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► **To cite this version:**

Claudia Fonte, Valentin Schmutz. Long time behavior of an age and leaky memory-structured neuronal population equation. 2021. hal-03249140

**HAL Id: hal-03249140**

**<https://hal.science/hal-03249140>**

Preprint submitted on 3 Jun 2021

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# Long time behavior of an age and leaky memory-structured neuronal population equation

Claudia Fonte, Valentin Schmutz

## Abstract

We study the asymptotic stability of a two-dimensional mean-field neuronal population equation, which is explicitly linked to spiking neuron models, and generalizes the time-elapsing neuron network model by the inclusion of a leaky memory variable. This additional variable can represent a slow fatigue mechanism, like spike frequency adaptation or short-term synaptic depression. Even though two-dimensional models are known to have emergent behaviors, like population bursts, which are not observed in one-dimensional models, we show that in the weak connectivity regime, two-dimensional models behave like one-dimensional models, i.e. they relax to a unique stationary state.

The proof is based on a version of Harris' ergodic theorem and a perturbation argument. The multidimensionality of the model and delay effects require careful consideration.

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## 1 Introduction

Multidimensional mean-field models in theoretical neuroscience are notoriously challenging to analyse [34, 40, 1, 26] but their study is a necessary step towards understanding how the multiple

timescales present at the single neuron level [33, 37] affect the dynamics of large networks of neurons.

One-dimensional mean-field population equations for spiking neurons with ‘escape noise’ [16, Ch. 9] have been a subject of mathematical study since the seminal papers of Pakdaman, Perthame and Salort [28, 29, 30], providing rigorous foundations to earlier works in theoretical neuroscience [42, 17, 14, 15]. The fact that these population equations correspond to the mean-field limit of large networks of interacting neurons has been proved in [8, 12, 4]. All these population equations are derived from spiking neuron models that are of the ‘renewal’ type (with the exception of the model with fatigue of [30]), which means that they can capture the effect of neuronal refractoriness but neglect slower neuronal timescales, like that of spike frequency adaptation and short-term synaptic plasticity.

To take into account slow neuronal timescales, state-of-the-art phenomenological spiking neuron models are multidimensional [22, 37] or kernel-based [38, 31, 32] (and see [16, Ch. 6.4]). Recently, one of the author showed that, for a large class of multidimensional spiking neuron models, the mean-field limit can be characterized by an ‘age and leaky memory’-structured evolution PDE in  $1 + d$  spatial dimensions,  $d$  being the number of ‘leaky memory’ variables [36].

The aim of this work is to study the long time behavior of the  $1 + 1$  dimensional case (i.e. one age and one leaky memory variable) of the equation proposed in [36].

## 1.1 The age- and leaky memory-structured model

The population model we consider describes the evolution of a density  $\rho_t$  over the state-space  $(a, m) \in \mathbb{R}_+ \times \mathbb{R}_+^*$ , where  $a$  and  $m$  are the ‘age’ and ‘leaky memory’ variables of the neuron, and  $\rho_t(a, m)$  represents the density of neurons in state  $(a, m)$  at time  $t$ .

The nonlinear evolution problem for the density  $\rho_t$ , for the initial datum  $u_0$ , writes

$$\partial_t \rho_t + \nabla \cdot (b \rho_t) = -f(a, m, \varepsilon x_t) \rho_t, \quad (1a)$$

$$\rho_t(0, m) = p(t, \gamma^{-1}(m)), \quad (1b)$$

$$x_t = \int_0^t \int_0^\infty \int_0^\infty h(t-s, a, m) f(a, m, \varepsilon x_s) \rho_s(a, m) da dm ds, \quad (1c)$$

$$\rho_0 = u_0. \quad (1d)$$

The dynamics of the model can be decomposed in three elements: the behavior of neurons between spikes, the spike-triggered jumps and the interaction between neurons.

Between spikes, neurons are transported along the vector field  $b(a, m) = (1, -\lambda m)$ , with  $\lambda > 0$  ( $\nabla \cdot$  denotes the divergence operator over the state-space).

Neurons spike at a rate  $f(a, m, \varepsilon x_t)$ , where  $f : \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R} \rightarrow \mathbb{R}_+$  is the ‘firing rate function’. When a neuron spikes, its age  $a$  is reset to 0 and its leaky memory variable  $m$  jumps to  $\gamma(m)$ , where  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  is the ‘jump mapping’ and is assumed to be a strictly increasing  $\mathcal{C}^1$ -diffeomorphism. As a consequence, the border condition (1b) has a simple interpretation: the density of neurons in state  $(0, m)$  at time  $t$  is equal to the the marginal density of neurons which have their leaky memory variable in state  $\gamma^{-1}(m)$  and spike at time  $t$ . This quantity is denoted  $p(t, \gamma^{-1}(m))$  and has the explicit expression

$$p(t, \gamma^{-1}(m)) := 1_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \varepsilon x_t) \rho_t(a, \gamma^{-1}(m)) da. \quad (1e)$$

The indicator function  $1_{m > \gamma(0)}$  reflects the fact that  $m$  is always strictly positive and the term

$|(\gamma^{-1})'(m)|$  is necessary to guarantee the conservation of the total mass of neurons. Indeed, formally,

$$\begin{aligned} \partial_t \int \rho_t &= \int 1_{m > \gamma(0)} |(\gamma^{-1})'(m)| \int_0^\infty f(a, \gamma^{-1}(m), \varepsilon x_t) \rho_t(a, \gamma^{-1}(m)) da dm - \int f(a, m, \varepsilon x_t) \rho_t \\ &= 0, \end{aligned}$$

by a change of variable.

Lastly, neurons interact through the ‘total postsynaptic potential’  $x_t$ , which integrates the past spiking activity of the population, filtered by the ‘interaction function’  $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ .  $x_t$ , weighted by the connection strength  $\varepsilon \in \mathbb{R}$ , influences the firing rate  $f$ . If we write  $N(t)$  the mean firing rate

$$N(t) := \int_0^\infty \int_0^\infty f(a, m, \varepsilon x_t) \rho_t(a, m) da dm,$$

and if we take  $h$  independent of  $a$  and  $m$ , then  $x_t$  takes the form

$$x_t = \int_0^t h(t-s) N(s) ds,$$

and  $h$  is simply a delay kernel, as in the age-structured model of [28]. In our formulation,  $h$  in Eq. (1c) allows to model more general interactions. For example, in Sec. 1.2.2, we show that by choosing  $h(t, a, m) = \hat{h}(t)(1-m)$ , we can include the effects of a classical short-term synaptic plasticity model [39].

## 1.2 Motivation

The model (1) extends the age-structured model of Pakdaman, Perthame and Salort [28] by the addition of a leaky memory variable which can accumulate over spikes (as opposed to the age variable which is reset to 0 at each spike) and hence introduces a slow timescale in the population dynamics. Such a slow timescale is typically used to account for some form of fatigue mechanism, which can act on the spiking activity (spike frequency adaptation) or on synaptic transmission (short-term synaptic depression). Slow fatigue at the single neuron level can lead to nontrivial emergent behaviors at the population level, like population bursts [41, 18, 13] (see Fig. 1), which are not observed in the age- or voltage-structured models of [28] and [8]. Even though some population equations have been successfully used in the computational neuroscience literature to study emergent behaviors in networks of neurons with fatigue [18, 13], these population equations were obtained at the cost of a timescale separation approximation, making them inexact. In contrast, the model (1) is an exact mean-field limit [36].

This model is also directly related to standard models of spike frequency adaptation and short-term synaptic depression.

### 1.2.1 Spike frequency adaptation

The recent spike history of a neuron can modulate its firing rate  $f$ , leading to spike frequency adaptation [2]. If  $h$  is independent of  $a$  and  $m$  and if  $\gamma(m) = m + \hat{\Gamma}$ , for a fixed  $\hat{\Gamma} > 0$ , (1) becomes

$$\partial_t \rho_t + \nabla \cdot (b \rho_t) = -f(a, m, \varepsilon x_t) \rho_t, \quad (2a)$$

$$\rho_t(0, m) = 1_{m > \hat{\Gamma}} \int_0^\infty f(a, m - \hat{\Gamma}, \varepsilon x_t) \rho_t(a, m - \hat{\Gamma}) da, \quad (2b)$$

$$x_t = \int_0^t h(t-s) \int_0^\infty \int_0^\infty f(a, m, \varepsilon x_s) \rho_s(a, m) da dm ds, \quad (2c)$$

$$\rho_0 = u_0. \quad (2d)$$

If  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a bounded function such that  $\lim_{a \rightarrow +\infty} \eta(a) = 0$  ( $\eta$  is the ‘refractory kernel’ [16, Sec. 9.3]), we can define  $f$  more explicitly:

$$f(a, m, \tilde{x}) := \hat{f}(\eta(a) - m + \tilde{x}), \quad (2e)$$

where  $\hat{f} : \mathbb{R} \rightarrow \mathbb{R}_+$  is typically a non-decreasing function. Since  $m$  makes jumps of size  $\hat{\Gamma} > 0$  at each spike and decays at exponential speed  $-\lambda$  between spikes,  $m$  accumulates over spikes, which decreases the firing rate  $f$  (Eq. (2e)), leading to spike frequency adaptation [2]. More specifically, Eq. (2) is a population equation for adaptive SRM<sub>0</sub> (Spike Response Model) neurons [21, 16].

Populations of spiking neurons with spike frequency adaptation exhibit self-sustained population bursts when the connectivity strength is sufficiently strong [41, 18, 13]. In Fig. 1, we show simulations of (2) for two different connectivity strengths  $\varepsilon$ . For large  $\varepsilon$ , we observe self-sustained bursts, whereas for small  $\varepsilon$ , we observe relaxation to a stationary state.

### 1.2.2 Short-term synaptic depression

The recent spike history of a presynaptic neuron can modulate the synaptic transmission, leading to short-term synaptic plasticity [43]. We will consider here the case of depressive synapses and use the model of [39] (with a change of variable for convenience). In this case, the state-space is  $(a, m) \in \mathbb{R}_+ \times ]0, 1[$ . Taking  $f$  independent of  $m$ ,  $h$  of the the form  $h(t, a, m) := \hat{h}(t)(1 - m)$  and  $\gamma(m) := 1 - v + vm$  for a fixed  $v \in ]0, 1[$ , (1) becomes

$$\partial_t \rho_t + \nabla \cdot (b \rho_t) = -f(a, m, \varepsilon x_t) \rho_t, \quad (3a)$$

$$\rho_t(0, m) = 1_{m > \gamma(0)} \frac{1}{v} \int_0^\infty f(a, \varepsilon x_t) \rho_t(a, \gamma^{-1}(m)) da, \quad (3b)$$

$$x_t = \int_0^t \hat{h}(t-s) \int_0^1 \int_0^\infty (1-m) f(a, \varepsilon x_s) \rho_s(a, m) da dm ds, \quad (3c)$$

$$\rho_0 = u_0. \quad (3d)$$

Note that the term  $\frac{1}{v}$  on the RHS of Eq. (3b) simply comes from the fact that  $|(\gamma^{-1})'(m)| = \frac{1}{v}$ , for all  $m \in ]0, 1[$ . Here, at each spike,  $m$  makes strictly positive jumps which size tends to 0 as  $m$  tends to 1 ( $\gamma(1) = 1$ ) and decays at exponential speed  $-\lambda$  between spikes. If  $m$  is close to 1, synaptic transmission is weak because of the factor  $(1 - m)$  in Eq. (3c). This is an implementation of the purely depressive Tsodyks-Markram model [39], a standard model of short-term synaptic depression.

As observed in [35], the stationary state of populations of neurons with short-term synaptic plasticity can be described by a simple formula, which we prove in Sec. 4.3.

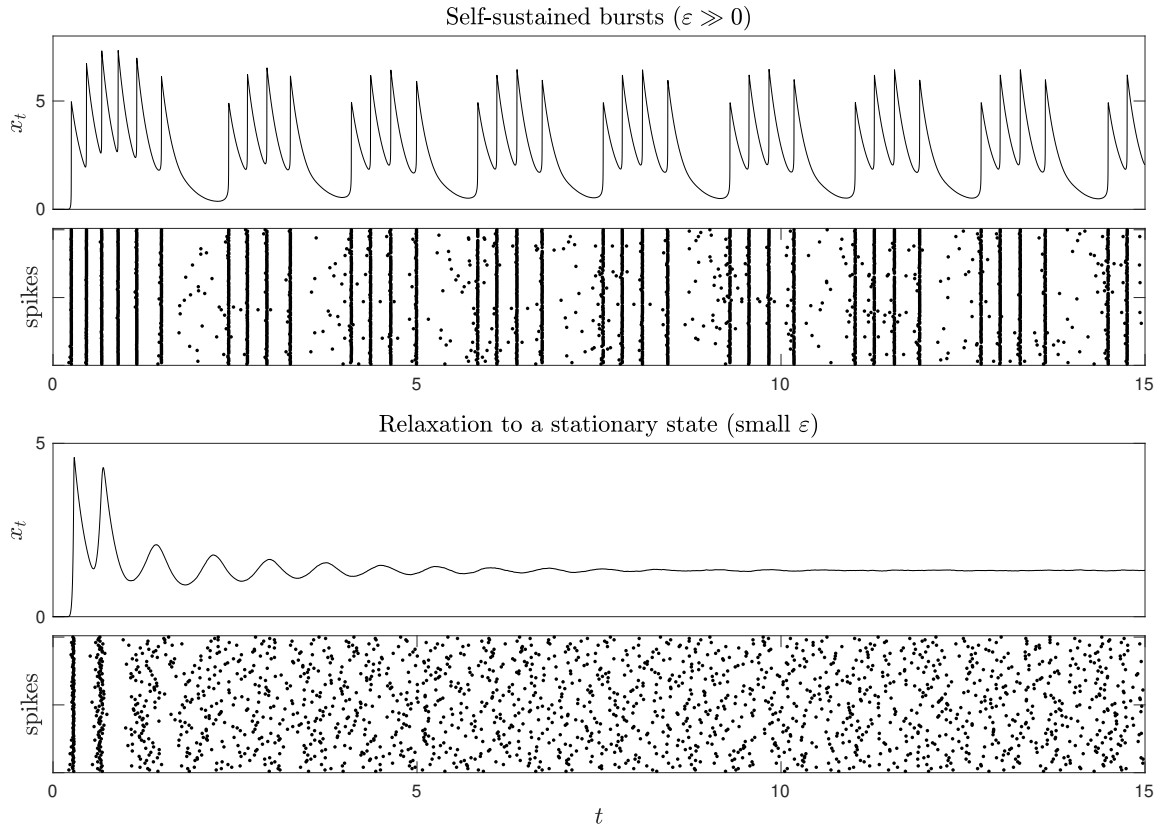


Figure 1: **Simulations of Eq. (2) showing self-sustained bursts for strong connectivity strength ( $\varepsilon \gg 0$ ) and relaxation to a stationary state for weak connectivity strength (small  $\varepsilon$ ).** Simulations of a network of  $5 \cdot 10^5$  adaptive SRM<sub>0</sub> neurons approximating Eq. (2) with identical parameters (except for  $\varepsilon$ ) and identical initial conditions. The raster plots below the plots for the time-evolution of the total postsynaptic potential  $x_t$  represent the spikes of 100 randomly selected neurons.

### 1.3 Assumptions and main results

The main result of this work is the exponential stability of the age and leaky memory-structured equation (1) in the weak connectivity regime (**Theorem 3**). Before proving the exponential stability, we first establish the well-posedness of (1) in the appropriate function space (**Theorem 1**) and show that stationary solutions exist and are unique for sufficiently weak connectivity (**Theorem 2**).

Here, we study the weak solutions to (1) for an initial datum in  $L_+^1 := L^1(\mathbb{R}_+ \times \mathbb{R}_+^*, \mathbb{R}_+)$  and write  $L_+^1(\mathbb{R}_+^*) := L^1(\mathbb{R}_+^*, \mathbb{R}_+)$ .

