows and it only suffices now to show that a such solution exists.

Step 2 : Construction of a regular solution. A bound property for the second derivative in x of a solution is difficult to obtain directly from the characterization in equation (3.25) because the diffusion coefficient of the stochastic differential equation (3.8) is not differentiable. Here we suggest another method. We start with the equation that would be satisfied by the first derivative in x of the solution we are aiming to construct. Let us then introduce the new system

$$\begin{cases} \partial_s g_s(x) + h(x,s)\partial_x g_s(x) + xQ[\nu](x,s)\partial_x^2 g_s(x) = -\partial_x \zeta[\nu](x,s) g_s(x),\ x>0, s\in[0,t) \\ g_{\big|_{s=t}} = \phi' \ \text{ sur } \mathbb{R}_+ \end{cases} \tag{3.31}$$

where

$$h(x,s) = \zeta[\nu](x,s) + \partial_x(xQ[\nu])(x,s)\,,\ \forall (s,x)\in[0,t]\times\mathbb{R}_+ \tag{3.32}$$

that is differentiable, Lipschitz continuous in x uniformly in s thanks to *(A.1), (A.2)* in Assumption 3.1, and satisfies

$$h(0,s) = \zeta[\nu](0,s) + Q[\nu](0,s) \geq 0, \forall s\in[0,t].$$

We also introduce the function defined by

$$g_s^\phi(x) = \mathbb{E}_{x,s}\left[\phi'(Y_t) e^{\int_s^t \partial_x\zeta[\nu](Y_u,u)du}\right], \forall x\geq 0, \forall s\leq t \tag{3.33}$$

where

$$dY_u = h(Y_u,u)du + \sqrt{2Y_u Q[\nu]\,(Y_u,u)}\,dW_u\,,\ \forall u\leq t. \tag{3.34}$$

Equation (3.34) admits a unique weak solution for any deterministic non-negative initial condition and initial time (see Lemma B.1 in Appendix B). It is shown in the same appendix that there is strong existence, and then we can choose the right continuous and completed canonical filtration of the given Brownian motion W that we will denote $(\mathcal{F}_t^W)_{t\in[0,T]}$. The notation $\mathbb{E}_{x,s}$ refers to the expectation conditionned by the initial time $s<t$ and the initial condition $Y_s = x \geq 0$. The function $(x,s)\mapsto g_s^\phi(x)$ is well defined because ϕ' and $\partial_x\zeta$ are bounded thanks to (A.1) in Assumption 3.1. In addition, it is continuous on $[0,t]\times\mathbb{R}_+$ and satisfies for all $x,y\geq 0$ and $s_1,s_2\leq t$,

$$\left|g_{s_1}^\phi(x) - g_{s_2}^\phi(y)\right| \leq C_T\left(\|\phi'\|_\infty + \|\phi''\|_\infty\right)\left\{|x-y| + (1+x)^{1/2}|s_1-s_2|^{1/2} + (1+x)|s_1-s_2|\right\} \tag{3.35}$$

(see Lemma B.1 in Appendix B). We construct a classical solution for (3.31) by restricting the problem on increasingly greater intervals over which it is uniformly parabolic. Let us then introduce two monotonous sequences $(\varepsilon_n)_n, (m_n)_n$ of positive real numbers that converge towards 0 and ∞ respectively, and satisfy $\varepsilon_0 < m_0$. It follows from the property (3.4) and *(A.5)* in Assumption 3.1 that

$$\forall n\in\mathbb{N}\,,\ \exists a_n>0\,,\ Q[\nu](x,s)\geq a_n\,,\ \forall (x,s)\in[\varepsilon_n,m_n]\times[0,t].$$

For each n, we consider the restricted system

$$\begin{cases} \partial_s g_s(x) + h(x,s)\partial_x g_s(x) + xQ[\nu](x,s)\partial_x^2 g_s(x) = -\partial_x\zeta[\nu](x,s)g_s(x),\ (s,x)\in[0,t)\times(\varepsilon_n,m_n) \\ g_s(x) = g_s^\phi(x)\,,\ (s,x)\in[0,t)\times\{\varepsilon_n,m_n\} \\ g_{\big|_{s=t}} = \phi' \ \text{ sur } [\varepsilon_n,m_n], \end{cases} \tag{3.36}$$

that has Lipschitz continuous coefficients (see Lemma 3.1), and compatible continuous boundary and final values. It then admits a unique classical solution in $\mathcal{C}([0,t]\times[\varepsilon_n,m_n],\mathbb{R})\cap\mathcal{C}^{1,2}([0,t)\times(\varepsilon_n,m_n),\mathbb{R})$ thanks to [17], Theorem 3.6 p138, that is characterized by the Feynman-Kac formula

$$g_s^n(x)=\mathbb{E}_{x,s}\left\{\phi'(Y_t)e^{\int_s^t\partial_x\zeta[\nu](Y_u,u)du}1_{\{\tau_n=t\}}\right\}+\mathbb{E}_{x,s}\left\{g^\phi_{\tau_n}(Y_{\tau_n})e^{\int_s^{\tau_n}\partial_x\zeta[\nu](Y_u,u)du}1_{\{\tau_n<t\}}\right\} \tag{3.37}$$

(see [17], Theorem 10, p147). The stopping time τ_n represents the minimum between the final time t and the hitting time of the boundary $\{\varepsilon_n,m_n\}$ by the stochastic process (Y_u). Further, it follows from (3.33) that

$$\begin{aligned}&\mathbb{E}_{x,s}\left\{g^\phi_{\tau_n}(Y_{\tau_n})e^{\int_s^{\tau_n}\partial_x\zeta[\nu](Y_u,u)du}1_{\{\tau_n<t\}}\right\}\\&\qquad=\mathbb{E}_{x,s}\left\{\mathbb{E}_{Y_{\tau_n},\tau_n}\left[\phi'(Y_t)e^{\int_{\tau_n}^t\partial_x\zeta[\nu](Y_u,u)du}\right]e^{\int_s^{\tau_n}\partial_x\zeta[\nu](Y_u,u)du}1_{\{\tau_n<t\}}\right\}\end{aligned}$$

and thanks to the strong Markov property,

$$\begin{aligned}&\mathbb{E}_{x,s}\left\{g^\phi_{\tau^n}(Y_{\tau_n})e^{\int_s^{\tau_n}\partial_x\zeta[\nu](Y_u,u)du}1_{\{\tau_n<t\}}\right\}\\&\qquad=\mathbb{E}_{x,s}\left\{\mathbb{E}_{x,s}\left[\phi'(Y_t)e^{\int_{\tau_n}^t\partial_x\zeta[\nu](Y_u,u)du}\middle|\mathcal{F}^W_{\tau_n}\right]e^{\int_s^{\tau_n}\partial_x\zeta[\nu](Y_u,u)du}1_{\{\tau_n<t\}}\right\}\\&\qquad=\mathbb{E}_{x,s}\left\{\phi'(Y_t)e^{\int_s^t\partial_x\zeta[\nu](Y_u,u)du}1_{\{\tau_n<t\}}\right\}.\end{aligned}$$

As $\tau_n\le t$ a.s, that implies with (3.37) that $g^n=g^\phi$ on $[0,t]\times[\varepsilon_n,m_n]$. As n becomes larger and larger, we deduce that $g^\phi\in\mathcal{C}([0,t]\times\mathbb{R}_+,\mathbb{R})\cap\mathcal{C}^{1,2}([0,t)\times(0,\infty),\mathbb{R})$ and is a classical solution of (3.31). Furthermore, it follows from its definition that

$$|g_s^\phi(x)|\le\|\phi'\|_\infty e^{\|\partial_x\zeta\|_\infty(t-s)}\,,\ \forall(s,x)\in[0,t]\times\mathbb{R}_+, \tag{3.38}$$

and the inequality (3.35) implies that for all $x>0$ and $s\in[0,t)$,

$$\left|\partial_x g_s^\phi(x)\right|=\lim_{y\to x}\left|\frac{g_s^\phi(y)-g_s^\phi(x)}{y-x}\right|\le C_T\left(\|\phi'\|_\infty+\|\phi''\|_\infty\right). \tag{3.39}$$

Then let us introduce for a given $\theta>0$ the function defined for all $(s,x)\in[0,t)\times\mathbb{R}_+$ by

$$f_s^{\theta,\phi}(x)=\phi(\theta)+\int_s^t\left\{\zeta[\nu](\theta,u)g_u^\phi(\theta)+\theta Q[\nu](\theta,u)\partial_x g_u(\theta)\right\}du+\int_\theta^x g_s^\phi(y)dy, \tag{3.40}$$

that is in $\mathcal{C}^{0,1}([0,t]\times\mathbb{R}_+,\mathbb{R})\cap\mathcal{C}^{1,3}([0,t)\times(0,\infty),\mathbb{R})$ and satisfies

$$\partial_x f^{\theta,\phi}=g^\phi\ \text{ and }\ f^{\theta,\phi}_{|s=t}=\phi.$$

Then thanks to (3.38), (3.39) we have the bound

$$|\partial_x f_s^{\theta,\phi}(x)|+|\partial_x^2 f_s^{\theta,\phi}(x)|\le C_T\left(\|\phi'\|_\infty+\|\phi''\|_\infty\right),\forall x>0,\forall s<t \tag{3.41}$$

that corresponds to the condition (3.24) with some $p>0$. The parameter θ is chosen positive to ensure that it is possible to swap the integral and time derivative symbols in

the last term of (3.40), given that we know nothing about $\partial_s g_s^\phi$ in the neighbourhood of 0. Hence for all $(s,x) \in [0,t) \times (0,\infty)$,

$$\begin{aligned}
\partial_s f_s^{\theta,\phi}(x) &= -\left\{\zeta[\nu](\theta,s)g_s^\phi(\theta) + \theta Q[\nu](\theta,s)\partial_x g_s(\theta)\right\} + \int_\theta^x \partial_s g_s^\phi(y)dy \\
&= -\left\{\zeta[\nu](\theta,s)g_s^\phi(\theta) + \theta Q[\nu](\theta,s)\partial_x g_s(\theta)\right\} - \\
&\qquad \int_\theta^x \left\{h(y,s)\partial_y g_s^\phi(y) + yQ[\nu](y,s)\partial_y^2 g_s^\phi(y) + \partial_y \zeta[\nu](y,s)g_s^\phi(y)\right\} dy \\
&= -\left\{\zeta[\nu](\theta,s)g_s^\phi(\theta) + \theta Q[\nu](\theta,s)\partial_x g_s(\theta)\right\} - \int_\theta^x \partial_y \left\{\zeta[\nu]g^\phi + yQ[\nu]\partial_y g^\phi\right\}(y,s)dy \\
&= -\left\{\zeta[\nu](x,s)g_s^\phi(x) + xQ[\nu](x,s)\partial_x g_s^\phi(x)\right\}.
\end{aligned}$$

That implies that the function $(s,x) \mapsto \partial_s f_s^{\theta,\phi}(x)$ can be continuously extended on the straight line $x = 0$ by setting

$$\partial_s f_s^{\theta,\phi}(0) = -\zeta[\nu](0,s)g_s^\phi(0)\,,\ \forall s \le t, \tag{3.42}$$

and then satisfies the bound

$$\left|\partial_s f_s^{\theta,\phi}(x)\right| \le C\left(\|\phi'\|_\infty + \|\phi''\|_\infty\right)(1+x)\,,\ \forall (s,x) \in [0,t) \times \mathbb{R}_+ \tag{3.43}$$

thanks to (3.38), (3.39). In addition,

$$\partial_s f_s^{\theta,\phi}(x) + \zeta[\nu](x,s)\partial_x f_s^{\theta,\phi}(x) + xQ[\nu](x,s)\partial_x^2 f_s^{\theta,\phi}(x)1_{\{x>0\}} = 0,\ \forall s \in [0,t),\ \forall x \ge 0 \tag{3.44}$$

with the final condition $f_{|s=t}^{\theta,\phi} = \phi$ on $\mathbb{R}_+$. The function $(s,x) \mapsto f_s^{\theta,\phi}(x)$ is then in $\mathcal{C}([0,t] \times \mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}^{1,1}([0,t) \times \mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}^{1,2}([0,t) \times (0,\infty), \mathbb{R})$ and is a solution of the problem (3.23) that satisfies the bound (3.24). Hence the **Step 1** here above holds and the Feynman-Kac characterization (3.25) follows so that the solution is independant of the choice of θ. Let us now denote it by $f_s^\phi(x)$, then

$$|f_s^\phi(x)| \le \|\phi\|_\infty\,,\ \forall (s,x) \in [0,t] \times \mathbb{R}_+ \tag{3.45}$$

and the bound (3.26) follows thanks to the inequality (3.41). □

We now state the following result whose proof is similar to the one of Lemma 3.4.

Lemma 3.6. *Let $(x,r) \mapsto g(x,r)$ be a fonction from $\mathbb{R}_+ \times [0,\bar{R}]$ to $\mathbb{R}$ that is Lipschitz continuous in x uniformly in r, and $\nu^1, \nu^2 \in \mathcal{C}([0,T], (\mathcal{M}_F(\mathbb{R}_+), w))$ be two solutions of (3.7), then*

$$\sup_{(x,s)\in\mathbb{R}_+\times[0,t]} \left|g[\nu^1](x,s) - g[\nu^2](x,s)\right| \le C\|g\|_{Lip} \int_0^t |||\nu_u^1 - \nu_u^2|||du, \forall t \in [0,T]. \tag{3.46}$$

The uniqueness result can then be stated.

Theorem 3.3. *Under Assumptions 2.1, 2.2 and 3.1, the system (2.18) admits a unique solution in $\mathcal{C}([0,T], (\mathcal{M}_F(\mathbb{R}_+) \times \mathbb{R}, w \otimes |.|))$ that satisfies*

$$\sup_{0\le t\le T} \langle \nu_t, 1+x \rangle < +\infty.$$

Proof. The existence and the bound follow from Theorem 2.1. It then suffices to show that the solution is unique. Let $\phi \in \mathcal{C}_b^2(\mathbb{R}_+, \mathbb{R})$ and $(\nu^1, R^1), (\nu^2, R^2) \in \mathcal{C}([0,T], (\mathcal{M}_F(\mathbb{R}_+) \times \mathbb{R}, w \otimes |.|))$ be two solutions of (2.18) such that $\sup_{0\le t\le T}\langle \nu_t^1 + \nu_t^2, 1+x\rangle < +\infty$. We denote by $f_s^\phi(x)$ the solution of (3.23) in the previous theorem for $\nu = \nu^1$ and the final condition ϕ. As in (3.28), we regularize this solution by setting

$$f_s^{n,\phi}(x) = \int_0^\infty f_s^\phi(x+y)\rho_n(y)dy\,,\ \forall (s,x) \in [0,t) \times \mathbb{R}_+.$$

That allows us to approximate $f_s^\phi(x)$ and its derivatives by a sequence of functions that are smooth enough. Let us recall that $f^{n,\phi} \in \mathcal{C}([0,t]\times\mathbb{R}_+,\mathbb{R}) \cap \mathcal{C}^{1,\infty}([0,t)\times\mathbb{R}_+,\mathbb{R})$ for all $n \ge 1$. It follows from the bounds (3.26), (3.43) that for all $x \ge 0$, $s < t$,

$$|\partial_s f_s^{n,\phi}(x)| \le \int_0^\infty |\partial_s f_s^\phi(x+y)|\rho_n(y)dy \le C\left(\|\phi'\| + \|\phi''\|\right)(1+x) \tag{3.47}$$

and

$$|f_s^{n,\phi}(x)| + |\partial_x f_s^{n,\phi}(x)| + |\partial_x^2 f_s^{n,\phi}(x)| \le \int_0^\infty \left\{|f_s^\phi(x+y)| + |\partial_x f_s^\phi(x+y)| + |\partial_x^2 f_s^\phi(x+y)|\right\}\rho_n(y)dy.$$

We deduce that

$$|f_s^{n,\phi}(x)| + |\partial_x f_s^{n,\phi}(x)| + |\partial_x^2 f_s^{n,\phi}(x)| \le C_T\|\phi\|. \tag{3.48}$$

Hence for each $j = 1, 2$, we have

$$\begin{aligned}\langle \nu_t^j, f_t^{n,\phi}\rangle = \langle \nu_0, f_0^{n,\phi}\rangle &+ \int_0^t\int_0^\infty \left\{\partial_s f_s^{n,\phi}(x) + \zeta[\nu^j](x,s)\partial_x f_s^{n,\phi}(x) + xQ[\nu^j](x,s)\partial_x^2 f_s^{n,\phi}(x)\right\}\nu_s^j(dx)ds \\ &+ \int_0^t\int_0^\infty \left\{b[\nu^j](x,s)\mathcal{G}[f_s^{n,\phi}](x) - d(x)f_s^{n,\phi}(x)\right\}\nu_s^j(dx)ds,\end{aligned}$$

which implies that

$$\begin{aligned}\langle \nu_t^1 - \nu_t^2, f_t^{n,\phi}\rangle &= \int_0^t\int_0^\infty \left\{\partial_s f_s^{n,\phi}(x) + \zeta[\nu^1](x,s)\partial_x f_s^{n,\phi}(x) + xQ[\nu^1](x,s)\partial_x^2 f_s^{n,\phi}(x)\right\}(\nu_s^1 - \nu_s^2)(dx)ds \\ &+ \int_0^t\int_0^\infty \left\{(\zeta[\nu^1] - \zeta[\nu^2])(x,s)\partial_x f_s^{n,\phi}(x) + (Q[\nu^1] - Q[\nu^2])(x,s)x\partial_x^2 f_s^{n,\phi}(x)\right\}\nu_s^2(dx)ds \\ &+ \int_0^t\int_0^\infty \left\{b[\nu^1](x,s)\mathcal{G}[f_s^{n,\phi}](x) - d(x)f_s^{n,\phi}(x)\right\}(\nu_s^1 - \nu_s^2)(dx)ds \\ &+ \int_0^t\int_0^\infty \left\{b[\nu^1] - b[\nu^2]\right\}(x,s)\mathcal{G}[f_s^{n,\phi}](x)\nu_s^2(dx)ds.\end{aligned}$$

