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► **To cite this version:**

Philippe Balbiani, Joseph Boudou, Martín Diéguez, David Fernández-Duque. Bisimulations for intuitionistic temporal logics (2021). *Journal of Applied Logics - IfCoLoG Journal of Logics and their Applications*, 2021, 8 (8), pp.2265-2285. hal-03248077

HAL Id: hal-03248077

<https://hal.science/hal-03248077v1>

Submitted on 13 Jun 2022

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BISIMULATIONS FOR INTUITIONISTIC TEMPORAL LOGICS

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Abstract

We introduce bisimulations for the logic ITL^e with \circ ('next'), \mathcal{U} ('until') and \mathcal{R} ('release'), an intuitionistic temporal logic based on structures (W, \preceq, S) , where \preceq is used to interpret intuitionistic implication and S is a \preceq -monotone function used to interpret the temporal modalities. Our main results are that \diamond ('eventually'), which is definable in terms of \mathcal{U} , cannot be defined in terms of \circ and \square , and similarly that \square ('henceforth'), definable in terms of \mathcal{R} , cannot be defined in terms of \circ and \mathcal{U} , even over the smaller class of here-and-there models.

This research was partially supported by ANR-11-LABX-0040-CIMI within the program ANR-11-IDEX-0002-02.

1 Introduction

The definition and study of full combinations of modal [5] and intuitionistic [6, 23] logics can be quite challenging [30], and temporal logics, such as LTL [28], are no exception. Some intuitionistic analogues of temporal logics have been proposed, including logics with ‘past’ and ‘future’ tenses [9], or with ‘next’ [7, 19] and ‘henceforth’ [17]. We proposed an alternative formulation in [4], where we defined the logics ITL^e and ITL^p using semantics similar to those of *expanding* and *persistent* products of modal logics, respectively [13], and the tenses \circ (‘next’), \diamond (‘eventually’), and \square (‘henceforth’). ITL^e in particular differs from previous proposals (e.g. [9, 27]) in that we consider minimal frame conditions that allow for all formulas to be upward-closed under the intuitionistic preorder, which we denote \preceq . We then showed that ITL^e with \circ (‘next’), \diamond (‘eventually’), and \square (‘henceforth’) is decidable, thus obtaining the first intuitionistic analogue of LTL which contains the three tenses, is conservative over propositional intuitionistic logic, is interpreted over unbounded time, and is known to be decidable.

Note that both \diamond and \square are taken as primitives, in contrast with the classical case, where $\diamond\varphi$ may be defined by $\diamond\varphi \equiv \neg\square\neg\varphi$, whereas the latter equivalence is not intuitionistically valid. The same situation holds in the more expressive language with \mathcal{U} (‘until’): while the language with \circ and \mathcal{U} is equally expressive to classical monadic first-order logic with \leq over \mathbb{N} [12], \mathcal{U} admits a first-order definable intuitionistic dual, \mathcal{R} (‘release’), which cannot be defined in terms of \mathcal{U} using the classical definition. However, this is not enough to conclude that \mathcal{R} cannot be defined in a different way. Thus, while in [4] we explored the question of decidability, here we will focus on *definability*; which of the modal operators can be defined in terms of the others?

Following Simpson [30] and other authors, we interpret the language of ITL^e using bi-relational structures, with a partial order \preceq to interpret intuitionistic implication, and a function or relation, which we denote S , representing the passage of time. Alternatively, one may consider topological interpretations [8], but we will not discuss those here. Various intuitionistic temporal logics have been considered, using variants of these semantics and different formal languages. The main contributions include:

- Davies’ intuitionistic temporal logic with \circ [7] was provided Kripke semantics and a complete deductive system by Kojima and Igarashi [19].
- Logics with \circ, \square were axiomatized by Kamide and Wansing [17], where \square was interpreted over bounded time.

- Nishimura [25] provided a sound and complete axiomatization for an intuitionistic variant of the propositional dynamic logic PDL.
- Balbiani and Diéguez axiomatized the here-and-there variant of LTL with \circ, \diamond, \square [2], here denoted ITL^{ht} .
- Fernández-Duque [10] proved the decidability of a logic based on topological semantics with \circ, \diamond and a universal modality.
- The authors [4] proved that the logic ITL^e with \circ, \diamond, \square has the strong finite model property and hence is decidable, yet the logic ITL^p , based on a more restrictive class of frames, does not enjoy the fmp.

In this paper, we extend ITL^e to include \mathcal{U} ('until') and \mathcal{R} ('release'). As is well-known, $\diamond\varphi \equiv \top\mathcal{U}\varphi$ and $\square\varphi \equiv \perp\mathcal{R}\varphi$; these equivalences remain valid in the intuitionistic setting, but many of the tenses are no longer inter-definable as in the classical case. To show this, we will introduce different notions of bisimulation which preserve formulas with \circ and each of $\diamond, \square, \mathcal{U}$ and \mathcal{R} . With this, we will show that \mathcal{R} (or even \square) may not be defined in terms of \mathcal{U} over the class of here-and-there models, while \diamond *can* be defined in terms of \square , and \mathcal{U} can be defined in terms of \mathcal{R} over this class. However, we show that over the wider class of expanding models, \diamond cannot be defined in terms of \square .

2 Syntax and semantics

We will work in sublanguages of the language \mathcal{L} given by the following grammar:

$$\varphi, \psi := p \mid \perp \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \circ\varphi \mid \diamond\varphi \mid \square\varphi \mid \varphi\mathcal{U}\psi \mid \varphi\mathcal{R}\psi$$

where p is an element of a countable set of propositional variables \mathbb{P} . All sublanguages we will consider include all Boolean operators and \circ , hence we denote them by displaying the additional connectives as a subscript; for example, $\mathcal{L}_{\diamond\square}$ denotes the \mathcal{U} -free, \mathcal{R} -free fragment. As an exception to this general convention, \mathcal{L}_{\circ} denotes the fragment without $\diamond, \square, \mathcal{U}$ or \mathcal{R} . As in the propositional case, $\neg\varphi \stackrel{\text{def}}{=} \varphi \rightarrow \perp$.

Given any formula φ , we define the *length* of φ (in symbols, $|\varphi|$) recursively as follows:

- $|p| = |\perp| = 0$;
- $|\phi \odot \psi| = 1 + |\phi| + |\psi|$, with $\odot \in \{\vee, \wedge, \rightarrow, \mathcal{R}, \mathcal{U}\}$;
- $|\odot\psi| = 1 + |\psi|$, with $\odot \in \{\neg, \circ, \square, \diamond\}$.

Broadly speaking, the length of a formula φ corresponds to the number of connectives appearing in φ .

2.1 Dynamic posets

Formulas of \mathcal{L} are interpreted over dynamic posets. A *dynamic poset* is a tuple $\mathcal{D} = (W, \preceq, S)$, where W is a non-empty set of states, \preceq is a partial order, and S is a function from W to W satisfying the *forward confluence* condition that for all $w, v \in W$, if $w \preceq v$ then $S(w) \preceq S(v)$. An *intuitionistic dynamic model*, or simply *model*, is a tuple $\mathcal{M} = (W, \preceq, S, V)$ consisting of a dynamic poset equipped with a valuation function V from W to sets of propositional variables that is \preceq -monotone, in the sense that for all $w, v \in W$, if $w \preceq v$ then $V(w) \subseteq V(v)$. In the standard way, we define $S^0(w) = w$ and, for all $k > 0$, $S^k(w) = S(S^{k-1}(w))$. Then we define the satisfaction relation \models inductively by:

1. $\mathcal{M}, w \models p$ iff $p \in V(w)$;
2. $\mathcal{M}, w \not\models \perp$;
3. $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$;
4. $\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$;
5. $\mathcal{M}, w \models \bigcirc \varphi$ iff $\mathcal{M}, S(w) \models \varphi$;
6. $\mathcal{M}, w \models \varphi \rightarrow \psi$ iff $\forall v \succeq w$, if $\mathcal{M}, v \models \varphi$, then $\mathcal{M}, v \models \psi$;
7. $\mathcal{M}, w \models \diamond \varphi$ iff there exists k s.t. $\mathcal{M}, S^k(w) \models \varphi$;
8. $\mathcal{M}, w \models \square \varphi$ iff for all k , $\mathcal{M}, S^k(w) \models \varphi$;
9. $\mathcal{M}, w \models \varphi \mathcal{U} \psi$ iff there exists $k \geq 0$ s.t. $\mathcal{M}, S^k(w) \models \psi$ and $\forall i \in [0, k)$, $\mathcal{M}, S^i(w) \models \varphi$;
10. $\mathcal{M}, w \models \varphi \mathcal{R} \psi$ iff for all $k \geq 0$, either $\mathcal{M}, S^k(w) \models \psi$, or $\exists i \in [0, k)$ s.t. $\mathcal{M}, S^i(w) \models \varphi$.