**Definition (Solutions).**  $(\rho, x) \in \mathcal{C}(\mathbb{R}_+, L_+^1) \times \mathcal{C}(\mathbb{R}_+)$  is a solution to (1), for the initial datum  $u_0 \in L_+^1$ , if

$$x_t = \int_0^t \int_0^\infty \int_0^\infty h(t-s, a, m) f(a, m, \varepsilon x_s) \rho_s(a, m) da dm ds, \quad \forall t \geq 0, \quad (4a)$$

and if for all  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*)$ ,

$$\begin{aligned} 0 &= \int_0^\infty \int_0^\infty u_0(a, m) \varphi(0, a, m) da dm \\ &+ \int_0^\infty \int_0^\infty \int_0^\infty \rho_t(a, m) \left\{ [\partial_t + \partial_a - \lambda m \partial_m] \varphi + (\varphi(t, 0, \gamma(m)) - \varphi(t, a, m)) f(a, m, \varepsilon x_t) \right\} da dm dt. \end{aligned} \quad (4b)$$

To prove the well-posedness of (1), we need some simple assumptions of the firing rate function  $f$  and the interaction function  $h$ :

**Assumption 1.**  $f$  is bounded and  $L_f$ -Lipschitz, i.e.

$$|f(a, m, x) - f(a^*, m^*, x^*)| \leq L_f (|a - a^*| + |m - m^*| + |x - x^*|),$$

and  $h$  is bounded and continuous.

Since our goal is the use Harris' theorem to study the long time behavior of (1), the well-posedness in  $L^1$  (which is treated in [36]) is not enough and we need the well-posedness in a weighted  $L^1$  space (where the weight satisfies a Lyapunov condition [23]) with a global-in-time estimate in the weighted  $L^1$  norm.

Using the weight function

$$w : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [1, \infty), \quad (a, m) \mapsto 1 + m,$$

we define the function space

$$L_+^1(w) := \left\{ g \in L^1(\mathbb{R}_+ \times \mathbb{R}_+^*, \mathbb{R}_+) \mid \|g\|_{L^1(w)} := \int_0^\infty \int_0^\infty g(a, m) w(a, m) da dm < \infty \right\}.$$

To obtain a global-in-time estimate in the  $L_+^1(w)$  norm, we further need that the jump mapping  $\gamma$  grows at most linearly. For convenience, we formulate this assumption in terms of the jump size function  $\Gamma$ :

**Assumption 2.** There exists  $\Gamma \in \mathcal{C}_b^1(\mathbb{R}_+^+, \mathbb{R}_+^*)$  such that  $\gamma(m) = m + \Gamma(m), \forall m \in \mathbb{R}_+^*$ .

**Theorem 1 (Well-posedness).** Grant Assumption 1. For any initial datum  $u_0 \in L_+^1$ , there exists a unique weak solution  $(\rho, x)$  to (1). This solution satisfies

- (i) ( $L^1$ -stability)  $\|\rho_t\|_{L^1} = \|u_0\|_{L^1}, \forall t > 0,$   
(ii) (Global bound in  $L^1_+(w)$ ) if, in addition, Assumption 2 holds and  $u_0 \in L^1_+(w)$ , then

$$\forall t > 0, \quad \|\rho_t\|_{L^1(w)} \leq \|u_0\|_{L^1(w)} e^{-\alpha t} + \frac{b}{\alpha}(1 - e^{-\alpha t}), \quad (5)$$

for some constants  $\alpha > 0$  and  $b \in \mathbb{R}$ .

In contrast to [36], the well-posedness proof presented here does not involve any probabilistic argument. The proof consists of two consecutive applications of Banach's fixed-point theorem, where a first fixed-point gives the unique solution to a linearized version of (1) which is then used in a second fixed-point treating the nonlinearity of (1).

The second step towards the exponential stability proof is the study of the existence and uniqueness of the stationary solutions to (1). For this step, we require:

**Assumption 3.**

- (i) There exists  $\Delta_{abs} > 0$  and  $\sigma > 0$  such that

$$f(a, m, x) \geq \sigma, \quad \forall (a, m, x) \in [\Delta_{abs}, +\infty[ \times \mathbb{R}_+^* \times \mathbb{R}.$$

- (ii) There exists  $C_\gamma \in ]0, 1]$  such that  $C_\gamma \leq \gamma' \leq 1$ .

- (iii)  $\bar{h}(a, m) = \int_0^\infty h(t, a, m) dt$  is bounded.

The first point of Assumption 3 sets a lower bound on the firing rate function  $f$  for any  $a \geq \Delta_{abs}$  and hence allows for an absolute refractory period  $\Delta_{abs} > 0$ , i.e. a period of time following a spike during which  $f = 0$  (which is an important neurodynamical feature [16, Sec. 1.1]). This assumption is also used in [3].

In the second point of Assumption 3, the lower bound  $0 < C_\gamma \leq \gamma'$  guarantees that  $\gamma$  is strictly increasing, which reflects the idea that  $m$  is a 'leaky memory' variable of the past neuronal activity. On the other hand, the upper bound  $\gamma' \leq 1$ , which can be rewritten in terms of the jump size function  $\Gamma$  as  $\Gamma' \leq 0$ , prevents the variable  $m$  from growing too fast and allows for a potential saturation of the memory, as in the example with short-term synaptic plasticity (3). The third point of Assumption 3 is purely technical.

We emphasize that the two examples shown above, spike frequency adaptation (2) and short-term synaptic depression (3), satisfy Assumption 3.

**Theorem 2** (Stationary solutions). *Grant Assumptions 1 and 3.*

- (i) There exists a stationary solution to (1).  
(ii) There exists  $\varepsilon^* > 0$  such that for all  $\varepsilon \in ]-\varepsilon^*, +\varepsilon^*[$ , the stationary solution to (1) is unique.

Over the course of this work, we obtained the existence of the stationary solution by two different approaches. The first approach is based on the Doeblin-Harris technique [19] and is similar to that of [3]. First, we show that when  $x_t$  is fixed and time-invariant in (1) (neurons are non-interacting), the system satisfies a Harris condition – this constitutes a key result of this work –, and we can use Harris' theorem to get the stationary solution. Then, we use the Lipschitz continuity the stationary solutions with respect to the fixed  $x$  to prove the existence of a stationary solution for arbitrary connectivity strengths  $\varepsilon$ . Finally, for  $\varepsilon$  small enough, we also get the uniqueness of the stationary solution, by Banach's fixed-point theorem.



The second approach relies on the fact that the stationary solutions solve an integral equation, for which we can show that a solution exists by the Schauder's fixed-point theorem. In the process, we get several estimates on the stationary solutions, namely that they are continuous, bounded, and exponentially decaying in  $m$ . However, this approach does not give uniqueness.

As mentioned above, the application of Harris' Theorem requires us to consider solutions in the weighted space  $L^1(w)$ . However, in the case where the state-space of the leaky memory variable  $m$  is bounded, the situation is simpler: we can use Doeblin's theorem in  $L^1$ . The following assumption guarantees that  $m$  stays in a bounded state-space:

**Assumption 4.** *There exists  $G > 0$  such that for all  $m \in \mathbb{R}_+$ ,  $\gamma(m) < G$ .*

Note that this assumption is satisfied in the example with short-term synaptic plasticity (3), with  $G = 1$ .

Finally, to study the exponential stability of (1), we need on exponential decay on  $h$ :

**Assumption 5.** *There exists  $\mathfrak{h}, C_h > 0$  such that  $h(t, a, m) \leq C_h e^{-\mathfrak{h}t}$ ,  $\forall (t, a, m)$ .*

By a perturbative argument similarly as in [24], we obtain our main result:

**Theorem 3** (Exponential stability in the weak connectivity regime). *Grant Assumptions 1, 3 and 5. For any  $W > 0$ , there exists  $\varepsilon_W^{**} > 0$  such that for  $\varepsilon \in ]-\varepsilon_W^{**}, +\varepsilon_W^{**}[$ , there exists  $C \geq 1$  and  $c_W > 0$  such that for all initial data  $u_0 \in L^1_+(w)$  with  $\|u_0\|_{L^1} = 1$  and  $\|u_0\|_{L^1(w)} \leq W$ , the solution  $(\rho, x)$  to (1) satisfies*

$$\|\rho_t - \rho_\infty\|_{L^1(w)} + |x_t - x_\infty| \leq C e^{-c_W t} \left( \|u_0 - \rho_\infty\|_{L^1(w)} + 1 \right), \quad \forall t \geq 0, \quad (6)$$

where  $(\rho_\infty, x_\infty)$  is the unique stationary solution given by Theorem 2 (ii).

If, in addition, we grant Assumption 4, then there exists  $\varepsilon^{**} > 0$  such that for all  $\varepsilon \in ]-\varepsilon^{**}, +\varepsilon^{**}[$ , there exists  $C' \geq 1$  and  $c > 0$  such that for all initial data  $u_0 \in L^1_+$  with  $\|u_0\|_{L^1} = 1$ ,

$$\|\rho_t - \rho_\infty\|_{L^1} + |x_t - x_\infty| \leq C' e^{-c t} \left( \|u_0 - \rho_\infty\|_{L^1} + 1 \right), \quad \forall t \geq 0. \quad (7)$$

From the neuronal modeling point of view, this result is not surprising: when the connection strength is weak enough, neurons do not synchronize and the population activity converges to a stationary state. This was already proved for simpler one-dimensional models (see below) and the addition of a leaky memory variable carrying the effect of spike frequency adaptation or short-term synaptic plasticity does not change this behavior.

## 1.4 Methodology discussion

The asymptotic stability of the age-structured model of [28] in the weak connectivity regime has been studied by the same authors in [28, 29] using entropy methods and assuming that  $f$  is a step-function, in [25, 24] using spectral analysis of semigroups in Banach spaces and in [3] using Doeblin's theorem. For the treatment of the strong connectivity regime, we refer to [28, 24].

On the closely related voltage-structured model [8], [7] also proved asymptotic stability in the weak connectivity regime using Laplace transform techniques. For this model, the nonlinear stability of the stationary solutions has been recently studied in [5] (see also [9]) and can identify Hopf bifurcations [6]. This type of analysis would be challenging in the case of (1) in general, as there is no representation formula for the stationary solutions when  $\varepsilon = 0$ .

Finally, Doeblin's theorem has also been used in [11] in the case of the 'threshold crossing' neuronal population equation of [27]. Note also that closely related methods have been used by probabilists to study the ergodicity of single neuron models [20, 10].

Our approach combines strategies from [25] and [3]. On the one hand, our proof is based on the application of Harris' theorem for the linear problem, like in [3]<sup>1</sup>, which simplifies the proof of [25]. On the other hand, we use an argument from in [25] to deal with delay effects, which are not considered in [3]. Note that our model is two-dimensional (by the addition of the leaky memory variable), whereas the aforementioned works only considered one-dimensional models.

## 1.5 Plan of the paper

The proof of Theorem 1 (Well-posedness) is presented in **Section 2**. In **Section 3**, we prove the exponential stability of (1) in the non-interacting case  $\varepsilon = 0$  using Harris' and Doeblin's theorems. The proof of Theorem 2 (Stationary solutions) is presented in **Section 4** which is divided in three parts: in the first part, we present a proof which uses the exponential stability of the non-interacting case; in the second part, we present an alternative proof for the existence of stationary solutions which does not involve the Doeblin-Harris method; and in the last part, we present proof for the formula of [35] in the case of short-term synaptic plasticity (3). Finally, **Section 5** is dedicated to the proof of Theorem 3 (Exponential stability in the weak connectivity regime).

## 2 Well-posedness

This section is dedicated to the proof of Theorem 1, which we decompose in several Lemmas. First, we verify the *a priori*  $L^1$ -stability of the solutions to (1), an important technical result we use later in the proof. Then, we introduce a linearized version of (1) and show that it is well-posed by an application of Banach's fixed-point theorem. Another Banach's fixed-point argument is used to treat the nonlinearity of (1) and concludes the proof of the well-posedness of (1) in  $L^1$ . Finally, we prove the global bound in  $L^1_+(w)$  of the solutions (point (ii) of Theorem 1), which we will need in order to apply Harris' theorem in the next Sections.

**Lemma 1** (*A priori*  $L^1$ -stability). *Grant Assumption 1. If  $(\rho, x)$  is a weak solution to (1) for the initial datum  $u_0 \in L^1_+$ , then*

$$\|\rho_t\|_{L^1} = \|u_0\|_{L^1}, \quad \forall t > 0.$$

*Proof.* By a standard cut-off in time argument, we have that for all  $T > 0$  and for all  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*)$ ,

$$\begin{aligned} & \int_0^\infty \int_0^\infty \rho_T(a, m) \varphi(T, a, m) da dm - \int_0^\infty \int_0^\infty u_0(a, m) \varphi(0, a, m) da dm = \\ & \int_0^T \int_0^\infty \int_0^\infty \rho_t(a, m) \left\{ [\partial_t + \partial_a - \lambda m \partial_m] \varphi + (\varphi(t, 0, \gamma(m)) - \varphi(t, a, m)) f(a, m, \varepsilon x_t) \right\} da dm dt. \end{aligned}$$

Let  $\chi$  be a function in  $\mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+^*, \mathbb{R}_+)$  such that

$$\chi(a, m) = 1, \quad \text{for all } a^2 + m^2 \leq 1.$$

For all  $n \in \mathbb{N}^*$ , we write  $\tilde{\varphi}^n \in \mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*)$  the classical solution to the transport equation

$$\partial_t \tilde{\varphi}^n(t, a, m) + \partial_a \tilde{\varphi}^n(t, a, m) - \lambda m \partial_m \tilde{\varphi}^n(t, a, m) = 0, \quad (8a)$$

$$\tilde{\varphi}^n(0, a, m) = \chi(a/n, m/n). \quad (8b)$$

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<sup>1</sup>To be precise, they use Doeblin's theorem.

Because of the finite speed of propagation of the transport equation, for all  $n$ , there exists a function  $\varphi^n \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*)$  such that  $\varphi^n(t, a, m) = \tilde{\varphi}^n(t, a, m)$ , for all  $(t, a, m) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+^*$ . Hence, for all  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} & \int_0^\infty \int_0^\infty \varphi^n(T, a, m) \rho_T(a, m) da dm - \int_0^\infty \int_0^\infty \varphi^n(0, a, m) u_0(a, m) da dm = \\ & \int_0^T \int_0^\infty \int_0^\infty \left\{ \partial_t \varphi^n + \partial_a \varphi^n - \lambda m \partial_m \varphi^n + (\varphi^n(t, 0, \gamma(m)) - \varphi^n(t, a, m)) f(a, m, \varepsilon x_t) \right\} \rho_t(a, m) da dm dt. \end{aligned}$$

As  $\varphi_n$  is a solution to Eq. (8a) on time  $[0, T]$ , we get

$$\begin{aligned} & \int_0^\infty \int_0^\infty \varphi^n(T, a, m) \rho_T(a, m) da dm - \int_0^\infty \int_0^\infty \varphi^n(0, a, m) u_0(a, m) da dm = \\ & \int_0^T \int_0^\infty \int_0^\infty \left\{ (\varphi^n(t, 0, \gamma(m)) - \varphi^n(t, a, m)) f(a, m, \varepsilon x_t) \right\} \rho_t(a, m) da dm dt. \end{aligned}$$

For all  $(t, a, m) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+^*$ ,  $\varphi^n(t, a, m) \xrightarrow[n \rightarrow \infty]{} 1$ , since the initial datum tends to 1 as  $n \rightarrow \infty$  (Eq. (8b)) and by finite speed of propagation. Thus, by dominated convergence, we get

$$\int_0^\infty \int_0^\infty \rho_T(a, m) da dm - \int_0^\infty \int_0^\infty u_0(a, m) da dm = 0. \quad (9)$$

Since  $\rho$  is nonnegative, this concludes the proof.  $\square$

Lemma 1 will allow us to prove the well-posedness of (1) by the means of fixed-point arguments. Let us first introduce a linearized version of Eq. (1): for all  $x \in \mathcal{C}(\mathbb{R}_+)$ , we consider the linear evolution problem

$$\partial_t \rho_t + \nabla \cdot (b \rho_t) = -f(a, m, \varepsilon x_t) \rho_t, \quad (10a)$$

$$\rho_t(0, m) = 1_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \varepsilon x_t) \rho_t(a, \gamma^{-1}(m)) da, \quad (10b)$$

$$\rho_0 = u_0. \quad (10c)$$

We can see Eq. (10) as the Kolmogorov forward equation of a time-dependent Markov process. Indeed, we can rewrite Eqs. (10a) and (10b) as

$$\partial_t \rho_t = \mathcal{L}_t \rho_t \quad (11)$$

where, for all suitable test function  $\phi : \mathbb{R}_+ \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ ,

$$\mathcal{L}_t^* \phi(a, m) = b(a, m) \nabla \cdot \phi(a, m) + [\phi(0, \gamma(m)) - \phi(a, m)] f(a, m, \varepsilon x_t). \quad (12)$$

$\mathcal{L}_t^*$  is the time-dependent generator of a piecewise deterministic Markov process with degenerate jumps.

