Multi-objective Unified qLPV Observer: Application to a Semi-active Suspension System

Gia Quoc Bao Tran, Thanh-Phong Pham, Olivier Sename

Multi-objective Unified qLPV Observer: Application to a Semi-active Suspension System

Gia Quoc Bao Tran [*] **Thanh-Phong Pham** [**] **Olivier Sename** [*]

[*] *Univ. Grenoble Alpes, CNRS, Grenoble INP*[⊥]*, GIPSA-lab, Grenoble, France* [⊥]*Institute of Engineering Univ. Grenoble Alpes*
(e-mail: gia-quoc-bao.tran@grenoble-inp.org)
[**] *Faculty of Electrical and Electronic Engineering, The University of Danang - University of Technology and Education, Danang, Vietnam*
(e-mail: ptphong@ute.udn.vn)

Abstract: This work concerns the design of a multi-objective unified qLPV observer for the state estimation problem in LPV systems with a parameter-dependent control input matrix. The standard forms of the system and the observer are first presented, where the observer matrices are functions of the estimated states (qLPV problem). The effects of bounded unknown input disturbances are decoupled from the estimation error thanks to the parameterization of the observer matrices. To treat the disturbance caused by inexact scheduling parameters, we introduce an upper bound on the parameter estimation error, which is considered uncertainty. Then, the effects of the control input and the random measurement noise on the estimation error are minimized using the $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ conditions, respectively, as a multi-objective optimization problem. In the solution of the LMI sets, the projection lemma is applied to reduce the high conservativeness that would otherwise lead to suboptimal results. Then this observer is applied to a semi-active automotive suspension system written in LPV form, using simulations with real data measured from our experiment test platform, and compared with the linear time-invariant $\mathcal{H}_\infty$ and $\mathcal{H}_2$ observers.

Keywords: Unified observer, linear parameter-varying systems, $\mathcal{H}_\infty$, generalized $\mathcal{H}_2$, projection lemma, semi-active suspension.

1. INTRODUCTION

Control of semi-active automotive suspension systems has been an exciting topic of automatic control, as these allow to enhance driving comfort with better performance than passive suspension and come with less energy consumption compared to active suspension. To effectively control and monitor automotive suspension systems, we need to know the damper force, which is of high importance for performance and diagnosis purposes. However, the direct measurement of this force is usually difficult and costly, which provokes the need for developing state observers to estimate it.

Different damper force estimation strategies have been introduced, including the Kalman filter (Koch et al., 2010), an $\mathcal{H}_\infty$ observer based on a nonlinear dynamic model of the electro-rheological (ER) damper (Vela et al., 2018), an LPV-$\mathcal{H}_\infty$ approach (Tudon-Martinez et al., 2018; Pham et al., 2019b), and recently a unified $\mathcal{H}_\infty$ observer for nonlinear Lipschitz condition (Pham et al., 2019). In the last-mentioned work, the authors establish the appropriate forms of the system and the observer, then thanks to a procedure called parameterization, the effects of bounded unknown input disturbances are decoupled from the state estimation error. The $\mathcal{H}_\infty$ conditions are then applied to minimize the effects of measurement noise on the estimation error. Note that we use throughout the paper $\mathcal{H}_\infty$ instead of $\mathcal{L}_2$-induced by the abuse of language.

Motivated by the fact that the generalized $\mathcal{H}_2$ condition can be very interesting for the unified observers, where the measurement noise is important and may induce a large bias on the estimation error, we propose here a multi-objective unified observer. Compared to $\mathcal{H}_\infty$, the generalized $\mathcal{H}_2$ framework is considered more suitable for the type of noise we are faced with in the considered application case (sensors are accelerometers). We extend from the existing work (Pham et al., 2019), where the authors propose an $\mathcal{H}_\infty$ (static) observer for nonlinear Lipschitz systems, into an LPV one that uses the multi-objective $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ condition. This observer is then applied to an automotive's semi-active suspension system, for which the damper nonlinearity is considered a varying parameter in a quasi-LPV model.

The main contribution of this work is the design of a multi-objective unified qLPV observer for LPV systems with a parameter-dependent control input. This observer is unified because it has a structure that allows its matrices to be parameterized, thus decoupling the effects of bounded unknown input disturbances from the estimation error. It is quasi-LPV (qLPV) since its matrices are scheduled depending on the set of parameters estimated using the system's states, which was not the case in our previous studies. Finally, it is a multi-objective observer, as the $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ conditions are used to minimize the effects of the control input and the measurement noise, respectively, on the estimation error through solving linear matrix inequality (LMI) sets. We model the parameter estimation error as bounded uncertainty. The solution of the observer design problem is presented as a three-step procedure wherein the projection lemma is applied that breaks the LMIs into ones with fewer variables so as to reduce the high level of conservativeness that would otherwise reduce the observer into a proportional (P) one.

1.1 Notation

In this work, $W^\top$ denotes the transpose of matrix W. Moreover, W^{-1} is the inverse of the square matrix W. The notation W^+ denotes any general inverse of W satisfying $WW^+W = W$, including the pseudo-inverse. Lastly, $W^\perp$ denotes the left null space of W, i.e., $W^\perp W = 0$.