As usual, a formula φ is *satisfiable over a class of models* Ω if there is a model $\mathcal{M} \in \Omega$ and a world w of \mathcal{M} so that $\mathcal{M}, w \models \varphi$, and *valid over* Ω if, for every world w of every model $\mathcal{M} \in \Omega$, $\mathcal{M}, w \models \varphi$. Satisfiability (validity) over the class of models based on an arbitrary dynamic poset will be called *satisfiability (validity)* for ITL^e , or *expanding domain linear temporal logic*.¹

The relation between dynamic posets and expanding products of modal logics is detailed in [4], where the following is also shown. Below, we use the notation $\llbracket \varphi \rrbracket = \{w \in W \mid \mathcal{M}, w \models \varphi\}$.

Lemma 2.1. *Let $\mathcal{D} = (W, \preceq, S)$, where (W, \preceq) is a poset and $S:W \rightarrow W$ is any function. Then, \mathcal{D} is a dynamic poset if and only if, for every valuation V on W and every formula φ , $\llbracket \varphi \rrbracket$ is \preceq -monotone, i.e., if $w \in \llbracket \varphi \rrbracket$ and $v \succeq w$, then $v \in \llbracket \varphi \rrbracket$.*

Proof. The left to right direction is proved by induction on φ . The case of $\varphi \in \mathbb{P}$ is proved by using the condition on V . The rest of the inductive steps are routine. For

¹Note that in [4] we used ‘ ITL^e ’ to denote the fragment of this logic without \mathcal{U}, \mathcal{R} .

instance, let us consider the case of $\varphi\mathcal{U}\psi$ and suppose that $v \succcurlyeq w$ and $w \in \llbracket \varphi\mathcal{U}\psi \rrbracket$. Then, there exists $k \geq 0$ such that $\mathcal{M}, S^k(w) \models \psi$ and for all $0 \leq j < k$, $\mathcal{M}, S^j(w) \models \varphi$. Since S is confluent, an easy induction shows that $S^i(v) \succcurlyeq S^i(w)$ for all $0 \leq i \leq k$. Therefore, by induction hypothesis, we get $\mathcal{M}, S^k(v) \models \psi$ and for all $0 \leq j < k$, $\mathcal{M}, S^j(v) \models \varphi$, hence $v \in \llbracket \varphi\mathcal{U}\psi \rrbracket$. For the converse direction we assume that $\mathcal{D} = (W, \preccurlyeq, S)$ and $w, v \in W$ such that $v \succcurlyeq w$ and $S(w) \not\preccurlyeq S(v)$. Take $p \in \mathbb{P}$ and define $V(u) = \{p\}$ if $S(w) \preccurlyeq u$, $V(u) = \emptyset$ otherwise. It is easy to see that V is \preccurlyeq -monotone, but $p \notin V(S(v))$ (because $S(w) \not\preccurlyeq S(v)$) and $p \in V(S(w))$ (because $S(w) \preccurlyeq S(w)$), from which it follows that $(\mathcal{D}, V), w \models \text{Op}$ but $(\mathcal{D}, V), v \not\models \text{Op}$. \square

This suggests that dynamic posets provide suitable semantics for intuitionistic LTL. Moreover, dynamic posets are convenient from a technical point of view:

Theorem 2.2 ([4]). *There exists a computable function B such that any formula $\varphi \in \mathcal{L}_{\diamond\Box}$ satisfiable (resp. falsifiable) on an arbitrary model is satisfiable (resp. falsifiable) on a model whose size is bounded by $B(|\varphi|)$.*

It follows that the $\mathcal{L}_{\diamond\Box}$ -fragment of ITL^e is decidable. Moreover, as we will see below, many of the familiar axioms of classical LTL are valid over the class of dynamic posets, making them a natural choice of semantics for intuitionistic LTL.

2.2 Persistent posets

Despite the advantages of dynamic posets, in the literature one typically considers a more restrictive class of frames, as we define them below.

Definition 2.3. *Let (W, \preccurlyeq) be a poset. If $S:W \rightarrow W$ is such that, whenever $v \succcurlyeq S(w)$, there is $u \succcurlyeq w$ such that $v = S(u)$, we say that S is backward confluent. If S is both forward and backward confluent, we say that it is persistent. A tuple (W, \preccurlyeq, S) where S is persistent is a persistent intuitionistic temporal frame, and the set of valid formulas over the class of persistent intuitionistic temporal frames is denoted ITL^p , or persistent domain LTL.*

As we will see, persistent frames do have some technical advantages over arbitrary dynamic posets. Nevertheless, they have a crucial disadvantage:

Theorem 2.4 ([4]). *The logic ITL^p does not have the finite model property, even for formulas in $\mathcal{L}_{\diamond\Box}$.*

2.3 Temporal here-and-there models

An even smaller class of models which, nevertheless, has many applications is that of temporal here-and-there models [2]. Some of the results we will present here apply to this class, so it will be instructive to review it. Recall that the logic of here-and-there is the maximal logic strictly between classical and intuitionistic propositional logic, given by a frame $\{0, 1\}$ with $0 \leq 1$. The logic of here-and-there is obtained by adding to intuitionistic propositional logic the axiom $p \vee (p \rightarrow q) \vee \neg q$.

A temporal here-and-there frame is a persistent frame that is ‘locally’ based on this frame. We can define here-and-there models using the following construction.

Definition 2.5. *Let T be a set and $f: T \rightarrow T$. We define a dynamic poset $\text{HT}(T, f) = (W, \leq, S)$, with $W = T \times \{0, 1\}$, $(t, i) \leq (s, j)$ if and only if $t = s$ and $i \leq j$, and $S(t, i) = (f(t), i)$.*

The prototypical example is the frame $\text{HT}(\mathbb{N}, f)$, where $f(n) = n + 1$. Note, however, that our definition allows for other values of T (see Figure 1). In [2], this logic is axiomatized, and it is shown that \Box cannot be defined in terms of \Diamond , a result we will strengthen here to show that \Box cannot be defined even in terms of \mathcal{U} . It is also claimed in [2] that \Diamond is not definable in terms of \Box over the class of here-and-there models, but as we will see in Proposition 6.3, this claim is incorrect.

3 Some valid and non-valid ITL^e -formulas

In this section we explore which axioms of classical LTL are still valid in our setting. We start by showing that the intuitionistic version of the interaction and induction axioms used in [2] remain valid in our setting. However, not all Fisher-Servi axioms [11], which are valid in the here-and-there LTL of [2], are valid in ITL^e .

Proposition 3.1. *The following formulas:*

- | | |
|--|---|
| 1. $\Box \perp \leftrightarrow \perp$ | 7. $\Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi)$; |
| 2. $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$; | 8. $\Box\varphi \leftrightarrow \varphi \wedge \Box\varphi$; |
| 3. $\Box(\varphi \vee \psi) \leftrightarrow (\Box\varphi \vee \Box\psi)$; | 9. $\varphi \vee \Box\Diamond\varphi \leftrightarrow \Diamond\varphi$; |
| 4. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$; | 10. $\Box(\varphi \rightarrow \Box\varphi) \rightarrow (\varphi \rightarrow \Box\varphi)$ |
| 5. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$; | 11. $\Box(\Box\varphi \rightarrow \varphi) \rightarrow (\Diamond\varphi \rightarrow \varphi)$. |
| 6. $\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$; | |

are ITL^e -valid.

Proof. Let us consider (10) and (11). For (10), let $\mathcal{M} = (W, \leq, S)$ be any ITL^e model and $w \in W$ be such that $\mathcal{M}, w \models \Box(\varphi \rightarrow \Box\varphi)$. Let $v \geq w$ be arbitrary and assume that $\mathcal{M}, v \models \varphi$. Then, by induction on i we obtain that $S^i(w) \leq S^i(v)$ for all i ; since $\mathcal{M}, S^i(w) \models \varphi \rightarrow \Box\varphi$ for all i , it follows that $\mathcal{M}, S^i(v) \models \varphi \rightarrow \Box\varphi$ for all i as well. Hence an easy induction shows that $\mathcal{M}, S^i(v) \models \varphi$ for all i , which means that $\mathcal{M}, v \models \Box\varphi$. Since w was arbitrary, we conclude that the formula (10) is valid.