The linearized equation (10) will play a special role in the following Sections and it therefore deserves its own Proposition:

**Proposition 1** (Well-posedness of the linearized equation (10)). *Grant Assumption 1. For any initial datum  $u_0 \in L_+^1$  and any  $x \in \mathcal{C}(\mathbb{R}_+)$ , there exists a unique weak solution  $\rho^x \in \mathcal{C}(\mathbb{R}_+, L_+^1)$  to Eq. (10). Furthermore,  $\rho^x$  satisfies*

(i) For all  $t > 0$  and for all  $m \in \mathbb{R}_+^*$ ,

$$\begin{aligned} \rho_t^x(0, m) &= 1_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \varepsilon x) \rho_t^x(a, \gamma^{-1}(m)) da, \\ \rho_t^x(a, m) &= \begin{cases} u_0(a-t, e^{\lambda t} m) \exp\left(\lambda t - \int_0^t f(a-t+s, e^{\lambda(t-s)} m, \varepsilon x) ds\right) & \text{if } a \geq t, \\ \rho_{t-a}^x(0, e^{\lambda a} m) \exp\left(\lambda a - \int_{t-a}^t f(a-t+s, e^{\lambda(t-s)} m, \varepsilon x) ds\right) & \text{if } 0 < a < t. \end{cases} \end{aligned}$$

(ii) For all  $t > 0$  and for all  $\phi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+^*)$ ,

$$\langle \rho_t^x, \phi \rangle = \langle u_0, \phi \rangle + \int_0^t \langle \rho_s^x, \mathcal{L}_x^* \phi \rangle ds. \quad (14)$$

*Proof.* Fix  $x \in \mathcal{C}(\mathbb{R}_+)$ . For all  $p \in \mathcal{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*))$  and  $u_0 \in L_+^1$ , we know, from the standard theory of transport equations, that there is a unique weak solution to

$$\begin{aligned} \partial_t \rho_t + \nabla \cdot (b \rho_t) &= -f(a, m, \varepsilon x_t) \rho_t, \\ \rho_t(0, m) &= p_t(m), \\ \rho_0 &= u_0, \end{aligned}$$

which we denote  $\rho^{x,p}$  and is given by the representation formula,

$$\rho_t^{x,p}(a, m) := \begin{cases} u_0(a-t, e^{\lambda t} m) \exp\left(\lambda t - \int_0^t f(a-t+s, e^{\lambda(t-s)} m, \varepsilon x_s) ds\right) & \text{if } a \geq t, \\ p_{t-a}(e^{\lambda a} m) \exp\left(\lambda a - \int_{t-a}^t f(a-t+s, e^{\lambda(t-s)} m, \varepsilon x_s) ds\right) & \text{if } 0 < a < t. \end{cases}$$

$\rho^{x,p}$  is in  $\mathcal{C}(\mathbb{R}_+, L^1)$ , since

$$\forall t \in \mathbb{R}_+, \quad \|\rho_t^{x,p}\|_{L^1} \leq \|u_0\|_{L^1} + \int_0^t \|p_s\|_{L^1} ds.$$

We have

$$\left( 1_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), x) \rho_t^{x,p}(a, \gamma^{-1}(m)) da \right)_{(t,m) \in \mathbb{R}_+ \times \mathbb{R}_+^*} \in \mathcal{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*))$$

since

$$\begin{aligned} \forall t \in \mathbb{R}_+, \quad \int_{\gamma(0)}^\infty \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), x) \rho_t^{x,p}(a, \gamma^{-1}(m)) da dm \\ \leq \|f\|_\infty \|\rho_t^{x,p}\|_{L^1} \leq \|f\|_\infty \left( \|u_0\|_{L^1} + \int_0^t \|p_s\|_{L^1} ds \right). \end{aligned}$$

Hence, we can define, for any  $T > 0$ , the operator  $\Phi_T^x$ :

$$\begin{aligned} \mathcal{C}([0, T], L_+^1(\mathbb{R}_+^*)) &\rightarrow \mathcal{C}([0, T], L_+^1(\mathbb{R}_+^*)) \\ p &\mapsto \left( 1_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), x) \rho_t^{x,p}(a, \gamma^{-1}(m)) da \right)_{(t,m) \in [0, T] \times \mathbb{R}_+^*}. \end{aligned}$$

For any  $p, q \in \mathcal{C}([0, T], L_+^1(\mathbb{R}_+^*))$ ,

$$\begin{aligned} \|\Phi_T^x(p) - \Phi_T^x(q)\|_{\mathcal{C}([0, T], L^1)} &\leq \|f\|_\infty \sup_{t \in [0, T]} \|\rho_t^{x, p} - \rho_t^{x, q}\|_{L^1} \\ &\leq \|f\|_\infty \int_0^T \|p_s - q_s\|_{L^1} ds \\ &\leq T \|f\|_\infty \|p - q\|_{\mathcal{C}([0, T], L^1)}. \end{aligned}$$

Therefore, if  $0 < T < \|f\|_\infty^{-1}$ ,  $\Phi_T^x$  is a contraction. By Banach's fixed-point theorem, there exists a unique  $\rho^x \in \mathcal{C}([0, T], L_+^1)$  solving Eq. (10). Since the choice of the contracting  $T$  does not depend on the initial datum, we can iterate the above argument on successive time intervals of length  $T$  and conclude that there exists a unique  $\rho^x \in \mathcal{C}(\mathbb{R}_+, L_+^1)$  solving Eq. (10) for which the formula (i) is satisfied. Then, (ii) follows from a standard cut-off in time argument.  $\square$

Now, we can prove the existence and uniqueness of a solution to the nonlinear problem (1) by the means of a second application of Banach's fixed-point theorem.

*Proof of the well-posedness of (1) in  $L^1$ .* For any  $x \in \mathcal{C}(\mathbb{R}_+)$ , we take the  $\rho^x$  given by Proposition 1. We have

$$\left( \int_0^t \int_{\mathbb{R}_+ \times \mathbb{R}_+^*} h(t-s) f(\varepsilon x_s) \rho_s^x dadmds \right)_{t \in \mathbb{R}_+} \in \mathcal{C}(\mathbb{R}_+)$$

since

$$\forall t \in \mathbb{R}_+, \quad \left| \int_0^t \int_{\mathbb{R}_+ \times \mathbb{R}_+^*} h(t-s) f(\varepsilon x_s) \rho_s^x dadmds \right| \leq \|h\|_\infty \|f\|_\infty \int_0^t \|\rho_s^x\|_{L^1} ds.$$

Hence, for any  $T > 0$ , we can define the operator

$$\begin{aligned} \Psi_T : \mathcal{C}([0, T]) &\rightarrow \mathcal{C}([0, T]) \\ x &\mapsto \left( \int_0^t \int_{\mathbb{R}_+ \times \mathbb{R}_+^*} h(t-s) f(\varepsilon x_s) \rho_s^x dadmds \right)_{t \in [0, T]}. \end{aligned}$$

For any  $x, y \in \mathcal{C}([0, T])$ , we have

$$\begin{aligned} \|\Psi_T(x) - \Psi_T(y)\|_{\mathcal{C}([0, T])} &\leq T \|h\|_\infty \sup_{t \in [0, T]} \int_{\mathbb{R}_+ \times \mathbb{R}_+^*} |f(\varepsilon x_t) \rho_t^x - f(\varepsilon y_t) \rho_t^y| dadm \\ &\leq T \|h\|_\infty \sup_{t \in [0, T]} \left( \varepsilon L_f \|x_t - y_t\|_{L^1} \|\rho_t^x\|_{L^1} + \|f\|_\infty \|\rho_t^x - \rho_t^y\|_{L^1} \right). \end{aligned}$$

By Grönwall's lemma,  $\|\rho_t^x\|_{L^1} \leq \|u_0\|_{L^1} \exp(\|f\|_\infty t)$ , since

$$\forall t \in [0, T], \quad \|\rho_t^x\|_{L^1} \leq \|u_0\|_{L^1} + \|f\|_\infty \int_0^t \|\rho_s^x\|_{L^1} ds.$$

On the other hand, we have, for all  $t \in [0, T]$ ,

$$\begin{aligned} \|\rho_t^x - \rho_t^y\|_{L^1} &\leq \|f\|_\infty \int_0^t \int_0^\infty \int_0^\infty \left| \rho_s^x(a, m) \exp\left(-\int_s^t f(u-s, e^{-\lambda(u-s)}\gamma(m), \varepsilon x_u) du\right) \right. \\ &\quad \left. \rho_s^y(a, m) \exp\left(-\int_s^t f(u-s, e^{-\lambda(u-s)}\gamma(m), \varepsilon y_u) du\right) \right| dadmds \\ &\leq \|f\|_\infty \int_0^t \|\rho_s^x - \rho_s^y\|_{L^1} ds + t\varepsilon \|f\|_\infty L_f \|x - y\|_{\mathcal{C}([0, T])} \int_0^t \|\rho_s^x\|_{L^1} ds. \end{aligned}$$

Hence, by Grönwall's lemma, for all  $t \in [0, T]$ ,

$$\|\rho_t^x - \rho_t^y\|_{L^1} \leq \varepsilon L_f \|u_0\|_{L^1} \frac{(\exp(\|f\|_\infty t) - 1)^2}{\|f\|_\infty} \|x - y\|_{\mathcal{C}([0, T])}.$$

Gathering the bounds, we get

$$\|\Psi_T(x) - \Psi_T(y)\|_{\mathcal{C}([0, T])} \leq T\varepsilon \|h\|_\infty L_f \|u_0\|_{L^1} \exp(\|f\|_\infty T) [1 + \exp(\|f\|_\infty T)] \|x - y\|_{\mathcal{C}([0, T])}.$$

For  $T$  small enough,  $\Psi_T$  is a contraction and, by Banach fixed-point theorem, has a unique fixed-point. Thus, there exists a unique solution  $(\rho, x) \in \mathcal{C}([0, T], L_+^1)$ . Since by Lemma 1,  $\|\rho_T\|_{L^1} = \|u_0\|_{L^1}$ , we can iterate this argument on successive time intervals of length  $T$  and conclude that there exists a unique solution in  $\mathcal{C}(\mathbb{R}_+, L_+^1)$ .  $\square$

To conclude the proof of Theorem 1, it remains to show the estimate Eq. (5). Under Assumption 2, the weight function

$$w : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [1, \infty), \quad (a, m) \mapsto 1 + m$$

defined above, satisfies  $w(a, m) \rightarrow \infty$  when  $m \rightarrow \infty$  and the Lyapunov condition on  $m$ :

$$\exists \alpha > 0, b \geq 0 \quad \text{such that} \quad \mathcal{L}_t^* w \leq -\alpha w + b. \quad (16)$$

Indeed, for all  $(t, a, m) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*$ ,

$$\mathcal{L}_t^* w(a, m) = -\lambda m + \Gamma(m) f(a, m, \varepsilon x_t) \leq -\lambda w(a, m) + \lambda + \|\Gamma\|_\infty \|f\|_\infty.$$

Importantly, the constants  $\alpha$  and  $b$  do not depend on  $x$ .

**Lemma 2** (Global bound in  $L_+^1(w)$ ). *Grant Assumptions 1 and 2. If the initial datum  $u_0$  is in  $L_+^1(w)$ , then  $\rho_t \in L_+^1(w)$  for all  $t \geq 0$ . Moreover,*

$$\forall t > 0, \quad \|\rho_t\|_{L^1(w)} \leq \|u_0\|_{L^1(w)} e^{-\alpha t} + \frac{b}{\alpha} (1 - e^{-\alpha t}), \quad (17)$$

where the constants  $\alpha$  and  $b$  are taken from the Lyapunov condition (16).

*Proof.* We divide the proof in two steps: first, we prove that the solution is stable in  $L_+^1(w)$  with a weaker and time dependent bound; then, we use this first bound to apply the dominated convergence theorem and obtain Eq. (17) by Grönwall's lemma.

**Step 1.** Fix any  $T > 0$ . Let  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}_+, \mathbb{R}_+)$  be a non-increasing function such that  $\chi(x) = 1$  if  $0 \leq x \leq 1$  and  $\chi(x) = 0$  if  $x > 2$ . For all  $n \in \mathbb{N}^*$ , let us write  $\varphi_k(a)\chi_n(m) := \chi(a/k)\chi(m/n)$ . For all  $n, k$ ,  $w\chi_n\varphi_k \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ . Hence,

$$\forall n \in \mathbb{N}^*, \quad \langle \rho_T, w\chi_n\varphi_k \rangle = \langle u_0, w\chi_n\varphi_k \rangle + \int_0^T \langle \rho_t, \mathcal{L}_x^*(w\chi_n\varphi_k) \rangle dt,$$

where

$$\begin{aligned} \mathcal{L}_x^*(w\chi_n\varphi_k) &= \partial_a(w\chi_n\varphi_k) - \lambda m \partial_m(w\chi_n\varphi_k) + (w(\gamma(m))\chi_n(\gamma(m))\varphi_k(0) - w\chi_n\varphi_k) f \\ &= w\chi_n \frac{1}{k} \chi'(a/k) + w\varphi_k \frac{1}{n} \chi'(m/n) - \lambda m w\chi_n + (w(\gamma(m))\chi_n(\gamma(m))\varphi_k - w\chi_n\varphi_k) f. \end{aligned}$$

From the  $L^1$ -stability and the fact that both  $w\partial_m\chi_n$  and  $w\chi_n$  are bounded, we take the limit  $k \rightarrow \infty$  with the dominated convergence theorem to obtain

$$\langle \rho_T, w\chi_n \rangle = \langle u_0, w\chi_n \rangle + \int_0^T \left\langle \rho_t, w \frac{1}{n} \chi'(m/n) - \lambda m w\chi_n + (w(\gamma(m))\chi_n(\gamma(m)) - w\chi_n) f \right\rangle dt. \quad (18)$$

From the properties of  $\chi$  and  $\gamma$ , we get

$$\begin{aligned} w(0, \gamma(m))\chi_n(\gamma(m)) &\leq w(0, m + \|\Gamma\|_\infty)\chi_n(\gamma(m)) \leq (1 + \|\Gamma\|_\infty)w(a, m)\chi_n(m), \\ \frac{1}{n}w\chi'(m/n) &\leq \frac{1 + 2n}{n} \|\chi'\|_\infty. \end{aligned}$$

Then, since  $f$  is bounded, there exists a constant  $C$ , which does not depend on  $n$ , such that

$$\langle \rho_T, w\chi_n \rangle \leq \langle u_0, w\chi_n \rangle + \int_0^T \langle \rho_t, C(w\chi_n(m) + 1) \rangle dt.$$

We can now apply Grönwall's lemma to obtain

$$\langle \rho_T, w\chi_n \rangle \leq \max(\langle u_0, w\chi_n \rangle C e^{Ct}, C).$$

It follows from Fatou's lemma that  $\rho_t = S_T^x u_0 \in L_+^1(w)$ .

**Step 2.** To improve the previous estimate, we come back to (18) and use dominated convergence in  $n$  (domination being guaranteed by the bound above) to get

$$\langle \rho_T, w \rangle = \langle u_0, w \rangle + \int_0^T \langle \rho_t, \mathcal{L}_x^* w \rangle dt.$$

By the Lyapunov condition (16),

$$\|\rho_T\|_{L^1(w)} \leq \|u_0\|_{L^1(w)} - \alpha \int_0^T \|\rho_t\|_{L^1(w)} + Tb.$$

Finally, Eq. (17) is obtained by Grönwall's lemma.  $\square$

**Remark.** Following the same steps as in the proof above, we can show that the bound Eq. (17) also holds for the linearized equation (10) and does not depend on  $x$  nor the constants  $\alpha$  and  $b$ .

### 3 Exponential stability in the non-interacting case

If  $x \in \mathcal{C}(\mathbb{R}_+)$  in the linearized equation (10) is time-invariant, i.e.  $x \equiv \tilde{x}$  for some  $\tilde{x} \in \mathbb{R}$ , then Eq. (10) can be seen as the dynamics of a non-interacting population of neurons. In this section, we prove the exponential stability in the non-interacting case using Harris' and Doeblin's theorems. This is the key result of this work and will allow us to prove the existence and uniqueness of the stationary solution to (1) (Section 4) and the exponential convergence to it (Section 5).

For  $\tilde{x} \in \mathbb{R}$ ,  $u_0 \in L^1$ , we denote  $\rho_t^{\tilde{x}}$  the unique solution to Eq. (10) for the initial datum  $u_0$  and  $x \equiv \tilde{x}$ , given by Proposition 1. We write, using the semigroup notation,

$$S_t^{\tilde{x}} u_0 := \rho_t^{\tilde{x}}, \quad \forall t \geq 0. \quad (19)$$

To show that the Eq. (19) is exponentially stable we will use Harris' theorem in the general case or Doeblin's theorem if Assumption 4 is granted. In both cases, the main technical difficulty is to verify the Doeblin minoration condition (Lemma 3) as the jumps of the process described by Eq. (12) are degenerate and the model is two-dimensional.