2. DEFINITION AND PARAMETERIZATION OF THE UNIFIED QLPV OBSERVER

2.1 Problem Formulation

Consider the following class of LPV systems with a parameter-dependent control input matrix

$$\Sigma : \begin{cases} \dot{x} = Ax + B(\rho)u + D_1 w_r \\ y = Cx + D_2 w_n, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w_r \in \mathbb{R}^l$ is the unknown disturbance input, $w_n \in \mathbb{R}^f$ is the measurement noise input, $y \in \mathbb{R}^p$ is the measured output, and ρ is the vector of varying parameters. As usually assumed for LPV systems, ρ is known or estimated, and is bounded, i.e., $\rho(t) \in [\underline{\rho}, \overline{\rho}], \forall t$.

Note that here the matrix A is supposed independent of varying parameters, which is true for some particular systems, including the semi-active suspension system with an ER damper we are considering in this study. Moreover, in such an application case, the parameter ρ is a function of some state variables. This is, therefore, a qLPV system.

Consider a state observer of the following form

$$\Sigma_{obs} : \begin{cases} \dot{z} = Nz + Jy + H(\hat{\rho})u + Mv \\ \dot{v} = Pz + Qy + Gv \\ \hat{x} = Rz + Sy, \end{cases} \tag{2}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimated state and $z, v \in \mathbb{R}^n$ are the state and auxiliary vectors of the observer. By designing the unified observer, we determine all the matrices N, J, M, P, Q, G, R, S, as well as $H(\hat{\rho})$ at all the vertices, such that the conditions related to the decoupling of the bounded unknown input disturbance and the minimization of the effects of the control input and random measurement noise on the estimation error are all satisfied. It can be seen that when M is null, the observer is reduced to a P observer. Therefore, to preserve the dynamics of the arbitrary state z, we aim to get M, P, Q, G as non-null matrices.

Let us define the dynamic error as

$$\epsilon = z - Tx, \tag{3}$$

where $T \in \mathbb{R}^{n\times n}$ is an arbitrary matrix.

Differentiating (3) with respect to time, using (1) and (2), and denoting $\zeta = (\epsilon \ v)^\top$ lead to

$$\begin{cases} \dot{\zeta} = \begin{pmatrix} N & M \\ P & G \end{pmatrix} \zeta + \begin{pmatrix} NT - TA + JC \\ PT + QC \end{pmatrix} x + \begin{pmatrix} TD_1 \\ 0 \end{pmatrix} \omega_r + \\ \quad + \begin{pmatrix} H(\hat{\rho}) - TB(\rho) \\ 0 \end{pmatrix} u + \begin{pmatrix} JD_2 \\ QD_2 \end{pmatrix} \omega_n \\ \hat{x} = (R \ 0)\,\zeta + (RT + SC)\,x + SD_2\omega_n. \end{cases} \tag{4}$$

Denoting $B(\rho) = B(\hat{\rho}) + (B(\rho) - B(\hat{\rho}))$, it then follows that if the following decoupling conditions are satisfied

$$NT - TA + JC = 0, \tag{5}$$
$$TD_1 = 0, \tag{6}$$
$$H(\hat{\rho}) - TB(\hat{\rho}) = 0, \tag{7}$$
$$PT + QC = 0, \tag{8}$$
$$RT + SC = I, \tag{9}$$

the system in (4) is reduced to

$$\begin{cases} \dot{\zeta} = \begin{pmatrix} N & M \\ P & G \end{pmatrix} \zeta + \begin{pmatrix} -T(B(\rho) - B(\hat{\rho})) \\ 0 \end{pmatrix} u + \begin{pmatrix} JD_2 \\ QD_2 \end{pmatrix} \omega_n \\ e = (R \ 0)\,\zeta + SD_2\omega_n, \end{cases} \tag{10}$$

where $e = \hat{x} - x$ is the state estimation error.

The multi-objective unified qLPV observer design problem is equivalent to determining the observer matrices N, J, $H(\hat{\rho})$, M, P, Q, G, R, S such that all the decoupling conditions (5)-(9) are satisfied, and such that the effects of the control input u and the measurement noise ω_n on the state estimation error e are minimized according to a multi-objective $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ condition.

2.2 Parameterization of Observer Matrices

In order to determine the observer matrices N, J, $H(\hat{\rho})$, M, P, Q, G, R, S of the proposed observer satisfying all the conditions (5)-(9), parameterization is performed by using the general solution of (5)-(9) as explained in (Gao et al., 2016) and (Pham et al., 2019).