For (11), let \mathcal{M} be as above and $w \in W$ be such that $\mathcal{M}, w \models \Box(\Box\varphi \rightarrow \varphi)$. Let $v \geq w$ be such that $\mathcal{M}, v \models \Diamond\varphi$, and let n be least so that $\mathcal{M}, S^n(v) \models \varphi$. If $n > 0$ then from $\Box\varphi \rightarrow \varphi$ we obtain $\mathcal{M}, S^{n-1}(v) \models \varphi$, contradicting the minimality of n . We conclude that $n = 0$, hence $\mathcal{M}, v \models \varphi$.

The proofs for the rest of formulas are standard. \square

Some of the well-known Fisher Servi axioms [11] are only valid on the class of persistent frames.

Proposition 3.2. *The formulas*

1. $(\Box\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi)$,
2. $(\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi)$

are not ITL^e -valid. However they are ITL^P -valid.

Proof. Let $\{p, q\}$ be a set of propositional variables and let us consider the ITL^e model $\mathcal{M} = (W, \leq, S, V)$ defined as: 1) $W = \{w, v, u\}$; 2) $S(w) = v$, $S(v) = v$ and $S(u) = u$; 3) $v \leq u$; 4) $V(p) = \{u\}$. Clearly, $\mathcal{M}, u \not\models p \rightarrow q$, so $\mathcal{M}, v \not\models p \rightarrow q$. By definition, $\mathcal{M}, w \not\models \Box(p \rightarrow q)$ and $\mathcal{M}, w \not\models \Box(\Box(p \rightarrow q))$; however, it can easily be checked that $\mathcal{M}, w \models \Box p \rightarrow \Box q$ and $\mathcal{M}, w \models \Diamond p \rightarrow \Box q$, so $\mathcal{M}, w \not\models (\Box p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$ and $\mathcal{M}, w \not\models (\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$.

Let us check their validity over the class of persistent frames. For (1), let $\mathcal{M} = (W, \leq, S, V)$ be an ITL^P model and w a world of \mathcal{M} such that $\mathcal{M}, w \models \Box\varphi \rightarrow \Box\psi$. Suppose that $v \geq S(w)$ satisfies $\mathcal{M}, v \models \varphi$. By backward confluence, there exists $u \geq w$ such that $v = S(u)$, so that $\mathcal{M}, u \models \Box\varphi$ and thus $\mathcal{M}, u \models \Box\psi$. But this means that $\mathcal{M}, v \models \psi$, and since $v \geq S(w)$ was arbitrary, $\mathcal{M}, S(w) \models \varphi \rightarrow \psi$, i.e. $\mathcal{M}, w \models \Box(\varphi \rightarrow \psi)$.

Similarly, for (2) let us assume that $\mathcal{M} = (W, \leq, S, V)$ is an ITL^P model and w a world of \mathcal{M} such that $\mathcal{M}, w \models \Diamond\varphi \rightarrow \Box\psi$. Consider arbitrary $k \in \mathbb{N}$, and suppose that $v \geq S^k(w)$ is such that $\mathcal{M}, v \models \varphi$. Then, it is readily checked that the composition of backward confluent functions is backward confluent, so that in particular S^k is backward confluent. This means that there is $u \geq w$ such that $S^k(u) = v$. But then, $\mathcal{M}, u \models \Diamond\varphi$, hence $\mathcal{M}, u \models \Box\psi$, and $\mathcal{M}, v \models \psi$. It follows that $\mathcal{M}, S^k(w) \models \varphi \rightarrow \psi$, and since k was arbitrary, $\mathcal{M}, w \models \Box(\varphi \rightarrow \psi)$. \square

We make a special mention of the schema $\Box(\Box\varphi \rightarrow \psi) \vee \Box(\Box\psi \rightarrow \varphi)$, which characterises the class of *weakly connected frames* [14] in classical modal logic. We say that a frame (W, R, V) is weakly connected iff it satisfies the following first-order property: for every $x, y, z \in W$, if $x R y$ and $x R z$, then either $y R z$, $y = z$, or $z R y$.

Proposition 3.3. *The axiom schema $\Box(\Box\varphi \rightarrow \psi) \vee \Box(\Box\psi \rightarrow \varphi)$ is not ITL^{ht} -valid.*

Proof. Let us consider the set of propositional variables $\{p, q\}$, $T = \{0, 1\}$, $f: T \rightarrow T$ be given by $f(x) = 1$, and let $\mathcal{M} = (W, \leq, S, V)$ be the here-and-there model based on $\text{HT}(T, f)$ with $V(p) = \{(0, 1), (1, 1)\}$ and $V(q) = \{(1, 0), (1, 1)\}$. The reader can check that $\mathcal{M}, (0, 0) \not\models \Box p \rightarrow q$ and $\mathcal{M}, (0, 1) \not\models \Box q \rightarrow p$. Consequently, $\mathcal{M}, w \not\models \Box(\Box p \rightarrow q) \vee \Box(\Box q \rightarrow p)$. \square

Finally, we show that $\Diamond\varphi$ (resp. $\Box\varphi$) can be defined in terms of \mathcal{U} (resp. \mathcal{R}) and the LTL axioms involving \mathcal{U} and \mathcal{R} are also valid in our setting:

Proposition 3.4. *The following formulas are ITL^e -valid:*

- | | |
|--|---|
| 1. $\varphi \mathcal{U} \psi \leftrightarrow \psi \vee (\varphi \wedge \circ(\varphi \mathcal{U} \psi))$; | 5. $\Diamond\varphi \leftrightarrow \top \mathcal{U} \varphi$; |
| 2. $\varphi \mathcal{R} \psi \leftrightarrow \psi \wedge (\varphi \vee \circ(\varphi \mathcal{R} \psi))$; | 6. $\Box\varphi \leftrightarrow \perp \mathcal{R} \varphi$; |
| 3. $\varphi \mathcal{U} \psi \rightarrow \Diamond\psi$; | 7. $\circ(\varphi \mathcal{U} \psi) \leftrightarrow \circ\varphi \mathcal{U} \circ\psi$; |
| 4. $\Box\psi \rightarrow \varphi \mathcal{R} \psi$; | 8. $\circ(\varphi \mathcal{R} \psi) \leftrightarrow \circ\varphi \mathcal{R} \circ\psi$. |

Proof. We consider some cases below. For (1), from left to right, let us assume that $\mathcal{M}, w \models \varphi \mathcal{U} \psi$. Therefore there exists $k \geq 0$ s.t. $\mathcal{M}, S^k(w) \models \psi$ and for all j satisfying $0 \leq j < k$, $\mathcal{M}, S^j(w) \models \varphi$. If $k = 0$ then $\mathcal{M}, w \models \psi$ while, if $k > 0$ it follows that $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, S(w) \models \varphi \mathcal{U} \psi$. Therefore $\mathcal{M}, w \models \psi \vee (\varphi \wedge \circ(\varphi \mathcal{U} \psi))$. From right to left, if $\mathcal{M}, w \models \psi$ then $\mathcal{M}, w \models \varphi \mathcal{U} \psi$ by definition. If $\mathcal{M}, w \models \varphi \wedge \circ(\varphi \mathcal{U} \psi)$ then $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, S(w) \models \varphi \mathcal{U} \psi$ so, due to the semantics, we conclude that $\mathcal{M}, w \models \varphi \mathcal{U} \psi$. In any case, $\mathcal{M}, w \models \varphi \mathcal{U} \psi$.