In addition, the operator $\mathcal{G}$ is an endomorphism of $(\mathcal{C}_b^2(\mathbb{R}_+,\mathbb{R}), \|\cdot\|)$. Indeed, it is obviously linear and satisfies for all $f \in \mathcal{C}_b^2(\mathbb{R}_+,\mathbb{R})$

$$\begin{aligned}\|\mathcal{G}[f]\|_\infty &= \sup_{x\ge 0}\left|-f(x) + 2\int_0^1 f(\alpha x)\bar{m}(d\alpha)\right| \le 3\|f\|_\infty, \\ \|\partial_x\mathcal{G}[f]\|_\infty &= \sup_{x\ge 0}\left|-f'(x) + 2\int_0^1 \alpha f'(\alpha x)\bar{m}(d\alpha)\right| \le 2\|f'\|_\infty, \\ \|\partial_x^2\mathcal{G}[f]\|_\infty &= \sup_{x\ge 0}\left|-f''(x) + 2\int_0^1 \alpha^2 f''(\alpha x)\bar{m}(d\alpha)\right| \le \kappa\|f''\|_\infty,\end{aligned}$$

with

$$\kappa = 1 + 2\int_0^1 \alpha^2 \bar{m}(d\alpha).$$

Then

$$\|\mathcal{G}[f]\| \le (3+\kappa)\,\|f\|\,,\ \forall f \in \mathcal{C}_b^2(\mathbb{R}_+,\mathbb{R}). \tag{3.49}$$

We deduce that for each $s<t$ and $n\ge 1$, the function $\mathbb{R}_+ \ni x \mapsto b[\nu^2](x,s)\mathcal{G}[f_s^{n,\phi}](x) - d(x)f_s^{n,\phi}(x)$ is in $\mathcal{C}_b^2(\mathbb{R}_+,\mathbb{R})$ and satisfies

$$\begin{aligned}
\|b[\nu^2](\cdot,s)\mathcal{G}[f_s^{n,\phi}] - d(\cdot)f_s^{n,\phi}\| &= \|b[\nu^2](\cdot,s)\mathcal{G}[f_s^{n,\phi}] - d(\cdot)f_s^{n,\phi}\|_\infty \\
&\quad + \|\partial_x\big(b[\nu^2](\cdot,s)\mathcal{G}[f_s^{n,\phi}] - d(\cdot)f_s^{n,\phi}\big)\|_\infty \\
&\quad + \|\partial_x^2\big(b[\nu^2](\cdot,s)\mathcal{G}[f_s^{n,\phi}] - d(\cdot)f_s^{n,\phi}\big)\|_\infty \\
&\le \|b\|_\infty\|\mathcal{G}[f_s^{n,\phi}]\|_\infty + \|d\|_\infty\|f_s^{n,\phi}\|_\infty + \|\partial_x b\|_\infty\|\mathcal{G}[f_s^{n,\phi}]\|_\infty \\
&\quad + \|b\|_\infty\|\partial_x\mathcal{G}[f_s^{n,\phi}]\|_\infty + \|\partial_x d\|_\infty\|f_s^{n,\phi}\|_\infty + \|d\|_\infty\|\partial_x f_s^{n,\phi}\|_\infty \\
&\quad + \|\partial_x^2 b\|_\infty\|\mathcal{G}[f_s^{n,\phi}]\|_\infty + 2\|\partial_x b\|_\infty\|\partial_x\mathcal{G}[f_s^{n,\phi}]\|_\infty + \|b\|_\infty\|\partial_x^2\mathcal{G}[f_s^{n,\phi}]\|_\infty \\
&\quad + \|\partial_x^2 d\|_\infty\|f_s^{n,\phi}\|_\infty + 2\|\partial_x b\|_\infty\|\partial_x f_s^{n,\phi}\|_\infty + \|d\|_\infty\|\partial_x^2 f_s^{n,\phi}\|_\infty \\
&\le C\|f_s^{n,\phi}\|.
\end{aligned}$$

Thanks to the uniform bound (3.48), we obtain

$$\|b[\nu^2](\cdot,s)\mathcal{G}[f_s^{n,\phi}] - d(x)f_s^{n,\phi}\| \le C_T\|\phi\|\,,\ \forall s\in[0,t).$$

It follows that

$$\begin{aligned}
\langle \nu_t^1 - \nu_t^2, f_t^{n,\phi}\rangle &\le \int_0^t\int_0^\infty \left\{\partial_s f_s^{n,\phi}(x) + \zeta[\nu^1](x,s)\partial_x f_s^{n,\phi}(x) + xQ[\nu^1](x,s)\partial_x^2 f_s^{n,\phi}(x)\right\}(\nu_s^1-\nu_s^2)(dx)ds \\
&\quad + \int_0^t\int_0^\infty \left\{(\zeta[\nu^1]-\zeta[\nu^2])(x,s)\partial_x f_s^{n,\phi}(x) + (Q[\nu^1]-Q[\nu^2])(x,s)x\partial_x^2 f_s^{n,\phi}(x)\right\}\nu_s^2(dx)ds \\
&\quad + C_T\|\phi\|\int_0^t |||\nu_s^1-\nu_s^2|||ds + \int_0^t\int_0^\infty \left\{b[\nu^1]-b[\nu^2]\right\}(x,s)\mathcal{G}[f_s^{n,\phi}](x)\nu_s^2(dx)ds.
\end{aligned}$$

By the dominated convergence theorem that can be applied because of the uniform bounds (3.47), (3.48), and thanks to the equation (3.44), we obtain as $n\to\infty$

$$\begin{aligned}
\langle \nu_t^1 - \nu_t^2, \phi\rangle &\le \int_0^t\int_0^\infty \left\{(\zeta[\nu^1]-\zeta[\nu^2])(x,s)\partial_x f_s^{\phi}(x) + (Q[\nu^1]-Q[\nu^2])(x,s)x\partial_x^2 f_s^{\phi}(x)1_{\{x>0\}}\right\}\nu_s^2(dx)ds \\
&\quad + C_T\|\phi\|\int_0^t |||\nu_s^1-\nu_s^2|||ds + \int_0^t\int_0^\infty \left\{b[\nu^1]-b[\nu^2]\right\}(x,s)\mathcal{G}[f_s^{\phi}](x)\nu_s^1(dx)ds.
\end{aligned} \tag{3.50}$$

We deduce from this inequality, the bound (3.26) and Lemma 3.6, that

$$\langle \nu_t^1 - \nu_t^2, \phi\rangle \le C_T\|\phi\|\left[1 + \sup_{0\le s\le T}\langle \nu_s^1+\nu_s^2, 1+x\rangle\right]\int_0^t |||\nu_s^1-\nu_s^2|||ds, \forall t\in[0,T]$$

and then

$$|||\nu_t^1-\nu_t^2||| \le C_T\left[1 + \sup_{0\le s\le T}\langle \nu_s^1+\nu_s^2, 1+x\rangle\right]\int_0^t |||\nu_s^1-\nu_s^2|||ds.$$

By the Gronwall lemma, we finally obtain $\nu_t^1=\nu_t^2$ for all $t\in[0,T]$. We conclude that $(\nu^1,R^1)=(\nu^2,R^2)$. □

4 Existence of a function solution

In this section, we use the mild formulation (3.12) to show that under suitable assumptions the solution of the decoupled equation (3.7) admits a density with respect to the Lebesgue measure. This density will then be the solution of (1.1) in the weak sense given by (2.18). We first make the following general assumptions.

Assumption 4.1.

(A.1) *There exists* $\underline{c} > 0$ *such that* $Q(x,r), \zeta(x,r) \geq \underline{c}$ *for all* $(x,r) \in \mathbb{R}_+ \times (0, \bar{R}]$.

(A.2) $Q(x,r)$ *is bounded, positive for all* $(x,r) \in (0,\infty) \times (0,\bar{R}]$ *and*

$$|Q(x,r) - Q(x,r')| \leq C(1+x^k)|r-r'|^\beta, \forall x \geq 0, \forall r, r' \in (0, \bar{R}]$$

with $C, k > 0$ *and* $\beta \in (0,1]$.

(A.3) $\zeta(x,r), Q(x,r) \in \mathcal{C}(\mathbb{R}_+ \times (0,\bar{R}], \mathbb{R})$ *are Lipschitz continuous in* $x \in \mathbb{R}_+$, *uniformly in* $r \in (0,\bar{R}]$.

We easily verify that the above assumption (A.2) implies that the decoupled function $Q[\nu]$ satisfies the same Hölder property in its time variable. Indeed, it follows from (3.1) that for all $x \geq 0$ and $s,t \in [0,T]$,

$$|Q[\nu](x,t) - Q[\nu](x,s)| = |Q(x,R_t) - Q(x,R_s)| \leq C(1+x^k)|R_t - R_s|^\beta \leq C(1+x^k)|t-s|^\beta.$$

For the sake of simplicity, we introduce the function

$$\tilde{\sigma}_t(y) = \sqrt{yQ[\nu](y,t)}\,, \forall (y,t) \in \mathbb{R}_+ \times [0,T] \tag{4.1}$$

that will be important in the sequel. We have the preliminary result

Proposition 4.1. *Under Assumption 4.1, there exists a unique non negative function* $p^\nu_{s,t}(x,y)$ *defined for all* $(t,x,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$ *and* $s < t$, *such that for all test functions* $\phi \in \mathcal{C}_b(\mathbb{R}_+, \mathbb{R})$,

$$P^\nu_{s,t-s}\phi(x) = \int_0^\infty p^\nu_{s,t}(x,y)\phi(y)dy. \tag{4.2}$$

In addition, if we set

$$c_{\alpha,k} = \alpha \vee \frac{(k \vee 1)}{2} \quad \text{and} \quad \eta_{\alpha,\beta,m} = \beta \wedge \frac{1}{2} \wedge \frac{m-3\alpha}{2m/\alpha} \tag{4.3}$$

for any $0 < \alpha < 1$ *and* $m > 3\alpha$, *then the function*

$$y \mapsto \tilde{\sigma}_t^m(y) p^\nu_{s,t}(x,y)$$

is, for any $(x,t) \in \mathbb{R}_+ \times (0,T]$ *and* $s < t$, *in the Besov space* $\mathcal{B}_{1,\infty}^{\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}}(\mathbb{R}_+^*)$ *(defined in Appendix C) and satisfies the bound*

$$\sup_{s<t\leq T} \left\| \tilde{\sigma}_t^m p^\nu_{s,t}(x,\cdot) \right\|_{\mathcal{B}_{1,\infty}^{\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}}(\mathbb{R}_+^*)} \leq C\left(1 + x^{c_{\alpha,k}+m/2}\right), \forall x \geq 0. \tag{4.4}$$

Proof. Basically, the left term in (4.2) is defined by

$$P^\nu_{s,t-s}\phi(x) = \mathbb{E}\left[\phi(X^x_{s,t})\right], \forall x \geq 0, \forall 0 \leq s < t \leq T.$$

The proof then consists in showing that the random variable $X^x_{s,t}$ for fixed $x \geq 0$ and $0 \leq s < t \leq T$ admits a density with respect to the Lebesgue measure. As the diffusion coefficient in equation (3.8) is degenerate at $x = 0$, we will first verify that there is no atom at this point and secondly show that there is a density on the open set $\mathbb{R}_+^*$.

Step 1 : No atom on the boundary $\{0\}$**.** Let us first notice that for all $x \geq 0$ and $0 \leq s < T$, it follows from the comparison theorem that $X^0_{s,t} \leq X^x_{s,t}$ a.s for all $t \in (s, T]$. It then suffices to show that $X^0_{s,t}$ does not have an atom at $\{0\}$.
Let us introduce for $h > 0$ the convex continuous and decreasing function

$$\chi_h(x) = \left(1 - \frac{x}{h}\right)_+.$$

Then as $h \to 0$, the sequence $(\chi_h)_{h>0}$ converges toward $1_{\{x=0\}}$. In addition, we have

$$\frac{1}{2} 1_{[0,h/2]} \leq \chi_h \leq 1_{[0,h]} \text{ on } \mathbb{R}_+ \tag{4.5}$$

and the function defined for fixed $t \in (0, T]$ by

$$u_h(x, s) = \mathbb{E}_{x,s}\left[\chi_h(X^x_{s,t})\right], \forall (x, s) \in \mathbb{R}^*_+ \times [0, t] \tag{4.6}$$

is a classical solution of the partial differential equation

$$\partial_s u_h + \zeta[\nu](x, s)\partial_x u_h + xQ[\nu](x, s)\partial^2_x u_h = 0 \text{ on } \mathbb{R}^*_+ \times [0, t). \tag{4.7}$$

Our aim is to show that $\mathbb{P}(X^0_{s,t} = 0) = \lim_{h\to 0} u_h(0, s) = 0$ for all $s \in [0, t]$. For that purpose, let us first notice that the function u_h is convex since its initial condition χ_h is also convex (see [22]). It follows thanks to (A.1) in Assumption 4.1 that

$$\partial_s u_h + \zeta[\nu](x, s)\partial_x u_h + \underline{c}\, x\partial^2_x u_h \leq 0 \text{ on } \mathbb{R}^*_+ \times [0, t). \tag{4.8}$$

It follows from the Maximum principle that $u_h \leq \bar{u}_h$ where

$$\bar{u}_h(x, s) = \mathbb{E}\left[\chi_h(Y^x_{s,t})\right]$$

with

$$dY_t = \zeta[\nu](Y_t, t)dt + \sqrt{2\underline{c}\, Y_t}\, dW_t.$$

The function $\bar{u}_h$ is, indeed, a classical solution of the partial differential equation

$$\partial_s \bar{u}_h + \zeta(x, s)\partial_x \bar{u}_h + \underline{c}\, x\partial^2_x \bar{u}_h = 0, \text{ on } \mathbb{R}^*_+ \times [0, t)$$

with the same terminal condition χ_h. Further, thanks to (A.1) in Assumption 4.1, it follows from the comparison theorem for stochastic differential equations that $Y^x_{s,t} \geq Z^x_{s,t}$ a.s, where

$$dZ_t = \underline{c}\, dt + \sqrt{2\underline{c}\, Z_t}\, dW_t. \tag{4.9}$$

Since χ_h is decreasing we obtain

$$\bar{u}_h(x, s) \leq \tilde{u}_h(x, s) := \mathbb{E}\left[\chi_h(Z^x_{s,t})\right],$$

and thanks to (4.5),

$$\frac{1}{2} F^x_{s,t}(h/2) \leq u_h(x, s) \leq \tilde{u}_h(x, s) \leq \tilde{F}^x_{s,t}(h) \tag{4.10}$$

where $F^x_{s,t}$ and $\tilde{F}^x_{s,t} = \tilde{F}^x_{t-s}$ are the cumulative distribution functions of the random variables $X^x_{s,t}$ and $Z^x_{s,t}$ respectively. Further it directly follows from the autosimilarity

property of the Brownian motion and the uniqueness in law of the solution of (4.9) given the initial condition, that

$$cZ^{x/c}_{t/c} = Z^x_t \text{ in law, } \forall c > 0, \forall t, x \in \mathbb{R}_+.$$

We deduce that specifically for $x = 0$,

$$\tilde{F}^0_t(y) = \tilde{F}^0_{t/c}(y/c), \forall t, y \geq 0, \forall c > 0$$

and then there exists a function $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\tilde{F}^0_t(y) = G(y/t), \forall y \geq 0, \forall t > 0. \tag{4.11}$$

Since $\tilde{F}^0$ is a weak solution of

$$\begin{cases} \partial_t \tilde{F}^0_t - \underline{c}\partial_y(y\partial_y \tilde{F}^0_t) + \underline{c}\partial_y \tilde{F}^0_t = 0 \\ (\tilde{F}^0_t(y))_{|t=0} = 1, \forall y \geq 0, \end{cases} \tag{4.12}$$

we consider the time changing $\bar{F}_t(y) = \tilde{F}_{t/\underline{c}}(y)$ and $\bar{G}(y) = G(\underline{c}\, y)$, then we have

$$\partial_t \bar{F}_t - \partial_y(y\partial_y \bar{F}_t) + \partial_y \bar{F}_t = 0$$

with the same initial condition as (4.12). One deduce with (4.11), that

$$-y\bar{G}'(y) + \bar{G}'(y) - (y\bar{G}'(y))' = 0$$

whose general solution is given by

$$\bar{G}(y) = a_1 + a_2 e^{-y}, \forall y > 0$$

with $a_1, a_2 \in \mathbb{R}$. Coming back to the cumulative distribution function of Z^0_t, we get

$$\tilde{F}^0_t(y) = \bar{G}\left(\frac{y}{\underline{c}\, t}\right) \xrightarrow[y\to 0]{} a_1 + a_2. \tag{4.13}$$

Since $(Z^0_t)_{t\in[s,T]}$ is a non-negative semi-martingale with a zero local time at 0 (see Lemma B.1, Appendix B), it follows from the Tanaka formula that

$$\int_0^t 1_{\{Z^0_u=0\}} dZ^0_u = \int_0^t 1_{\{Z^0_u=0\}} d(Z^0_u)^+ = \int_0^t 1_{\{Z^0_u=0\}} \left(1_{\{Z^0_u>0\}} dZ^0_u + \frac{1}{2} dL^0_u(Z^0)\right) = 0,$$

and then

$$0 = \int_0^t 1_{\{Z^0_u=0\}} \left[\underline{c}\, du + \sqrt{2\underline{c}\, Z^0_u}\, dW_u\right] = \underline{c} \int_0^t 1_{\{Z^0_u=0\}} du.$$

We deduce by taking the expectation, that

$$\int_0^t \mathbb{P}\big(Z^0_u = 0\big) du = 0\,,\ \forall t \in (s, T].$$

Noticing that $\mathbb{P}(Z^0_u = 0) = \tilde{F}^0_u(0)$, we conclude that $a_1 + a_2 = 0$ in (4.13). It follows thanks to (4.10) that

$$\mathbb{P}(X^0_{s,t} = 0) = \lim_{h\to 0} F^0_{s,t}(h/2) \leq 2 \lim_{h\to 0} \tilde{F}^0_{t-s}(h) = 0\,,\ \forall t \in (s, T].$$

Then for all test function $\phi \in \mathcal{C}_b(\mathbb{R}_+, \mathbb{R})$,

$$\mathbb{E}\left[\phi(X^x_{s,t})\right] = \phi(0) \underbrace{\mathbb{P}\left[X^x_{s,t} = 0\right]}_{=0} + \mathbb{E}\left[\phi(X^x_{s,t}) 1_{\{X^x_{s,t}>0\}}\right],$$

and then it suffices to show that the random variable $X^x_{s,t}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^*_+$, to conclude that it has a density on $\mathbb{R}_+$.