Firstly, equations (8) and (9) are rewritten as

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{11}$$

The equation (11) is solvable iff

$$rank \begin{pmatrix} T \\ C \\ 0 \\ I \end{pmatrix} = rank \begin{pmatrix} T \\ C \end{pmatrix} = n. \tag{12}$$

Let $E \in \mathbb{R}^{n\times n}$ be a full row rank arbitrary matrix s.t.

$$rank \begin{pmatrix} E \\ C \end{pmatrix} = rank \begin{pmatrix} T \\ C \end{pmatrix} = n. \tag{13}$$

Then there always exists a parameter matrix K s.t.

$$\begin{pmatrix} T \\ C \end{pmatrix} = \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix} \begin{pmatrix} E \\ C \end{pmatrix} \Leftrightarrow T = E - KC. \tag{14}$$

Consequently, equation (11) becomes

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix} \begin{pmatrix} E \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}, \tag{15}$$

whose exact solution is

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \left[\begin{pmatrix} 0 \\ I \end{pmatrix} \Sigma^+ - Y_m \left(I - \Sigma\Sigma^+ \right) \right] \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}, \tag{16}$$

where $\Sigma = \begin{pmatrix} E \\ C \end{pmatrix}$ and Y_m is a free matrix of appropriate dimension. This is equivalent to

$$\begin{aligned} P &= -Y_{m1}\beta_1,\ Q = -Y_{m1}\beta_2, \\ R &= \alpha_1 - Y_{m2}\beta_1,\ S = \alpha_2 - Y_{m2}\beta_2, \end{aligned} \tag{17}$$

where $Y_{m1} = (I\ 0)\, Y_m,\ Y_{m2} = (0\ I)\, Y_m,\ \alpha_1 = \Sigma^+ \begin{pmatrix} I \\ 0 \end{pmatrix}$, $\alpha_2 = \Sigma^+ \begin{pmatrix} K \\ I \end{pmatrix}, \beta_1 = (I - \Sigma\Sigma^+) \begin{pmatrix} I \\ 0 \end{pmatrix}, \beta_2 = (I - \Sigma\Sigma^+) \begin{pmatrix} K \\ I \end{pmatrix}$.
Besides, from the equations (6) and (14), we obtain

$$KCD_1 = ED_1, \tag{18}$$

which is solvable iff

$$rank \begin{pmatrix} ED_1 \\ CD_1 \end{pmatrix} = rank \left[\begin{pmatrix} E \\ C \end{pmatrix} D_1 \right] = rankD_1 = rankCD_1, \tag{19}$$

and the exact solution of (18) is

$$K = ED_1(CD_1)^+. \tag{20}$$

From the condition (7), we have

$$H(\hat{\rho}) = TB(\hat{\rho}) = (E - KC)B(\hat{\rho}) = (E - ED_1(CD_1)^+C)B(\hat{\rho}). \tag{21}$$

On the other hand, substituting (14) into the decoupling condition (5), we obtain

$$\begin{aligned} &N(E - KC) - (E - KC)A + JC = 0 \\ &\Leftrightarrow (N\ \ J - NK)\,\Sigma = (E - ED_1(CD_1)^+C)A, \end{aligned} \tag{22}$$

which can also be parameterized as

$$(N\ \ K_1)\,\Sigma = \Theta, \tag{23}$$

where

$$K_1 = J - NK, \Theta = (E - ED_1(CD_1)^+C)A, \tag{24}$$

and the solution set of (23) is given by

$$(N\ \ K_1) = \Theta\Sigma^+ - Y_{m3}(I - \Sigma\Sigma^+), \tag{25}$$

which is equivalent to

$$N = \alpha_3 - Y_{m3}\beta_1, \tag{26}$$

$$K_1 = \alpha_4 - Y_{m3}\beta_3, \tag{27}$$

where Y_{m3} is a free matrix of appropriate dimension and

$$\alpha_3 = \Theta\Sigma^+ \begin{pmatrix} I \\ 0 \end{pmatrix}, \alpha_4 = \Theta\Sigma^+ \begin{pmatrix} 0 \\ I \end{pmatrix}, \beta_3 = (I - \Sigma\Sigma^+) \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{28}$$

Remark 1. If the matrices P, Q, R, S, $H(\hat{\rho})$, N, and J can be chosen according to (17), (21), (26), and (24), respectively, then all the decoupling conditions (5)-(9) are satisfied.

As mentioned above, since the conditions (5)-(9) are satisfied, the system (4) is rewritten as (10) and the effects of the unknown disturbance input ω_r are effectively decoupled from the estimation error e.

2.3 Effect of the Parameter Estimation Error

The presence of inexact scheduling parameters is a key issue in observer design (Heemels et al. (2010); Chandra et al. (2017)). In our case, it is worth mentioning (as seen in the system (10)) that this uncertainty only affects the input matrix, which does not affect the stability of the estimation error.

Let us denote $B(\rho) - B(\hat{\rho})$ as the inexact varying parameter, which we treat as uncertainty. We assume, for simplicity, that this uncertainty is bounded by a constant matrix in a similar way as in (Gómez-Peñate et al. (2020)) as

$$|B(\rho) - B(\hat{\rho})| \leq B_\Delta. \tag{29}$$

Note that such an assumption may lead to conservative results. Therefore, in the future, more accurate representations of such effects might be considered as recently proposed in (Sato (2020)).