For (2), we work by contrapositive. From right to left, let us assume that $\mathcal{M}, w \not\models \varphi \mathcal{R} \psi$. Therefore there exists $k \geq 0$ s.t. $\mathcal{M}, S^k(w) \not\models \psi$ and for all j satisfying $0 \leq j < k$, $\mathcal{M}, S^j(w) \not\models \varphi$. If $k = 0$ then $\mathcal{M}, w \not\models \psi$ while, if $k > 0$ it follows that $\mathcal{M}, w \not\models \varphi$ and $\mathcal{M}, S(w) \not\models \varphi \mathcal{R} \psi$. In any case, $\mathcal{M}, w \not\models \psi \wedge (\varphi \vee \circ(\varphi \mathcal{R} \psi))$. From left to right, if $\mathcal{M}, w \not\models \psi$ then $\mathcal{M}, w \not\models \varphi \mathcal{R} \psi$ by definition. If $\mathcal{M}, w \not\models \varphi \vee \circ(\varphi \mathcal{R} \psi)$ then $\mathcal{M}, w \not\models \varphi$ and $\mathcal{M}, S(w) \not\models \varphi \mathcal{U} \psi$ so, due to the semantics of \mathcal{R} , we conclude that $\mathcal{M}, w \not\models \varphi \mathcal{R} \psi$. In any case, $\mathcal{M}, w \not\models \varphi \mathcal{R} \psi$.

For (7), from left to right, let us assume that $\mathcal{M}, w \models \circ(\varphi \mathcal{U} \psi)$. Therefore there exists $k \geq 0$ s.t. $\mathcal{M}, S^{k+1}(w) \models \psi$ and for all j satisfying $0 \leq j < k$, $\mathcal{M}, S^{j+1}(w) \models \varphi$. It follows from $\mathcal{M}, S^{k+1}(w) \models \psi$ that $\mathcal{M}, S^k(w) \models \circ\psi$, and from $\mathcal{M}, S^{j+1}(w) \models \varphi$

that $\mathcal{M}, S^j(w) \models \circ\varphi$ for all $j < k$. We conclude that $\mathcal{M}, w \models \circ\varphi\mathcal{U}\circ\psi$. Conversely, if $\mathcal{M}, w \models \circ\varphi\mathcal{U}\circ\psi$, then there is $k \geq 0$ so that $\mathcal{M}, S^k(w) \models \circ\psi$ and, for all $i < k$, $\mathcal{M}, S^i(w) \models \circ\varphi$. It follows that $\mathcal{M}, S^{k+1}(w) \models \psi$ and, for all $i < k$, $\mathcal{M}, S^{i+1}(w) \models \varphi$, witnessing that $\mathcal{M}, S(w) \models \varphi\mathcal{U}\psi$ and $\mathcal{M}, w \models \circ(\varphi\mathcal{U}\psi)$.

For (8), we proceed similarly, but work by contrapositive. From right to left, let us assume that $\mathcal{M}, w \not\models \circ(\varphi\mathcal{R}\psi)$. Therefore there exists $k \geq 0$ s.t. $\mathcal{M}, S^{k+1}(w) \not\models \psi$ and for all j satisfying $0 \leq j < k$, $\mathcal{M}, S^{j+1}(w) \not\models \varphi$. This implies that $\mathcal{M}, S^k(w) \not\models \circ\psi$ and for all j satisfying $0 \leq j < k$, $\mathcal{M}, S^j(w) \not\models \circ\varphi$, hence $\mathcal{M}, w \not\models \circ(\varphi\mathcal{R}\psi)$. Similarly, if $\mathcal{M}, w \not\models \circ(\varphi\mathcal{R}\psi)$ then any $k \geq 0$ so that $\mathcal{M}, S^k(w) \not\models \circ\psi$ and, for all $i < k$, $\mathcal{M}, S^i(w) \not\models \circ\varphi$ yields $\mathcal{M}, S^{k+1}(w) \not\models \psi$ and, for all $i < k$, $\mathcal{M}, S^{i+1}(w) \not\models \varphi$, witnessing that $\mathcal{M}, S(w) \not\models \varphi\mathcal{R}\psi$ and $\mathcal{M}, w \not\models \circ(\varphi\mathcal{R}\psi)$.

The proof of the remaining items is routine. \square

As in the classical case, over the class of persistent models we can ‘push down’ all occurrences of \circ to the propositional level. Say that a formula φ is in \circ -normal form if all occurrences of \circ are of the form $\circ^i p$, with p a propositional variable.

Theorem 3.5. *Given $\varphi \in \mathcal{L}$, there exists $\tilde{\varphi}$ in \circ -normal form such that $\varphi \leftrightarrow \tilde{\varphi}$ is valid over the class of persistent models.*

Proof. The claim can be proven by structural induction using the validities in Propositions 3.1, 3.2 and 3.4. \square

We remark that the only reason that this argument does not apply to arbitrary ITL^e models is the fact that $(\circ\varphi \rightarrow \circ\psi) \rightarrow \circ(\varphi \rightarrow \psi)$ is not valid in general (Proposition 3.2).

4 Bounded bisimulations for \diamond and \square

In this section we adapt the classical definition of bounded bisimulations for modal logic [3] to our case. To do so we combine the ordinary definition of bounded bisimulations with the work of [26] on bisimulations for propositional intuitionistic logic. Such work introduces extra conditions involving the partial order \leq . In our setting, we combine both approaches in order to define bisimulation for a language involving \diamond , \square and \circ as modal operators plus an intuitionistic \rightarrow . Since all languages we consider contain Booleans and \circ , it is convenient to begin with a ‘basic’ notion of bisimulation for this language.

Definition 4.1. *Given $n > 0$ and two ITL^e models \mathcal{M}_1 and \mathcal{M}_2 , a sequence of binary relations $\mathcal{Z}_n \subseteq \dots \subseteq \mathcal{Z}_0 \subseteq W_1 \times W_2$ is said to be a bounded \circ -bisimulation if for all $(w_1, w_2) \in W_1 \times W_2$ and for all $0 \leq i < n$, the following conditions are satisfied:*

ATOMS. If $w_1 \mathcal{Z}_i w_2$ then for all propositional variables p , $\mathcal{M}_1, w_1 \models p$ iff $\mathcal{M}_2, w_2 \models p$.

FORTH \rightarrow . If $w_1 \mathcal{Z}_{i+1} w_2$ then for all $v_1 \in W_1$, if $v_1 \succcurlyeq w_1$, there exists $v_2 \in W_2$ such that $v_2 \succcurlyeq w_2$ and $v_1 \mathcal{Z}_i v_2$.

BACK \rightarrow . If $w_1 \mathcal{Z}_{i+1} w_2$ then for all $v_2 \in W_2$ if $v_2 \succcurlyeq w_2$ then there exists $v_1 \in W_1$ such that $v_1 \succcurlyeq w_1$ and $v_1 \mathcal{Z}_i v_2$.

FORTH \circ . if $w_1 \mathcal{Z}_{i+1} w_2$ then $S(w_1) \mathcal{Z}_i S(w_2)$.

Note that there is not ‘back’ clause for \circ ; this is simply because S is a function, so its ‘forth’ and ‘back’ clauses are identical. Bounded \circ -bisimulations are useful because they preserve the truth of relatively small \mathcal{L}_\circ -formulas.

Lemma 4.2. *Given two ITL^e models \mathcal{M}_1 and \mathcal{M}_2 and a bounded \circ -bisimulation $\mathcal{Z}_n \sqsubseteq \dots \sqsubseteq \mathcal{Z}_0$ between them, for all $0 \leq i \leq n$ and $(w_1, w_2) \in W_1 \times W_2$, if $w_1 \mathcal{Z}_i w_2$ then for all $\varphi \in \mathcal{L}_\circ$ satisfying $|\varphi| \leq i^2$, $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.*

Proof. We proceed by induction on i . Let $0 \leq i \leq n$ be such that for all $j < i$ the lemma holds. Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $w_1 \mathcal{Z}_i w_2$ and let us consider $\varphi \in \mathcal{L}_\circ$ such that $|\varphi| \leq i$. The cases where φ is an atom or of the forms $\theta \wedge \psi$, $\theta \vee \psi$ are as in the classical case and we omit them. Thus we focus on the following:

CASE $\varphi = \theta \rightarrow \psi$. We proceed by contrapositive to prove the left-to-right implication. Note that in this case we must have $i > 0$.

Assume that $\mathcal{M}_2, w_2 \not\models \theta \rightarrow \psi$. Therefore there exists $v_2 \in W_2$ such that $v_2 \succcurlyeq w_2$, $\mathcal{M}_2, v_2 \models \theta$, and $\mathcal{M}_2, v_2 \not\models \psi$. By the BACK \rightarrow condition, it follows that there exists $v_1 \in W_1$ such that $v_1 \succcurlyeq w_1$ and $v_1 \mathcal{Z}_{i-1} v_2$. Since $|\theta| \leq i-1$ and $|\psi| \leq i-1$, by the induction hypothesis, it follows that $\mathcal{M}_1, v_1 \models \theta$ and $\mathcal{M}_1, v_1 \not\models \psi$. Consequently, $\mathcal{M}_1, w_1 \not\models \theta \rightarrow \psi$. The converse direction is proved in a similar way but using FORTH \rightarrow .