Step 2 : Existence of a density on $\mathbb{R}_+^*$. We adapt the proof developed by Romito [28], Proposition 3.1 in the case of non homogeneous and non bounded coefficients. It consists in showing the sufficient condition given in Lemma C.4 (see Appendix C) with a suitable function. For that purpose, we introduce for a given initial time $s \in [0,T)$, a fixed time $t \in (s,T]$ and $0 < \varepsilon < t-s$, the non negative function

$$\sigma_{t-\varepsilon,t}(y) = \sqrt{\frac{y}{\varepsilon}\int_{t-\varepsilon}^{t} Q[\nu](y,u)\,du}\,,\ \forall y \geq 0 \tag{4.14}$$

that satisfy with $\tilde{\sigma}_t$ defined in (4.1) the two following properties :

$$\begin{aligned} &\text{(P1)} : \tilde{\sigma}_t(y) + \sigma_{t-\varepsilon,t}(y) \leq C y^{1/2}, \forall y \geq 0, \\ &\text{(P2)} : |\tilde{\sigma}_t(x) - \tilde{\sigma}_t(y)| + |\sigma_{t-\varepsilon,t}(x) - \sigma_{t-\varepsilon,t}(y)| \leq C(1+x^{1/2})|x-y|^{1/2}, \forall x,y \geq 0. \end{aligned} \tag{4.15}$$

Let $m \geq 1$, $0 < \alpha < 1$ and $h \in [-1,1]$, our aim is to show that the inequality (C.7) holds for the random variable $X_{s,t}^x$ with the function $y \mapsto \tilde{\sigma}_t^m(y)$. Indeed, for any $\phi \in \mathscr{C}_b^\alpha(\mathbb{R})$,

$$\begin{aligned} \mathbb{E}\left[\tilde{\sigma}_t^m(X_{s,t}^x)\Delta_h^m\phi(X_{s,t}^x)\right] &= \mathbb{E}\left[\left\{\tilde{\sigma}_t^m(X_{s,t}^x) - \sigma_{t-\varepsilon,t}^m(X_{s,t}^x)\right\}\Delta_h^m\phi(X_{s,t}^x)\right] \\ &\quad + \mathbb{E}\left[\sigma_{t-\varepsilon,t}^m(X_{s,t}^x)\Delta_h^m\phi(X_{s,t}^x)\right]. \end{aligned} \tag{4.16}$$

Thanks to (A.2) in Assumption 4.1, we obtain

$$\begin{aligned} |\tilde{\sigma}_t^m(X_{s,t}^x) - \sigma_{t-\varepsilon,t}^m(X_{s,t}^x)| &\leq |\tilde{\sigma}_t(X_{s,t}^x) - \sigma_{t-\varepsilon,t}(X_{s,t}^x)| \sum_{j=0}^{m-1} |\sigma_{t-\varepsilon,t}(X_{s,t}^x)|^j |\tilde{\sigma}_t(X_{s,t}^x)|^{m-1-j} \\ &\leq C\left(\sqrt{\frac{X_{s,t}^x}{\varepsilon}\int_{t-\varepsilon}^{t}\left|Q[\nu](X_{s,t}^x,t) - Q[\nu](X_{s,t}^x,u)\right|du}\right)(X_{s,t}^x)^{\frac{m-1}{2}} \\ &\leq C\left(\sqrt{\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}(t-u)^\beta du}\right)\left(1+(X_{s,t}^x)^k\right)^{1/2}(X_{s,t}^x)^{\frac{m}{2}} \\ &\leq C\left(1+(X_{s,t}^x)^{k/2}\right)(X_{s,t}^x)^{\frac{m}{2}}\varepsilon^{\beta/2}. \end{aligned}$$

It follows from the moment property (3.9) and the property 1. of Lemma C.1 (see Appendix C) that the first term in the right side of (4.16) satisfies

$$\mathbb{E}\left[\left\{\tilde{\sigma}_t^m(X_{s,t}^x) - \sigma_{t-\varepsilon,t}^m(X_{s,t}^x)\right\}\Delta_h^m\phi(X_{s,t}^x)\right] \leq C\big(1+x^{\frac{m+k}{2}}\big)|h|^\alpha\|\phi\|_{\mathscr{C}_b^\alpha(\mathbb{R})}\varepsilon^{\beta/2}. \tag{4.17}$$

In order to get a good control on the second term of the same equation, we introduce the following new process

$$\forall u \in [s,t],\ Y_{s,u}^{\varepsilon,x} = \begin{cases} X_{s,u}^x & \text{if } u \leq t-\varepsilon, \\ X_{s,t-\varepsilon}^x + \displaystyle\int_{t-\varepsilon}^{u}\sqrt{2X_{s,t-\varepsilon}^x Q[\nu](X_{s,t-\varepsilon}^x,u)}\,dW_u & \text{if } u > t-\varepsilon. \end{cases}$$

This process is continuous and Gaussian on the interval $(t-\varepsilon, t]$ when it is conditioned by $X^x_{t-\varepsilon}$. Then,

$$\begin{aligned}\mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(X^x_{s,t})\Delta^m_h\phi(X^x_{s,t})\right] &= \mathbb{E}\left[\left\{\sigma^m_{t-\varepsilon,t}(X^x_{s,t}) - \sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\right\}\Delta^m_h\phi(X^x_{s,t})\right] \\ &+ \mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\left\{\Delta^m_h\phi(X^x_{s,t}) - \Delta^m_h\phi(Y^{\varepsilon,x}_{s,t})\right\}\right] \\ &+ \mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\Delta^m_h\phi(Y^{\varepsilon,x}_{s,t})\right].\end{aligned} \tag{4.18}$$

It now suffices to show that each term of this decomposition can be well controled. For the first one, it follows from (4.15) and the property 1. of Lemma C.1 (see Appendix C) that

$$\begin{aligned}&\mathbb{E}\left[\left\{\sigma^m_{t-\varepsilon,t}(X^x_{s,t}) - \sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\right\}\Delta^m_h\phi(X^x_{s,t})\right] \\ &\quad\le C|h|^\alpha\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\mathbb{E}\left[\left|\sigma^m_{t-\varepsilon,t}(X^x_{s,t}) - \sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\right|\right] \\ &\quad\le C|h|^\alpha\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\mathbb{E}\left[\left|\sigma_{t-\varepsilon,t}(X^x_{s,t}) - \sigma_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\right|\sum_{j=0}^{m-1}|\sigma_{t-\varepsilon,t}(X^x_{s,t})|^j|\sigma_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})|^{m-1-j}\right] \\ &\quad\le C|h|^\alpha\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\mathbb{E}\left[\left|\sigma_{t-\varepsilon,t}(X^x_{s,t}) - \sigma_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\right|\sup_{0\le u\le t}(X^x_{s,u})^{\frac{m-1}{2}}\right] \\ &\quad\le C|h|^\alpha\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\mathbb{E}\left[\left|X^x_{s,t} - X^x_{s,t-\varepsilon}\right|^{1/2}\sup_{0\le u\le t}\left(1+(X^x_{s,u})^{\frac{m}{2}}\right)\right].\end{aligned}$$

Thanks to the Cauchy-Schwartz inequality and the moment property (3.9),

$$\begin{aligned}&\mathbb{E}\left[\left\{\sigma^m_{t-\varepsilon,t}(X^x_{s,t}) - \sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\right\}\Delta^m_h\phi(X^x_{s,t})\right] \\ &\qquad\le C|h|^\alpha\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\mathbb{E}\left[\left|X^x_{s,t} - X^x_{s,t-\varepsilon}\right|\right]^{1/2}\mathbb{E}\left[\sup_{0\le u\le t}\left(1+(X^x_{s,u})^m\right)\right]^{1/2} \\ &\qquad\le C\big(1+x^{m/2}\big)|h|^\alpha\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\mathbb{E}\left[\left|X^x_{s,t} - X^x_{s,t-\varepsilon}\right|\right]^{1/2}.\end{aligned}$$

In addition, for all $s \le t-\varepsilon \le r \le t$,

$$\left|X^x_{s,r} - X^x_{s,t-\varepsilon}\right| \le \int_{t-\varepsilon}^r |\zeta[\nu](X^x_{s,u},u)|du + \sup_{t-\varepsilon\le r'\le r}\left|\int_{t-\varepsilon}^{r'}\sqrt{2X^x_{s,u}Q[\nu](X^x_{s,u},u)}\,dW_u\right|.$$

Then, thanks to the Burkholder-Davis-Gundy inequality and the moment property (3.9),

$$\begin{aligned}\mathbb{E}\left[\left|X^x_{s,r} - X^x_{s,t-\varepsilon}\right|\right] &\le C\left\{\varepsilon\mathbb{E}\left[\sup_{s\le u\le t}\left(1+X^x_{s,u}\right)\right] + \mathbb{E}\left[\left(\int_{t-\varepsilon}^r 2X^x_{s,u}Q[\nu](X^x_{s,u},u)du\right)^{1/2}\right]\right\} \\ &\le C\left\{\varepsilon\mathbb{E}\left[\sup_{s\le u\le t}\left(1+X^x_{s,u}\right)\right] + \varepsilon^{1/2}\mathbb{E}\left[\sup_{s\le u\le t}\left(X^x_{s,u}\right)^{1/2}\right]\right\} \\ &\le C\big(1+x\big)\varepsilon^{1/2}.\end{aligned} \tag{4.19}$$

That implies for the first term in the right side of (4.18) that

$$\mathbb{E}\left[\left\{\sigma^m_{t-\varepsilon,t}(X^x_{s,t}) - \sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\right\}\Delta^m_h\phi(X^x_{s,t})\right] \le C\big(1+x^{\frac{m+1}{2}}\big)|h|^\alpha\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\varepsilon^{1/4}. \tag{4.20}$$

The second one satisfies, thanks to (P1) in (4.15), the property 2. of Lemma C.1 (see Appendix C), the Hölder inequality and the moment property (3.9),

$$\begin{aligned}
&\mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\left\{\Delta^m_h\phi(X^x_{s,t})-\Delta^m_h\phi(Y^{\varepsilon,x}_{s,t})\right\}\right]\\
&\qquad\le C\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\mathbb{E}\left[\left|X^x_{s,t}-Y^{\varepsilon,x}_{s,t}\right|^\alpha\sup_{s\le u\le t}(X^x_{s,u})^{m/2}\right]\\
&\qquad\le C\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\mathbb{E}\left[\left|X^x_{s,t}-Y^{\varepsilon,x}_{s,t}\right|\right]^\alpha\mathbb{E}\left[\sup_{s\le u\le t}(X^x_{s,u})^{m/2(1-\alpha)}\right]^{1-\alpha}\\
&\qquad\le C\big(1+x^{m/2}\big)\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\mathbb{E}\left[\left|X^x_{s,t}-Y^{\varepsilon,x}_{s,t}\right|\right]^\alpha
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left|X^x_{s,t}-Y^{\varepsilon,x}_{s,t}\right|&\le\int_{t-\varepsilon}^t|\zeta[\nu](X^x_{s,u},u)|du\\
&\quad+\sup_{t-\varepsilon\le t'\le t}\left|\int_{t-\varepsilon}^{t'}\left[\sqrt{2X^x_{s,u}Q[\nu](X^x_{s,u},u)}-\sqrt{2X^x_{s,t-\varepsilon}Q[\nu](X^x_{s,t-\varepsilon},u)}\right]dW_u\right|,
\end{aligned}$$

and thanks to the Burkholder-Davis-Gundy, the Hölder inequalities and (3.9),

$$\begin{aligned}
\mathbb{E}\left[\left|X^x_{s,t}-Y^{\varepsilon,x}_{s,t}\right|\right]&\le C\varepsilon\mathbb{E}\left[\sup_{s\le u\le t}\left(1+X^x_{s,u}\right)\right]\\
&\quad+\mathbb{E}\left[\left(\int_{t-\varepsilon}^t\left|\sqrt{2X^x_{s,u}Q[\nu](X^x_{s,u},u)}-\sqrt{2X^x_{s,t-\varepsilon}Q[\nu](X^x_{s,t-\varepsilon},u)}\right|^2du\right)^{1/2}\right]\\
&\le C\varepsilon\mathbb{E}\left[\sup_{s\le u\le t}\left(1+X^x_{s,u}\right)\right]+C\mathbb{E}\left[\sup_{s\le u\le t}\left(1+X^x_{s,u}\right)^{1/2}\left(\int_{t-\varepsilon}^t\left|X^x_{s,u}-X^x_{s,t-\varepsilon}\right|du\right)^{1/2}\right]\\
&\le C\varepsilon\mathbb{E}\left[\sup_{s\le u\le t}\left(1+X^x_{s,u}\right)\right]+C\mathbb{E}\left[\sup_{s\le u\le t}\left(1+X^x_{s,u}\right)\right]^{1/2}\mathbb{E}\left[\int_{t-\varepsilon}^t\left|X^x_{s,u}-X^x_{s,t-\varepsilon}\right|du\right]^{1/2}\\
&\le C\big(1+x^{1/2}\big)\left\{\big(1+x^{1/2}\big)\varepsilon+\mathbb{E}\left[\int_{t-\varepsilon}^t\left|X^x_{s,u}-X^x_{s,t-\varepsilon}\right|du\right]^{1/2}\right\}.
\end{aligned}$$

The last term in the right side is well controled thanks to (4.19), so that

$$\mathbb{E}\left[\int_{t-\varepsilon}^t\left|X^x_{s,u}-X^x_{s,t-\varepsilon}\right|du\right]=\int_{t-\varepsilon}^t\mathbb{E}\left[\left|X^x_{s,u}-X^x_{s,t-\varepsilon}\right|\right]du\le C\big(1+x\big)\varepsilon^{3/2},$$

that implies for the second term in the decomposition (4.18) that

$$\mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\left\{\Delta^m_h\phi(X^x_{s,t})-\Delta^m_h\phi(Y^{\varepsilon,x}_{s,t})\right\}\right]\le C\big(1+x^{\alpha+m/2}\big)\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\varepsilon^{3\alpha/4}. \tag{4.21}$$

Finally, by conditioning by $X^x_{t-\varepsilon}$, the third and last term of that decomposition satisfies

$$\begin{aligned}
&\mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\Delta^m_h\phi(Y^{\varepsilon,x}_{s,t})\right]\\
&\qquad=\mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\Delta^m_h\phi\left(X^x_{s,t-\varepsilon}+\int_{t-\varepsilon}^t\sqrt{2X^x_{s,t-\varepsilon}Q[\nu](X^x_{s,t-\varepsilon},u)}dW_u\right)\right]\\
&\qquad=\mathbb{E}\left\{\mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(y)\Delta^m_h\phi\left(y+\int_{t-\varepsilon}^t\sqrt{2yQ[\nu](y,u)}dW_u\right)\right]_{\big|y=X^x_{s,t-\varepsilon}}\right\}
\end{aligned}$$

The expression inside the second expectation is zero if $y = 0$, and for $y > 0$ it satisfies

$$\int_s^t \sqrt{2yQ[\nu](y,u)}dW_u \sim \mathcal{N}(0, 2\varepsilon\sigma^2_{t-\varepsilon,t}(y)).$$