It follows that the minimization of the effects of u and ω_n on e in the system in (10) is achieved if it is also achieved with the system

$$\begin{cases} \dot{\zeta} &= \begin{pmatrix} N & M \\ P & G \end{pmatrix} \zeta + \begin{pmatrix} -TB_\Delta \\ 0 \end{pmatrix} u + \begin{pmatrix} JD_2 \\ QD_2 \end{pmatrix} \omega_n \\ &=: \mathbb{A}_1\zeta + \mathbb{B}_{u,1}u + \mathbb{B}_{n,1}\omega_n \\ e &= \begin{pmatrix} R & 0 \end{pmatrix} \zeta + SD_2\omega_n =: \mathbb{C}_1\zeta + \mathbb{D}_1\omega_n, \end{cases} \tag{30}$$

where

$$\mathbb{A}_1 = \begin{pmatrix} N & M \\ P & G \end{pmatrix} = A_{11} - ZA_{12}, \tag{31}$$

$$\mathbb{B}_{u,1} = \begin{pmatrix} -TB_\Delta \\ 0 \end{pmatrix}, \tag{32}$$

$$\mathbb{B}_{n,1} = \begin{pmatrix} JD_2 \\ QD_2 \end{pmatrix} = B_{n,11} - ZB_{n,12}, \tag{33}$$

where $A_{11} = \begin{pmatrix} \alpha_3 & 0 \\ 0 & 0 \end{pmatrix}$, $A_{12} = \begin{pmatrix} \beta_1 & 0 \\ 0 & -I \end{pmatrix}$, $B_{n,11} = \begin{pmatrix} \Theta\Sigma^+ \begin{pmatrix} K \\ I_p \end{pmatrix} D_2 \\ 0 \end{pmatrix}$, $B_{n,12} = \begin{pmatrix} \beta_2 D_2 \\ 0 \end{pmatrix}$, and $Z = \begin{pmatrix} Y_{m3} & M \\ Y_{m1} & G \end{pmatrix}$.
Note that the estimation error e is independent of the matrix R. The following matrices are obtained

$$\mathbb{C}_1 = (R\ 0) = (\alpha_1\ 0), \tag{34}$$

$$\mathbb{D}_1 = SD_2 = (\alpha_2 - Y_{m2}\beta_2)D_2. \tag{35}$$

Note that all the matrices A_{11}, A_{12}, $\mathbb{B}_{u,1}$, $B_{n,11}$, $B_{n,12}$, $\mathbb{C}_1$, $\mathbb{D}_1$ are known and finding all the observer matrices reduces to finding Z, which is discussed in the following part.

3. MULTI-OBJECTIVE DESIGN OF THE UNIFIED QLPV OBSERVER

First, to apply the generalized $\mathcal{H}_2$ condition, the parameterized estimation error system has to be strictly proper, i.e., $\mathbb{D}_1 = 0$. The free matrix Y_{m2} is thus chosen s.t. $(\alpha_2 - Y_{m2}\beta_2)D_2 = 0$. E.g., we can let

$$Y_{m2} = (\alpha_2 D_2)(\beta_2 D_2)^+. \tag{36}$$

Now that the system is strictly proper, the multi-objective unified qLPV observer design problem is to find Z s.t.

- The system (30) is stable for $u(t) = 0$ and $\omega_n(t) = 0$;
- $\left\| \frac{e(t)}{u(t)} \right\|_\infty < \gamma_\infty$ and $\left\| \frac{e(t)}{\omega_n(t)} \right\|_2 < \gamma_2$;
- The poles of the observer, i.e., of the matrix $\mathbb{A}_1$ in (31), are sufficiently fast to ensure an efficient estimation.

The multi-objective problem is solved considering a single-objective one defined as a combination of the two considered norms as

$$[\alpha \cdot \gamma_\infty + (1 - \alpha) \cdot \gamma_2],$$

where $\alpha \in [0, 1]$ is a constant, to be minimized for $u(t) \neq 0$ and $\omega_n(t) \neq 0$.

Proposition. The multi-objective unified qLPV observer design problem is solved, given a constant $\alpha \in [0,1]$, as

$$min\left[\alpha \cdot \gamma_\infty + (1-\alpha)\cdot \gamma_2\right],$$

s.t. there exists $X = X^\top > 0$ satisfying

$$\begin{cases} \begin{pmatrix} -A_{12} \\ 0 \end{pmatrix}^{\perp} \begin{pmatrix} A_{11}^\top X + XA_{11} + \mathbb{C}_1^\top \mathbb{C}_1 & X\mathbb{B}_{u,1} \\ \mathbb{B}_{u,1}^\top X & -\gamma_\infty^2 I \end{pmatrix} \begin{pmatrix} -A_{12} \\ 0 \end{pmatrix}^{\perp\top} < 0 \\ \begin{pmatrix} -A_{12} \\ -B_{n,12} \end{pmatrix}^{\perp} \begin{pmatrix} A_{11}^\top X + XA_{11} & XB_{n,11} \\ B_{n,11}^\top X & -I \end{pmatrix} \begin{pmatrix} -A_{12} \\ -B_{n,12} \end{pmatrix}^{\perp\top} < 0 \\ \begin{pmatrix} X & \mathbb{C}_1^\top \\ \mathbb{C}_1 & \gamma_2^2 I \end{pmatrix} > 0. \end{cases} \tag{37}$$

Then, given the found X and a positive constant x_α (the pole placement decay rate), Y is found by solving both LMIs below, including the pole placement condition to improve the numerical efficiency of the observer to be embedded.

$$\begin{pmatrix} A_{11}^\top X + XA_{11} - A_{12}^\top Y^\top - YA_{12} & XB_{n,11} - YB_{n,12} \\ B_{n,11}^\top X - B_{n,12}^\top Y^\top & -I \end{pmatrix} < 0, \tag{38}$$

$$A_{11}^\top X + XA_{11} - A_{12}^\top Y^\top - YA_{12} + 2x_\alpha X < 0. \tag{39}$$

Finally, Z in (31) is given by $Z = X^{-1}Y$.