CASE $\varphi = \circ\psi$. Once again we have that $i > 0$. Assume that $\mathcal{M}_1, w_1 \models \circ\psi$, so that $\mathcal{M}_1, S(w_1) \models \psi$. By FORTH \circ , $S_1(w_1) \mathcal{Z}_{i-1} S_2(w_2)$. Moreover, $|\psi| \leq i-1$, so that by the induction hypothesis, $\mathcal{M}_2, S(w_2) \models \psi$, and $\mathcal{M}_2, w_2 \models \circ\psi$. The right-to-left direction is analogous. \square

Next, we will extend the notion of a bounded \circ -bisimulation to include other tenses. Let us begin with \diamond .

²Although not optimal, we use the length of the formula in this lemma for the sake of simplicity. More precise measures like counting the number of modalities and implications could be equally used.

Definition 4.3. Given $n > 0$ and two ITL^e models \mathcal{M}_1 and \mathcal{M}_2 , a bounded \circ -bisimulation $\mathcal{Z}_n \subseteq \dots \subseteq \mathcal{Z}_0 \subseteq W_1 \times W_2$ is said to be a bounded \diamond -bisimulation if for all $(w_1, w_2) \in W_1 \times W_2$ and for all $0 \leq i < n$, if $w_1 \mathcal{Z}_{i+1} w_2$, then the following conditions are satisfied:

FORTH \diamond . For all $k_1 \geq 0$ there exist $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that $S^{k_2}(w_2) \succcurlyeq v_2$, $v_1 \succcurlyeq S^{k_1}(w_1)$ and $v_1 \mathcal{Z}_i v_2$.

BACK \diamond . For all $k_2 \geq 0$ there exist $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that $S^{k_1}(w_1) \succcurlyeq v_1$, $v_2 \succcurlyeq S^{k_2}(w_2)$ and $v_1 \mathcal{Z}_i v_2$.

The reader will notice that the clauses for \diamond involve the intuitionistic partial order, even though this is not involved in the semantics of \diamond . However, this will give us more flexibility in designing bisimulations. The reason it works is that if k_1 is so that $S^{k_1}(w_1)$ witnesses that $\diamond\varphi$ is true on w_1 , then φ will also be true on any $v_1 \succcurlyeq S^{k_1}(w_1)$ by the monotonicity of intuitionistic truth. Similarly, if $S^{k_2}(w_2) \succcurlyeq v_2$ and φ holds on v_2 , then it will also hold on $S^{k_2}(w_2)$. Thus we do not need $S^{k_1}(w_1)$ and $S^{k_2}(w_2)$ to be directly connected by the bisimulation; rather, it is sufficient for v_1, w_1 to act as ‘proxies’. As was the case of Lemma 4.2, if two worlds are related by a bounded \diamond -bisimulation, then they satisfy the same \mathcal{L}_\diamond -formulas of small length.

Lemma 4.4. Given two ITL^e models \mathcal{M}_1 and \mathcal{M}_2 and a bounded \diamond -bisimulation $\mathcal{Z}_i \subseteq \dots \subseteq \mathcal{Z}_0$ between them, for all $0 \leq i \leq n$ and $(w_1, w_2) \in W_1 \times W_2$, if $w_1 \mathcal{Z}_i w_2$, then for all $\beta \varphi \in \mathcal{L}_\diamond$ satisfying $|\varphi| \leq i$, $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.

Proof. We proceed by induction on n . Let $0 \leq i \leq n$ be such that for all $j < i$ the lemma holds. Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $w_1 \mathcal{Z}_i w_2$ and let us consider $\varphi \in \mathcal{L}_\diamond$ such that $|\varphi| \leq i$. We only consider the case where $\varphi = \diamond\psi$, as other cases are covered by Lemma 4.2.

From left to right, if $\mathcal{M}_1, w_1 \models \diamond\psi$ then there exists $k_1 \geq 0$ such that $\mathcal{M}_1, S^{k_1}(w_1) \models \psi$. By FORTH \diamond , there exists $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that $S^{k_2}(w_2) \succcurlyeq v_2$, $v_1 \succcurlyeq S^{k_1}(w_1)$ and $v_1 \mathcal{Z}_{i-1} v_2$. By \preccurlyeq -monotonicity, $\mathcal{M}_1, v_1 \models \psi$. Then, by the induction hypothesis and the fact that $|\psi| \leq i - 1$, it follows that $\mathcal{M}_2, v_2 \models \psi$, thus by \preccurlyeq -monotonicity once again, $\mathcal{M}_2, S^{k_2}(w_2) \models \psi$, so that $\mathcal{M}_2, w_2 \models \diamond\psi$. The converse direction is proved similarly by using BACK \diamond . \square

We can define bounded \square -bisimulations in a similar way.

³We remind the reader that, as per our convention, \mathcal{L}_\diamond is the $\square, \mathcal{U}, \mathcal{R}$ -free fragment. A similar comment applies to other sublanguages of \mathcal{L} mentioned below.

Definition 4.5. A bounded \circ -bisimulation $\mathcal{Z}_n \subseteq \dots \subseteq \mathcal{Z}_0 \subseteq W_1 \times W_2$ is said to be a bounded \square -bisimulation if for all $(w_1, w_2) \in W_1 \times W_2$ and for all $0 \leq i < n$, if $w_1 \mathcal{Z}_{i+1} w_2$, then:

FORTH \square . For all $k_2 \geq 0$ there exist $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ s.t. $S^{k_2}(w_2) \succcurlyeq v_2$, $v_1 \succcurlyeq S^{k_1}(w_1)$ and $v_1 \mathcal{Z}_i v_2$.

BACK \square . For all $k_1 \geq 0$ there exist $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ s.t. $S^{k_1}(w_1) \succcurlyeq v_1$, $v_2 \succcurlyeq S^{k_2}(w_2)$ and $v_1 \mathcal{Z}_i v_2$.

The intuition for the role of v_1, v_2 in the clauses for \square is similar to that of \diamond , except that now we have to transfer *negative* information. If $\square\varphi$ fails at w_1 , there will be $k_1 \geq 0$ so that φ fails on $S^{k_1}(w_1)$; but then, φ will forcibly fail on any $v_1 \preccurlyeq S^{k_1}(w_1)$. Similarly, if φ fails on $v_2 \succcurlyeq S^{k_2}(w_2)$, φ will fail on $S^{k_2}(w_2)$ as well, witnessing that $\square\varphi$ fails on w_2 .

Lemma 4.6. Given two ITL^e models \mathcal{M}_1 and \mathcal{M}_2 and a bounded \square -bisimulation $\mathcal{Z}_n \subseteq \dots \subseteq \mathcal{Z}_0$ between them, for all $(w_1, w_2) \in W_1 \times W_2$ and $0 \leq i \leq n$, if $w_1 \mathcal{Z}_i w_2$ then for all $\varphi \in \mathcal{L}_{\square}$ such that $|\varphi| \leq i$, then $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.

Proof. We proceed by induction on i . Let $i \geq 0$ be such that for all $j < i$ the lemma holds. Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $w_1 \mathcal{Z}_i w_2$ and let us consider $\varphi \in \mathcal{L}_{\square}$ such that $|\varphi| \leq i$. Note that the cases for atoms as well as propositional and \circ connectives are proved as in Lemma 4.2, so we only consider $\varphi = \square\psi$.

For the left-to-right implication, we work by contrapositive, and assume that $\mathcal{M}_2, w_2 \not\models \square\psi$. Then, there exists $k_2 \geq 0$ such that $\mathcal{M}_2, S^{k_2}(w_2) \not\models \psi$. By FORTH \square , there exist $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ s.t. $S^{k_2}(w_2) \succcurlyeq v_2$, $v_1 \succcurlyeq S^{i_1}(w_1)$ and $v_1 \mathcal{Z}_{i-1} v_2$. As in the proof of Lemma 4.4, by \preccurlyeq -monotonicity, the induction hypothesis and the fact that $|\psi| \leq i - 1$, it follows that $\mathcal{M}_1, v_1 \not\models \psi$; thus $\mathcal{M}_1, S^{k_1}(w_1) \not\models \psi$, and again by \preccurlyeq -monotonicity $\mathcal{M}_1, w_1 \not\models \square\psi$. The converse direction follows a similar reasoning but using BACK \square . \square

5 Bounded bisimulations for \mathcal{U} and \mathcal{R}

In this section we adapt the bisimulations defined for a language with *until* and *since* [18] presented by Kurtonina and de Rijke [20] to our case. As with bisimulations for \diamond and \square , we modify the standard clauses so that witnesses for \mathcal{U} or \mathcal{R} do not have to be directly connected, and, instead, it suffices for suitable ‘proxy’ worlds to be connected by the bisimulation. Let us begin with bounded bisimulations for \mathcal{U} .