Thanks to properties 3. and 4. of Lemma C.1 (see Appendix C), that implies

$$\begin{aligned}
\mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(y)\Delta^m_h\phi\left(y + \int_s^t \sqrt{2yQ[\nu](y,u)}dW_u\right)\right] &= \sigma^m_{t-\varepsilon,t}(y)\int_{-\infty}^{+\infty} \Delta^m_h\phi\,(y+z)\, g^{\varepsilon,y}_t(z)dz \\
&= \sigma^m_{t-\varepsilon,t}(y)\int_{-\infty}^{+\infty} \phi\,(y+z)\,\Delta^m_{-h} g^{\varepsilon,y}_t(z)dz \\
&\leq \sigma^m_{t-\varepsilon,t}(y)\|\phi\|_\infty \|\Delta^m_{-h} g^{\varepsilon,y}_t\|_{L^1} \\
&\leq \sigma^m_{t-\varepsilon,t}(y)|h|^m\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\|\partial^m_z g^{\varepsilon,y}_t\|_{L^1}
\end{aligned}$$

where

$$g^{\varepsilon,y}_t(z) = \frac{1}{2\sigma_{t-\varepsilon,t}(y)\sqrt{\pi\varepsilon}}\exp\left[-\frac{x^2}{4\varepsilon\sigma^2_{t-\varepsilon,t}(y)}\right].$$

Further,

$$\|\partial^m_z g^{\varepsilon,y}_t\|_{L^1} \leq \frac{C_m}{\varepsilon^{m/2}\sigma^m_{t-\varepsilon,t}(y)}$$

(see Lemma D.1 in Appendix D), and then

$$\mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(y)\Delta^m_h\phi\left(y + \int_s^t \sqrt{2yQ[\nu](y,u)}dW_u\right)\right] \leq C_m|h|^m\|\phi\|_{\mathcal{C}^\alpha_b(\mathbb{R})}\varepsilon^{-m/2}.$$

The following control on the third term of the decomposition (4.18) follows :

$$\mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(X^x_{s,t-\varepsilon})\Delta^m_h\phi(Y^{\varepsilon,x}_{s,t})\right] \leq C|h|^m\|\phi\|_{\mathcal{C}^\alpha_b(\mathbb{R})}\varepsilon^{-m/2}. \tag{4.22}$$

Mixing (4.20), (4.21) and (4.22), it follows from the decomposition (4.18) that

$$\mathbb{E}\left[\sigma^m_{t-\varepsilon,t}(X^x_{s,t})\Delta^m_h\phi(X^x_{s,t})\right] \leq C\|\phi\|_{\mathcal{C}^\alpha_b(\mathbb{R})}\left[|h|^\alpha\varepsilon^{1/4} + \varepsilon^{3\alpha/4} + |h|^m\varepsilon^{-m/2}\right],$$

and then with (4.17), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\tilde{\sigma}^m_t(X^x_{s,t})\Delta^m_h\phi(X^x_{s,t})\right] \\
&\quad\leq C\big(1 + x^{\alpha+m/2} + x^{\frac{m+1}{2}} + x^{\frac{m+k}{2}}\big)\|\phi\|_{\mathscr{C}^\alpha_b(\mathbb{R})}\left[|h|^\alpha\varepsilon^{\frac{1}{2}(\beta\wedge\frac{1}{2})} + \varepsilon^{3\alpha/4} + |h|^m\varepsilon^{-m/2}\right].
\end{aligned}$$

We choose for $h \neq 0$

$$\varepsilon = (t-s)\frac{|h|^{\frac{2m}{m+3\alpha/2}}}{1+|h|^{\frac{2m}{m+3\alpha/2}}} \in]0, t-s[.$$

Then since $|h| \leq 1$, we get with (4.3)

$$
\begin{aligned}
&\mathbb{E}\left[\tilde{\sigma}_t^m(X_{s,t}^x)\Delta_h^m\phi(X_{s,t}^x)\right] \\
&\quad\leq C\big(1+x^{c_{\alpha,k}+m/2}\big)\|\phi\|_{\mathscr{C}_b^\alpha(\mathbb{R})}\left[|h|^\alpha\left((t-s)\frac{|h|^{\frac{2m}{m+3\alpha/2}}}{1+|h|^{\frac{2m}{m+3\alpha/2}}}\right)^{\frac{1}{2}(\beta\wedge\frac{1}{2})}\right. \\
&\qquad\left.+\left((t-s)\frac{|h|^{\frac{2m}{m+3\alpha/2}}}{1+|h|^{\frac{2m}{m+3\alpha/2}}}\right)^{3\alpha/4}+|h|^m\left((t-s)\frac{|h|^{\frac{2m}{m+3\alpha/2}}}{1+|h|^{\frac{2m}{m+3\alpha/2}}}\right)^{-m/2}\right] \\
&\quad\leq C\big(1+x^{c_{\alpha,k}+m/2}\big)\|\phi\|_{\mathscr{C}_b^\alpha(\mathbb{R})}\left[(t-s)^{\frac{1}{2}(\beta\wedge\frac{1}{2})}|h|^{\alpha+\frac{1}{2}(\beta\wedge\frac{1}{2})\frac{2m}{m+3\alpha/2}}\right. \\
&\qquad\left.+(t-s)^{3\alpha/4}|h|^{\frac{3\alpha}{4}\frac{2m}{m+3\alpha/2}}+\left(\frac{2}{t-s}\right)^{m/2}|h|^{m-\frac{m}{2}\frac{2m}{m+3\alpha/2}}\right] \\
&\quad\leq C\big(1+x^{c_{\alpha,k}+m/2}\big)\|\phi\|_{\mathscr{C}_b^\alpha(\mathbb{R})}\left[|h|^{\alpha+\frac{1}{2}(\beta\wedge\frac{1}{2})\frac{2m}{m+3\alpha/2}}+|h|^{\frac{3\alpha}{4}\frac{2m}{m+3\alpha/2}}+|h|^{m-\frac{m}{2}\frac{2m}{m+3\alpha/2}}\right] \\
&\quad\leq C\big(1+x^{c_{\alpha,k}+m/2}\big)\|\phi\|_{\mathscr{C}_b^\alpha(\mathbb{R})}\left[|h|^{\alpha+(\beta\wedge\frac{1}{2})\frac{2m}{2m+3\alpha}}+|h|^{\frac{3\alpha m}{2m+3\alpha}}\right] \\
&\quad\leq C\big(1+x^{c_{\alpha,k}+m/2}\big)\|\phi\|_{\mathscr{C}_b^\alpha(\mathbb{R})}\left[|h|^{\alpha+(\beta\wedge\frac{1}{2})\frac{2m}{2m+3\alpha}}+|h|^{\alpha+\alpha\frac{m-3\alpha}{2m+3\alpha}}\right].
\end{aligned}
$$

It follows with (4.3) that for $m > 3\alpha$,

$$
\mathbb{E}\big[\tilde{\sigma}_t^m(X_{s,t}^x)\Delta_h^m\phi(X_{s,t}^x)\big] \leq C\big(1+x^{c_{\alpha,k}+m/2}\big)\|\phi\|_{\mathscr{C}_b^\alpha(\mathbb{R})}|h|^{\alpha+\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}} \tag{4.23}
$$

and for $m > 3\alpha$, we have

$$
\alpha < \alpha + \frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m} < m.
$$

We deduce from Lemma C.4 (see Appendix C) and hypothesis (A.2) in Assumption 4.1 that the random variable $X_{s,t}^x$ admits a density with respect to the Lebesgue measure on $\mathbb{R}_+^*$. In addition, if we denote by $y \mapsto p_{s,t}^\nu(x,y)$ its density, then the function $y \mapsto \tilde{\sigma}_t^m(y)p_{s,t}^\nu(x,y)$ is in the Besov space $\mathcal{B}_{1,\infty}^{\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}}(\mathbb{R}_+^*)$ and, as the right term in (4.23) is uniform in time and $c_{\alpha,k} \geq 0$, it satisfies the bound (4.4). □

This result shows that despite the degenerescence of the diffusion coefficient in (3.8), the law of the solution of this equation is regularized. This property is very important because it induces a regularization of the measure ν_t as we will show in the following result.

Theorem 4.1. *Under the assumptions of Theorems 3.1 or 3.3, and Assumption 4.1, the measure ν_t admits a density $x \mapsto u_t(x)$ with respect to the Lebesgue measure for each $t > 0$. In addition, the couple (u, R) is characterized by the mild system*

$$
\begin{cases}
\forall t \in (0,T], \forall x > 0, \\
u_t(x) = \big\langle \nu_0, p_{0,t}^u(\cdot,x)\big\rangle + \displaystyle\int_0^t\int_0^\infty \big[b(x',R_s)\mathcal{G}[p_{s,t}^u(\cdot,x)](x') - d(x')p_{s,t}^u(x',x)\big]\,u_s(x')dx'ds \\
R_t = R_0 + \displaystyle\int_0^t \left\{D(r_{in}-R_s) - \lambda\int_0^\infty b(x',R_s)u_s(x')dx'\right\}ds
\end{cases} \tag{4.24}
$$

and is the unique solution of (1.1) in $\mathbb{L}^\infty((0,T],\mathbb{L}^1(\mathbb{R}_+,(1+x)dx)\times\mathbb{R})$*, in the sense given by (2.18) for the initial condition* (ν_0,R_0)*.*

Furthermore, if $k\in[0,2)$ *and there exist* $0<\alpha<1$ *and* $m>3\alpha$ *such that*

$$c_{\alpha,k}+m/2\leq 1, \tag{4.25}$$

then we have $\tilde{\sigma}^m u\in\mathbb{L}^\infty\big((0,T],\mathcal{B}^{\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}}_{1,\infty}(\mathbb{R}_+^*)\big)$*.*

The notation p^u in the above mild system (4.24) refers to the density p^ν defined in Proposition 4.1.

Proof of Theorem 4.1. For a given non negative function $\phi\in\mathcal{C}_b(\mathbb{R}_+,\mathbb{R}_+)$ it follows from the mild formulation that

$$\begin{aligned}\langle\nu_t,\phi\rangle &= \int_0^\infty\int_0^\infty \phi(y)p^\nu_{0,t}(x,y)dy\nu_0(dx)\\ &\quad+\int_0^t\int_0^\infty\Big\{b[\nu](x,s)\Big[-\int_0^\infty\phi(y)p^\nu_{s,t}(x,y)dy+2\int_0^1\int_0^\infty\phi(y)p^\nu_{s,t}(\alpha x,y)dy\bar{m}(d\alpha)\Big]\\ &\qquad\qquad-d(x)\int_0^\infty\phi(y)p^\nu_{s,t}(x,y)dy\Big\}\nu_s(dx)ds\\ &\leq\int_0^\infty\underbrace{\Big(\int_0^\infty p^\nu_{0,t}(x,y)\nu_0(dx)+2\bar{b}\int_0^t\int_0^\infty\int_0^1 p^\nu_{s,t}(\alpha x,y)\bar{m}(d\alpha)\nu_s(dx)ds\Big)}_{H^\nu_t(y)}\phi(y)dy\end{aligned}$$

In addition, the function $y\mapsto H^\nu_t(y)$ is non negative and belongs to $\mathbb{L}^1(\mathbb{R}_+)$. Indeed,

$$\begin{aligned}\int_0^\infty H^\nu_t(y)dy &= \int_0^\infty\int_0^\infty p^\nu_{0,t}(x,y)dy\,\nu_0(dx)\\ &\qquad+2\bar{b}\int_0^t\int_0^\infty\int_0^1\int_0^\infty p^\nu_{s,t}(\alpha x,y)dy\,\bar{m}(d\alpha)\nu_s(dx)ds\\ &=\langle\nu_0,1\rangle+2\bar{b}\int_0^t\langle\nu_s,1\rangle ds<+\infty.\end{aligned}$$

We deduce that for all $t>0$, the measure ν_t is absolutely continuous with respect to the Lebesgue measure. We denote by u_t its density, then Theorems 3.1 and 3.3 imply that $u\in\mathbb{L}^\infty((0,T],\mathbb{L}^1(\mathbb{R}_+,(1+x)dx))$. The notation p^u therefore makes sense. The mild system (4.24) directly follows from the mild formulation (3.12) and the

representation (4.2). Indeed, for all non negative functions $\phi \in \mathcal{C}_b(\mathbb{R}_+, \mathbb{R})$ and $t > 0$,

$$\begin{aligned}
\int_0^\infty u_t(x)\phi(x)dx &= \int_0^\infty \int_0^\infty \phi(y)p^u_{0,t}(x,y)dy\nu_0(dx) \\
&+ \int_0^t \int_0^\infty \Big\{ b(x,R_s)\Big[-\int_0^\infty \phi(y)p^u_{s,t}(x,y)dy + 2\int_0^1 \int_0^\infty \phi(y)p^u_{s,t}(\alpha x,y)dy\bar{m}(d\alpha)\Big] \\
&\qquad - d(x)\int_0^\infty \phi(y)p^u_{s,t}(x,y)dy\Big\}u_s(x)dxds \\
&= \int_0^\infty \left(\int_0^\infty p^u_{0,t}(x,y)\nu_0(dx)\right)\phi(y)dy \\
&\quad + \int_0^\infty \Big(\int_0^t \int_0^\infty \Big\{ b(x,R_s)\Big[-p^u_{s,t}(x,y) + 2\int_0^1 p^u_{s,t}(\alpha x,y)\bar{m}(d\alpha)\Big] \\
&\qquad - d(x)p^u_{s,t}(x,y)\Big\}u_s(x)dxds\Big)\phi(y)dy.
\end{aligned}$$

We obtain the mild system (4.24), that implies for all $(x,t) \in \mathbb{R}_+^* \times (0,T]$, that

$$\begin{aligned}
\tilde{\sigma}^m_t(x)u_t(x) &= \langle \nu_0, \tilde{\sigma}^m_t(x)p^u_{0,t}(\cdot,x)\rangle \\
&+ \int_0^t \int_0^\infty \left[b(x',R_s)\mathcal{G}[\tilde{\sigma}^m_t(x)p^u_{s,t}(\cdot,x)](x') - d(x')\tilde{\sigma}^m_t(x)p^u_{s,t}(x',x)\right]u_s(x')dx'ds.
\end{aligned}$$

Then, it follows from Lemma C.2 that

$$\begin{aligned}
\|\tilde{\sigma}^m_t u_t\|_{\mathcal{B}^{\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}}(\mathbb{R}_+^*)} &\le \int_0^\infty \left\|\tilde{\sigma}^m_t p^u_{0,t}(x',\cdot)\right\|_{\mathcal{B}^{\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}}(\mathbb{R}_+^*)}\nu_0(dx') \\
&+ \int_0^t \int_0^\infty \Big\{ b(x',R_s)\Big[\left\|\tilde{\sigma}^m_t p^u_{s,t}(x',\cdot)\right\|_{\mathcal{B}^{\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}}(\mathbb{R}_+^*)} \\
&\qquad + 2\int_0^1 \left\|\tilde{\sigma}^m_t p^u_{s,t}(\alpha' x',\cdot)\right\|_{\mathcal{B}^{\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}}(\mathbb{R}_+^*)}\bar{m}(d\alpha')\Big] \\
&\qquad + d(x')\left\|\tilde{\sigma}^m_t p^u_{s,t}(x',\cdot)\right\|_{\mathcal{B}^{\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}}(\mathbb{R}_+^*)}\Big\}u_s(x')dx'ds.
\end{aligned}$$

Thanks to (4.4), this implies that for $\alpha \in]0,1[$ and $m > 3\alpha$

$$\begin{aligned}
\sup_{0<t\le T}\|\tilde{\sigma}^m_t u_t\|_{\mathcal{B}^{\frac{2m}{2m+3\alpha}\eta_{\alpha,\beta,m}}(\mathbb{R}_+^*)} &\le \int_0^\infty \left(1 + x^{c_{\alpha,k}+m/2}\right)\nu_0(dx) \\
&\quad + C\int_0^T \int_0^\infty \left(1 + x^{c_{\alpha,k}+m/2}\right)u_s(x)dxds.
\end{aligned}$$

The right term in this inequality is finite if (4.25) holds. Furthermore it suffices to replace ν_t by $u_t(x)dx$ for all $t > 0$ in the limiting problem (2.18) to get u as the unique weak function solution of (1.1). □

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Appendix

A Proof of Theorem 2.1

A.1 Step 1 : Control of tails

Since the individual trait is unbounded, we need to get a control on the measure valued process tails. We then have the following lemma that is adapted from [21].