**Lemma 3** (Doeblin minoration condition). *Grant Assumptions 1 and 3. Fix any  $x \in \mathbb{R}$ . For all  $R > 0$ , there exists  $T > 0$  and a positive non-zero measure  $\nu$  such that*

$$\forall u_0 \in L_+^1, \quad S_T^{\tilde{x}} u_0 \geq \nu \int_{\mathbb{R}_+ \times ]0, R]} u_0 \, dadm. \quad (20)$$

*Proof.* We proceed in two steps. First (**Step 1**), we choose a time  $T > 0$  and a rectangle  $[0, \bar{a}] \times [\underline{m}, \bar{m}] \subset \mathbb{R}_+ \times \mathbb{R}_+^*$  (with nonzero Lebesgue measure) and show that the density  $S_T^{\tilde{x}} u_0 \in L^1$  has a lower bound on  $[0, \bar{a}] \times [\underline{m}, \bar{m}]$  which depends on a Lebesgue integral in  $\mathbb{R}_+^2$  involving  $u_0$ . Then (**Step 2**), we perform a change of variable to express this lower bound in terms of  $\int_{\mathbb{R}_+ \times ]0, R]} u_0 \, dadm$ . The proof only relies on the expression of  $S_t^{\tilde{x}} u_0$  given by the method of characteristics (see Proposition 1) and this allows treating a typically probabilistic question – the Doeblin minoration condition – from a transport point of view. This is possible because  $S_t^{\tilde{x}}$  is the stochastic (mass-conservative) semigroup of a piecewise deterministic Markov process.

The constants  $\Delta_{\text{abs}}$ ,  $\sigma$  and  $C_\gamma$  are taken from Assumption 3.

**Step 1:**

Fix  $R > 0$ . Since  $\gamma(e^{-\lambda \Delta_{\text{abs}}} \gamma(0)) > \gamma(0)$  and  $\gamma(e^{-\lambda t} \gamma(e^{-\lambda \Delta_{\text{abs}}} R)) \rightarrow \gamma(0)$  as  $t \rightarrow \infty$ , there exists  $\bar{a} > 0$  and  $T > \bar{a} + \Delta_{\text{abs}}$  such that

$$\underline{m} =: \gamma(e^{-\lambda(T-\bar{a}-\Delta_{\text{abs}})} \gamma(e^{-\lambda \Delta_{\text{abs}}} R)) < e^{-\lambda \bar{a}} \gamma(e^{-\lambda \Delta_{\text{abs}}} \gamma(0)) =: \bar{m}. \quad (21)$$

Eq. (21) has the following heuristic interpretation: if we see  $S_t^{\tilde{x}}$  as the stochastic semigroup of the piecewise deterministic Markov process defined by the generator Eq. (12), for any initial point  $(a_0, m_0) \in \mathbb{R}_+ \times ]0, R]$  and any landing point  $(a, m) \in [0, \bar{a}] \times [\underline{m}, \bar{m}]$  at time  $T$ , there is a ‘possible’ trajectory going from  $(a_0, m_0)$  to  $(a, m)$ , with exactly two jumps (spikes). Since the trajectories of the process are determined by the jump times, we will exploit the fact that these ‘possible’ trajectories correspond to jump times with strictly positive probability density. Below, we take a transport point of view on this probabilistic argument.



For all  $(a, m) \in [0, \bar{a}] \times [\underline{m}, \bar{m}]$ ,

$$\begin{aligned}
(S_T^{\tilde{x}} u_0)(a, m) &\geq 1_{\{a < T\}} (S_{T-a}^{\tilde{x}} u_0)(0, e^{\lambda a} m) \exp \left( \lambda a - \int_{T-a}^T f(a - T + s, e^{\lambda(T-s)} m, \tilde{x}) ds \right) \\
&\geq 1_{\{a < T\}} e^{-\|f\|_\infty T} e^{\lambda a} (S_{T-a}^{\tilde{x}} u_0)(0, e^{\lambda a} m) \\
&\geq 1_{\{a < T\}} e^{-\|f\|_\infty T} \sigma e^{\lambda a} \left| (\gamma^{-1})'(e^{\lambda a} m) \right| \int_{\Delta_{\text{abs}}}^{\infty} (S_{T-a}^{\tilde{x}} u_0)(a', \gamma^{-1}(e^{\lambda a} m)) da' \\
&= 1_{\{a < T\}} e^{-\|f\|_\infty T} \sigma \left| \frac{d}{dm} \gamma^{-1}(e^{\lambda a} m) \right| \int_{\Delta_{\text{abs}}}^{\infty} \underbrace{(S_{T-a}^{\tilde{x}} u_0)(a', \gamma^{-1}(e^{\lambda a} m))}_{(\star)} da'.
\end{aligned}$$

Above, we went back in time to the last jump time  $T - a$ . Let us notice that  $\gamma^{-1}(e^{\lambda a} m) \geq \gamma^{-1}(e^{\lambda a} \underline{m}) > 0$ . We can therefore define

$$a_{a,m}^* := \frac{1}{\lambda} \left( \log \gamma(0) - \log \gamma^{-1}(e^{\lambda a} m) \right).$$

Note that  $a_{a,m}^*$  satisfies  $\gamma^{-1}(e^{\lambda a_{a,m}^*} \gamma^{-1}(e^{\lambda a} m)) = 0$ . In other words,  $a_{a,m}^*$  is the minimal time between the last and second last jumps for a trajectory landing at  $(a, m)$  at time  $T$ . We can easily verify that, by our choice of  $\{T, \bar{a}, \underline{m}, \bar{m}\}$ ,  $\Delta_{\text{abs}} \leq a_{a,m}^* < T - a - \Delta_{\text{abs}}$ . This guarantees that it is possible to make two jumps in  $[0, T]$  and land at  $(a, m)$  at time  $T$  while respecting the absolute refractoriness of the neuron (i.e. there needs to be a time interval  $\geq \Delta_{\text{abs}}$  between jumps). This allows us to go further back in time to the second last jump.

For all  $a' \in [a_{a,m}^*, T - a - \Delta_{\text{abs}}]$ ,

$$(\star) \geq 1_{\{a' < T-a\}} e^{-\|f\|_\infty T} \sigma \left| (\gamma^{-1})'(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)) \right| e^{\lambda a'} \int_{\Delta_{\text{abs}}}^{\infty} \underbrace{(S_{T-a-a'}^{\tilde{x}} u_0)(a'', \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)))}_{(\star\star)} da''.$$

Then, we can go further back to time 0 to get  $u_0$ :

$$(\star\star) \geq 1_{\{a'' \geq T-a-a'\}} e^{-\|f\|_\infty T} e^{\lambda(T-a-a')} u_0(a'' - (T - a - a'), e^{\lambda(T-a-a')} \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m))).$$

Putting all the lower bounds together, we get

$$\begin{aligned}
(S_T^{\tilde{x}} u_0)(a, m) &\geq 1_{\{a < T\}} e^{-3\|f\|_\infty T} \sigma^2 \\
&\int_{a_{a,m}^*}^{T-a-\Delta_{\text{abs}}} \int_{T-a-a'}^{\infty} \left| \frac{d}{dm} e^{\lambda(T-a-a')} \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)) \right| \\
&\quad u_0(a'' - (T - a - a'), e^{\lambda(T-a-a')} \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m))) da'' da'.
\end{aligned}$$

Since  $\gamma' \leq 1$  (Assumption 3),

$$\left| \frac{d}{dm} e^{\lambda(T-a-a')} \gamma^{-1}(e^{\lambda a'} \gamma^{-1}(e^{\lambda a} m)) \right| \geq e^{\lambda T}.$$

Thus,

$$(S_T^{\tilde{x}}u_0)(a, m) \geq 1_{\{a < T\}} e^{(\lambda - 3\|f\|_\infty)T} \sigma^2 \int_{a_{a,m}^*}^{T-a-\Delta_{\text{abs}}} \int_0^\infty u_0(a_0, e^{\lambda(T-a-a')}\gamma^{-1}(e^{\lambda a'}\gamma^{-1}(e^{\lambda a}m))) da_0 da'. \quad (22)$$

We have obtained that on  $[0, \bar{a}] \times [\underline{m}, \bar{m}]$ , the density  $(S_T^{\tilde{x}}u_0)$  is lower bounded by a constant depending on a Lebesgue integral on  $\mathbb{R}_+^2$  involving  $u_0$ .

**Step 2:**

Now, we want express the lower bound Eq. (22) in terms of  $\int_{\mathbb{R}_+ \times ]0, R]} u_0 dadm$  by a change of variable. Let us define the function  $\psi_{a,m}^T$ :

$$\psi_{a,m}^T : [a_{a,m}^*, T - a - \Delta_{\text{abs}}] \rightarrow \mathbb{R}_+, \quad a' \mapsto e^{\lambda(T-a-a')}\gamma^{-1}(e^{\lambda a'}\gamma^{-1}(e^{\lambda a}m)).$$

We verify that  $(\psi_{a,m}^T)' > 0$ :

For all  $a' \in [a_{a,m}^*, T - a]$ ,

$$(\psi_{a,m}^T)'(a') = \lambda e^{\lambda(T-a-a')} \left\{ (\gamma^{-1})'(e^{\lambda a'}\gamma^{-1}(e^{\lambda a}m)) e^{\lambda a'}\gamma^{-1}(e^{\lambda a}m) - \gamma^{-1}(e^{\lambda a'}\gamma^{-1}(e^{\lambda a}m)) \right\}. \quad (23)$$

As  $\Gamma > 0$  and  $\gamma' \leq 1$  (Assumption 3), we have

$$\begin{aligned} (\psi_{a,m}^T)'(a') &> \lambda e^{\lambda(T-a-a')} \left\{ (\gamma^{-1})'(e^{\lambda a'}\gamma^{-1}(e^{\lambda a}m)) e^{\lambda a'}\gamma^{-1}(e^{\lambda a}m) - e^{\lambda a'}\gamma^{-1}(e^{\lambda a}m) \right\} \\ &= \lambda e^{\lambda(T-a)}\gamma^{-1}(e^{\lambda a}m) \left\{ \underbrace{(\gamma^{-1})'(e^{\lambda a'}\gamma^{-1}(e^{\lambda a}m))}_{\geq 1} - 1 \right\} \geq 0. \end{aligned}$$

Therefore,  $\psi_{a,m}^T$  is a strictly increasing  $C^1$ -diffeomorphism from  $[a_{a,m}^*, T - a - \Delta_{\text{abs}}]$  to  $[\psi_{a,m}^T(a_{a,m}^*), \psi_{a,m}^T(T - a - \Delta_{\text{abs}})]$ . We can now rewrite Eq. (22):

$$\begin{aligned} (S_T^{\tilde{x}}u_0)(a, m) &\geq e^{(\lambda - 3\|f\|_\infty)T} \sigma^2 \int_{a_{a,m}^*}^{T-a-\Delta_{\text{abs}}} \int_0^\infty u_0(a_0, \psi_{a,m}^T(a')) da_0 da' \\ &= e^{(\lambda - 3\|f\|_\infty)T} \sigma^2 \int_{\psi_{a,m}^T(a_{a,m}^*)}^{\psi_{a,m}^T(T-a-\Delta_{\text{abs}})} \int_0^\infty u_0(a_0, m_0) \left| ((\psi_{a,m}^T)^{-1})'(m_0) \right| da_0 dm_0. \end{aligned}$$

Going back to Eq. (23), and using the fact that there exists  $C_\gamma$  such that  $C_\gamma \leq \gamma' \leq 1$  (Assumption 3), we have, for all  $a' \in [a_{a,m}^*, T - a - \Delta_{\text{abs}}]$ ,

$$(\psi_{a,m}^T)'(a') \leq \lambda e^{\lambda(T-a-a')} C_\gamma^{-1} e^{\lambda a'}\gamma^{-1}(e^{\lambda a}m) \leq \lambda e^{\lambda T} C_\gamma^{-1} \bar{m}.$$

Hence,

$$(S_T^{\tilde{x}}u_0)(a, m) \geq \frac{e^{-3\|f\|_\infty T} \sigma^2 C_\gamma}{\lambda \bar{m}} \int_{\psi_{a,m}^T(a_{a,m}^*)}^{\psi_{a,m}^T(T-a-\Delta_{\text{abs}})} \int_0^\infty u_0(a_0, m_0) da_0 dm_0.$$

In addition, by our choice of  $\{T, \bar{a}, \underline{m}, \bar{m}\}$ , we have

$$\begin{aligned} \psi_{a,m}^T(a_{a,m}^*) &= 0, \\ \psi_{a,m}^T(T - a - \Delta_{\text{abs}}) &= e^{\lambda \Delta_{\text{abs}}}\gamma^{-1}(e^{\lambda(T-a-\Delta_{\text{abs}})}\gamma^{-1}(e^{\lambda a}m)) > R. \end{aligned}$$

Therefore,

$$(S_T^{\tilde{x}}u_0)(a, m) \geq \frac{e^{-3\|f\|_\infty T} \sigma^2 C_\gamma}{\lambda \bar{m}} \int_0^R \int_0^\infty u_0(a_0, m_0) da_0 dm_0.$$

Since we have supposed that  $(a, m) \in [0, \bar{a}] \times [\underline{m}, \bar{m}]$ , this concludes the proof.  $\square$

With the Lyapunov condition (16) and the Doeblin minoration condition (20), we can apply a version of Harris' theorem:

**Theorem 4** (Harris). *Grant Assumptions 1 and 3. For all  $\tilde{x} \in \mathbb{R}$ , there exists a unique  $\rho_\infty^{\tilde{x}} \in L_+^1(w)$  with  $\|\rho_\infty^{\tilde{x}}\|_{L^1} = 1$  such that  $S_t^{\tilde{x}} \rho_\infty^{\tilde{x}} = \rho_\infty^{\tilde{x}}$ , for all  $t \geq 0$ , and there exists  $K \geq 1$  and  $\mathfrak{a} > 0$  such that for all initial data  $u_0 \in L_+^1(w)$  with  $\|u_0\|_{L^1} = 1$ ,*

$$\left\| S_t^{\tilde{x}} u_0 - \rho_\infty^{\tilde{x}} \right\|_{L^1(w)} \leq K e^{-\mathfrak{a}t} \left\| u_0 - \rho_\infty^{\tilde{x}} \right\|_{L^1(w)}, \quad \forall t \geq 0. \quad (24)$$

Furthermore, by Lemma 2, we have that  $\left\| \rho_\infty^{\tilde{x}} \right\|_{L^1(w)} \leq \frac{b}{\alpha}$ , where the constants  $\alpha$  and  $b$  are taken from the Lyapunov condition (16).

*Proof.* This is a classic result which proof can be found in the work of Hairer and Mattingly [19].  $\square$

If, in addition, Assumption 4 holds, we can simply apply Doeblin's theorem:

**Theorem 5** (Doeblin). *Grant Assumptions 1, 3 and 4. For all  $\tilde{x} \in \mathbb{R}$ , there exists a unique  $\rho_\infty^{\tilde{x}} \in L_+^1$  with  $\|\rho_\infty^{\tilde{x}}\|_{L^1} = 1$  such that  $S_t^{\tilde{x}} \rho_\infty^{\tilde{x}} = \rho_\infty^{\tilde{x}}$ , for all  $t \geq 0$ , and there exists  $K \geq 1$  and  $\mathfrak{a} > 0$  such that for all initial data  $u_0 \in L_+^1$  with  $\|u_0\|_{L^1} = 1$ ,*

$$\left\| S_t^{\tilde{x}} u_0 - \rho_\infty^{\tilde{x}} \right\|_{L^1} \leq K e^{-\mathfrak{a}t} \left\| u_0 - \rho_\infty^{\tilde{x}} \right\|_{L^1}, \quad \forall t \geq 0. \quad (25)$$

*Proof.* See, for example, Theorem 2.3 in [3].  $\square$

We say that  $\rho_\infty^{\tilde{x}}$  is the *invariant probability measure* of the semigroup  $(S_t^{\tilde{x}})_{t \in \mathbb{R}_+}$ . Note that both theorems imply the next corollary.