Definition 5.1. Given $n \in \mathbb{N}$ and two ITL^e models \mathcal{M}_1 and \mathcal{M}_2 , a bounded \circ -bisimulation $\mathcal{Z}_n \subseteq \dots \subseteq \mathcal{Z}_0 \subseteq W_1 \times W_2$ is said to be a bounded \mathcal{U} -bisimulation iff for all $(w_1, w_2) \in W_1 \times W_2$, and for all $0 \leq i < n$ if $w_1 \mathcal{Z}_{i+1} w_2$:

FORTH \mathcal{U} . For all $k_1 \geq 0$ there exist $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

1. $S^{k_2}(w_2) \succcurlyeq v_2$, $v_1 \succcurlyeq S^{k_1}(w_1)$ and $v_1 \mathcal{Z}_i v_2$, and
2. for all $j_2 \in [0, k_2)$ there exist $j_1 \in [0, k_1)$ and $(u_1, u_2) \in W_1 \times W_2$ such that $u_1 \succcurlyeq S^{j_1}(w_1)$, $S^{j_2}(w_2) \succcurlyeq u_2$ and $u_1 \mathcal{Z}_i u_2$.

BACK \mathcal{U} . For all $k_2 \geq 0$ there exist $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

1. $S^{k_1}(w_1) \succcurlyeq v_1$, $v_2 \succcurlyeq S^{k_2}(w_2)$ and $v_1 \mathcal{Z}_i v_2$, and
2. for all $j_1 \in [0, k_1)$ there exist $j_2 \in [0, k_2)$ and $(u_1, u_2) \in W_1 \times W_2$ such that $u_2 \succcurlyeq S^{j_2}(w_2)$, $S^{j_1}(w_1) \succcurlyeq u_1$ and $u_1 \mathcal{Z}_i u_2$.

As was the case before, the following lemma states that two bounded \mathcal{U} -bisimilar models agree on small $\mathcal{L}_{\mathcal{U}}$ formulas.

Lemma 5.2. Given two ITL^e models \mathcal{M}_1 and \mathcal{M}_2 and a bounded \mathcal{U} -bisimulation $\mathcal{Z}_n \subset \dots \subset \mathcal{Z}_0$ between them, for all $0 \leq m \leq n$ and $(w_1, w_2) \in W_1 \times W_2$, if $w_1 \mathcal{Z}_m w_2$ then for all $\varphi \in \mathcal{L}_{\mathcal{U}}$ such that $|\varphi| \leq m$, $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.

Proof. Once again, proceed by induction on n . Let $m \leq n$ be such that for all $k < m$ the lemma holds. Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $w_1 \mathcal{Z}_m w_2$ and let us consider $\varphi \in \mathcal{L}_{\mathcal{U}}$ such that $|\varphi| \leq m$. As before, we only consider the ‘new’ case, where $\varphi = \theta \mathcal{U} \psi$. From left to right, assume that $\mathcal{M}_1, w_1 \models \theta \mathcal{U} \psi$. Then, there exists $i_1 \geq 0$ such that $\mathcal{M}_1, S^{i_1}(w_1) \models \psi$ and for all j_1 satisfying $0 \leq j_1 < i_1$, $\mathcal{M}_1, S^{j_1}(w_1) \models \theta$. By FORTH \mathcal{U} , there exist $i_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that 1. $S^{i_2}(w_2) \succcurlyeq v_2$, $v_1 \succcurlyeq S^{i_1}(w_1)$ and $v_1 \mathcal{Z}_{m-1} v_2$; 2. for all j_2 satisfying $0 \leq j_2 < i_2$ there exist $j_1 \in [0, i_1)$ and $(u_1, u_2) \in W_1 \times W_2$ s. t. $u_1 \succcurlyeq S^{j_1}(w_1)$, $S^{j_2}(w_2) \succcurlyeq u_2$ and $u_1 \mathcal{Z}_{m-1} u_2$.

From the first item, \preceq -monotonicity, the fact that $|\psi| \leq m - 1$, and the induction hypothesis, it follows that $\mathcal{M}_2, S^{i_2}(w_2) \models \psi$. Take any j_2 satisfying $0 \leq j_2 < i_2$. By the second item, the fact that $|\theta| \leq m - 1$, and the induction hypothesis, we conclude that $\mathcal{M}_2, S^{j_2}(w_2) \models \theta$ so $\mathcal{M}_2, w_2 \models \theta \mathcal{U} \psi$. The right-to-left direction is symmetric (but using BACK \mathcal{U}). \square

Finally, we define bounded bisimulations for \mathcal{R} .

Definition 5.3. A bounded \circ -bisimulation $\mathcal{Z}_n \subseteq \dots \subseteq \mathcal{Z}_0 \subseteq W_1 \times W_2$ is said to be a bounded \mathcal{R} -bisimulation if for all $(w_1, w_2) \in W_1 \times W_2$ and for all $0 \leq i < n$, if $w_1 \mathcal{Z}_{i+1} w_2$ then :

FORTH \mathcal{R} . For all $k_2 \geq 0$ there exist $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

1. $S^{k_2}(w_2) \succcurlyeq v_2$, $v_1 \succcurlyeq S^{k_1}(w_1)$ and $v_1 \mathcal{Z}_i v_2$, and
2. for all j_1 satisfying $0 \leq j_1 < k_1$ there exist j_2 such that $0 \leq j_2 < k_2$ and $(u_1, u_2) \in W_1 \times W_2$ s. t. $u_1 \succcurlyeq S^{j_1}(w_1)$, $S^{j_2}(w_2) \succcurlyeq u_2$ and $u_1 \mathcal{Z}_i u_2$.

BACK \mathcal{R} . For all $k_1 \geq 0$ there exist $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

1. $S^{k_1}(w_1) \succcurlyeq v_1$, $v_2 \succcurlyeq S^{k_2}(w_2)$ and $v_1 \mathcal{Z}_i v_2$, and
2. for all j_2 satisfying $0 \leq j_2 < k_2$ there exist j_1 such that $0 \leq j_1 < k_1$ and $(u_1, u_2) \in W_1 \times W_2$ s. t. $u_2 \succcurlyeq S^{j_2}(w_2)$, $S^{j_1}(w_1) \succcurlyeq u_1$ and $u_1 \mathcal{Z}_i u_2$.

Once again, we obtain a corresponding bisimulation lemma for $\mathcal{L}_{\mathcal{R}}$.

Lemma 5.4. *Given two ITL^e models \mathcal{M}_1 and \mathcal{M}_2 and a bounded \mathcal{R} -bisimulation $\mathcal{Z}_n \subseteq \dots \subseteq \mathcal{Z}_0$ between them, for all $0 \leq m \leq n$ and $(w_1, w_2) \in W_1 \times W_2$, if $w_1 \mathcal{Z}_m w_2$ then for all $\varphi \in \mathcal{L}_{\mathcal{U}}$ such that $|\varphi| \leq m$, $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.*

Proof. As before, we proceed by induction on n ; the critical case where $\varphi = \theta \mathcal{R} \psi$ follows by a combination of the reasoning for Lemmas 4.6 and Lemma 4.6. Details are left to the reader. \square

6 Definability and undefinability of modal operators

In this section, we explore the question of when it is that the basic connectives can or cannot be defined in terms of each other. It is known that, classically, \diamond and \square are interdefinable, as are \mathcal{U} and \mathcal{R} ; we will see that this is not the case intuitionistically. On the other hand, \mathcal{U} (and hence \mathcal{R}) is not definable in terms of \diamond, \square in the classical setting [18], and this result immediately carries over to the intuitionistic setting, as the class of classical LTL models can be seen as the subclass of that of dynamic posets where the partial order is the identity.