Lemma A.1. *Let $T > 0$, if a subsequence of $\left\{ \left(\nu_t^K\right)_{t\geq 0}, K > 1 \right\}$ converges in law in the Skorohod space $\mathbb{D}([0,T], (\mathcal{M}_F(\mathbb{R}_+), v))$ then its limit $(\nu_t)_{t\in[0,T]}$ satisfies*

$$\mathbb{E}\left\{ \sup_{0\leq t\leq T} \left[\langle \nu_t, 1\rangle^{1+\varrho} + \langle \nu_t, x\rangle \right] \right\} < \infty. \tag{A.1}$$

In addition, there exists a decreasing sequence of inscreasing functions $(f_n)_{n\geq 1} \subset \mathcal{C}_b^2(\mathbb{R}_+, \mathbb{R})$ such that $f_n = 0$ on $[0, n-1]$ and $f_n = 1$ on $[n, \infty)$, which satisfies

$$\lim_{n\to\infty} \limsup_{K\to\infty} \mathbb{E}\left\{ \sup_{0\leq t\leq T} \langle \nu_t^K, f_n\rangle \right\} = 0. \tag{A.2}$$

Proof of Lemma A.1. In order to show (A.1), we approximate the functions $x \mapsto 1$ and $x \mapsto x$ by an increasing sequence of continuous and compact supported functions in order to use the convergence assumption for the vague topology. We then introduce the uniformly bounded sequence of continuous functions defined for $n \in \mathbb{N}^\star$ by

$$\phi_n(x) = \begin{cases} 1 & \text{if } 0 \leq x < n-1, \\ n-x & \text{if } n-1 \leq x < n, \\ 0 & \text{if } x \geq n. \end{cases}$$

that converges increasingly pointwise towards the constant function $x \mapsto 1$. It follows from the Fatou lemma that holds thanks to the convergence of the sequence of measure valued process for the vague topology, that we have for all $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}\left\{ \sup_{0\leq t\leq T} \left[\langle \nu_t, \phi_n\rangle^{1+\varrho} + \langle \nu_t, x\phi_n\rangle \right] \right\} &\leq \liminf_{K\to\infty} \mathbb{E}\left\{ \sup_{0\leq t\leq T} \left[\left\langle \nu_t^K, \phi_n\right\rangle^{1+\varrho} + \left\langle \nu_t^K, x\phi_n\right\rangle \right] \right\} \\ &\leq \liminf_{K\to\infty} \mathbb{E}\left\{ \sup_{0\leq t\leq T} \left[\left\langle \nu_t^K, 1\right\rangle^{1+\varrho} + \left\langle \nu_t^K, x\right\rangle \right] \right\}. \end{aligned}$$

As $n \to \infty$ in this inequality, we obtain by monotony

$$\mathbb{E}\left\{\sup_{0\le t\le T}\left[\langle \nu_t, 1\rangle^{1+\varrho} + \langle \nu_t, x\rangle\right]\right\} \le \sup_K \mathbb{E}\left\{\sup_{0\le t\le T}\left[\left\langle \nu_t^K, 1\right\rangle^{1+\varrho} + \left\langle \nu_t^K, x\right\rangle\right]\right\} < \infty.$$

In order to show (A.2), we approximate the indicator of the outside of the integers limited compacts by a non-increasing and uniformly bounded sequence of regular fonctions, in order to control the tails of the distributions $\{\mathcal{Z}^K, K > 1\}$. Let us introduce the sequence $(f_n)_n$ such that $f_0 = 1$, and for each $n \ge 1$ the function f_n is null on the interval $[0, n-1]$ and identically equal to 1 on $[n, \infty)$. The restriction of f_n on $[n-1, n]$ is constructed so that the function f_n is non-decreasing and twice differentiable with its derivatives uniformly bounded. One can for exemple consider the polynomial $w(x) = x^3(6x^2 - 15x + 10)$ and set

$$f_n(x) = w(0 \vee (x - n + 1) \wedge 1)\ ,\ \forall n \ge 1 \tag{A.3}$$

which are indeed in $\mathcal{C}_b^2(\mathbb{R}_+, \mathbb{R})$ with uniformly bounded derivatives. It then follows that for all $s \le t \le T$

$$\begin{aligned}
\left\langle \nu_s^K, f_n\right\rangle &\le \left\langle \nu_0^K, f_n\right\rangle + M_s^{K,f_n} + \int_0^s \int_0^\infty \left\{\zeta(x, R_u^K) f_n'(x) + xQ(x, R_s^K) f_n''(x)\right\} \nu_u^K(dx)du \\
&\quad + \int_0^s \int_0^\infty \left\{b(x, R_u^K) \int_0^1 \left[f_n(\alpha x) + f_n((1-\alpha)x)\right] m(d\alpha)\right\} \nu_u^K(dx)du \\
&\le \left\langle \nu_0^K, f_n\right\rangle + A_T^{K,f_n} + \bar{b} \int_0^s \left\langle \nu_u^K, f_n\right\rangle du + \sup_{0\le u\le T} \left|M_u^{K,f_n}\right|
\end{aligned}$$

where

$$A_T^{K,f_n} := C \int_0^T \left\langle \nu_u^K, (1+x)(\partial_x f_n + |\partial_{xx} f_n|)\right\rangle du.$$

The sequence $(A_T^{K,f_n})_{K>1}$ is uniformly integrable according to K and converges in law towards the random variable $C\int_0^T \langle \nu_u, (1+x)(\partial_x f_n + |\partial_{xx} f_n|)\rangle\, du$ as $K \to \infty$ because the function $x \mapsto (1+x)(\partial_x f_n + |\partial_{xx} f_n|)(x)$ is continuous with compact support. We have

$$\mathbb{E}\left\{\sup_{0\le s\le t} \left\langle \nu_s^K, f_n\right\rangle\right\} \le \mathbb{E}\left\{\left\langle \nu_0^K, f_n\right\rangle + A_T^{K,f_n} + \sup_{0\le u\le T} \left|M_u^{K,f_n}\right|\right\} + \bar{l} \int_0^t \mathbb{E}\left\{\sup_{0\le u\le s} \left\langle \nu_u^K, f_n\right\rangle\right\} ds$$

that induces thanks to Gronwall's lemma that

$$\mathbb{E}\left\{\sup_{0\le s\le t} \left\langle \nu_s^K, f_n\right\rangle\right\} \le \mathbb{E}\left\{\left\langle \nu_0^K, f_n\right\rangle + A_T^{K,f_n} + \sup_{0\le u\le T} \left|M_u^{K,f_n}\right|\right\} e^{\bar{b}t}.$$

As $K \to \infty$, it follows from (2.15) and the Doob inequality that the right term with the martingale converges towards 0. In addition the remaining terms in the expectation are uniformly integrable and converge in law, then

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{K\to\infty} \mathbb{E}\left\{\sup_{0\le s\le T} \left\langle \nu_s^K, f_n\right\rangle\right\} \\
&\qquad \le e^{\bar{b}T} \lim_{n\to\infty} \mathbb{E}\left\{\langle \nu_0, f_n\rangle + C\int_0^T \langle \nu_u, (1+x)(\partial_x f_n + |\partial_{xx} f_n|)\rangle\, du\right\} = 0.
\end{aligned}$$

□

A.2 Step 2 : Tightness

Given the above Lemma A.1, the tightness of the sequence of laws $\left\{\mathcal{Z}^K = \mathcal{L}(\nu^K, R^K), K > 1\right\}$ follows. It is adapted from [2], Theorem 7.4.

Proposition A.1. *The sequence of laws* $\left\{\mathcal{Z}^K = \mathcal{L}(\nu^K, R^K), K > 1\right\}$ *is tight in the space of probability measures* $\mathcal{P}(\mathbb{D}([0,T], (\mathcal{M}_F(\mathbb{R}_+) \times \mathbb{R}, w \otimes |.|)))$ *for all* $T > 0$.

Proof. Let us first endow the measure space $\mathcal{M}_F(\mathbb{R}_+)$ with its vague topology and prove the tightness property for the sequence $\left\{\mathcal{Z}^K = \mathcal{L}\left(\nu^K, R^K\right), K > 1\right\}$. By noticing that $\mathcal{C}_c^2(\mathbb{R}_+, \mathbb{R})$ is a dense subspace of $\mathcal{C}_0(\mathbb{R}_+, \mathbb{R})$ for the topology of uniform convergence on compact sets, it suffices according to [27] to show that for each test function $f \in \mathcal{C}_c^2(\mathbb{R}_+, \mathbb{R})$ the sequence of processes $\left\{\left(\left\langle \nu_t^K, f\right\rangle, R_t^K\right)_{t\geq 0}, K > 1\right\}$ is tight in the Skorohod space $\mathbb{D}([0,T], \mathbb{R}^2)$. First, remember that we have the decomposition

$$\left\langle \nu_t^K, f\right\rangle = \left\langle \nu_0^K, f\right\rangle + V_t^{K,f} + M_t^{K,f}$$

and thanks to (2.15),

$$\lim_{K\to\infty} \mathbb{E}\left\langle M^{K,f}\right\rangle_t = 0\,, \forall t \geq 0.$$

It then directly follows from Doob's inequality that the sequence of martingales $\left\{M^{K,f}, K > 1\right\}$ converges in $\mathbb{L}^2$ toward 0 locally uniformly in time. This sequence is then tight in the Skorohod space $\mathbb{D}([0,T], \mathbb{R})$, and all we need to prove now is the tightness of the other sequences $\{V^{K,f}, K > 1\}$ and $\{R^K, K > 1\}$. For that purpose, we use the Aldous criterion (see [1]), that consists in showing the uniform controls

$$\text{(a)} \ \sup_K \mathbb{E}\left[\sup_{0\leq t\leq T}\left|V_t^{K,f}\right|\right] < \infty,$$

$$\text{(b)} \ \sup_K \mathbb{E}\left[\sup_{0\leq t\leq T}\left|R_t^K\right|\right] < \infty$$

and showing that the Aldous condition holds for $(R_t^K, V_t^{K,f})_{t\geq 0}$. The assertions (a) and (b) are immediate. Indeed,

$$R_t^K \leq r \vee R_0 = \bar{R}\,, \forall t \geq 0$$

and thanks to the boundedness of the test function,

$$\sup_{0\leq t\leq T}\left|V_t^{K,f}\right| \leq C_f \sup_{0\leq t\leq T}\left[\left\langle \nu_t^K, 1\right\rangle + \left|M_t^{K,f}\right|\right].$$

Let us now consider $\varepsilon > 0$ and a stopping time τ that satisfies $\tau + \epsilon \leq T$, then

$$\begin{aligned}\mathbb{E}\left(\left|R_{\tau+\varepsilon}^K - R_\tau^K\right|\right) &\leq \mathbb{E}\left(\int_\tau^{\tau+\varepsilon}\left|D(r_{in} - R_s^K) - \lambda\int_0^\infty b(x, R_s^K)\nu_s^K(dx)\right| ds\right)\\ &\leq \varepsilon\left[D(r_{in} + \bar{R}) + \lambda\bar{b}\mathbb{E}\left(\sup_{0\leq t\leq T}\left\langle \nu_t^K, 1\right\rangle\right)\right].\end{aligned}$$

By a similar computation, we obtain

$$\mathbb{E}\left(\left|V_{\tau+\varepsilon}^{K,f} - V_\tau^{K,f}\right|\right) \leq C\varepsilon\mathbb{E}\left[\sup_{0\leq t\leq T}\left\langle \nu_t^K, 1 + x\right\rangle\right],$$

that implies the Aldous condition thanks to Lemma 2.1. The sequence of law $(\mathcal{Z}^K)_K$ is then tight and by the Prokhorov's theorem one can extract from each sub-sequence, a sub-sub-sequence that converges in $\mathcal{P}(\mathbb{D}([0,T],(\mathcal{M}_F(\mathbb{R}_+)\times\mathbb{R},v\otimes|.|)))$. Let us denote by $\mathcal{Z}$ a limit value of this sequence, and for simplicity of the notation, $(\mathcal{Z}^K)_K$ a sub-sequence that converges towards $\mathcal{Z}$. For each $K>1$ we have

$$\sup_{0\le s\le T}\sup_{f\in L^\infty(\mathbb{R}_+),\|f\|_\infty\le 1}\left|\langle\nu_t^K,f\rangle-\langle\nu_{t-}^K,f\rangle\right|\le 1/K. \tag{A.4}$$

We then deduce from the continuity of the mapping $\mu\mapsto\sup_{0\le t\le T}|\langle\mu_t,f\rangle-\langle\mu_{t-},f\rangle|$ for $f\in\mathcal{C}_c(\mathcal{E}\times\mathbb{R}_+,\mathbb{R})$, that the limiting law $\mathcal{Z}$ only charges the set $\mathcal{C}([0,T],(\mathcal{M}_F(\mathbb{R}_+)\times\mathbb{R},v\otimes|.|))$. We are aiming to prove the same result for the weak topology on $\mathcal{M}_F(\mathbb{R}_+)$ and this is where the sequence of functions introduced in Lemma A.1 is usefull.

Let us denote by (ν,R) a process of law $\mathcal{Z}$, then according to the above argument its trajectories are in the space $\mathcal{C}([0,T],(\mathcal{M}_F(\mathbb{R}_+)\times\mathbb{R},v\otimes|.|))$. As in [21], it follows from the Fatou lemma that for $n,l\in\mathbb{N}$,

$$\mathbb{E}\left[\sup_{0\le t\le T}\langle\nu_t,(1-f_l)f_n\rangle\right]\le\liminf_{K\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}\langle\nu_t^K,(1-f_l)f_n\rangle\right]\le\limsup_{K\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}\langle\nu_t^K,f_n\rangle\right].$$

As $l\to\infty$ it follows from Beppo-Levi's theorem applied to the left term, that

$$\mathbb{E}\left[\sup_{0\le t\le T}\langle\nu_t,f_n\rangle\right]\le\limsup_{K\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}\langle\nu_t^K,f_n\rangle\right]$$

and then by the Lemma A.1,

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}\langle\nu_t,f_n\rangle\right]\le\lim_{n\to\infty}\limsup_{K\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}\langle\nu_t^K,f_n\rangle\right]=0.$$

One can therefore extract a sub-sequence $\left(\sup_{0\le t\le T}\langle\nu_t,f_{n_k}\rangle\right)_k$ that converges almost surely towards 0, and then the limit law $\mathcal{Z}$ only charges the set $\mathcal{C}([0,T],(\mathcal{M}_F(\mathbb{R}_+)\times\mathbb{R},w\otimes|.|))$. We deduce that the convergence of $\{\mathcal{Z}^K,K>1\}$ towards $\mathcal{Z}$ for the weak topology on $\mathcal{M}_F(\mathbb{R}_+)$ holds if the sequence of total masses $\left\{\langle\nu^K,1\rangle,K>1\right\}$ converges in law towards $\langle\nu,1\rangle$ in the Skorohod space $\mathbb{D}([0,T],\mathbb{R})$ (see [23]). So let us consider a Lipschitz continuous function $\Phi:\mathbb{D}([0,T],\mathbb{R})\to\mathbb{R}$, and $n\in\mathbb{N}$.

$$\begin{aligned}\left|\mathbb{E}\left[\Phi\left(\langle\nu^K,1\rangle\right)-\Phi\left(\langle\nu,1\rangle\right)\right]\right|&\le\mathbb{E}\left|\Phi\left(\langle\nu^K,1\rangle\right)-\Phi\left(\langle\nu^K,1-f_n\rangle\right)\right|\\&+\left|\mathbb{E}\left[\Phi\left(\langle\nu^K,1-f_n\rangle\right)-\Phi\left(\langle\nu,1-f_n\rangle\right)\right]\right|\\&+\mathbb{E}\left|\Phi\left(\langle\nu,1-f_n\rangle\right)-\Phi\left(\langle\nu,1\rangle\right)\right|\\&\le C_\Phi\mathbb{E}\left\{\sup_{0\le t\le T}\langle\nu_t^K,f_n\rangle+\sup_{0\le t\le T}\langle\nu_t,f_n\rangle\right\}\\&+\left|\mathbb{E}\left[\Phi\left(\langle\nu^K,1-f_n\rangle\right)-\Phi\left(\langle\nu,1-f_n\rangle\right)\right]\right|\end{aligned}$$

The function $1-f_n$ being continuous with a compact support and the sequence $\{\nu^K,K>1\}$ converging in law toward the limit process ν in $\mathbb{D}([0,T],(\mathcal{M}_F(\mathbb{R}_+)\times\mathbb{R},v\otimes|.|))$, we have

$$\lim_{K\to\infty}\left|\mathbb{E}\left[\Phi\left(\langle\nu^K,1-f_n\rangle\right)-\Phi\left(\langle\nu,1-f_n\rangle\right)\right]\right|=0.$$

It follows that for all $n \in \mathbb{N}$,

$$\limsup_{K\to\infty} \left|\mathbb{E}\left[\Phi\left(\left\langle \nu^K, 1\right\rangle\right) - \Phi\left(\langle \nu, 1\rangle\right)\right]\right| \\ \leq C_\Phi \left[\limsup_{K\to\infty} \mathbb{E}\left\{\sup_{0\leq t\leq T} \left\langle \nu_t^K, f_n\right\rangle\right\} + \mathbb{E}\left\{\sup_{0\leq t\leq T} \langle \nu_t, f_n\rangle\right\}\right].$$

Thanks to Lemma A.1, that implies that as $n \to \infty$ in the right term,

$$\limsup_{K\to\infty} \left|\mathbb{E}\left[\Phi\left(\left\langle \nu^K, 1\right\rangle\right) - \Phi\left(\langle \nu, 1\rangle\right)\right]\right| = 0.$$

The sub-sequence $\{(\langle \nu_t^K, 1\rangle)_{t\in[0,T]}, K > 1\}$ then converges in law towards the process $(\langle \nu_t, 1\rangle)_{t\in[0,T]}$, and then the subsequence $(\mathcal{Z}^K)_K$ converges towards $\mathcal{Z}$ in the space of probability distributions $\mathcal{P}(\mathbb{D}([0,T], (\mathcal{M}_F(\mathbb{R}_+) \times \mathbb{R}, w \otimes |.|)))$. That is enough to conclude that our initial sequence of laws is tight. □

A.3 Step 3 : Identification of a limiting process

We are here interested in the characterization of a process $(\nu_t, R_t)_{t\in[0,T]}$ whose law is a limiting value of the sequence $\{\mathcal{Z}^K, K > 1\}$. Given the operator defined in (2.17), we have the following result

Proposition A.2. *Let us denote by $\mathcal{Z}$ a limiting value of the sequence of distributions $\{\mathcal{Z}^K, K > 1\}$. Then a process (ν, R) of law $\mathcal{Z}$ satisfies (2.18).*