Proof. The proof uses the projection lemma in a three-step procedure. We need to solve for $X = X^\top > 0$ minimizing the given convex combination of $\gamma_\infty > 0$, and $\gamma_2 > 0$ using the set of LMIs formed hereafter. First, the $\mathcal{H}_\infty$ condition is used to minimize the effect of u, i.e., the control input, on the estimation error

$$\begin{pmatrix} \mathbb{A}_1^\top X + X\mathbb{A}_1 + \mathbb{C}_1^\top \mathbb{C}_1 & X\mathbb{B}_{u,1} \\ \mathbb{B}_{u,1}^\top X & -\gamma_\infty^2 I \end{pmatrix} < 0. \tag{40}$$

Note that here in this multi-objective problem we need $\mathbb{D}_1 = 0$ (not required by $\mathcal{H}_\infty$) so as to satisfy the $\mathcal{H}_2$ condition. The same X appears in the generalized $\mathcal{H}_2$ condition used to minimize the effect of ω_n, i.e., the random measurement noise, on the estimation error, that is $\mathbb{D}_1 = 0$ (already satisfied using (36)) and that (Scherer and Weiland (2000))

$$\begin{pmatrix} \mathbb{A}_1^\top X + X\mathbb{A}_1 & X\mathbb{B}_{n,1} \\ \mathbb{B}_{n,1}^\top X & -I \end{pmatrix} < 0, \begin{pmatrix} X & \mathbb{C}_1^\top \\ \mathbb{C}_1 & \gamma_2^2 I \end{pmatrix} > 0. \tag{41}$$

Substituting the equations for $\mathbb{A}_1$, $\mathbb{B}_{u,1}$, and $\mathbb{B}_{n,1}$ above into conditions (40) and (41) gives

$$\begin{cases} \begin{pmatrix} A_{11}^\top X + XA_{11} - A_{12}^\top Z^\top X - XZA_{12} + \mathbb{C}_1^\top \mathbb{C}_1 & X\mathbb{B}_{u,1} \\ \mathbb{B}_{u,1}^\top X & -\gamma_\infty^2 I \end{pmatrix} < 0 \\ \begin{pmatrix} A_{11}^\top X + XA_{11} - A_{12}^\top Z^\top X - XZA_{12} & XB_{n,11} - XZB_{n,12} \\ B_{n,11}^\top X - B_{n,12}^\top Z^\top X & -I \end{pmatrix} < 0 \\ \begin{pmatrix} X & \mathbb{C}_1^\top \\ \mathbb{C}_1 & \gamma_2^2 I \end{pmatrix} > 0. \end{cases} \tag{42}$$

Letting $Y = XZ$, we obtain the following four-variable LMI set

$$\begin{cases} \begin{pmatrix} A_{11}^\top X + XA_{11} - A_{12}^\top Y^\top - YA_{12} + \mathbb{C}_1^\top \mathbb{C}_1 & X\mathbb{B}_{u,1} \\ \mathbb{B}_{u,1}^\top X & -\gamma_\infty^2 I \end{pmatrix} < 0 \\ \begin{pmatrix} A_{11}^\top X + XA_{11} - A_{12}^\top Y^\top - YA_{12} & XB_{n,11} - YB_{n,12} \\ B_{n,11}^\top X - B_{n,12}^\top Y^\top & -I \end{pmatrix} < 0 \\ \begin{pmatrix} X & \mathbb{C}_1^\top \\ \mathbb{C}_1 & \gamma_2^2 I \end{pmatrix} > 0. \end{cases} \tag{43}$$

It is worth noting that, if we try to solve the above LMI set for X, Y and γ_∞ and γ_2 simultaneously, the matrices M, P, Q and G may be reduced to zero due to the high conservativeness caused by the too numerous decision variables, which leads to the observer being a P observer, as stated in (Pham et al., 2019). In order to get a "full-dynamics" observer, so to prevent M from being null, we here apply the projection lemma (Skelton et al., 1998). The LMI set (43) is rewritten in the mentioned form as

$$\begin{cases} \left[\begin{pmatrix} I \\ 0 \end{pmatrix} Y \begin{pmatrix} -A_{12} & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} I \\ 0 \end{pmatrix} Y \begin{pmatrix} -A_{12} & 0 \end{pmatrix}\right]^\top + \\ + \begin{pmatrix} A_{11}^\top X + XA_{11} + \mathbb{C}_1^\top \mathbb{C}_1 & X\mathbb{B}_{u,1} \\ \mathbb{B}_{u,1}^\top X & -\gamma_\infty^2 I \end{pmatrix} < 0 \\ \left[\begin{pmatrix} I \\ 0 \end{pmatrix} Y \begin{pmatrix} -A_{12} & -B_{n,12} \end{pmatrix}\right] + \left[\begin{pmatrix} I \\ 0 \end{pmatrix} Y \begin{pmatrix} -A_{12} & -B_{n,12} \end{pmatrix}\right]^\top + \\ + \begin{pmatrix} A_{11}^\top X + XA_{11} & XB_{n,11} \\ B_{n,11}^\top X & -I \end{pmatrix} < 0 \\ \begin{pmatrix} X & \mathbb{C}_1^\top \\ \mathbb{C}_1 & \gamma_2^2 I \end{pmatrix} > 0. \end{cases} \tag{44}$$