Interdefinability of modal operators can vary within intermediate logics. For example, \wedge, \vee and \rightarrow are basic connectives in propositional intuitionistic logic, but in the intermediate logic of here-and-there [15], \wedge [1, 2] and \rightarrow [1] are basic operators while \vee is definable in terms of \rightarrow and \wedge [22]. In first-order here-and-there [21], the quantifier \exists is definable in terms of \forall and \rightarrow [24] while \forall is not definable in terms of the other operators. In the modal case, Simpson [30] shows that modal operators are not interdefinable in the logic IK and Balbiani and Diéguez [2] proved the same result for the linear time temporal extension of here-and-there. This last proof is adapted to show that modal operators are not definable in ITL^e . Note, however, that here we correct the claim of [2] stating that \diamond is not here-and-there definable in terms of \square .

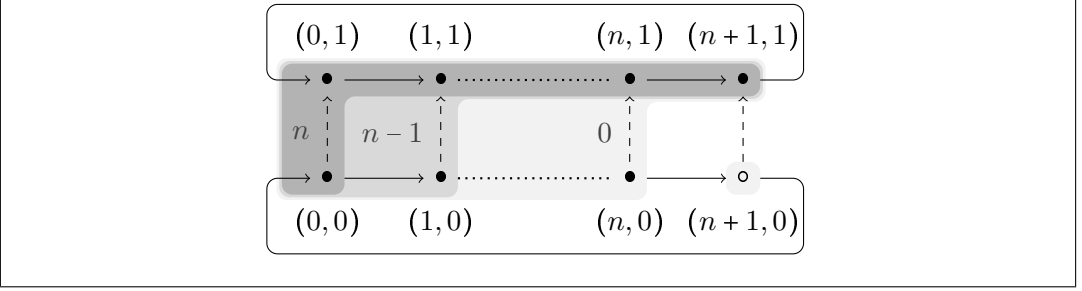


Figure 1: The here-and-there model \mathcal{H}_n . Black dots satisfy the atom p , white dots do not; all other atoms are false everywhere. Dashed lines indicate \leq and solid lines indicate S . The \sim_i -equivalence classes are shown as grey regions.

Let us begin by studying the definability of \square in terms of \circ and \mathcal{U} . Below, if $\mathcal{L}' \subseteq \mathcal{L}$, $\varphi \in \mathcal{L}$ and Ω is a class of models, we say that φ is \mathcal{L}' -definable over Ω if there is $\varphi' \in \mathcal{L}'$ such that $\Omega \models \varphi \leftrightarrow \varphi'$.

Theorem 6.1. *The connective \square is not $\mathcal{L}_{\mathcal{U}}$ -definable, even over the class of finite here-and-there models.*

Proof. Assume for the sake of contradiction that $\diamond p$ can be expressed as a \mathcal{U} -free formula φ with $|\varphi| = n > 0$. Let $T = \{0, \dots, n+1\}$ and $f: T \rightarrow T$ be given by $y = f(x)$ if and only if $y \equiv x+1 \pmod{n+2}$. Then consider a here-and-there model $\mathcal{H}_n = (W, \leq, S, V)$ based on $\text{HT}(T, f)$ and with $V(p) = W \setminus \{(n+1, 0)\}$. For $k \leq n$, define $(i, j) \sim_k (i', j')$ if $(i, j) = (i', j')$ or

$$\max\{i(1-j), i'(1-j')\} \leq n-k$$

(see Figure 1). Clearly, $(\mathcal{H}_n, (0, 0)) \not\models \diamond p$, while $(\mathcal{H}_n, (0, 1)) \models \diamond p$. Let us check now that $(\sim_k)_{k \leq n}$ is a bounded \mathcal{U} -bisimulation. It is easy to check that the sequence is increasing under inclusion. Moreover, \sim_k is symmetric (indeed, an equivalence relation) for each k , so by symmetry, we only check the FORTH clauses.

ATOMS : Assume that $0 \leq k \leq n$ and $x \sim_k y$. Since $(n+1)(1-0) > n-k$, either $x = y$ (so the two satisfy the same atoms) or $x, y \neq (n+1, 0)$, so the two also satisfy the same atoms (namely, $\{p\}$).

FORTH \rightarrow : Let k satisfy $0 \leq k < n$ and let us assume $(i_1, j_1) \sim_{k+1} (i_2, j_2)$ and $(i_1, j_1) \leq (i'_1, j'_1)$. If $(i_1, j_1) = (i_2, j_2)$, then $(i'_2, j'_2) \stackrel{\text{def}}{=} (i'_1, j'_1)$ witnesses that the clause holds, so we assume otherwise. Let us define $(i'_2, j'_2) \stackrel{\text{def}}{=} (i_2, 1)$. Then, $(i_2, j_2) \leq (i'_2, j'_2)$ and $\max\{i'_1(1-j'_1), i'_2(1-j'_2)\} = \max\{i'_1(1-j'_1), 0\} = i'_1(1-j'_1) \leq n-k$, meaning that $(i'_1, j'_1) \sim_k (i'_2, j'_2)$, as required.

FORTH \circ : Let k satisfy $0 \leq k < n$ and let us consider $(i_1, j_1) \sim_{k+1} (i_2, j_2)$. If $(i_1, j_1) = (i_2, j_2)$, then also $S(i_1, j_1) = S(i_2, j_2)$, so we assume otherwise. We claim that for $\ell \in \{1, 2\}$, $f(i_\ell)(1 - j_\ell) \leq n - k$. If $j_\ell = 1$ this is obvious, otherwise from the definition of \sim_{k+1} we obtain $i_\ell < n - k$ so that $f(i_\ell) = i_\ell + 1 \leq n - k$. We conclude that $\max\{(f(i_1)(1 - j_1), f(i_2 + 1)(1 - j_2))\} \leq n - k$, so that $S(i_1, j_1) \sim_k S(i_2, j_2)$, as required.

FORTH \mathcal{U} : Let k satisfy $0 \leq k < n$, and let us suppose that $(i_1, j_1) \sim_{k+1} (i_2, j_2)$. Assume moreover that $(i_1, j_1) \neq (i_2, j_2)$, as the other case is easy to check. Fix $k_1 \geq 0$ and define $(i'_1, j'_1) = S^{k_1}(i_1, j_1)$. Let us define $k_2 = 0$, $v_1 = (i_1, 1)$, and $v_2 = (i_2, j_2)$, so that $S^{k_2}(i_2, j_2) = (i_2, j_2)$. Since $\max\{i_1(1 - 1), i_2(1 - j_2)\} = i_2(1 - j_2) < n - k$, we have that $v_1 \sim_k v_2$ and satisfy Condition 1. Note also that the Condition 2 holds vacuously because of $[0, k_2) = \emptyset$.

Consequently, $(\sim_m)_{m \leq n}$ is a bounded \mathcal{U} -bisimulation. By using Lemma 5.2 and the fact that $(0, 0) \sim_n (0, 1)$ we get that $(0, 0)$ and $(0, 1)$ satisfy the same \mathcal{U} -free formulas ψ with $|\psi| \leq n$. However, $(\mathcal{H}_n, (0, 0)) \not\models \varphi$ and $(\mathcal{H}_n, (0, 1)) \models \varphi$: a contradiction. \square

As a consequence:

Corollary 6.2. *The connective \mathcal{R} is not definable in terms of \circ and \mathcal{U} , even over the class of persistent models.*

Proof. If we could define $q \mathcal{R} p$, then we could also define $\Box p \equiv \perp \mathcal{R} p$. \square

Proposition 6.3. *Over the class of here-and-there models, \diamond is \mathcal{L}_\Box -definable. To be precise, $\diamond p$ is equivalent to*

$$\varphi = (\Box(p \rightarrow \Box(p \vee \neg p)) \wedge \Box(\Box(p \vee \neg p) \rightarrow p \vee \neg p \vee \Box \Box \neg p)) \rightarrow (\Box(p \vee \neg p) \wedge \neg \Box \neg p).$$