**Corollary 1.** *Grant Assumptions 1 and 3. For all  $\tilde{x} \in \mathbb{R}$ , there exists a unique  $\rho_\infty^{\tilde{x}} \in L_+^1(w)$  with  $\|\rho_\infty^{\tilde{x}}\|_{L^1} = 1$  solving*

$$\partial_a \rho_\infty^{\tilde{x}}(a, m) - \lambda \partial_m (m \rho_\infty^{\tilde{x}}(a, m)) = -f(a, m, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m), \quad (26a)$$

$$\rho_\infty^{\tilde{x}}(0, m) = 1_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \tilde{x}) \rho_\infty^{\tilde{x}}(a, \gamma^{-1}(m)) da, \quad (26b)$$

in the weak sense. Furthermore, we have that  $\rho_\infty^{\tilde{x}} \in \mathcal{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \cap L^\infty(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*))$ .

## 4 Stationary solutions for arbitrary connectivity strength

In this section, we study the stationary solutions to (1), namely the solution to

$$\partial_a \rho_\infty(a, m) - \lambda \partial_m (m \rho_\infty(a, m)) = -f(a, m, \varepsilon x_\infty) \rho_\infty(a, m), \quad (27a)$$

$$\rho_\infty(0, m) = 1_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \varepsilon x_\infty) \rho_\infty(a, \gamma^{-1}(m)) da, \quad (27b)$$

$$x_\infty = \int_0^\infty \int_0^\infty \bar{h}(a, m) f(a, m, \varepsilon x_\infty) \rho_\infty(a, m) da dm. \quad (27c)$$

**Definition.**  $(\rho_\infty, x_\infty) \in L_+^1(w) \cap \mathcal{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \cap L^\infty(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \times \mathbb{R}_+$  is a *stationary solution* to (1) if  $\|\rho_\infty\|_{L^1} = 1$  and if it solves Eq. (27) in the weak sense.

#### 4.1 Existence and uniqueness using the Doeblin-Harris method.

We present two important Lipschitz continuity results, which will allow us to prove the existence (and the uniqueness when  $\varepsilon$  is small) of stationary solutions. The following lemma plays the same role as Theorem 4.5 in [3]:

**Lemma 4** (Lipschitz continuity at finite  $T$ ). *Grant Assumptions 1 and 3. For all initial data  $u_0 \in L^1_+(w)$  and for all  $T > 0$ , there exists a constant  $C_{T\|u_0\|_{L^1(w)}} > 0$  such that*

$$\forall \widetilde{x}_1, \widetilde{x}_2 \in \mathbb{R}, \quad \left\| S_T^{\widetilde{x}_1} u_0 - S_T^{\widetilde{x}_2} u_0 \right\|_{L^1(w)} \leq C_{T\|u_0\|_{L^1(w)}} |\widetilde{x}_1 - \widetilde{x}_2|. \quad (28)$$

*Proof.* For all  $t > 0$ ,

$$\begin{aligned} & \left\| S_t^{\widetilde{x}_1} u_0 - S_t^{\widetilde{x}_2} u_0 \right\|_{L^1(w)} \\ &= \int_0^\infty \int_t^\infty \left| u_0(a-t, e^{\lambda t} m) \exp \left( \lambda t - \int_0^t f(a-t+s, e^{\lambda(t-s)} m, \widetilde{x}_1) ds \right) \right. \\ & \quad \left. - u_0(a-t, e^{\lambda t} m) \exp \left( \lambda t - \int_0^t f(a-t+s, e^{\lambda(t-s)} m, \widetilde{x}_2) ds \right) \right| w(a, m) dadm \\ & \quad + \int_0^\infty \int_0^t \left| \rho_{t-a}^{\widetilde{x}_1}(0, e^{\lambda a} m) \exp \left( \lambda a - \int_{t-a}^t f(a-t+s, e^{\lambda(t-s)} m, \widetilde{x}_1) ds \right) \right. \\ & \quad \left. - \rho_{t-a}^{\widetilde{x}_2}(0, e^{\lambda a} m) \exp \left( \lambda a - \int_{t-a}^t f(a-t+s, e^{\lambda(t-s)} m, \widetilde{x}_2) ds \right) \right| w(a, m) dadm \\ & =: Q_1 + Q_2. \end{aligned}$$

$$\begin{aligned} Q_1 &= \int_0^\infty \int_0^\infty u_0(a, m) \left| \exp \left( - \int_0^t f(a+s, e^{-\lambda s} m, \widetilde{x}_1) ds \right) - \exp \left( - \int_0^t f(a+s, e^{-\lambda s} m, \widetilde{x}_2) ds \right) \right| \\ & \quad w(a+t, e^{-\lambda t} m) dadm \\ & \leq \int_0^\infty \int_0^\infty u_0(a, m) \left( \int_0^t \left| f(a+s, e^{-\lambda s} m, \widetilde{x}_1) - f(a+s, e^{-\lambda s} m, \widetilde{x}_2) \right| ds \right) w(a+t, e^{-\lambda t} m) dadm \\ & \leq t L_f |\widetilde{x}_1 - \widetilde{x}_2| \int_0^\infty \int_0^\infty u_0(a, m) w(a+t, e^{-\lambda t} m) dadm \\ & \leq t L_f \|u_0\|_{L^1(w)} |\widetilde{x}_1 - \widetilde{x}_2|, \end{aligned}$$

where in the last inequality we used

$$w(a+t, e^{-\lambda t} m) \leq w(a, m), \quad \forall a \geq 0, m \geq 0. \quad (29)$$

$$\begin{aligned} Q_2 &= \int_0^\infty \int_0^t \left| \rho_{t-a}^{\widetilde{x}_1}(0, m) \exp \left( - \int_{t-a}^t f(a-t+s, e^{\lambda(t-s-a)} m, \widetilde{x}_1) ds \right) \right. \\ & \quad \left. - \rho_{t-a}^{\widetilde{x}_2}(0, m) \exp \left( - \int_{t-a}^t f(a-t+s, e^{\lambda(t-s-a)} m, \widetilde{x}_2) ds \right) \right| w(a, e^{-\lambda a} m) dadm. \end{aligned}$$

By changes of variables,

$$\begin{aligned}
Q_2 &= \int_0^\infty \int_0^t \left| \rho_s^{\widetilde{x}_1}(0, m) \exp\left(-\int_0^{t-s} f(u, e^{-\lambda u} m, \widetilde{x}_1) du\right) \right. \\
&\quad \left. - \rho_s^{\widetilde{x}_2}(0, m) \exp\left(-\int_0^{t-s} f(u, e^{-\lambda u} m, \widetilde{x}_2) du\right) \right| w(t-s, e^{-\lambda(t-s)} m) ds dm \\
&\leq \int_0^\infty \int_0^t \rho_s^{\widetilde{x}_1}(0, m) \left| \exp\left(-\int_0^{t-s} f(u, e^{-\lambda u} m, \widetilde{x}_1) du\right) - \exp\left(-\int_0^{t-s} f(u, e^{-\lambda u} m, \widetilde{x}_2) du\right) \right| \\
&\quad w(t-s, e^{-\lambda(t-s)} m) ds dm \\
&\quad + \int_0^\infty \int_0^t \left| \rho_s^{\widetilde{x}_1}(0, m) - \rho_s^{\widetilde{x}_2}(0, m) \right| w(t-s, e^{-\lambda(t-s)} m) ds dm \\
&=: Q_{2,1} + Q_{2,2}
\end{aligned}$$

$$\begin{aligned}
Q_{2,1} &\leq t \|f\|_\infty L_f |\widetilde{x}_1 - \widetilde{x}_2| \int_0^t \int_0^\infty \int_0^\infty \left| (\gamma^{-1})'(m) \right| \rho_s^{\widetilde{x}_1}(a, \gamma^{-1}(m)) w(t, m) da dm ds \\
&\leq t \|f\|_\infty L_f |\widetilde{x}_1 - \widetilde{x}_2| \int_0^t \int_0^\infty \int_0^\infty \rho_s^{\widetilde{x}_1}(a, m) w(t, m + \|\Gamma\|_\infty) da dm ds \\
&\leq t(1 + \|\Gamma\|_\infty) \|f\|_\infty L_f |\widetilde{x}_1 - \widetilde{x}_2| \int_0^t \left\| \rho_s^{\widetilde{x}_1} \right\|_{L^1(w)} ds,
\end{aligned}$$

where in the last inequality we used

$$w(t, m + \|\Gamma\|_\infty) = 1 + m + \|\Gamma\|_\infty \leq (1 + \|\Gamma\|_\infty) w(a, m), \quad \forall a \geq 0, m \geq 0. \quad (30)$$

By Lemma 2,

$$Q_{2,1} \leq t^2 (1 + \|\Gamma\|_\infty) \|f\|_\infty L_f \left( \|u_0\|_{L^1(w)} + \frac{b}{\alpha} \right) |\widetilde{x}_1 - \widetilde{x}_2|.$$

$$\begin{aligned}
Q_{2,2} &\leq \|f\|_\infty \int_0^t \int_0^\infty \int_0^\infty \left| (\gamma^{-1})'(m) \right| \left| \rho_s^{\widetilde{x}_1}(a, \gamma^{-1}(m)) - \rho_s^{\widetilde{x}_2}(a, \gamma^{-1}(m)) \right| w(t, m) da dm ds \\
&\leq \|f\|_\infty \int_0^t \int_0^\infty \int_0^\infty \left| \rho_s^{\widetilde{x}_1}(a, m) - \rho_s^{\widetilde{x}_2}(a, m) \right| w(t, m + \|\Gamma\|_\infty) da dm ds \\
&\leq (1 + \|\Gamma\|_\infty) \|f\|_\infty \int_0^t \left\| S_s^{\widetilde{x}_1} u_0 - S_s^{\widetilde{x}_2} u_0 \right\|_{L^1(w)} ds,
\end{aligned}$$

where again, in the last inequality, we used Eq. (30). Fix  $T > 0$ . Gathering the bounds for  $Q_1$ ,  $Q_{2,1}$  and  $Q_{2,2}$  we see that there exists constants  $C > 0$  and  $C'_{T, \|u_0\|_{L^1(w)}} > 0$  such that, for all  $t \in [0, T]$ ,

$$\left\| S_t^{\widetilde{x}_1} u_0 - S_t^{\widetilde{x}_2} u_0 \right\|_{L^1(w)} \leq C \int_0^t \left\| S_s^{\widetilde{x}_1} u_0 - S_s^{\widetilde{x}_2} u_0 \right\|_{L^1(w)} ds + t C'_{T, \|u_0\|_{L^1(w)}} |\widetilde{x}_1 - \widetilde{x}_2|.$$

By Grönwall's lemma, for all  $t \in [0, T]$ ,

$$\left\| S_t^{\widetilde{x}_1} u_0 - S_t^{\widetilde{x}_2} u_0 \right\|_{L^1(w)} \leq \frac{C'_{T, \|u_0\|_{L^1(w)}} |\widetilde{x}_1 - \widetilde{x}_2|}{C} (\exp(Ct) - 1). \quad (31)$$

Since Eq. (31) holds for all  $t \in [0, T]$ , this achieves the proof.  $\square$

**Lemma 5** (Lipschitz continuity at  $T = \infty$ ). *Grant Assumptions 1 and 3. Writing  $\rho_{\infty}^{\tilde{x}} \in L_+^1(w)$  the invariant probability measure given by Theorem 4 for any  $\tilde{x} \in \mathbb{R}$ , the function*

$$\Upsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \Upsilon(x) = \int_0^{\infty} \int_0^{\infty} \bar{h}(a, m) f(a, m, \varepsilon x) \rho_{\infty}^{\varepsilon x}(a, m) da dm$$

is Lipschitz and there exists  $C > 0$  such that

$$\forall x_1, x_2 \in \mathbb{R}_+, \quad |\Upsilon(x_1) - \Upsilon(x_2)| \leq |\varepsilon| C |x_1 - x_2|.$$

*Proof.* Since  $f$  is Lipschitz in  $x$ , we have, for any  $x_1, x_2 \in \mathbb{R}_+$ ,

$$\begin{aligned} |\Upsilon(x_1) - \Upsilon(x_2)| &\leq \|\bar{h}\|_{\infty} \left\{ \|f\|_{\infty} \|\rho_{\infty}^{\varepsilon x_1} - \rho_{\infty}^{\varepsilon x_2}\|_{L^1} + L_f |\varepsilon| |x_1 - x_2| \right\} \\ &\leq \|\bar{h}\|_{\infty} \left\{ \|f\|_{\infty} \|\rho_{\infty}^{\varepsilon x_1} - \rho_{\infty}^{\varepsilon x_2}\|_{L^1(w)} + L_f |\varepsilon| |x_1 - x_2| \right\}, \end{aligned}$$

from where we only need to bound the first term on the right hand side. We can use Theorem 4 and Lemma 4: for any  $T \in \mathbb{R}_+$ ,

$$\begin{aligned} \|\rho_{\infty}^{\varepsilon x_1} - \rho_{\infty}^{\varepsilon x_2}\|_{L^1(w)} &= \|S_T^{\varepsilon x_1} \rho_{\infty}^{\varepsilon x_1} - S_T^{\varepsilon x_1} \rho_{\infty}^{\varepsilon x_2} + S_T^{\varepsilon x_1} \rho_{\infty}^{\varepsilon x_2} - S_T^{\varepsilon x_2} \rho_{\infty}^{\varepsilon x_2}\|_{L^1(w)} \\ &\leq K e^{-\mathfrak{a}T} \|\rho_{\infty}^{\varepsilon x_1} - \rho_{\infty}^{\varepsilon x_2}\|_{L^1(w)} + C_T |\varepsilon| |x_1 - x_2|, \end{aligned}$$

where  $K$  and  $\mathfrak{a}$  are the exponential stability constants of Theorem 4. Choosing  $T$  such that  $K e^{-\mathfrak{a}T} = 1/2$ , we get

$$\|\rho_{\infty}^{\varepsilon x_1} - \rho_{\infty}^{\varepsilon x_2}\|_{L^1(w)} \leq 2C_T |\varepsilon| |x_1 - x_2|.$$

Gathering the bounds concludes the proof.  $\square$

**Theorem 6** (Stationary solutions). *Grant Assumptions 1 and 3. We have*

(i) *There exists a stationary solution to (1).*

(ii) *There exists  $\varepsilon^* > 0$  such that for all  $\varepsilon \in ]-\varepsilon^*, +\varepsilon^*[$ , the stationary solution to (1) is unique.*

*Proof.* For all  $\tilde{x} \in \mathbb{R}$ , let us write  $\rho_{\infty}^{\tilde{x}} \in L_+^1(w)$  the unique invariant measure given by Theorem 4 and let us also take the function  $\Upsilon$  from Lemma 5. By Corollary 1,  $(\rho_{\infty}, x_{\infty}) \in L_+^1(w) \cap \mathcal{C}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \cap L^{\infty}(\mathbb{R}_+, L_+^1(\mathbb{R}_+^*)) \times \mathbb{R}_+$  is a weak solution to Eq. (27) if and only if  $\rho_{\infty} = \rho_{\infty}^{\varepsilon x_{\infty}}$  and  $x_{\infty}$  is a fixed-point of  $\Upsilon$ . Hence, the study of the existence and the uniqueness of stationary solutions is reduced to the study of the existence and the uniqueness of the fixed-point of  $\Upsilon$ .

Since for all  $x \in \mathbb{R}_+$ ,  $\|\rho_{\infty}^{\varepsilon x}\|_{L^1} = 1$ , we have that for all  $x \in \mathbb{R}_+$ ,  $\Upsilon(x) \leq \|\bar{h}\|_{\infty} \|f\|_{\infty}$ . Therefore, the set  $[0, \|\bar{h}\|_{\infty} \|f\|_{\infty}]$  (which is compact and convex) is stable by  $\Upsilon$ . Then, the continuity of  $\Upsilon$  guarantees the existence of a fixed-point, which proves (i).

To obtain (ii), we observe that the Lipschitz constant of  $\Upsilon$  is  $|\varepsilon|C$ : if we take  $|\varepsilon| < \varepsilon^* := C^{-1}$ ,  $\Upsilon$  is a contraction and we can apply Banach's fixed-point theorem to conclude.  $\square$

## 4.2 Alternative proof for the existence using Schauder's fixed-point theorem

We include here an alternative proof for the existence of a stationary solution, which is interesting for two reasons: on the one hand, it does not rely on the Harris-Doeblin method, and on the other hand, it provides some estimates on the stationary solutions.