From the projection lemma, the resulting LMI set that we need to solve for X, γ_∞ and γ_2 are

$$\begin{cases} \begin{pmatrix} I \\ 0 \end{pmatrix}^{\perp} \begin{pmatrix} A_{11}^\top X + XA_{11} + \mathbb{C}_1^\top \mathbb{C}_1 & X\mathbb{B}_{u,1} \\ \mathbb{B}_{u,1}^T X & -\gamma_\infty^2 I \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}^{\perp\top} < 0 \\ \begin{pmatrix} -A_{12} \\ 0 \end{pmatrix}^{\perp} \begin{pmatrix} A_{11}^\top X + XA_{11} + \mathbb{C}_1^\top \mathbb{C}_1 & X\mathbb{B}_{u,1} \\ \mathbb{B}_{u,1}^\top X & -\gamma_\infty^2 I \end{pmatrix} \begin{pmatrix} -A_{12} \\ 0 \end{pmatrix}^{\perp\top} < 0 \\ \begin{pmatrix} I \\ 0 \end{pmatrix}^{\perp} \begin{pmatrix} A_{11}^\top X + XA_{11} & XB_{n,11} \\ B_{n,11}^\top X & -I \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}^{\perp\top} < 0 \\ \begin{pmatrix} -A_{12} \\ -B_{n,12} \end{pmatrix}^{\perp} \begin{pmatrix} A_{11}^\top X + XA_{11} & XB_{n,11} \\ B_{n,11}^\top X & -I \end{pmatrix} \begin{pmatrix} -A_{12} \\ -B_{n,12} \end{pmatrix}^{\perp\top} < 0 \\ \begin{pmatrix} X & \mathbb{C}_1^\top \\ \mathbb{C}_1 & \gamma_2^2 I \end{pmatrix} > 0, \end{cases} \tag{45}$$

where the first and third LMIs reduce to $-\gamma_\infty^2 < 0$ and $-1 < 0$, respectively, which are trivial inequalities. This means we only must solve the other three LMIs for X, γ_∞ and γ_2 as given in (37). Then, Y is found from the second equation of (43) (to prioritize the generalized $\mathcal{H}_2$ condition), together with a pole placement constraint, which leads to both LMIs (38) and (39). Finally, $Z = X^{-1}Y$. The proof is completed. □

Remark 2. Using the projection lemma, we solve for the variables consecutively instead of simultaneously. This approach reduces the high conservativeness, thus preventing the matrices M, P, Q, and G from being null, and so the observer is of full scale instead of being reduced to a P observer.

4. FORCE OBSERVER DESIGN FOR THE SEMI-ACTIVE SUSPENSION SYSTEM

4.1 System Modeling

The proposed observer is applied to a semi-active suspension system used to mitigate the vertical oscillations in automotive vehicles with better performance than passive suspension and less energy consumption compared to active suspension (Savaresi et al., 2010; Unger et al., 2013). The main objective is to estimate the damper force, which is achieved by estimating the system's states. This problem is of high importance not only for diagnosis purposes (evaluating the state of health of the damper) but also for control purposes when a local force control of a semi-active damper is included in the suspension control architecture.

The system's LPV state-space representation has the form (1). Taken measurements from the accelerometers as input, the observer estimates $\hat{x} \in \mathbb{R}^5$ from which the estimated damper force is given as

$$\hat{F}_d = k_0\hat{x}_1 + c_0(\hat{x}_2 - \hat{x}_4) + \hat{x}_5. \qquad (46)$$

The observer is implemented according to the following block diagram:

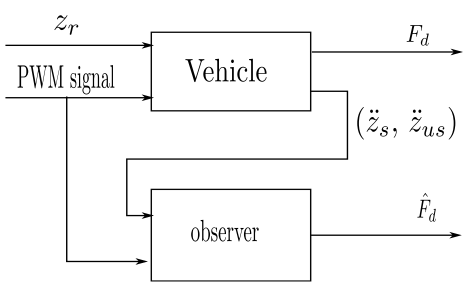

Figure 1. Implementation of the observer.

The complete system description and parameters are found in (Pham et al., 2019). Note that in the automotive industry, two accelerometers are more economical and easily installed compared to a displacement sensor.

4.2 Observer Design and Analysis

The proposed observer is designed for the described system in the same way as in (Pham et al., 2019a). Here, the bound on inexact scheduling parameter estimation uncertainty is assumed equivalent to $|\rho - \hat{\rho}| \leq 0.1$ and the decay rate for pole placement is $x_\alpha = 2$.

It is known that in multi-objective optimization (here $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$), there is a trade-off between both objectives. In our case, if α is close to 0, the minimization of the combined norm is oriented towards the generalized $\mathcal{H}_2$ norm, thus decreasing the influence of ω_n on the estimation error while increasing the one of u, and vice-versa when α is close to 1. To balance the trade-off, we have chosen $\alpha = 0.6$. The resulting norms are $\gamma_\infty = 1.12$ and $\gamma_2 = 3.8$. The frequency-domain plot below shows the effects of u and ω_n on the estimation errors of the five states, which are greatly attenuated.