Proof. First and foremost, let $f \in \mathcal{C}_c^2(\mathbb{R}_+, \mathbb{R})$ be a test function and $t \in [0,T]$ a fixed time. We introduce the following real-valued functions defined for (μ, r) in the subset of $\mathcal{C}([0,T], (\mathcal{M}_F(\mathbb{R}_+) \times \mathbb{R}, w \otimes |.|))$ constitued of processes that satisfy $|r| \leq \bar{R}$

$$\Phi_t^f(\mu, r) = \langle \mu_t, f\rangle - \langle \mu_0, f\rangle - \int_0^t \int_0^\infty \left\{\zeta(x, r_s) f'(x) + xQ(x, r_s) f''(x)\right\} \mu_s(dx)ds \\ - \int_0^t \int_0^\infty \left\{b(x, r_s)\mathcal{G}[f](x) - d(x)f(x)\right\}\mu_s(dx)ds$$

$$\Phi_t(\mu, r) = r_t - r_0 - \int_0^t \left\{D(r_{in} - r_s) - \lambda \int_0^\infty b(x, r_s)\mu_s(dx)\right\} ds.$$

We are aiming to show that

$$\mathbb{E}\left[\left|\Phi_t^f(\nu, R)\right|^2 + |\Phi_t(\nu, R)|\right] = 0. \tag{A.5}$$

For this purpose, we first note that the function Φ_t defined as above is continuous and

$$\Phi_t\left(\nu^K, R^K\right) = 0.$$

It then follows by the Fatou lemma that

$$\mathbb{E}\left|\Phi_t\left(\nu, R\right)\right| \leq \liminf_{K\to\infty} \mathbb{E}\left|\Phi_t\left(\nu^K, R^K\right)\right| = 0.$$

Futhermore, the terms in the integrals in the expression of the function Φ_t^f are continuous and bounded, then this function is continuous. In addition, it follows from the decomposition (2.12) that

$$\Phi_t^f\left(\nu^K, R^K\right) = M_t^{K,f}$$

and then thanks to the Fatou lemma and Doob's inequality,

$$\mathbb{E}\left[\left(\Phi_t^f(\nu,R)\right)^2\right] \le \liminf_{K\to\infty}\mathbb{E}\left[\left(\Phi_t^f\left(\nu^K,R^K\right)\right)^2\right]$$
$$\le \liminf_{K\to\infty}\mathbb{E}\left[\sup_{0\le s\le t}\left|M_s^{K,f}\right|^2\right] \le C\liminf_{K\to\infty}\mathbb{E}\left[\left\langle M^{K,f}\right\rangle_t\right].$$

We now deduce from (2.15) that

$$\mathbb{E}\left[\left(\Phi_t^f(\nu,R)\right)^2\right] \le \liminf_{K\to\infty}\frac{C}{K}\mathbb{E}\left[\sup_{0\le t\le T}\left\langle \nu_t^K, 1+x\right\rangle\right] = 0. \tag{A.6}$$

Let us now consider $f\in\mathcal{C}_b^2(\mathbb{R}_+,\mathbb{R})$ and the $(f_n)_n$ defined in (A.3) that is, with the sequence of its derivatives of first and second order, uniformly bounded. We deduce from (A.6) and (A.1) that a.s,

$$\Phi_t^f(\nu,R) = \lim_{n\to\infty}\Phi_t^{(1-f_n)f}(\nu,R) = 0.$$

□

B Some properties of a specific SDE

Let us introduce two continuous functions $h(x,s), q(x,s)\in\mathcal{C}(\mathbb{R}_+\times[0,T],\mathbb{R})$ that satisfy the following assumption.

Assumption B.1.

(A.1) *Boundary condition : for all* $s\le[0,T]$,

$$h(0,s)\ge 0.$$

(A.2) *Lipschitz condition : for all* $x,y\ge 0$ *and* $s\in[0,T]$,

$$|h(x,s)-h(y,s)|+|q(x,s)-q(y,s)|\le C|x-y|.$$

(A.3) *Bound property : for all* $x\ge 0$ *and* $s\in[0,T]$,

$$0\le q(x,s)\le\bar{q}.$$

Then we have the following results

Lemma B.1. *There is weak existence and uniqueness, and strong existence for the stochastic differential equation*

$$dZ_u = h(Z_u,u)du + \sqrt{2Z_u q(Z_u,u)}dW_u, \forall u\le T. \tag{B.1}$$

The solution $(Z_{s,u}^x)_{u\in[s,T]}$ *that starts at time* $s<T$ *at the position* $x\ge 0$*, has a zero local time at* 0*, is almost surely non negative at any time and satifies the moment estimate*

$$\mathbb{E}\left\{\sup_{s\le u\le T}\left(Z_{s,u}^x\right)^p\right\} \le C_{p,T}(1+x^p),\ \forall p>0. \tag{B.2}$$

In addition, for all $x_1,x_2\ge 0$ *and* $s_1,s_2\le T$*, we have for all* $u\le T$

$$\mathbb{E}\left(\left|Z_{s_1,u}^{x_1}-Z_{s_2,u}^{x_2}\right|\right) \le C_T\left\{|x_1-x_2|+(1+x_1+x_2)|s_1-s_2|+|1+x_1+x_2|^{1/2}|s_1-s_2|^{1/2}\right\}, \tag{B.3}$$

where $(Z_{s_1,u}^{x_1})_{u\in[s_1,T]}, (Z_{s_2,u}^{x_2})_{u\in[s_2,T]}$ *are assumed to be built with the same Brownian motion, and satisfies* $Z_{s_j,u}^{x_j}=x_j$ *for all* $u\le s_j$ *with* $j=1,2$.

Lemma B.2. *Let $p(x) \in \mathcal{C}_b^1(\mathbb{R}_+, \mathbb{R})$ and $a(x,s) \in \mathcal{C}_b(\mathbb{R}_+ \times [0,T], \mathbb{R})$ such that*

$$|a(x,s) - a(y,s)| \leq \bar{a}|x-y|, \, \forall x, y \geq 0, \forall s \leq T,$$

then for a given $t \leq T$, the function defined by

$$w_s(x) = \mathbb{E}\left[p(Z_{s,t}^x) e^{\int_s^t a(Z_{s,u}^x, u)du}\right], \, \forall (s,x) \in [0,t] \times \mathbb{R}_+ \tag{B.4}$$

satisfy for all $s_1, s_2 \leq t$ and $x_1, x_2 \geq 0$ the following inequality

$$|w_{s_1}(x_1) - w_{s_2}(x_2)| \leq C_T \left(\|p\|_\infty + \|p'\|_\infty\right) \Big\{ |x_1 - x_2| + (1+x_1)^{1/2}|s_1 - s_2|^{1/2} + (1+x_1)|s_1 - s_2| \Big\}. \tag{B.5}$$

B.1 Proof of Lemma B.1

We split this proof into the several steps.

Existence and uniqueness : According to [19], Theorem 1.1 it suffices to show that we have weak existence and pathwise uniqueness to conclude that there is weak existence and uniqueness, and strong existence. In addition it follows from the previous paragraph that this solution will be well defined at any time. The stochastic differential equation (B.1) is non homogeneous with continuous coefficients, then the coupled process $(Z_u, u)_{u \geq 0}$ satisfies an homogeneous stochastic differential equation with continuous coefficients. We then have weak existence for each initial condition $x \geq 0$ and initial time $s \in [0,T]$ thanks to [19], Theorem 2.3, p173. Furthermore, for $x, y \geq 0$ and $s \leq u \leq T$ we have by hypothesis

$$|h(x,u) - h(y,u)| \leq C|x-y|,$$

and thanks to the inequality $|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a-b|}$, $\forall a, b \geq 0$,

$$\begin{aligned} \left|\sqrt{xq(x,u)} - \sqrt{yq(y,u)}\right| &\leq |xq(x,u) - yq(y,u)|^{1/2} \\ &= |(x-y)q(x,u) + y(q(x,u) - q(y,u))|^{1/2} \\ &\leq (\overline{q} + Cy)^{1/2} |x-y|^{1/2}. \end{aligned}$$

It follows that for all $N > 0$ and $s \leq u \leq T$,

$$\left|\sqrt{xq(x,u)} - \sqrt{yq(y,u)}\right| \leq (\overline{q} + CN)^{1/2} |x-y|^{1/2}, \, \forall x, y \in [0,N]$$

that implies pathwise uniqueness by a localisation argument (see [33]).

Moment estimate and positiveness : Notice that the solution that we constructed in the previous paragraph is well defined until it becomes negative or explodes. In order to get well defined terms for all times, we introduce the new equation

$$d\tilde{Z}_u = h(|\tilde{Z}_u|, u)du + \sqrt{2|\tilde{Z}_u| q(|\tilde{Z}_u|, u)} \, dW_u, \forall u \leq T \tag{B.6}$$

for which there is weak existence and pathwise uniqueness by an argument similar to the one in the previous paragraph, but only until explosion time. It follows from the continuity of the function $h(x,s)$ and hypothesis (A.2) of Assumption B.1 that

$$|h(x,s)| \leq C(1+x), \forall x \geq 0, \forall s \leq T.$$

For given $x \geq 0$ and $s \leq t$, we consider the sequence of stopping times defined by

$$\lambda_N = \inf\{v \geq s : |\tilde{Z}^x_{s,v}| > N\} \wedge T,\ N > 1$$

that converges increasingly. Then for $p \geq 2$ and $v \in [s,T]$ we have

$$|\tilde{Z}^x_{s,v\wedge\lambda_N}|^p \leq C_p \left\{ x^p + \int_s^{v\wedge\lambda_N} |h(|\tilde{Z}^x_{s,u}|,u)|^p du + \left| \int_s^{v\wedge\lambda_N} \sqrt{2|\tilde{Z}^x_{s,u}| q(|\tilde{Z}^x_{s,u}|,u)}\, dW_u \right|^p \right\}$$

that implies that

$$\begin{aligned}
\sup_{s\leq u\leq v\wedge\lambda_N} |\tilde{Z}^x_{s,u}|^p &\leq C_p \Bigg\{ x^p + \int_s^{v\wedge\lambda_N} |h(|\tilde{Z}^x_{s,u}|,u)| du \\
&\qquad + \sup_{s\leq u\leq v} \left(\left| \int_s^{u\wedge\lambda_N} \sqrt{2|\tilde{Z}^x_{s,u}| q(|\tilde{Z}^x_{s,u}|,u)}\, dW_u \right|^p \right) \Bigg\} \\
&\leq C_p \Bigg\{ x^p + \int_s^{v\wedge\lambda_N} \left[1 + |\tilde{Z}^x_{s,u}|^p \right] du \\
&\qquad + \sup_{s\leq u\leq v} \left(\left| \int_s^{u\wedge\lambda_N} \sqrt{2|\tilde{Z}^x_{s,u}| q(|\tilde{Z}^x_{s,u}|,u)}\, dW_u \right|^p \right) \Bigg\}.
\end{aligned}$$

It follows from the Doob inequality that

$$\begin{aligned}
&\mathbb{E}\left\{ \sup_{s\leq u\leq v} \left(\left| \int_s^{u\wedge\lambda_N} \sqrt{2|\tilde{Z}^x_{s,u}| q(|\tilde{Z}^x_{s,u}|,u)}\, dW_u \right|^p \right) \right\} \\
&\qquad \leq C_p \mathbb{E}\left\{ \left(\int_s^{v\wedge\lambda_N} 2|\tilde{Z}^x_{s,u}| q(|\tilde{Z}^x_{s,u}|,u) du \right)^{p/2} \right\} \\
&\qquad \leq 2^{p/2} C_p \overline{Q}^{p/2} \mathbb{E}\left\{ \left(\int_s^{v\wedge\lambda_N} |\tilde{Z}^x_{s,u}| du \right)^{p/2} \right\} \\
&\qquad \leq 2^{p/2} C_p \overline{Q}^{p/2} \mathbb{E}\left\{ \left(\int_s^{v} |\tilde{Z}^x_{s,u\wedge\lambda_N}| du \right)^{p/2} \right\} \\
&\qquad \leq 2^{p/2} C_p \overline{Q}^{p/2} \mathbb{E}\left\{ \int_s^{v} |\tilde{Z}^x_{s,u\wedge\lambda_N}|^{p/2} du \right\} \\
&\qquad \leq 2^{p/2} C_p \overline{Q}^{p/2} \mathbb{E}\left\{ \int_s^{v} \left[1 + |\tilde{Z}^x_{s,u\wedge\lambda_N}|^p \right] du \right\}
\end{aligned}$$

and then

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\leq u\leq v\wedge\lambda_N} |\tilde{Z}^x_{s,u}|^p \right] &\leq C_p \left\{ x^p + \mathbb{E}\left(\int_s^{v} \left[1 + |\tilde{Z}^x_{s,u\wedge\lambda_N}|^p \right] du \right) \right\} \\
&\leq C_p \left\{ x^p + \int_s^{v} \left[1 + \mathbb{E}\left\{ \sup_{s\leq z\leq u\wedge\lambda_N} |\tilde{Z}^x_{s,z}|^p \right\} \right] du \right\}.
\end{aligned}$$

Hence, we obtain for all $v \in [s,T]$

$$1+\mathbb{E}\left[\sup_{s\le u\le v\wedge\lambda_N}|\tilde{Z}^x_{s,u}|^p\right]\le C_p\left\{1+x^p+\int_s^v\left[1+\mathbb{E}\left\{\sup_{s\le z\le u\wedge\lambda_N}|\tilde{Z}^x_{s,z\wedge\lambda_N}|^p\right\}\right]du\right\},$$

and thanks to the Gronwall lemma,

$$\mathbb{E}\left[\sup_{s\le u\le v\wedge\lambda_N}|\tilde{Z}^x_{s,u}|^p\right]\le C_p\left(1+x^p\right)e^{C_p(v-s)}.$$

If we assume that there exists $T_0<T$ such that

$$\mathbb{P}\left\{\lim_N\lambda_N<T_0\right\}>0,$$

then for all $v\in[T_0,T]$, we will have

$$N^p\,\mathbb{P}\left\{\lim_N\lambda_N<T_0\right\}\le\mathbb{E}\left[\sup_{s\le u\le v\wedge\lambda_N}|\tilde{Z}^x_{s,u}|^p\right]\le C_p\left(1+x^p\right)e^{C_p(v-s)}.$$

That is impossible because the left side term tends towards infinity as $N\to\infty$. It follows that $\lambda_N\xrightarrow{a.s}T$ and by the monotone convergence theorem,

$$\mathbb{E}\left[\sup_{s\le u\le v}|\tilde{Z}^x_{s,u}|^p\right]\le C_p\left(1+x^p\right)e^{C_p(v-s)},\ \forall v\in[s,T].$$

It then suffices to take $v=T$. The case $0<p<2$ is obviously solved thanks to the Hölder inequality

$$\mathbb{E}\left[\sup_{s\le u\le T}|\tilde{Z}^x_{s,u}|^p\right]\le\left(\mathbb{E}\left[\sup_{s\le u\le T}|\tilde{Z}^x_{s,u}|^2\right]\right)^{p/2}\le C_{p,T}(1+x^p).$$

The process $(\tilde{Z}^x_{s,u})_{u\in[s,T]}$ is then well defined at any time and satisfies a moment estimate similar to (B.2). We will now show that this process is almost surely non negative, and then by uniqueness this will imply that it is a modification of the process $(Z^x_{s,u})_{u\in[s,T]}$. Indeed, the following decomposition holds

$$\tilde{Z}^x_{s,t}=(\tilde{Z}^x_{s,t})_+-(\tilde{Z}^x_{s,t})_-,\forall t\in[s,T] \tag{B.7}$$

and it follows from the Tanaka formula that

$$(\tilde{Z}^x_{s,t})_-=-\int_s^t 1_{\{\tilde{Z}^x_{s,t}\le 0\}}d\tilde{Z}^x_{s,t}+\frac{1}{2}L^0_t(\tilde{Z}^x_{s,\cdot}). \tag{B.8}$$

For $\varepsilon>0$,

$$\int_s^t 1_{\{0<\tilde{Z}^x_{s,u}\le\varepsilon\}}\frac{d\langle\tilde{Z}^x_{s,\cdot}\rangle_u}{\tilde{Z}^x_{s,u}}=2\int_s^t 1_{\{0<\tilde{Z}^x_{s,u}\le\varepsilon\}}q(\tilde{Z}^x_{s,u},u)du<\infty\text{ p.s,}$$

and the the local time $L^0_t(Y^x_{s,\cdot})$ is zero almost surely (see [26], Ch. IX, Lemma 3.3). Equation (B.8) becomes

$$(\tilde{Z}^x_{s,t})_-=-\int_s^t 1_{\{\tilde{Z}^x_{s,u}\le 0\}}h(-\tilde{Z}^x_{s,u},u)du-\int_s^t 1_{\{\tilde{Z}^x_{s,u}\le 0\}}\sqrt{-2\tilde{Z}^x_{s,u}q(-\tilde{Z}^x_{s,u},u)}\,dW_u$$

where the second term in the right side of the equality is a true martingale thanks to the moment estimate that we previously stated. It then follows from the hypothesis (A.1) and (A.2) in Assumption B.1 that

$$\begin{aligned}
\mathbb{E}\left[(\tilde{Z}^x_{s,t})_-\right] &= -\mathbb{E}\left(\int_s^t 1_{\{\tilde{Z}^x_{s,u}\leq 0\}} h(-\tilde{Z}^x_{s,u},u)du\right) \\
&\leq -\mathbb{E}\left(\int_s^t 1_{\{\tilde{Z}^x_{s,u}\leq 0\}}\left[h(-\tilde{Z}^x_{s,u},u)-h(0,u)\right]du\right) \\
&\leq C\mathbb{E}\left(\int_s^t 1_{\{\tilde{Z}^x_{s,u}\leq 0\}}\left|\tilde{Z}^x_{s,u}\right|du\right) \\
&\leq C\int_s^t \mathbb{E}\left[(\tilde{Z}^x_{s,u})_-\right]du
\end{aligned}$$

and hence by the Gronwall lemma,

$$\mathbb{E}\left[(\tilde{Z}^x_{s,t})_-\right] = 0, \forall t\in[s,T]$$

thus,

$$\forall t\in[s,T]\,,\,(\tilde{Z}^x_{s,t})_- = 0 \text{ p.s.}$$

We deduce that the stochastic process $(\tilde{Z}^x_{s,t})_{t\in[s,T]}$ is almost surely non negative at any time, and it is possible to remove the absolute values in (B.6) to conclude by a uniqueness argument that

$$\forall t\in[s,T],\ \tilde{Z}^x_{s,t} = Z^x_{s,t} \text{ p.s.}$$

The stochastic process $(Z^x_{s,t})_{t\in[s,T]}$ is then almost surely non negative at any time, satisfies the moment estimate (B.2) and has a zero local time at 0.