For any  $(\tilde{u}, \tilde{x}) \in L^1_+(\gamma(0), +\infty) \times \mathbb{R}$ , consider the transport equation

$$\begin{aligned} \partial_a \varrho(a, m) - \lambda \partial_m (m \varrho(a, m)) &= -f(a, m, \tilde{x}) \varrho(a, m), \\ \varrho(0, m) &= \tilde{u}(m). \end{aligned}$$

It has a unique weak solution  $\rho_{\infty}^{\tilde{u}, \tilde{x}} \in \mathcal{C}(\mathbb{R}_+, L^1_+(\mathbb{R}^*)) \cap L^\infty(\mathbb{R}_+, L^1_+(\mathbb{R}^*))$  given by the method of characteristics: for all  $(a, m) \in \mathbb{R}_+ \times \mathbb{R}^*$ ,

$$\rho_{\infty}^{\tilde{u}, \tilde{x}}(a, m) = \tilde{u}(e^{\lambda a} m) \exp \left( \lambda a - \int_0^a f(s, e^{\lambda(a-s)} m, \tilde{x}) ds \right). \quad (32)$$

We can now define the operator  $\Phi := (\Phi_1, \Phi_2)$  on  $L^1_+(\gamma(0), +\infty) \times \mathbb{R}$  where, for all  $(\tilde{u}, \tilde{x}) \in L^1_+(\gamma(0), +\infty) \times \mathbb{R}$ ,

$$\Phi_1(\tilde{u}, \tilde{x})(m) := 1_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \tilde{x}) \rho_{\infty}^{\tilde{u}, \tilde{x}}(a, \gamma^{-1}(m)) da, \quad (33a)$$

$$\Phi_2(\tilde{u}, \tilde{x}) := \int_0^\infty \int_0^\infty \bar{h}(a, m) f(a, m, \tilde{x}) \rho_{\infty}^{\tilde{u}, \tilde{x}}(a, m) da dm. \quad (33b)$$

$(\rho_\infty, x_\infty)$  is a stationary solution if and only if it is a fixed-point of  $\Phi$ . Whence, we get the *a priori* estimates:

**Lemma 6.** *Grant Assumptions 1 and 3. There exists  $\theta \in ]0, 1[$  such that for all  $(\tilde{u}, \tilde{x}) \in L^1_+(\gamma(0), +\infty) \times \mathbb{R}$ ,*

$$(i) \quad \|\Phi_1(\tilde{u}, \tilde{x})\|_{L^1} = \|\tilde{u}\|_{L^1}.$$

$$(ii) \quad \text{For all } m \in \mathbb{R}^*, |\Phi_1(\tilde{u}, \tilde{x})(m)| \leq 1_{m > \gamma(0)} \frac{\|f\|_\infty}{\lambda \gamma^{-1}(m)} \|\tilde{u}\|_{L^1}.$$

(iii)

$$\int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm \leq \max \left( \int_0^\infty \tilde{u}(m) m dm, \frac{\gamma(0)}{1 - \theta} \|\tilde{u}\|_{L^1} \right).$$

$$(iv) \quad \text{For all } \beta \in ]0, \frac{\min(f)}{\lambda}[,$$

$$\int_{\gamma(0)}^\infty \frac{\Phi_1(\tilde{u}, \tilde{x})(m)}{\gamma^{-1}(m)^\beta} dm \leq \frac{\|f\|_\infty}{\lambda \gamma(0)^\beta} \left( \frac{\min(f)}{\lambda} - \beta \right) \|\tilde{u}\|_{L^1}.$$

$$(v) \quad \Phi_2(\tilde{u}, \tilde{x}) \leq \|\bar{h}\|_\infty \|\tilde{u}\|_{L^1}.$$

*Proof.* (i) By changes of variables on  $m$ ,

$$\begin{aligned} \|\Phi_1(\tilde{u}, \tilde{x})\|_{L^1} &= \int_0^\infty \int_0^\infty f(a, m, \tilde{x}) \tilde{u}(e^{\lambda a} m) \exp \left( \lambda a - \int_0^a f(s, e^{\lambda(a-s)} m, \tilde{x}) ds \right) da dm \\ &= \int_0^\infty \tilde{u}(m) \underbrace{\int_0^\infty f(a, e^{-\lambda a} m, \tilde{x}) \exp \left( - \int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds \right) da}_{=1 \quad (\text{by Assumption 3 (i)})} dm. \end{aligned}$$

(ii)

$$\begin{aligned}
|\Phi_1(\tilde{u}, \tilde{x})(m)| &\leq 1_{m>\gamma(0)} \|f\|_\infty \int_0^\infty \tilde{u}(e^{\lambda a} \gamma^{-1}(m)) \exp(\lambda a) da \\
&= 1_{m>\gamma(0)} \frac{\|f\|_\infty}{\lambda \gamma^{-1}(m)} \int_0^\infty \tilde{u}(e^{\lambda a} \gamma^{-1}(m)) \gamma^{-1}(m) \lambda \exp(\lambda a) da \\
&= 1_{m>\gamma(0)} \frac{\|f\|_\infty}{\lambda \gamma^{-1}(m)} \underbrace{\int_{\gamma^{-1}(m)}^\infty \tilde{u}(y) dy}_{\leq \|\tilde{u}\|_{L^1}},
\end{aligned}$$

where for the last equality we used the change of variable  $y = e^{\lambda a} \gamma^{-1}(m)$ .

(iii) Performing the same change of variable as for (i) and using the fact that  $\gamma(m) \leq \gamma(0) + m$ ,  $\forall m \in \mathbb{R}_+$  (since  $\gamma' \leq 1$ ), we have

$$\begin{aligned}
&\int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm \\
&= \int_0^\infty \tilde{u}(m) \int_0^\infty \gamma(e^{-\lambda a} m) f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da dm \\
&\leq \int_0^\infty \tilde{u}(m) m \underbrace{\int_0^\infty e^{-\lambda a} f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da}_{=: \vartheta(m)} dm + \gamma(0) \|\tilde{u}\|_{L^1}.
\end{aligned}$$

There exists  $\theta \in ]0, 1[$  such that for all  $m \in \mathbb{R}_+^*$ ,  $\vartheta(m) < 1$ :

Fix  $\epsilon > 0$ .

$$\begin{aligned}
\vartheta(m) &\leq \int_0^\epsilon f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da \\
&\quad + \int_\epsilon^\infty e^{-\lambda \epsilon} f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da \\
&= 1 - (1 - e^{-\lambda \epsilon}) \int_\epsilon^\infty f(a, e^{-\lambda a} m, \tilde{x}) \exp\left(-\int_0^a f(s, e^{-\lambda s} m, \tilde{x}) ds\right) da \\
&= 1 - (1 - e^{-\lambda \epsilon}) \exp\left(-\int_0^\epsilon f(s, e^{-\lambda s} m, \tilde{x}) ds\right) \\
&\leq 1 - (1 - e^{-\lambda \epsilon}) \exp(-\|f\|_\infty \epsilon) =: \theta < 1.
\end{aligned}$$

Whence,

$$\int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm \leq \theta \int_0^\infty \tilde{u}(m) m dm + \gamma(0) \|\tilde{u}\|_{L^1}.$$

To see that

$$\int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm \leq \max\left(\int_0^\infty \tilde{u}(m) m dm, \frac{\gamma(0)}{1-\theta} \|\tilde{u}\|_{L^1}\right),$$

we can distinguish three cases: if  $\int_0^\infty \tilde{u}(m) m dm = \infty$ , the inequality is trivial; if  $\frac{\gamma(0)}{1-\theta} \|\tilde{u}\|_{L^1} \leq \int_0^\infty \tilde{u}(m) m dm < +\infty$ , then

$$\begin{aligned}
\int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m) m dm &\leq \int_0^\infty \tilde{u}(m) m dm - (1-\theta) \int_0^\infty \tilde{u}(m) m dm + \gamma(0) \|\tilde{u}\|_{L^1} \\
&\leq \int_0^\infty \tilde{u}(m) m dm;
\end{aligned}$$



and finally if  $\int_0^\infty \tilde{u}(m)mdm < \frac{\gamma(0)}{1-\theta}\|\tilde{u}\|_{L^1}$ , then

$$\int_0^\infty \Phi_1(\tilde{u}, \tilde{x})(m)mdm \leq \theta \frac{\gamma(0)}{1-\theta}\|\tilde{u}\|_{L^1} + \gamma(0)\|\tilde{u}\|_{L^1} = \frac{\gamma(0)}{1-\theta}\|\tilde{u}\|_{L^1}.$$

(iv)

$$\begin{aligned} \int_{\gamma(0)}^\infty \frac{\Phi_1(\tilde{u}, \tilde{x})(m)}{\gamma^{-1}(m)^\beta} dm &= \int_0^\infty \int_0^\infty \frac{1}{m} f(a, m, \tilde{x}) \tilde{u}(e^{\lambda a} m) \exp\left(\lambda a - \int_0^a f(s, e^{\lambda(a-s)} m, \tilde{x})\right) dadm \\ &\leq \|f\|_\infty \int_0^\infty \int_0^\infty \frac{1}{m^\beta} \tilde{u}(e^{\lambda a} m) \exp(\lambda a - \min(f)a) dadm, \end{aligned}$$

and the change of variable  $y = e^{\lambda a} m$ ,

$$\begin{aligned} &= \|f\|_\infty \int_0^\infty \int_m^\infty \frac{1}{\lambda m^{1+\beta}} \tilde{u}(y) \exp\left(-\min(f) \frac{1}{\lambda} \ln\left(\frac{y}{m}\right)\right) dy dm \\ &= \frac{\|f\|_\infty}{\lambda} \int_0^\infty \int_m^\infty m^{\min(f)/\lambda - 1 - \beta} \tilde{u}(y) y^{-\min(f)/\lambda} dy dm, \end{aligned}$$

using Fubini's theorem and since  $\min(f)/\lambda - \beta > 0$ ,

$$\begin{aligned} &= \frac{\|f\|_\infty}{\lambda} \int_0^\infty \tilde{u}(y) y^{-\min(f)/\lambda} \underbrace{\int_0^y m^{\min(f)/\lambda - 1 - \beta} dm}_{= \frac{y^{\min(f)/\lambda - \beta}}{\min(f)/\lambda - \beta}} dy \\ &= \frac{\|f\|_\infty}{\lambda} \left(\frac{\min(f)}{\lambda} - \beta\right) \int_0^\infty \tilde{u}(y) y^{-\beta} dy. \end{aligned}$$

Finally, it is easy to check that  $\int_0^\infty \tilde{u}(y) y^{-\beta} dy \leq \gamma(0)^{-\beta} \|\tilde{u}\|_{L^1}$ .

(v) Use Eq. (33b) and see the proof of (i).  $\square$

By these estimates, we see that there exists  $\beta, C_1, C_2, C_3, C_4 > 0$  such that the set  $\mathcal{C} \times B \in L^1(\cdot, \gamma(0), +\infty) \times \mathbb{R}$ , where

$$\mathcal{C} := \left\{ u \in L^1_+(\cdot, \gamma(0), +\infty) \mid \|u\|_{L^1} \leq 1; u \leq \frac{C_1}{\gamma^{-1}(\cdot)} \text{ a.e.; } \int_0^\infty u(m)mdm \leq C_2; \int_{\gamma(0)}^\infty \frac{u(m)}{\gamma^{-1}(m)^\beta} dm \leq C_3 \right\}$$

and  $B := [-C_4, +C_4]$ , is stable by the operator  $\Phi$ .

In order to apply Schauder's fixed-point theorem, we will need

**Lemma 7.** *Grant Assumptions 1 and 3.  $\mathcal{C}$  is convex, closed and compact for the weak topology  $\sigma(L^1, L^\infty)$ .*

*Proof.* It is easy to verify that  $\mathcal{C}$  is convex. Since  $\mathcal{C}$  is convex, it suffices to show that it is strongly closed to show that it is weakly closed. Let  $u_n$  be a sequence of elements of  $\mathcal{C}$  which converge strongly to  $u \in L^1(\cdot, \gamma(0), +\infty)$ . By the strong convergence,  $\|u\|_{L^1} \leq 1$ . We can extract

a subsequence  $u_{n_k}$  such that  $u_{n_k}$  converges to  $u$  a.e. Taking the pointwise limit, we have that  $u \leq \frac{C_1}{\gamma^{-1}(\cdot)}$  a.e. Furthermore, by Fatou's lemma,

$$\int_{\gamma(0)}^{\infty} u(m)mdm \leq \liminf_{k \rightarrow +\infty} \int_{\gamma(0)}^{\infty} u_{n_k}(m)mdm \leq C_2$$

and

$$\int_{\gamma(0)}^{\infty} \frac{u(m)}{\gamma^{-1}(m)^\beta} dm \leq \liminf_{k \rightarrow +\infty} \int_{\gamma(0)}^{\infty} \frac{u_{n_k}(m)}{\gamma^{-1}(m)^\beta} dm \leq C_3.$$

Hence,  $\mathcal{C}$  is strongly closed.

To show that  $\mathcal{C}$  is weakly compact, we will show that

- a.  $\sup_{u \in \mathcal{C}} \|u\|_{L^1} < \infty$ ,
- b.  $\forall \epsilon > 0, \exists R > 0$  such that  $\int_R^\infty u(m)dm < \epsilon$  for all  $u \in \mathcal{C}$ ,
- c.  $\mathcal{C}$  is equi-integrable, i.e.  $\forall \epsilon > 0, \exists \delta > 0$  such that for all Borel set  $A \subset \mathbb{R}_+$  with  $|A| \leq \delta$  and for all  $u \in \mathcal{C}$ ,  $\int_A u(m)dm \leq \epsilon$ ,

and use Dunford-Pettis theorem. (a.) is clearly verified. (b.) is also verified since for all  $R > 0$ ,  $\int_R^\infty u(m)dm \leq \frac{1}{R} \int_0^\infty u(m)mdm \leq \frac{C_2}{R}$ . To show (c.), let us first observe that for all  $\delta_1 > 0$ ,

$$\int_{\gamma(0)}^{\gamma(0)+\delta_1} u(m)dm \leq \gamma^{-1}(\gamma(0) + \delta_1)^\beta \int_{\gamma(0)}^{\infty} \frac{u(m)}{\gamma^{-1}(m)^\beta} dm \leq \gamma^{-1}(\gamma(0) + \delta_1)^\beta C_3.$$

For any  $\epsilon > 0$ , let us choose  $\delta_1 > 0$  such that  $\gamma^{-1}(\gamma(0) + \delta_1)^\beta C_3 \leq \frac{\epsilon}{2}$ . Then, for all Borel set  $A \subset \mathbb{R}_+$  with  $|A| \leq \delta$ ,

$$\int_A u(m)dm \leq \int_{\gamma(0)}^{\gamma(0)+\delta_1} u(m)dm + \int_{A \setminus [0, \gamma(0)+\delta_1]} u(m)dm \leq \frac{\epsilon}{2} + \delta \frac{C_1}{\gamma^{-1}(\gamma(0) + \delta_1)}.$$

Hence, we can choose  $\delta = \min\left(\delta_1, \frac{\epsilon \gamma^{-1}(\gamma(0)+\delta_1)}{2C_1}\right)$  and (c.) is verified. By Dunford-Pettis theorem,  $\mathcal{C}$  is weakly relatively compact. Finally, since  $\mathcal{C}$  is weakly closed,  $\mathcal{C}$  is weakly compact.  $\square$

We can now give an alternative proof of the existence of stationary solutions to (1) for arbitrary connectivity strength  $\varepsilon$ :

*Proof of Theorem 2 (i).* We verify that the operator  $\Phi$  is weakly continuous: For any sequence  $(u_n, x_n) \rightarrow (u, x)$  in  $\mathcal{C} \times \mathbb{R}$  and for any  $\varphi \in L^\infty(\mathbb{R}_+)$ ,

$$\left| \int (\Phi_1(u_n, x_n) - \Phi_1(u, x))\varphi(m)dm \right| \leq Q_1^n + Q_2^n + Q_3^n,$$

where

$$\begin{aligned} Q_1^n &:= \left| \int_0^\infty \int_0^\infty (u_n(e^{\lambda a} m) - u(e^{\lambda a} m))\varphi(\gamma(m))e^{\lambda a} f(a, m, x) e^{-\int_0^a f(\tau, e^{\lambda(a-\tau)} m, x) d\tau} da dm \right|, \\ Q_2^n &:= \|\varphi\|_\infty \int_0^\infty \int_0^\infty u_n(e^{\lambda a} m) e^{\lambda a} |f(a, m, x) - f(a, m, x_n)| e^{-\int_0^a f(\tau, e^{\lambda(a-\tau)} m, x) d\tau} da dm, \\ Q_3^n &:= \|\varphi\|_\infty \int_0^\infty \int_0^\infty u_n(e^{\lambda a} m) e^{\lambda a} f(a, m, x_n) \left| e^{-\int_0^a f(\tau, e^{\lambda(a-\tau)} m, x) d\tau} - e^{-\int_0^a f(\tau, e^{\lambda(a-\tau)} m, x_n) d\tau} \right| da dm. \end{aligned}$$

Making the change of variable,  $ydy = e^{\lambda a}mdm$  in  $Q_1$  we get

$$Q_1^n = \left| \int_0^\infty (u_n(y) - u(y)) \int_0^\infty \varphi(\gamma(ye^{-\lambda a})) f(a, ye^{-\lambda a}, x) e^{-\int_0^a f(\tau, e^{-\lambda\tau}y, x) d\tau} da dm \right|.$$

Since  $u_n$  converges to  $u$  in  $\sigma(L^1, L^\infty)$  and

$$\begin{aligned} \int_0^\infty \varphi(\gamma(ye^{-\lambda a})) f(a, ye^{-\lambda a}, x) e^{-\int_0^a f(\tau, e^{-\lambda\tau}y, x) d\tau} da \\ \leq \|\varphi\|_\infty \int_0^\infty f(a, ye^{-\lambda a}, x) e^{-\int_0^a f(\tau, e^{-\lambda\tau}y, x) d\tau} da = \|\varphi\|_\infty, \end{aligned}$$

$Q_1^n$  converges to 0. On the other hand, since  $f$  is bounded and Lipschitz,  $Q_2^n, Q_3^n \leq \|u_n\|_{L^1} C|x_n - x| \leq C|x_n - x|$ . Whence,  $\Phi_1$  is a continuous operator with respect to the weak topology  $\sigma(L^1, L^\infty)$ .