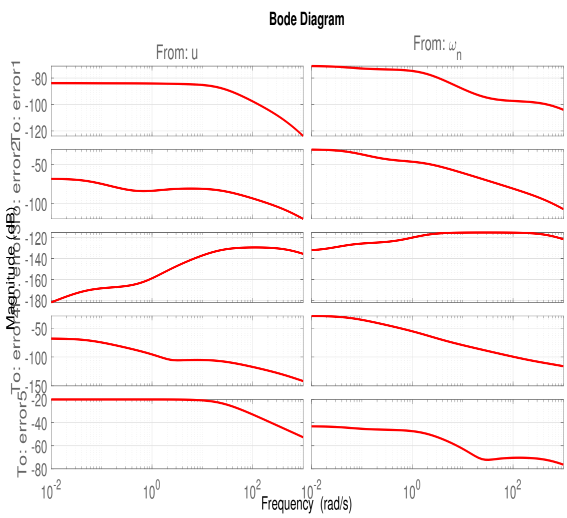

Figure 2. Bode plot of the estimation error system (10).

5. APPLICATION RESULTS

In this part, the performance of the designed multi-objective unified qLPV observer is examined using simulations with data obtained from a real testbed. It is also compared with that of the previously proposed linear time-invariant (LTI) $\mathcal{H}_\infty$ and $\mathcal{H}_2$ ones in (Pham et al., 2019a). We denote, for brevity:

- Our observer as observer 1;
- The LTI $\mathcal{H}_\infty$ observer as observer 2;
- The LTI $\mathcal{H}_2$ observer as observer 3.

5.1 Simulation Results

The simulation scenario is as follows:

- A chirp road profile of amplitude 4 mm and frequency varying from 0 to 5 Hz in 10 s is used;
- The control input u is given by the ADD output-feedback controller ($u \in [0.1, 0.35]$ in this case).

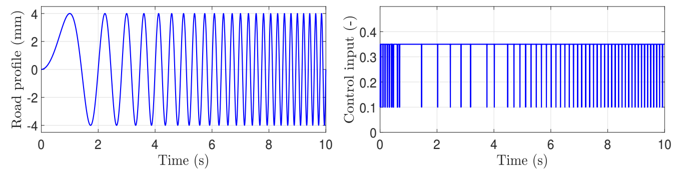

Figure 3. Simulation scenario: Road profile (left) and control input (right).

The simulation results are as follows:

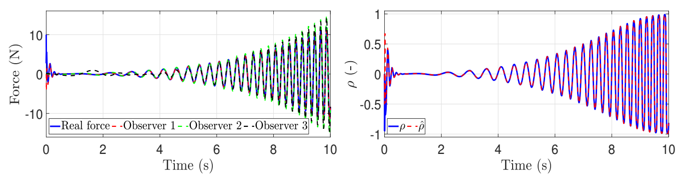

Figure 4. Simulation results: Damper force (left) and varying parameter estimation (right).

5.2 Simulation with Real Data Results

Experiments are performed with INOVE, our test platform at GIPSA-lab, which is equipped with semi-active suspension systems and force sensors (see http://www.gipsa-lab.fr/projet/inove/ for more details).

The experiment scenario is as follows:

- An ISO road profile of type C is used;
- The control input u is constant at 0.1.

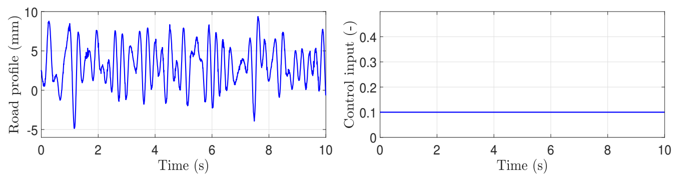

Figure 5. Simulation with real data scenario: Road profile (left) and control input (right).

We collect the resulting sensor measurements and perform simulations with these real data. The results are as follows:

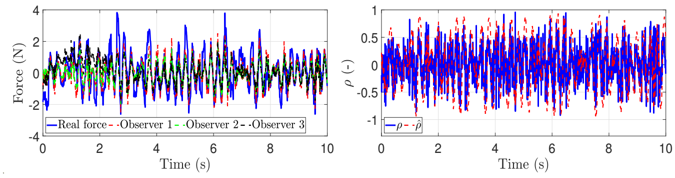

Figure 6. Simulation with real data results: Damper force (left) and varying parameter estimation (right).

5.3 Result Analysis

To compare the observers' performance in damper force estimation, the root-mean-square error (RMSE) between the real and estimated force is calculated for each case and shown in Table 1.

Table 1. Damper force estimation RMSE (N)

Observer	Simulation	Simulation with real data
Observer 1	0.3692	0.8295
Observer 2	0.9651	0.8767
Observer 3	0.9588	0.9478

Evidently, our observer provides a more accurate damper force estimation performance where the RMSE is smaller with the same initial conditions, road profile, and control input. Especially for this observer, as the effects of the unknown disturbance input are effectively decoupled from the estimation error, the force estimation performance does not depend on the road profile, which is not the case for the LTI observers (without parameterization).