Dependence on initial conditions : Thanks to the strong existence, it is possible to construct two solutions $(Z^{x_1}_{s_1,v})_{v\in[s_1,T]}$ and $(Z^{x_2}_{s_2,v})_{v\in[s_2,T]}$ on the same filtered probability space and with the same Brownian motion, for all $x_1,x_2\geq 0$ and $0\leq s_1\leq s_2\leq T$. Those solutions are well extended on $[0,T]$ if we set

$$Z^{x_j}_{s_j,v} = x_j, \forall v\in[0,s_j], j=1,2.$$

Hence, it follows from the Tanaka formula that for all $s_2\leq v\leq T$,

$$\left|Z^{x_1}_{s_1,v}-Z^{x_2}_{s_2,v}\right| = \left|Z^{x_1}_{s_1,s_2}-x_2\right| + \int_{s_2}^v \text{sign}(Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u})d(Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}) + l^0_v(Z^{x_1}_{s_1,\cdot}-Z^{x_2}_{s_2,\cdot}) \tag{B.9}$$

where $l^0_v(Z^{x_1}_{s_1,\cdot}-Z^{x_2}_{s_2,\cdot})$ is the local time of the process $Z^{x_1}_{s_1,\cdot}-Z^{x_2}_{s_2,\cdot}$ at 0 up to time v. Furthermore, thanks to the inequality $|\sqrt{a}-\sqrt{b}|\leq\sqrt{|a-b|}\,,\forall a,b\geq 0$ again, we have

for $\varepsilon > 0$,

$$\begin{aligned}
&\int_{s_2}^{v} 1_{\{0<Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}\le\varepsilon\}} \frac{d\left\langle Z^{x_1}_{s_1,\cdot}-Z^{x_2}_{s_2,\cdot}\right\rangle_u}{Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}} \\
&\quad= \int_{s_2}^{v} 1_{\{0<Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}\le\varepsilon\}} \frac{2\left|\sqrt{Z^{x_1}_{s_1,u}q(Z^{x_1}_{s_1,u},u)}-\sqrt{Z^{x_2}_{s_2,u}q(Z^{x_2}_{s_2,u},u)}\right|^2}{Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}}du \\
&\quad\le 2\int_{s_2}^{v} 1_{\{0<Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}\le\varepsilon\}} \frac{\left|Z^{x_1}_{s_1,u}q(Z^{x_1}_{s_1,u},u)-Z^{x_2}_{s_2,u}q(Z^{x_2}_{s_2,u},u)\right|}{Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}}du \\
&\quad\le 2\int_{s_2}^{v} 1_{\{0<Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}\le\varepsilon\}} \frac{\left|(Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u})q(Z^{x_1}_{s_1,u},u)+Z^{x_2}_{s_2,u}(q(Z^{x_1}_{s_1,u},u)-q(Z^{x_2}_{s_2,u},u))\right|}{Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}}du \\
&\quad\le 2\int_{s_2}^{v} 1_{\{0<Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}\le\varepsilon\}} (\overline{q}+CZ^{x_2}_{s_2,u})du.
\end{aligned}$$

That implies that for all $v \in [s_2, T]$

$$\int_{s_2}^{v} 1_{\{0<Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}\le\varepsilon\}} \frac{d\left\langle Z^{x_1}_{s_1,\cdot}-Z^{x_2}_{s_2,\cdot}\right\rangle_u}{Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u}} \le 2\int_{s_2}^{v} \left(\overline{q}+CZ^{x_2}_{s_2,u}\right) du < \infty \text{ a.s.}$$

We then deduce thanks to [26], Ch. IX, lemma 3.3 that the local time $l^0_v(Z^{x_1}_{s_1,\cdot}-Z^{x_2}_{s_2,\cdot})$ is zero almost surely, and it follows from (B.9) that

$$\begin{aligned}
\mathbb{E}\left|Z^{x_1}_{s_1,v}-Z^{x_2}_{s_2,v}\right| &\le \mathbb{E}\left(\left|Z^{x_1}_{s_1,s_2}-x_2\right|\right) \\
&\quad+\mathbb{E}\int_{s_2}^{v} \operatorname{sign}(Z^{x_1}_{s_1,u}-Z^{x_2}_{s_2,u})\left[h(Z^{x_1}_{s_1,u},u)-h(Z^{x_2}_{s_2,u},u)\right]du \\
&\le \mathbb{E}\left(\left|Z^{x_1}_{s_1,s_2}-x_2\right|\right)+A\int_{s_2}^{v}\mathbb{E}\left(\left|Z^{x_1}_{s_1,v}-Z^{x_2}_{s_2,v}\right|\right)du
\end{aligned}$$

which implies by the Gronwall lemma that

$$\mathbb{E}\left|Z^{x_1}_{s_1,v}-Z^{x_2}_{s_2,v}\right| \le \mathbb{E}\left(\left|Z^{x_1}_{s_1,s_2}-x_2\right|\right)e^{A(v-s_2)},\ \forall v\in[s_2,T]. \tag{B.10}$$

We are now interested in what happens for $s_1 \le v \le s_2$. Indeed,

$$\begin{aligned}
\mathbb{E}\left(\left|Z^{x_1}_{s_1,v}-Z^{x_2}_{s_2,v}\right|\right) &= \mathbb{E}\left\{\left|x_1+\int_{s_1}^{v}h(Z^{x_1}_{s_1,u},u)du+\int_{s_1}^{v}\sqrt{2Z^{x_1}_{s_1,u}q(Z^{x_1}_{s_1,u},u)}\,dW_u-x_2\right|\right\} \\
&\le \mathbb{E}\left\{|x_1-x_2|+\int_{s_1}^{v}|h(Z^{x_1}_{s_1,u},u)|du+\left|\int_{s_1}^{v}\sqrt{2Z^{x_1}_{s_1,u}q(Z^{x_1}_{s_1,u},u)}\,dW_u\right|\right\} \\
&\le |x_1-x_2|+C\int_{s_1}^{s_2}\left[1+\mathbb{E}(Z^{x_1}_{s_1,u})\right]du+\mathbb{E}\left\{\sup_{s_1\le v\le s_2}\left|\int_{s_1}^{v}\sqrt{2Z^{x_1}_{s_1,u}q(Z^{x_1}_{s_1,u},u)}\,dW_u\right|\right\}
\end{aligned}$$

and thanks to the Doob inequality,

$$\begin{aligned}
\mathbb{E}\left\{\sup_{s_1\le v\le s_2}\left|\int_{s_1}^{v}\sqrt{2Z^{x_1}_{s_1,u}q(Z^{x_1}_{s_1,u},u)}\,dW_u\right|\right\} &\le \mathbb{E}\left\{\left|\int_{s_1}^{s_2}2Z^{x_1}_{s_1,u}q(Z^{x_1}_{s_1,u},u)\,du\right|^{1/2}\right\} \\
&\le \sqrt{2\overline{q}\mathbb{E}\left(\int_{s_1}^{s_2}Z^{x_1}_{s_1,u}du\right)} \\
&\le \sqrt{2\overline{q}|s_1-s_2|\mathbb{E}\left(\sup_{s_1\le u\le T}Z^{x_1}_{s_1,u}\right)}.
\end{aligned}$$

Hence, we have for all $v \in [s_1, s_2]$,

$$\mathbb{E}\left(\left|Z^{x_1}_{s_1,v} - Z^{x_2}_{s_2,v}\right|\right) \leq |x_1 - x_2| + C\int_{s_1}^{s_2}\left[1 + \mathbb{E}(Z^{x_1}_{s_1,u})\right]du + \sqrt{2\overline{q}|s_1 - s_2|\mathbb{E}\left(\sup_{s_1\leq u\leq T} Z^{x_1}_{s_1,u}\right)}$$

which implies that

$$\mathbb{E}\left(\left|Z^{x_1}_{s_1,v} - Z^{x_2}_{s_2,v}\right|\right) \leq |x_1 - x_2| + C_1(1+x_1)|s_1 - s_2| + C_2|1+x_1|^{1/2}|s_1 - s_2|^{1/2}, \forall v \in [s_1, s_2]. \tag{B.11}$$

We finally obtain without any order on $s_1, s_2 \in [0, T], x_1, x_2 \geq 0$ that for all $v \in [s_1 \wedge s_2, T]$

$$\mathbb{E}\left(\left|Z^{x_1}_{s_1,v} - Z^{x_2}_{s_2,v}\right|\right) \leq C_T\left\{|x_1 - x_2| + (1 + x_1 + x_2)|s_1 - s_2| + |1 + x_1 + x_2|^{1/2}|s_1 - s_2|^{1/2}\right\}$$

that is easily extended for $v \in [0, T]$.

B.2 Proof of Lemma B.2

Let us first introduce the decomposition

$$|w_{s_1}(x_1) - w_{s_2}(x_2)| \leq |w_{s_1}(x_1) - w_{s_2}(x_1)| + |w_{s_2}(x_1) - w_{s_2}(x_2)| \tag{B.12}$$

thanks to the triangular inequality. Furthermore, by considering that $s_1 \leq s_2$, we have

$$\begin{aligned}
|w_{s_1}(x_1) - w_{s_2}(x_1)| &\leq \mathbb{E}\left(\left|p(Z^{x_1}_{s_1,t})e^{\int_{s_1}^t a(Z^{x_1}_{s_1,u},u)du} - p(Z^{x_1}_{s_2,t})e^{\int_{s_2}^t a(Z^{x_1}_{s_2,u},u)du}\right|\right) \\
&\leq \mathbb{E}\left(\left|p(Z^{x_1}_{s_1,t}) - p(Z^{x_1}_{s_2,t})\right| e^{\int_{s_1}^t a(Z^{x_1}_{s_1,u},u)du}\right) \\
&\qquad + \mathbb{E}\left(|p(Z^{x_1}_{s_1,t})|\left|e^{\int_{s_1}^t a(Z^{x_1}_{s_1,u},u)du} - e^{\int_{s_2}^t a(Z^{x_1}_{s_2,u},u)du}\right|\right) \\
&\leq e^{\|a\|_\infty(t-s_1)}\|p'\|_\infty\mathbb{E}\left(\left|Z^{x_1}_{s_1,t} - Z^{x_1}_{s_2,t}\right|\right) \\
&\qquad + \|p\|_\infty\left\{\mathbb{E}\left(\left|e^{\int_{s_1}^t a(Z^{x_1}_{s_1,u},u)du} - e^{\int_{s_1}^t a(Z^{x_1}_{s_2,u},u)du}\right|\right)\right. \\
&\qquad\qquad \left. + \mathbb{E}\left(\left|e^{\int_{s_1}^t a(Z^{x_1}_{s_2,u},u)du} - e^{\int_{s_2}^t a(Z^{x_1}_{s_2,u},u)du}\right|\right)\right\} \\
&\leq e^{\|a\|_\infty(t-s_1)}\left[\|p'\|_\infty\mathbb{E}\left(\left|Z^{x_1}_{s_1,t} - Z^{x_1}_{s_2,t}\right|\right)\right. \\
&\left. + \|p\|_\infty\left\{\mathbb{E}\left(\left|e^{\int_{s_1}^t\left[a(Z^{x_1}_{s_1,u},u) - a(Z^{x_1}_{s_2,u},u)\right]du} - 1\right|\right) + \mathbb{E}\left(\left|e^{-\int_{s_1}^{s_2} a(Z^{x_1}_{s_2,u},u)du} - 1\right|\right)\right\}\right].
\end{aligned}$$

Let us set for all $s_1 \leq v \leq t$

$$V^{x_1}_{s_1,s_2}(v) = e^{\int_{s_1}^v\left[a(Z^{x_1}_{s_1,u},u) - a(Z^{x_1}_{s_2,u},u)\right]du} - 1,$$

then

$$V^{x_1}_{s_1,s_2}(v) = \int_{s_1}^v\left[a(Z^{x_1}_{s_1,u},u) - a(Z^{x_1}_{s_2,u},u)\right]V^x_{s_1,s_2}(u)du + \int_{s_1}^v\left[a(Z^{x_1}_{s_1,u},u) - a(Z^{x_1}_{s_2,u},u)\right]du$$

that implies that,

$$\left|V^{x_1}_{s_1,s_2}(v)\right| \leq 2\|a\|_\infty\int_{s_1}^v\left|V^x_{s_1,s_2}(u)\right|du + \bar{a}\int_{s_1}^t\left|Z^{x_1}_{s_1,u} - Z^{x_1}_{s_2,u}\right|du$$

and thanks to the Gronwall lemma,

$$\left|V^{x_1}_{s_1,s_2}(v)\right| \leq \bar{a}\, e^{2\|a\|_\infty (v-s_1)} \int_{s_1}^t \left|Z^{x_1}_{s_1,u} - Z^{x_1}_{s_2,u}\right| du\,,\ \forall v \in [s_1,t].$$

We show by a similar argument that

$$\left|e^{-\int_{s_1}^{s_2} a(Y^x_{s_2,u},u)du} - 1\right| \leq e^{\|a\|_\infty |s_1-s_2|}\|a\|_\infty |s_1-s_2|.$$

Hence, one can write for all $x_1 \geq 0, s_1 \leq s_2 \leq t$

$$\begin{aligned}|w_{s_1}(x_1) - w_{s_2}(x_1)| \leq e^{\|a\|_\infty (t-s_1)} \Bigg\{ & \|p'\|_\infty \mathbb{E}\left(\left|Z^{x_1}_{s_1,t} - Z^{x_1}_{s_2,t}\right|\right) \\ & + \|p\|_\infty \left[\bar{a}\, e^{2\|a\|_\infty (t-s_1)} \int_{s_1}^t \mathbb{E}\left(\left|Z^{x_1}_{s_1,u} - Z^{x_1}_{s_2,u}\right|\right) du + e^{\|a\|_\infty |s_1-s_2|}\|a\|_\infty |s_1-s_2|\right] \Bigg\}\end{aligned}$$

and thanks to Lemma B.1, we have for all $s_1, s_2 \leq t$

$$|w_{s_1}(x_1) - w_{s_2}(x_1)| \leq C_T\left(\|p\|_\infty + \|p'\|_\infty\right)\left\{(1+x_1)^{1/2}|s_1-s_2|^{1/2} + (1+x_1)|s_1-s_2|\right\}. \tag{B.13}$$

In addition, for $x_1, x_2 \geq 0$ and $s_2 \leq t$,

$$\begin{aligned}|w_{s_2}(x_1) - w_{s_2}(x_2)| &\leq \mathbb{E}\left(\left|p(Z^{x_1}_{s_2,t}) e^{\int_{s_2}^t a(Z^{x_1}_{s_2,u},u)du} - p(Z^{x_2}_{s_2,t}) e^{\int_s^t a(Z^{x_2}_{s_2,u},u)du}\right|\right) \\ &\leq \mathbb{E}\left(\left|p(Z^{x_1}_{s_2,t}) - p(Z^{x_2}_{s_2,t})\right| e^{\int_{s_2}^t a(Z^{x_1}_{s_2,u},u)du}\right) \\ &\quad + \mathbb{E}\left(|p(Z^{x_2}_{s_2,t})| e^{\int_s^t a(Z^{x_2}_{s_2,u},u)du}\left|e^{\int_{s_2}^t \left[a(Z^{x_1}_{s_2,u},u) - a(Z^{x_2}_{s_2,u},u)\right]du} - 1\right|\right) \\ &\leq e^{\|a\|_\infty (t-s_2)} \Bigg\{ \|p'\|_\infty \mathbb{E}\left(\left|Z^{x_1}_{s_2,t} - Z^{x_2}_{s_2,t}\right|\right) \\ &\qquad + \|p\|_\infty \mathbb{E}\left(\left|e^{\int_s^t \left[a(Z^{x_1}_{s_2,u},u) - a(Z^{x_2}_{s_2,u},u)\right]du} - 1\right|\right) \Bigg\}.\end{aligned}$$

Let us now set for all $s_2 \leq v \leq t$

$$U^{x_1,x_2}_{s_2,v} = e^{\int_{s_2}^v \left[a(Y^{x_1}_{s_2,u},u) - a(Z^{x_2}_{s_2,u},u)\right]du} - 1,$$

then

$$U^{x_1,x_2}_{s_2,v} = \int_{s_2}^v \left[a(Z^{x_1}_{s_2,u},u) - a(Z^{x_2}_{s_2,u},u)\right] U^{x_1,x_2}_{s_2,u} du + \int_{s_2}^v \left[a(Z^{x_1}_{s_2,u},u) - a(Z^{x_2}_{s_2,u},u)\right] du$$

that implies that

$$\left|U^{x_1,x_2}_{s_2,v}\right| \leq 2\|a\|_\infty \int_{s_2}^v \left|U^{x_1,x_2}_{s_2,u}\right| du + \bar{a}\int_{s_2}^t \left|Z^{x_1}_{s_2,u} - Z^{x_2}_{s_2,u}\right| du$$

and thanks to the Gronwall lemma,

$$\left|U^{x_1,x_2}_{s_2,v}\right| \leq \bar{a}\, e^{2\|a\|_\infty (v-s_2)} \int_{s_2}^t \left|Z^{x_1}_{s_2,u} - Z^{x_2}_{s_2,u}\right| du\,,\ \forall v \in [s_2,t].$$

Hence for all $x_1, x_2 \geq 0, s_2 \leq t$ we have

$$|w_{s_2}(x_1) - w_{s_2}(x_2)| \leq e^{\|a\|_\infty (t-s_2)} \Big\{ \|p'\|_\infty \mathbb{E}\left(\left|Z^{x_1}_{s_2,t} - Z^{x_2}_{s_2,t}\right|\right) + \bar{a}\, e^{2\|a\|_\infty (t-s)} \|p\|_\infty \int_{s_2}^{t} \mathbb{E}\left(\left|Z^{x_1}_{s_2,u} - Z^{x_2}_{s_2,u}\right|\right) du \Big\}$$

and thanks to Lemma B.1,

$$|w_{s_2}(x_1) - w_{s_2}(x_2)| \leq C_T \left(\|p\|_\infty + \|p'\|_\infty\right) |x_1 - x_2|, \forall x_1, x_2 \geq 0, \forall s_2 \leq t. \quad \text{(B.14)}$$

Finally, the inequality (B.5) follows from (B.12), (B.13) and (B.14).