The continuity of  $\Phi_2$  is shown analogously, taking  $\varphi = h$  ( $h$  is a bounded).

Since  $\mathcal{C}$  is stable by  $\Phi$ , convex and weakly compact (Lemma 7), we can apply Schauder's fixed-point theorem to obtain the existence of a fixed-point, which gives the existence of a stationary solution.  $\square$

**Corollary 2.** *Grant Assumptions 1 and 3. If  $f$  is of class  $\mathcal{C}^k$ , then  $u(m)$  is a function of class  $\mathcal{C}^k$  for all  $m > \gamma(0)$ . Consequently, the stationary solutions of (1) are of class  $\mathcal{C}^k$ .*

*Proof.* If  $(u, \tilde{x})$  is a fixed-point of  $\Phi$ , then

$$\begin{aligned} u(m) = 1_{m > \gamma(0)} \left| (\gamma^{-1})'(m) \right| \int_0^\infty f(a, \gamma^{-1}(m), \tilde{x}) u(e^{\lambda a} \gamma^{-1}(m)) \\ \exp \left( \lambda a - \int_0^a f(s, e^{\lambda(a-s)} \gamma^{-1}(m), \tilde{x}) ds \right) da. \quad (34) \end{aligned}$$

Making the change of variable  $y = e^{\lambda a} \gamma^{-1}(m)$  in  $a$ , as in the estimate (ii) of Lemma 6, we obtain

$$u(m) = 1_{m > \gamma(0)} \frac{|(\gamma^{-1})'(m)|}{\lambda \gamma^{-1}(m)} \int_{\gamma^{-1}(m)}^\infty f(g(y, m), y, \tilde{x}) u(y) \exp \left( - \int_0^{g(y, m)} f(s, e^s y, \tilde{x}) ds \right) dy, \quad (35)$$

where  $g(y, m) = \ln \frac{y}{\lambda(\gamma^{-1}(m))}$ . We conclude with a bootstrap argument: if  $u$  is  $L^1$ , then the right hand side of Eq. (35) is a continuous function of  $m$ , meaning that  $u$  is continuous. But if  $u$  is continuous, then the right hand side is of class  $\mathcal{C}^1$ , etc.  $\square$

**Corollary 3.** *Grant Assumptions 1 and 3. There exist a constant  $C > 0$ , such that the stationary solution  $\rho_\infty$  satisfies,*

$$\rho_\infty(a, m) \leq \frac{C e^{-\sigma(a - \Delta_{abs})}}{m}. \quad (36)$$

*Proof.* From the previous theorem it follows that there is  $C$  such that  $u(m) \leq C/m$ , which together with (32) implies

$$\rho_\infty(a, m) \leq \overline{C} \frac{e^{-\int_0^a f(s, e^{\lambda(a-s)} \gamma^{-1}(m), \tilde{x}) ds}}{m}. \quad (37)$$

The estimate follows from Assumption 3 (i).  $\square$

### 4.3 Formula in the case of short-term synaptic depression

In general, there is no explicit formula for the invariant probability measure solving Eq. (26). However, in the case of short-term synaptic depression Eq. (3), we can derive an explicit expression for the total postsynaptic potential

$$X(\tilde{x}) := \int_0^\infty \hat{h}(t) \int_0^1 \int_0^\infty (1-m)f(a, \tilde{x})\rho_\infty^{\tilde{x}}(a, m)dadm dt, \quad (38)$$

for any  $\tilde{x} \in \mathbb{R}$ . This fact has been reported in the theoretical neuroscience literature [35]; we provide here a rigorous and analytic justification for it.

For all  $\tilde{x} \in \mathbb{R}$ , let us introduce the quantities

$$\begin{aligned} I^{\tilde{x}} &:= \int_0^\infty a f(a, \tilde{x}) \exp\left(-\int_0^a f(s, \tilde{x}) ds\right) da = \int_0^\infty \exp\left(-\int_0^a f(s, \tilde{x}) ds\right) da, \\ P^{\tilde{x}}(\lambda) &:= \int_0^\infty e^{-\lambda a} f(a, \tilde{x}) \exp\left(-\int_0^a f(s, \tilde{x}) ds\right) da. \end{aligned}$$

$I^{\tilde{x}}$  can be interpreted as the mean inter-spike interval of a neuron receiving a constant input  $\tilde{x}$ .  $P^{\tilde{x}}(\lambda)$  can be seen as the Laplace transform of the inter-spike interval distribution of that neuron, evaluated in  $\lambda$ .

**Proposition 2.** *Grant Assumptions 1 and 3. For all  $\tilde{x} \in \mathbb{R}$ ,*

$$X(\tilde{x}) = \int_0^\infty \hat{h}(t) dt \frac{1}{I^{\tilde{x}}} \left\{ \frac{1 - P^{\tilde{x}}(\lambda)}{1 - \nu P^{\tilde{x}}(\lambda)} \right\}.$$

*Proof.* Using the method of characteristics (i.e. combining Eqs. (26b) and (32)), we have

$$\begin{aligned} 1 &= \int_0^1 \int_0^\infty \rho_\infty^{\tilde{x}}(a, m) dadm = \int_0^1 \int_0^\infty 1_{e^{\lambda a} m < 1} \rho_\infty^{\tilde{x}}(0, e^{\lambda a} m) \exp\left(\lambda a - \int_0^a f(s, \tilde{x}) ds\right) dadm \\ &= \int_0^1 \int_0^\infty \rho_\infty^{\tilde{x}}(0, m) \exp\left(-\int_0^a f(s, \tilde{x}) ds\right) dadm = I^{\tilde{x}} \int_0^1 \rho_\infty^{\tilde{x}}(0, m) dm. \end{aligned}$$

Whence,

$$\int_0^1 \int_0^\infty f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm = \int_0^1 \rho_\infty^{\tilde{x}}(0, m) dm = \frac{1}{I^{\tilde{x}}}.$$

On the other hand,

$$\begin{aligned} &\int_0^1 \int_0^\infty m f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm \\ &= \int_0^1 \int_0^\infty 1_{e^{\lambda a} m < 1} m f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(0, e^{\lambda a} m) \exp\left(\lambda a - \int_0^a f(s, \tilde{x}) ds\right) dadm \\ &= \int_0^1 \int_0^\infty e^{-\lambda a} m f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(0, m) \exp\left(-\int_0^a f(s, \tilde{x}) ds\right) dadm \\ &= P^{\tilde{x}}(\lambda) \int_0^1 m \rho_\infty^{\tilde{x}}(0, m) dm \end{aligned}$$

and

$$\begin{aligned}
\int_0^1 m \rho_\infty^{\tilde{x}}(0, m) dm &= \int_0^1 m 1_{m>1-v} \frac{1}{v} \int_0^\infty f(a, \tilde{x}) \rho_\infty^{\tilde{x}} \left( a, 1 - \frac{1-m}{v} \right) dadm \\
&= \int_0^1 (1-v+vm) \int_0^\infty f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm \\
&= \frac{1-v}{I^{\tilde{x}}} + v P^{\tilde{x}}(\lambda) \int_0^1 m \rho_\infty^{\tilde{x}}(0, m) dm.
\end{aligned}$$

Whence,

$$\int_0^1 m \rho_\infty^{\tilde{x}}(0, m) dm = \frac{1-v}{I^{\tilde{x}}(1-vP^{\tilde{x}}(\lambda))}$$

and

$$\int_0^1 \int_0^\infty m f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm = \frac{P^{\tilde{x}}(\lambda)(1-v)}{I^{\tilde{x}}(1-vP^{\tilde{x}}(\lambda))}.$$

Finally, we have

$$\begin{aligned}
X(\tilde{x}) &= \int_0^\infty \hat{h}(t) dt \left\{ \int_0^1 \int_0^\infty f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm - \int_0^1 \int_0^\infty m f(a, \tilde{x}) \rho_\infty^{\tilde{x}}(a, m) dadm \right\} \\
&= \int_0^\infty \hat{h}(t) dt \frac{1}{I^{\tilde{x}}} \left\{ 1 - \frac{P^{\tilde{x}}(\lambda)(1-v)}{(1-vP^{\tilde{x}}(\lambda))} \right\} = \int_0^\infty \hat{h}(t) dt \frac{1}{I^{\tilde{x}}} \left\{ \frac{1-P^{\tilde{x}}(\lambda)}{1-vP^{\tilde{x}}(\lambda)} \right\}.
\end{aligned}$$

□

## 5 Exponential stability in the weak connectivity regime

To study the long time behavior of Eq. (1) in the weak connectivity regime, we proceed by a perturbation of the non-interacting case (19), taking  $\tilde{x} = \varepsilon x_\infty$ , where  $x_\infty$  is given by the unique stationary solution to Eq. (1) when  $\varepsilon \in ]-\varepsilon^*, +\varepsilon^*[$  ( $\varepsilon^*$  is taken from Theorem 2 (ii)). In this section, we keep the small  $\varepsilon$  fixed and we work under Assumptions 1, 3 and 5. We roughly follow the same line of argument as [24, Sec. 5].

For convenience, we first rewrite (1) in a more formal and compact form:

$$\partial_t \rho_t = -\partial_a \rho_t + \lambda \partial_m (m \rho_t) - f(\varepsilon x_t) \rho_t + \delta_0^a (\gamma \circ \Pi)_* (f(\varepsilon x_t) \rho_t), \quad (39a)$$

$$x_t = \int_0^t \int f(t-s) f(\varepsilon x_s) \rho_s dadm ds, \quad (39b)$$

$$\rho_0 = u_0, \quad (39c)$$

where  $\delta_0^a$  indicates that (singular) mass enters in  $a = 0^2$ ,  $\Pi : (a, m) \mapsto m$  is the projection on  $m$  and  $*$  denotes the pushforward measure. To write Eq. (39) as an evolution equation, we introduce an auxiliary transport equation on  $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*$

$$\begin{aligned}
\partial_t \zeta_t &= -\partial_s \zeta_t + \delta_0^s f(\varepsilon x_t) \rho_t, \\
\zeta_0 &= 0,
\end{aligned}$$

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<sup>2</sup> $\delta_0^a$  should not be confused with the Dirac distribution  $\delta_{0=a}$ . Using  $\delta_{0=a}$ , by integration by parts of weak solutions, Eq. (39a) should write

$$\partial_t \rho_t = -\partial_a \rho_t + \lambda \partial_m (m \rho_t) - f(\varepsilon x_t) \rho_t + \delta_{0=a} \left\{ (\gamma \circ \Pi)_* (f(\varepsilon x_t) \rho_t) - \rho_t(0, \cdot) \right\}.$$

which solution is given by the method of characteristics:

$$\zeta_t(s) = 1_{s \leq t} f(\varepsilon x_{t-s}) \rho_{t-s}, \quad \forall (t, s) \in \mathbb{R}_+^* \times \mathbb{R}_+.$$

Using the auxiliary equation, Eq. (39) is equivalent to

$$\partial_t(\rho_t, \zeta_t) = \left( -\partial_a \rho_t + \lambda \partial_m(m \rho_t) - f(\varepsilon x_t) \rho_t + \delta_0^a (\gamma \circ \Pi)_* (f(\varepsilon x_t) \rho_t), -\partial_s \zeta_t + \delta_0^s f(\varepsilon x_t) \rho_t \right), \quad (40a)$$

$$(\rho_0, \zeta_0) = (u_0, 0), \quad (40b)$$

where  $x_t := \int_0^\infty \int h(s) \zeta_t(s) dadm ds$ .

By Theorem 2, for all  $\varepsilon \in ]-\varepsilon^*, +\varepsilon^*[$ , there exists a unique stationary solution  $(\rho_\infty, x_\infty)$  and we have

$$-\partial_a \rho_\infty + \lambda \partial_m(m \rho_\infty) - f(\varepsilon x_\infty) \rho_\infty + \delta_0^a (\gamma \circ \Pi)_* (f(\varepsilon x_\infty) \rho_\infty) = 0. \quad (41)$$

Now, we write Eq. (40) as the sum of a linear equation and a perturbation:

$$\partial_t(\rho_t, \zeta_t) = \Lambda(\rho_t, \zeta_t) + (Z_t^{(1)}, Z_t^{(2)}), \quad (42a)$$

$$(\rho_0, \zeta_0) = (u_0, 0), \quad (42b)$$

where

$$\Lambda(\rho_t, \zeta_t) := \left( -\partial_a \rho_t + \lambda \partial_m(m \rho_t) - f(\varepsilon x_\infty) \rho_t + \delta_0^a (\gamma \circ \Pi)_* (f(\varepsilon x_\infty) \rho_t), -\partial_s \zeta_t + \delta_0^s f(\varepsilon x_\infty) \rho_t \right),$$

$$Z_t^{(1)} := [f(\varepsilon x_\infty) - f(\varepsilon x_t)] \rho_t + \delta_0^a (\gamma \circ \Pi)_* ([f(\varepsilon x_t) - f(\varepsilon x_\infty)] \rho_t),$$

$$Z_t^{(2)} := \delta_0^s [f(\varepsilon x_t) - f(\varepsilon x_\infty)] \rho_t.$$

Let us put  $\zeta_\infty(s) := f(\varepsilon x_\infty) \rho_\infty$ ,  $\forall s \in \mathbb{R}_+$ . Then, using Eq. (41), by the linearity of the operator  $\Lambda$  and writing  $\bar{\rho}_t := \rho_t - \rho_\infty$  and  $\bar{\zeta}_t := \zeta_t - \zeta_\infty$ , we get

$$\partial_t(\bar{\rho}_t, \bar{\zeta}_t) = \Lambda(\bar{\rho}_t, \bar{\zeta}_t) + (Z_t^{(1)}, Z_t^{(2)}), \quad (43a)$$

$$(\bar{\rho}_0, \bar{\zeta}_0) = (u_0 - \rho_\infty, -\zeta_\infty). \quad (43b)$$

Writing  $(S_t^\Lambda)_{t \in \mathbb{R}_+}$  the semigroup associated with the operator  $\Lambda$ , we have, by Duhamel's formula,

$$(\bar{\rho}_t, \bar{\zeta}_t) = S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0) + \int_0^t S_{t-s}^\Lambda(Z_s^{(1)}, Z_s^{(2)}) ds, \quad \forall t \geq 0. \quad (44)$$

Let us define the weighted space

$$L_+^1(\mu) := \left\{ \zeta \in L^1(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*, \mathbb{R}_+) \mid \int_0^\infty \|\zeta(s)\|_{L^1} \|h\|_\infty e^{-hs} ds < \infty \right\}.$$

Note that, for all  $t \geq 0$ ,

$$\begin{aligned} |x_t - x_\infty| &= \left| \int_0^\infty \int h(s) \zeta_t(s) dadm ds - \int_0^\infty \int h(s) \zeta_\infty(s) dadm ds \right| \\ &\leq \int_0^\infty \|h\|_\infty e^{-hs} \|\zeta_t(s) - \zeta_\infty(s)\|_{L^1} ds = \|\bar{\zeta}_t\|_{L^1(\mu)}. \end{aligned}$$

Also, we have, for all  $t \geq 0$ ,

$$\left\| Z_t^{(1)} \right\|_{L^1} \leq |\varepsilon| 2L_f \|\rho_t\|_{L^1} |x_t - x_\infty| \leq |\varepsilon| 2L_f \|\bar{\zeta}_t\|_{L^1(\mu)}, \quad (45a)$$

$$\left\| Z_t^{(1)} \right\|_{L^1(w)} \leq |\varepsilon| 2L_f \|\rho_t\|_{L^1(w)} |x_t - x_\infty| \leq |\varepsilon| 2L_f \left( \|u_0\|_{L^1(w) + \frac{b}{\alpha}} \right) \|\bar{\zeta}_t\|_{L^1(\mu)}, \quad (45b)$$

$$\left\| Z_t^{(2)} \right\|_{L^1(\mu)} \leq |\varepsilon| \|h\|_\infty L_f \|\rho_t\|_{L^1} |x_t - x_\infty| \leq |\varepsilon| \|h\|_\infty L_f \|\bar{\zeta}_t\|_{L^1(\mu)}, \quad (45c)$$

where we have used Theorem 1 (ii) in the first line and Lemma 1 in the second.