6. CONCLUSION

This work proposes a multi-objective unified qLPV observer for LPV systems with a parameter-dependent control input matrix. Such an observer not only decouples the effects of bounded unknown input disturbance from the estimation error, but it minimizes the combined effects of the control input and the measurement noise on this error as well. The matrices of the unified observer are obtained by solving a multi-objective problem. This observer is then applied to an ER semi-active automotive suspension system for damper force estimation. Simulations with real data from the vehicle test platform confirm the effectiveness of this observer (compared to LTI ones).

REFERENCES

Chandra, K.P.B., Alwi, H., and Edwards, C. (2017). Fault Detection in Uncertain LPV Systems with Imperfect Scheduling Parameter using Sliding Mode Observers. *European Journal of Control*, 34, 1 – 15. doi: https://doi.org/10.1016/j.ejcon.2016.12.001.

Gao, N., Darouach, M., Voos, H., and Alma, M. (2016). New Unified $\mathcal{H}_\infty$ Dynamic Observer Design for Linear Systems with Unknown Inputs. *Automatica*, 65, 43–52. doi:https://doi.org/10.1016/j.automatica.2015.10.052.

Gómez-Peñate, S., López-Estrada, F.R., Valencia-Palomo, G., Rotondo, D., and Guerrero-Sánchez, M.E. (2020). Actuator and Sensor Fault Estimation based on a Proportional Multiple-integral Sliding Mode Observer for Linear Parameter Varying Systems with Inexact Scheduling Parameters. *International Journal of Robust and Nonlinear Control.* doi: https://doi.org/10.1002/rnc.5371.

Heemels, W.P.M.H., Daafouz, J., and Millerioux, G. (2010). Observer-Based Control of Discrete-Time LPV Systems With Uncertain Parameters. *IEEE Transactions on Automatic Control*, 55(9), 2130–2135. doi: https://doi.org/10.1109/TAC.2010.2051072.

Koch, G., Kloiber, T., and Lohmann, B. (2010). Nonlinear and Filter Based Estimation for Vehicle Suspension Control. In *49th IEEE Conference on Decision and Control (CDC)*, 5592–5597. doi: https://doi.org/10.1109/CDC.2010.5718052.

Pham, T.P., Sename, O., and Dugard, L. (2019). Unified $\mathcal{H}_\infty$ Observer for a Class of Nonlinear Lipschitz Systems: Application to a Real ER Automotive Suspension. *IEEE Control Systems Letters*, 3(4), 817–822. doi: https://doi.org/10.1109/LCSYS.2019.2919813.

Pham, T.P., Sename, O., and Dugard, L. (2019a). Comparative Study of Three Robust Observers for Automotive Damper Force Estimation. In *IOP Conference Series: Materials Science and Engineering*, volume 707, 012014. IOP Publishing. doi: https://doi.org/10.1088/1757-899x/707/1/012014.

Pham, T.P., Sename, O., and Dugard, L. (2019b). Real-time Damper Force Estimation of Vehicle Electrorheological Suspension: A NonLinear Parameter Varying Approach. *IFAC-PapersOnLine*, 52(28), 94–99. doi: https://doi.org/10.1016/j.ifacol.2019.12.354.

Sato, M. (2020). One-shot Design of Performance Scaling Matrices and Observer-based Gain-scheduled Controllers Depending on Inexact Scheduling Parameters. *Systems & Control Letters*, 137, 104632. doi: https://doi.org/10.1016/j.sysconle.2020.104632.

Savaresi, S., Poussot-Vassal, C., Spelta, C., Sename, O., and Dugard, L. (2010). *Semi-Active Suspension Control for Vehicles.* Elsevier - Butterworth Heinemann. doi: https://doi.org/10.1016/C2009-0-63839-3.

Scherer, C. and Weiland, S. (2000). Linear Matrix Inequalities in Control. *Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands*, 3(2).

Skelton, R., Iwasaki, T., and Grigoriadis, K. (1998). A Unified Algebraic Approach to Linear Control Design, 1998. *Taylor & Francis.* doi: https://doi.org/10.1201/9781315136523.

Tudon-Martinez, J.C., Hernandez-Alcantara, D., Sename, O., Morales-Menendez, R., and de J. Lozoya-Santos, J. (2018). Parameter-Dependent $\mathcal{H}_\infty$ Filter for LPV Semi-Active Suspension Systems. *IFAC-PapersOnLine*, 51(26), 19–24. doi: https://doi.org/10.1016/j.ifacol.2018.11.174.

Unger, A., Schimmack, F., Lohmann, B., and Schwarz, R. (2013). Application of LQ-based Semi-active Suspension Control in a Vehicle. *Control Engineering Practice*, 21(12), 1841–1850. doi: https://doi.org/10.1016/j.conengprac.2013.06.006.

Vela, A., Alcántara, D., Menendez, R., Sename, O., and Dugard, L. (2018). $\mathcal{H}_\infty$ Observer for Damper Force in a Semi-Active Suspension. *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 51(11), 764–769. doi: https://doi.org/10.1016/j.ifacol.2018.08.411.