C Difference operator and Besov spaces $B^s_{1,\infty}(\mathbb{R})$

For $\alpha \in]0,1[$, we denote by $\mathscr{C}^\alpha_b(\mathbb{R})$ the Hölder-Zygmund space that is the set of real valued functions on $\mathbb{R}$ such that

$$\|f\|_{\mathscr{C}^\alpha_b(\mathbb{R})} = \|f\|_\infty + \sup_{x,y\in\mathbb{R}, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

We also introduce the difference operator defined for all $f : \mathbb{R} \to \mathbb{R}$, for given $m \geq 1$ integer and $h \in \mathbb{R}$ as

$$\forall x \in \mathbb{R},\ \Delta^1_h f(x) = f(x+h) - f(x)\,;\ \Delta^m_h f(x) = \Delta^1_h(\Delta^{m-1}_h f)(x),\ \text{pour } m > 1.$$

that is, by a recurrence argument,

$$\Delta^m_h f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jh) \quad \text{(C.1)}$$

with the following properties.

Lemma C.1. *Let $m \geq 1$ be an integer, $\alpha \in]0,1[$, $h \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, then :*

1. *If $f \in \mathscr{C}^\alpha_b(\mathbb{R})$,*
$$\|\Delta^m_h f\|_\infty \leq C_m |h|^\alpha \|f\|_{\mathscr{C}^\alpha_b(\mathbb{R})}. \quad \text{(C.2)}$$
2. *If $f \in \mathscr{C}^\alpha_b(\mathbb{R})$,*
$$|\Delta^m_h f(x) - \Delta^m_h f(y)| \leq C_m \|f\|_{\mathscr{C}^\alpha_b(\mathbb{R})} |x-y|^\alpha, \forall x, y \in \mathbb{R}. \quad \text{(C.3)}$$
3. *If $f \in \mathcal{C}^m(\mathbb{R}, \mathbb{R})$,*
$$\|\Delta^m_h f\|_{\mathbb{L}^1} \leq C_m |h|^m \|\partial^m_x f\|_{\mathbb{L}^1} \quad \text{(C.4)}$$
4. *If $f \in \mathcal{C}_b(\mathbb{R}, \mathbb{R})$ and $g : \mathbb{R} \to \mathbb{R}$ is an integrable function, then for all $a \in \mathbb{R}$,*
$$\int_{\mathbb{R}} \Delta^m_h f(x+a) g(x) dx = \int_{\mathbb{R}} f(x+a) \Delta^m_{-h} g(x) dx \quad \text{(C.5)}$$

Proof. For $0 < \alpha < 1$, $h \in \mathbb{R}$ and an integer $m \geq 1$,

1. If $f \in \mathscr{C}_b^\alpha(\mathbb{R})$, we proceed by a recurrence argument on $m \geq 1$. Indeed,
$$\|\Delta_h^1 f\|_\infty = \sup_{x\in\mathbb{R}} |f(x+h) - f(x)| \leq |h|^\alpha \|f\|_{\mathscr{C}_b^\alpha(\mathbb{R})}$$
and if we assume that the property holds at range m, we get
$$\|\Delta_h^{m+1} f\|_\infty = \sup_{x\in\mathbb{R}} |\Delta_h^m f(x+h) - \Delta_h^m f(x)| \leq 2C_m |h|^\alpha \|f\|_{\mathscr{C}_b^\alpha(\mathbb{R})}$$
that corresponds to the property with $C_{m+1} = 2C_m$. We deduce that it holds for each integer $m \geq 1$.

2. If $f \in \mathscr{C}_b^\alpha(\mathbb{R})$, then thanks to (C.1) we get for all $x, y \in \mathbb{R}$,
$$|\Delta_h^m f(x) - \Delta_h^m f(y)| \leq \sum_{j=0}^m \binom{m}{j} |f(x+jh) - f(y+jh)| \leq 2^m \|f\|_{\mathscr{C}_b^\alpha(\mathbb{R})} |x-y|^\alpha.$$

3. We proceed by a recurrence argument to show firstly that
$$(\mathbf{P}): \ \forall m \geq 1, \ \Delta_h^m f(x) = h^m \int_0^m H_m(t) \partial_x^m f(x+th) dt, \ \text{if } f \in \mathcal{C}^m(\mathbb{R},\mathbb{R})$$
where H_m is bounded and does not depend on f. Indeed for $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$, then
$$\begin{aligned}\Delta_h^{m+1} f(x) &= f(x+h) - f(x) \\ &= \int_x^{x+h} f'(t) dt \\ &= h \int_0^1 f'(x+th) dt\end{aligned}$$
and hence $H_1(t) = 1$. We now assume that the property holds at range m, then if $f \in \mathcal{C}^{m+1}(\mathbb{R},\mathbb{R})$,
$$\begin{aligned}\Delta_h^{m+1} f(x) &= \Delta_h^m f(x+h) - \Delta_h^m f(x) \\ &= h^m \int_0^m H_m(t) \left[\partial_x^m f(x+(1+t)h) - \partial_x^m f(x+th)\right] dt \\ &= h^m \int_0^m H_m(t) \int_{x+th}^{x+(1+t)h} \partial_x^{m+1} f(u) du dt \\ &= h^{m+1} \int_0^m H_m(t) \int_t^{1+t} \partial_x^{m+1} f(x+uh) du dt \\ &= h^{m+1} \int_0^m H_m(t) \int_0^{m+1} \partial_x^{m+1} f(x+uh) 1_{\{u\in[t,1+t]\}} du dt \\ &= h^{m+1} \int_0^{m+1} \underbrace{\left(\int_0^m H_m(t) 1_{\{u\in[t,1+t]\}} dt\right)}_{=H_{m+1}(u)} \partial_x^{m+1} f(x+uh) du\end{aligned}$$
where the function H_{m+1} is bounded and does not depend on f. We deduce that the property $(\mathbf{P})$ holds, and then for all $f \in \mathcal{C}^m(\mathbb{R},\mathbb{R})$,
$$\|\Delta_h^m f\|_{\mathbb{L}^1} \leq |h|^m \|H_m\|_\infty \int_{\mathbb{R}} \int_0^m |\partial_x^m f(x+th)| dt dx = m \|H_m\|_\infty |h|^m \|\partial_x^m f\|_{\mathbb{L}^1}.$$

4. Let $f \in \mathcal{C}_b(\mathbb{R}, \mathbb{R})$, and $g : \mathbb{R} \to \mathbb{R}$ an integrable function and $a \in \mathbb{R}$, then by changing variables in the integral

$$\begin{aligned}\int_{\mathbb{R}} \Delta_h^m f(x+a)g(x)dx &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\int_{\mathbb{R}} f(x+a+jh)g(x)dx \\ &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\int_{\mathbb{R}} f(x+a)g(x-jh)dx \\ &= \int_{\mathbb{R}} f(x+a)\Delta_{-h}^m g(x)dx\end{aligned}$$

□

The difference operator allows to define the Besov spaces $\mathcal{B}^s_{1,\infty}(\mathbb{R})$ as the set of function $f : \mathbb{R} \to \mathbb{R}$ that satisfy for $m > s$

$$\|f\|_{\mathcal{B}^s_{1,\infty}} := \|f\|_{\mathbb{L}^1(\mathbb{R})} + \sup_{|h|\leq 1} |h|^{-s}\|\Delta_h^m f\|_{\mathbb{L}^1(\mathbb{R})} < \infty$$

(see Triebel [31] Theorem 2.5.12, or [32] Theorem 2.6.1). The following result follows

Lemma C.2. *Let $(E, \mathcal{F}_E, \mu)$ be a measured space where μ is a positive measure. Let $(x, y) \mapsto f(x, y)$ an integrable function on $\mathbb{R} \times E$, then if we denote by*

$$\Phi_f(x) = \int_E f(x, y)\mu(dy), \forall x \geq 0$$

we have for all $s > 0$,

$$\|\Phi_f\|_{\mathcal{B}^s_{1,\infty}(\mathbb{R})} \leq \int_E \|f(\cdot, y)\|_{\mathcal{B}^s_{1,\infty}(\mathbb{R})}\mu(dy). \tag{C.6}$$

Proof. Let $s > 0$, then it follows from Fubini theorem that

$$\|\Phi\|_{\mathbb{L}^1(\mathbb{R})} = \int_{\mathbb{R}} \left|\int_E f(x, y)\mu(dy)\right| dx \leq \int_E \|f(\cdot, y)\|_{\mathbb{L}^1(\mathbb{R})}\mu(dy).$$

Similarly, for any $m > s$ and $h \in [-1, 1]$,

$$\|\Delta_h^m \Phi\|_{\mathbb{L}^1(\mathbb{R})} = \int_{\mathbb{R}} \left|\int_E \left(\Delta_h^m f(\cdot, y)\right)(x)\mu(dy)\right| dx \leq \int_E \|\Delta_h^m f(\cdot, y)\|_{\mathbb{L}^1(\mathbb{R})}\mu(dy).$$

The result directly follows by taking the supremum in h and summing the above inequalities. □

Furthermore, the above definition of Besov spaces allows a sufficient condition for the existence of a density for random variables given by the following result

Lemma C.3 ([28], Lemma A.1)**.** *Let X be a real valued random variable. If there are an integer $m \geq 1$, a real number $\theta > 0$, a real $\alpha > 0$, with $\alpha < \theta < m$, and a constant $K > 0$ such that for every $\phi \in \mathscr{C}_b^\alpha(\mathbb{R})$ and $h \in \mathbb{R}$ with $|h| \leq 1$,*

$$\mathbb{E}\left[\Delta_h^m \phi(X)\right] \leq K|h|^\theta \|\phi\|_{\mathscr{C}_b^\alpha(\mathbb{R})},$$

then X has a density f_X with respect to the Lebesgue measure on $\mathbb{R}$. Moreover $f_X \in \mathcal{B}^{\theta-\alpha}_{1,\infty}(\mathbb{R})$ and

$$\|f\|_{\mathcal{B}^{\theta-\alpha}_{1,\infty}} \lesssim 1 + K.$$

The consequence that follows gives a sufficient condition that will is used in the paper

Lemma C.4. *Let X be a real valued random variable. For a given non negative real valued function $\sigma(x)$ such that $\sigma(X) \in \mathbb{L}^1$ with $\mathbb{E}[\sigma(X)] \neq 0$, if there exists an integer $m \geq 1$, a real number $\theta > 0$, a real $\alpha > 0$ with $\alpha < \theta < m$, and a constant $K > 0$ such that for every $\phi \in \mathscr{C}_b^\alpha(\mathbb{R})$ and $h \in \mathbb{R}$ with $|h| \leq 1$,*

$$\mathbb{E}\left[\sigma(X)\Delta_h^m \phi(X)\right] \leq K|h|^\theta \|\phi\|_{\mathscr{C}_b^\alpha(\mathbb{R})}, \tag{C.7}$$

then X admits a density f_X on $\{x : \sigma(x) \neq 0\}$. In addition, if we denote by f_X its density on this set, then the function $x \mapsto \sigma(x) f_X(x)$ is in the Besov space $\mathcal{B}_{1,\infty}^{\theta-\alpha}(\mathbb{R})$ and satisfies the bound

$$\|\sigma f_X\|_{\mathcal{B}_{1,\infty}^{\theta-\alpha}(\mathbb{R})} \lesssim K + \mathbb{E}\left[\sigma(X)\right] \tag{C.8}$$

Proof. Let us denote by $\mu(dx)$ the law of X and set

$$\nu(dx) = \sigma(x)\mu(dx)$$

that is a non negative finite measure by assumption. Then it follows from (C.7) that

$$\mathbb{E}\left[\sigma(X)\Delta_h^m \phi(X)\right] = \|\nu\|_{\mathbb{L}^1} \int_0^\infty \Delta_h^m \phi(x) \frac{\nu(dx)}{\|\nu\|_{\mathbb{L}^1}} = \|\nu\|_{\mathbb{L}^1} \mathbb{E}\left[\Delta_h^m \phi(Y)\right]$$

where Y is a real valued random variable of law $\nu(dy)/\|\nu\|_{\mathbb{L}^1}$. It follows from (C.7) that

$$\mathbb{E}\left[\Delta_h^m \phi(Y)\right] \leq \frac{K}{\|\nu\|_{\mathbb{L}^1}} |h|^\theta \|\phi\|_{\mathscr{C}_b^\alpha(\mathbb{R})}$$

that implies thanks to Lemma C.3 that the probability measure $\nu(dy)/\|\nu\|_{\mathbb{L}^1}$ admits a density $g_X \in \mathcal{B}_{1,\infty}^{\theta-\alpha}(\mathbb{R})$ with respect to the Lebesgue measure on $\mathbb{R}$ that satisies the bound

$$\|g_X\|_{\mathcal{B}_{1,\infty}^{\theta-\alpha}(\mathbb{R})} \lesssim 1 + \frac{K}{\|\nu\|_{\mathbb{L}^1}}.$$

Then $\mu(dx)$ admits a density f_X on the set $\{x : \sigma(x) \neq 0\}$ that is defined by

$$f_X(x) = \frac{\|\nu\|_{\mathbb{L}^1}}{\sigma(x)} g_X(x)$$

and the bound (C.8) follows. □

D $\mathbb{L}^1$ estimates of derivatives of the Gaussian density

We are interested in the density of a centered Gaussian random variable with variance σ^2, that is

$$g_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \forall x \in \mathbb{R}.$$

We have the following result

Lemma D.1. *For each integer $m \geq 1$,*

$$\|\partial_x^m g_\sigma\|_{\mathbb{L}^1} \leq \frac{C_m}{\sigma^m}$$

where the constant C_m does not depend on σ.

Proof. Because of its exponential factor, let us set

$$\partial_x^m g_\sigma(x) = \frac{P_m^\sigma(x)}{\sigma^m} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \forall x \in \mathbb{R}, \forall m \geq 1.$$

Then $P_1^\sigma(x) = -x/\sigma$ and

$$P_{m+1}^\sigma = \sigma \partial_x P_m^\sigma - \frac{x}{\sigma} P_m^\sigma$$

that implies that each P_m^σ is a polynomial, thus

$$P_m^\sigma(x) = \sum_{k=0}^{m} A_k^m(\sigma) x^k$$

with

$$\begin{aligned}
&A_0^{m+1}(\sigma) = \sigma A_1^m(\sigma), \\
&A_k^{m+1}(\sigma) = \sigma(k+1) A_{k+1}^m(\sigma) - \frac{1}{\sigma} A_{k-1}^m(\sigma),\ k = 1, ..., m-1 \\
&A_m^{m+1}(\sigma) = -\frac{1}{\sigma} A_{m-1}^m(\sigma) \\
&A_{m+1}^{m+1}(\sigma) = -\frac{1}{\sigma} A_m^m(\sigma)
\end{aligned}$$

We easily verify by a recurrence argument on $m \geq 1$ that

$$\forall m \geq 1,\ |A_k^m| \leq \frac{C_m}{\sigma^k} \text{ pour } k = 1, ..., m$$

and then

$$\|\partial_x^m g_\sigma\|_{\mathbb{L}^1} = \frac{1}{\sigma^m} \int_{\mathbb{R}} |P_m^\sigma(x)|\, g_\sigma(x) dx \leq \frac{1}{\sigma^m} \sum_{k=0}^{m} |A_k^m(\sigma)| \int_{\mathbb{R}} |x|^k\, g_\sigma(x) dx \leq \frac{C_m}{\sigma^m}$$

□