**Lemma 8.** *Grant Assumptions 1, 3 and 5 and take  $(\bar{\rho}_0, \bar{\zeta}_0)$  as in Eq. (43). There exists  $K_1 \geq 1$  and  $\mathbf{a}_1 > 0$  such that, for all initial data  $u_0 \in L^1_+(w)$  with  $\|u_0\|_{L^1} = 1$ ,*

$$\left\| S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0) \right\|_{L^1(w) \times L^1(\mu)} \leq K_1 e^{-\mathbf{a}_1 t} \|(\bar{\rho}_0, \bar{\zeta}_0)\|_{L^1(w) \times L^1(\mu)}, \quad \forall t \geq 0. \quad (46)$$

*If in addition, we grant Assumption 4, then there exists  $K_2 \geq 1$  and  $\mathbf{a}_2 > 0$  such that, for all initial data  $u_0 \in L^1_+$  with  $\|u_0\|_{L^1} = 1$ ,*

$$\left\| S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0) \right\|_{L^1 \times L^1(\mu)} \leq K_2 e^{-\mathbf{a}_2 t} \|(\bar{\rho}_0, \bar{\zeta}_0)\|_{L^1 \times L^1(\mu)}, \quad \forall t \geq 0. \quad (47)$$

*Proof.* We write  $(S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)^{(1)}, S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)^{(2)}) := S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)$  the first and second component of  $S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)$ .

By Theorem 4, there exists  $K \geq 0$  and  $\mathbf{a} > 0$  such that,

$$\left\| S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)^{(1)} \right\|_{L^1(w)} \leq K e^{-\mathbf{a}t} \|\bar{\rho}_0\|_{L^1(w)}, \quad \forall t \geq 0.$$

Then,

$$\begin{aligned} \left\| S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)^{(2)} \right\|_{L^1(\mu)} &= \int_0^t \left\| f(\varepsilon x_\infty) S_{t-s}^\Lambda(\bar{\rho}_0, \bar{\zeta}_0)^{(1)} \right\|_{L^1} C_h e^{-\mathbf{h}s} ds \\ &\quad + \int_t^\infty \left\| \bar{\zeta}_0(s) \right\|_{L^1} C_h e^{-\mathbf{h}s} ds \\ &\leq C_h \left\{ \|f\|_\infty K \int_0^t e^{-\mathbf{a}(t-s)} e^{-\mathbf{h}s} ds \|\bar{\rho}_0\|_{L^1(w)} + e^{-\mathbf{h}t} \|\bar{\zeta}_0\|_{L^1(\mu)} \right\}. \end{aligned}$$

Gathering the bounds on the two components and observing that the function  $t \mapsto \int_0^t e^{-\mathbf{a}(t-s)} e^{-\mathbf{h}s} ds$  decays exponentially, we can conclude that there exists  $K_1 \geq 1$  and  $\mathbf{a}_1 > 0$  such that Eq. (46) holds.

For Eq. (47) we use Theorem 5 and follow the same argument.  $\square$

We can now prove our main result:

*Proof of Theorem 3.* By Duhamel's formula (44), Eq. (46) in Lemma 8 and the bounds Eqs. (45), for all  $t \geq 0$ ,

$$\begin{aligned} \left\| (\bar{\rho}_t, \bar{\zeta}_t) \right\|_{L^1(w) \times L^1(\mu)} &\leq \left\| S_t^\Lambda(\bar{\rho}_0, \bar{\zeta}_0) \right\|_{L^1(w) \times L^1(\mu)} + \int_0^t \left\| S_{t-s}^\Lambda(Z_s^{(1)}, Z_s^{(2)}) \right\|_{L^1(w) \times L^1(\mu)} ds \\ &\leq K_1 e^{-\mathbf{a}_1 t} \|(\bar{\rho}_0, \bar{\zeta}_0)\|_{L^1(w) \times L^1(\mu)} + K_1 \int_0^t e^{-\mathbf{a}_1(t-s)} \left\| (Z_s^{(1)}, Z_s^{(2)}) \right\|_{L^1(w) \times L^1(\mu)} ds \\ &\leq K_1 e^{-\mathbf{a}_1 t} \|(\bar{\rho}_0, \bar{\zeta}_0)\|_{L^1(w) \times L^1(\mu)} + |\varepsilon| \tilde{C}_W \int_0^t e^{-\mathbf{a}_1(t-s)} \left\| (\bar{\rho}_s, \bar{\zeta}_s) \right\|_{L^1(w) \times L^1(\mu)} ds \\ &=: \mathcal{Q}(t), \end{aligned}$$

where  $\tilde{C}_K$  is a constant depending on  $W$ . We have, for all  $t \geq 0$ ,

$$\begin{aligned} \frac{d}{dt} \mathcal{Q}(t) &= -\mathbf{a}_1 \mathcal{Q}(t) + |\varepsilon| \tilde{C}_W \|(\bar{\rho}_t, \bar{\zeta}_t)\|_{L^1(w) \times L^1(\mu)} \\ &\leq \left(-\mathbf{a}_1 + |\varepsilon| \tilde{C}_W\right) \mathcal{Q}(t). \end{aligned}$$

Whence, by Grönwall's lemma,

$$\forall t \geq 0, \quad \mathcal{Q}(t) \leq K_1 \|(\bar{\rho}_0, \bar{\zeta}_0)\|_{L^1(w) \times L^1(\mu)} \exp\left(\left(-\mathbf{a}_1 + |\varepsilon| \tilde{C}_W\right) t\right).$$

For all  $t \geq 0$ , we have

$$\|\rho_t - \rho_\infty\|_{L^1(w)} + |x_t - x_\infty| \leq \|(\bar{\rho}_t, \bar{\zeta}_t)\|_{L^1(w) \times L^1(\mu)} \leq \mathcal{Q}(t)$$

and

$$\|\bar{\zeta}_0\|_{L^1(\mu)} \leq \int_0^\infty \|f(\varepsilon x_\infty) \rho_\infty\|_{L^1} C_h e^{-\mathfrak{h}s} ds \leq \frac{\|f\|_\infty C_h}{\mathfrak{h}}.$$

Whence, choosing  $\varepsilon_W^{**} := \frac{\mathfrak{a}_1}{\tilde{C}_W} \wedge \varepsilon^*$ , we easily see that there exists  $C \geq 1$  and  $c_W > 0$  such that Eq. (6) holds.

For Eq. (7) we use Eq. (47) instead of Eq. (46) and follow the same argument.  $\square$

## Acknowledgements

We thank Stéphane Mischler for supervising this work and Wulfram Gerstner for many useful discussions. This research has been funded by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 754362 and the Swiss National Science Foundation (grant no. 200020\_184615).

## References

- [1] Manuel Beiran and Srdjan Ostojic. Contrasting the effects of adaptation and synaptic filtering on the timescales of dynamics in recurrent networks. *PLoS computational biology*, 15(3):e1006893, 2019.
- [2] Jan Benda and Andreas VM Herz. A universal model for spike-frequency adaptation. *Neural computation*, 15(11):2523–2564, 2003.
- [3] José A Cañizo and Havva Yoldaş. Asymptotic behaviour of neuron population models structured by elapsed-time. *Nonlinearity*, 32(2):464, 2019.
- [4] Julien Chevallier. Mean-field limit of generalized Hawkes processes. *Stochastic Processes and their Applications*, 127(12):3870–3912, 2017.
- [5] Quentin Cormier. A mean-field model of integrate-and-fire neurons: non-linear stability of the stationary solutions. *arXiv preprint arXiv:2002.08649*, 2020.
- [6] Quentin Cormier, Etienne Tanré, and Romain Veltz. Hopf bifurcation in a mean-field model of spiking neurons. *arXiv preprint arXiv:2008.11116*, 2020.



- [7] Quentin Cormier, Etienne Tanré, and Romain Veltz. Long time behavior of a mean-field model of interacting neurons. *Stochastic Processes and their Applications*, 130(5):2553–2595, 2020.
- [8] Anna De Masi, Antonio Galves, Eva Löcherbach, and Errico Presutti. Hydrodynamic limit for interacting neurons. *Journal of Statistical Physics*, 158(4):866–902, 2015.
- [9] Audric Drogoul and Romain Veltz. Exponential stability of the stationary distribution of a mean field of spiking neural network. *Journal of Differential Equations*, 270:809–842, 2021.
- [10] Aline Duarte, Eva Löcherbach, and Guilherme Ost. Stability, convergence to equilibrium and simulation of non-linear Hawkes processes with memory kernels given by the sum of Erlang kernels. *ESAIM: Probability and Statistics*, 23:770–796, 2019.
- [11] Grégory Dumont and Pierre Gabriel. The mean-field equation of a leaky integrate-and-fire neural network: measure solutions and steady states. *Nonlinearity*, 33(12):6381, 2020.
- [12] Nicolas Fournier and Eva Löcherbach. On a toy model of interacting neurons. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 52:1844–1876, 2016.
- [13] Richard Gast, Helmut Schmidt, and Thomas R Knösche. A mean-field description of bursting dynamics in spiking neural networks with short-term adaptation. *Neural Computation*, 32(9):1615–1634, 2020.
- [14] Wulfram Gerstner. Time structure of the activity in neural network models. *Physical review E*, 51(1):738, 1995.
- [15] Wulfram Gerstner. Population dynamics of spiking neurons: fast transients, asynchronous states, and locking. *Neural computation*, 12(1):43–89, 2000.
- [16] Wulfram Gerstner, Werner M Kistler, Richard Naud, and Liam Paninski. *Neuronal dynamics: From single neurons to networks and models of cognition*. Cambridge University Press, 2014.
- [17] Wulfram Gerstner and J Leo van Hemmen. Associative memory in a network of ‘spiking’ neurons. *Network: Computation in Neural Systems*, 3(2):139–164, 1992.
- [18] Guido Gigante, Maurizio Mattia, and Paolo Del Giudice. Diverse population-bursting modes of adapting spiking neurons. *Physical Review Letters*, 98(14):148101, 2007.
- [19] Martin Hairer and Jonathan C Mattingly. Yet another look at Harris’ ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, pages 109–117. Springer, 2011.
- [20] Reinhard Höpfner, Eva Löcherbach, and Michèle Thiellien. Ergodicity and limit theorems for degenerate diffusions with time periodic drift. application to a stochastic Hodgkin-Huxley model. *ESAIM: Probability and Statistics*, 20:527–554, 2016.
- [21] Renaud Jolivet, Alexander Rauch, Hans-Rudolf Lüscher, and Wulfram Gerstner. Predicting spike timing of neocortical pyramidal neurons by simple threshold models. *Journal of computational neuroscience*, 21(1):35–49, 2006.
- [22] Ryota Kobayashi, Yasuhiro Tsubo, and Shigeru Shinomoto. Made-to-order spiking neuron model equipped with a multi-timescale adaptive threshold. *Frontiers in computational neuroscience*, 3:9, 2009.

- [23] Sean P Meyn and Richard L Tweedie. Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Advances in Applied Probability*, pages 518–548, 1993.
- [24] Stéphane Mischler, Cristobal Quiñinao, and Qilong Weng. Weak and strong connectivity regimes for a general time elapsed neuron network model. *Journal of Statistical Physics*, 173(1):77–98, 2018.
- [25] Stéphane Mischler and Qilong Weng. Relaxation in time elapsed neuron network models in the weak connectivity regime. *Acta Applicandae Mathematicae*, 157(1):45–74, 2018.
- [26] Samuel P Muscinelli, Wulfram Gerstner, and Tilo Schwalger. How single neuron properties shape chaotic dynamics and signal transmission in random neural networks. *PLoS computational biology*, 15(6):e1007122, 2019.
- [27] Ahmet Omurtag, Bruce W. Knight, and Lawrence Sirovich. On the simulation of large populations of neurons. *Journal of computational neuroscience*, 8(1):51–63, 2000.
- [28] Khashayar Pakdaman, Benoît Perthame, and Delphine Salort. Dynamics of a structured neuron population. *Nonlinearity*, 23(1):55, 2009.
- [29] Khashayar Pakdaman, Benoît Perthame, and Delphine Salort. Relaxation and self-sustained oscillations in the time elapsed neuron network model. *SIAM Journal on Applied Mathematics*, 73(3):1260–1279, 2013.
- [30] Khashayar Pakdaman, Benoît Perthame, and Delphine Salort. Adaptation and fatigue model for neuron networks and large time asymptotics in a nonlinear fragmentation equation. *The Journal of Mathematical Neuroscience*, 4(1):14, 2014.
- [31] Jonathan W Pillow, Jonathon Shlens, Liam Paninski, Alexander Sher, Alan M Litke, EJ Chichilnisky, and Eero P Simoncelli. Spatio-temporal correlations and visual signalling in a complete neuronal population. *Nature*, 454(7207):995, 2008.
- [32] Christian Pozzorini, Skander Mensi, Olivier Hagens, Richard Naud, Christof Koch, and Wulfram Gerstner. Automated high-throughput characterization of single neurons by means of simplified spiking models. *PLoS Comput Biol*, 11(6):e1004275, 2015.
- [33] Christian Pozzorini, Richard Naud, Skander Mensi, and Wulfram Gerstner. Temporal whitening by power-law adaptation in neocortical neurons. *Nature neuroscience*, 16(7):942, 2013.
- [34] Magnus JE Richardson. Dynamics of populations and networks of neurons with voltage-activated and calcium-activated currents. *Physical Review E*, 80(2):021928, 2009.
- [35] Sandro Romani, Daniel J Amit, and Gianluigi Mongillo. Mean-field analysis of selective persistent activity in presence of short-term synaptic depression. *Journal of Computational Neuroscience*, 20(2):201, 2006.
- [36] Valentin Schmutz. Mean-field limit of Age and Leaky memory dependent Hawkes processes. *arXiv preprint arXiv:2003.13810*, 2020.
- [37] Corinne Teeter, Ramakrishnan Iyer, Vilas Menon, Nathan Gouwens, David Feng, Jim Berg, Aaron Szafer, Nicholas Cain, Hongkui Zeng, Michael Hawrylycz, et al. Generalized leaky integrate-and-fire models classify multiple neuron types. *Nature communications*, 9(1):1–15, 2018.

- [38] Wilson Truccolo, Uri T Eden, Matthew R Fellows, John P Donoghue, and Emery N Brown. A point process framework for relating neural spiking activity to spiking history, neural ensemble, and extrinsic covariate effects. *Journal of neurophysiology*, 93(2):1074–1089, 2005.
- [39] Misha Tsodyks, Klaus Pawelzik, and Henry Markram. Neural networks with dynamic synapses. *Neural computation*, 10(4):821–835, 1998.
- [40] Sebastian Vellmer and Benjamin Lindner. Theory of spike-train power spectra for multidimensional integrate-and-fire neurons. *Physical Review Research*, 1(2):023024, 2019.
- [41] C van Vreeswijk and David Hansel. Patterns of synchrony in neural networks with spike adaptation. *Neural computation*, 13(5):959–992, 2001.
- [42] Hugh R Wilson and Jack D Cowan. Excitatory and inhibitory interactions in localized populations of model neurons. *Biophysical journal*, 12(1):1–24, 1972.
- [43] Robert S Zucker and Wade G Regehr. Short-term synaptic plasticity. *Annual review of physiology*, 64(1):355–405, 2002.