Proof. Let $\mathcal{M} = (T \times \{0, 1\}, \leq, S, V)$ be a here-and-there model with $S(t, i) = (f(t), i)$ (see Section 2.3). Before proving that φ is equivalent to $\diamond p$, we give some intuition. Essentially, φ contemplates three different ways that $\diamond p$ could hold in (\mathcal{M}, x) , where $x = (x_1, x_2)$. It may be that $\Box(p \vee \neg p)$ holds, in which case (\mathcal{M}, x) behaves essentially as a classical model, at least for formulas whose only variable is p . In this case, $\diamond p$ holds iff $\neg \Box \neg p$ holds, as in the standard classical semantics. If $\Box(p \vee \neg p)$ fails, then \mathcal{M} does not behave classically; for some n , $S^n(x)$ falsifies $p \vee \neg p$. For φ to be true, we then need for either $\Box(p \rightarrow \Box(p \vee \neg p))$ or $\Box(\Box(p \vee \neg p) \rightarrow p \vee \neg p \vee \Box \Box \neg p)$ to fail. The formula $\Box(p \rightarrow \Box(p \vee \neg p))$ will fail exactly when there is m such that $S^m(x)$ satisfies p (hence x satisfies $\diamond p$), and \mathcal{M} does *not* behave classically after m ; that is, there is $n > m$ so that $S^n(x)$ falsifies $p \vee \neg p$. Meanwhile, $\Box(\Box(p \vee \neg p) \rightarrow$

$p \vee \neg p \vee \bigcirc \square \neg p$) will fail exactly when there is m such that $S^m(x)$ satisfies p but \mathcal{M} behaves classically after m ; in other words, $S^n(x)$ falsifies $p \vee \neg p$ only for $n < m$. In this case, $\bigcirc \square (p \vee \neg p) \rightarrow p \vee \neg p \vee \bigcirc \square \neg p$ will be falsified exactly at the *greatest* such n .

Now for the proof. Assume that $x = (x_1, x_2)$ is such that $(\mathcal{M}, x) \models \diamond p$. To check that $(\mathcal{M}, x) \models \varphi$, let $x' \succeq x$, so that $x' = (x_1, x'_2)$ with $x'_2 \geq x_2$, and consider the following cases.

CASE $(\mathcal{M}, x') \models \square (p \vee \neg p)$. In this case, it is easy to see that we also have $(\mathcal{M}, x') \models \neg \square \neg p$ given that $(\mathcal{M}, x) \models \diamond p$.

CASE $(\mathcal{M}, x') \not\models \square (p \vee \neg p)$. Using the assumption that $(\mathcal{M}, x) \models \diamond p$, choose k such that $(\mathcal{M}, (f^k(x_1), x_2)) \models p$ and consider two sub-cases.

1. Suppose there is $k' > k$ such that $(\mathcal{M}, (f^{k'}(x_1), x'_2)) \not\models p \vee \neg p$. Then, it follows that $(\mathcal{M}, (f^k(x_1), x'_2)) \not\models p \rightarrow \square p \vee \neg p$ and hence $(\mathcal{M}, x') \not\models \square (p \rightarrow \square (p \vee \neg p))$.
2. If there is not such k' , then there must be a maximal $k' < k$ such that $(\mathcal{M}, (f^{k'}(x_1), x'_2)) \not\models p \vee \neg p$ (otherwise, we would be in CASE $(\mathcal{M}, x') \models \square (p \vee \neg p)$). It is easily verified that

$$(\mathcal{M}, (f^{k'}(x_1), x'_2)) \not\models \bigcirc \square (p \vee \neg p) \rightarrow p \vee \neg p \vee \bigcirc \square \neg p,$$

and hence

$$(\mathcal{M}, x') \not\models \square (\bigcirc \square (p \vee \neg p) \rightarrow p \vee \neg p \vee \bigcirc \square \neg p).$$

Note that the above direction does not use any properties of here-and-there models, and works over arbitrary expanding models. However, we need these properties for the other implication. Suppose that $(\mathcal{M}, x) \models \varphi$. If $(\mathcal{M}, x) \models \square (p \vee \neg p) \wedge \neg \square \neg p$, then it is readily verified that $(\mathcal{M}, x) \models \diamond p$. Otherwise,

$$(\mathcal{M}, x) \not\models \square (p \rightarrow \square (p \vee \neg p)) \wedge \square (\bigcirc \square (p \vee \neg p) \rightarrow p \vee \neg p \vee \bigcirc \square \neg p).$$

If $(\mathcal{M}, x) \not\models \square (p \rightarrow \square (p \vee \neg p))$, then there is k such that

$$(\mathcal{M}, (f^k(x_1), x_2)) \not\models p \rightarrow \square (p \vee \neg p).$$

This is only possible if $x_2 = 0$ and $(\mathcal{M}, (f^k(x_1), x_2)) \models p$, so that $(\mathcal{M}, x) \models \diamond p$. Similarly, if

$$(\mathcal{M}, x) \not\models \square (\bigcirc \square (p \vee \neg p) \rightarrow p \vee \neg p \vee \bigcirc \square \neg p),$$

then there is k such that $(\mathcal{M}, (f^k(x_1), x_2)) \not\models \bigcirc \square (p \vee \neg p) \rightarrow p \vee \neg p \vee \bigcirc \square \neg p$. This is only possible if $x_2 = 0$, $(\mathcal{M}, (f^k(x_1), x_2)) \models \bigcirc \square (p \vee \neg p)$ and $(\mathcal{M}, (f^k(x_1), x_2)) \not\models \bigcirc \square \neg p$. But from this it easily can be seen that there is $k' > k$ with $(\mathcal{M}, (f^{k'}(x_1), x_2)) \models p$, hence $(\mathcal{M}, x) \models \diamond p$. \square

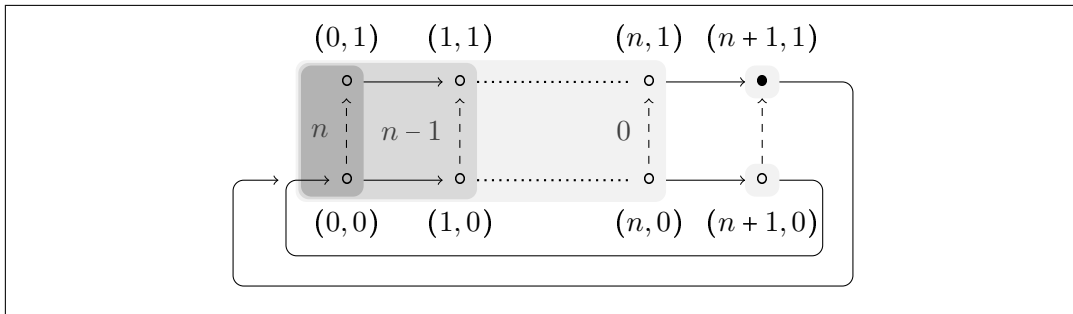


Figure 2: The expanding model \mathcal{E}_n . Notation is as in Figure 1.

Corollary 6.4. *Over the class of here-and-there models, $p\mathcal{U}q$ is $\mathcal{L}_{\mathcal{R}}$ -definable using the equivalence $p\mathcal{U}q \equiv (q\mathcal{R}(p \vee q)) \wedge \diamond q$.*

Hence, if we want to prove the undefinability of \diamond in terms of other operators, we must turn to a wider class of models, as we will do next.

Theorem 6.5. *The operator \diamond cannot be defined in terms of \square over the class of finite expanding models.*

Proof. Given $n > 0$, consider a model $\mathcal{E}_n = (W, \leq, S, V)$ with $W = \{0, \dots, n+1\} \times \{0, 1\}$, $(i, j) \leq (i', j')$ if $i = i'$ and $j \leq j'$, $S(i, j) = (i+1, j)$ if $i \leq n$, $S(n+1, j) = (0, 0)$, and $V(p) = \{(n+1, 1)\}$. For $m \leq n$, define $(i, j) \sim_m (i', j')$ if either $(i, j) = (i', j')$, or $\max\{i, i'\} \leq n - m$. Then, it can easily be checked that $(\mathcal{M}, (0, 0)) \not\models \diamond p$, $(\mathcal{M}, (0, 1)) \models \diamond p$, and $(0, 0) \sim_m (0, 1)$.

It remains to check that $(\sim_m)_{m \leq n}$ is a bounded \square -bimulation. We focus on the \square clauses, and by symmetry, prove only BACK \square . Suppose that $(i_1, j_1) \sim_m (i_2, j_2)$ and fix $k_1 \geq 0$. Let $(i'_1, j'_1) = S^{k_1}(i_1, j_1)$. Choose $k_2 > n+1$ such that $i_2 + k_2 \equiv i'_1 \pmod{n+1}$, and let $(i'_2, j'_2) = S^{k_2}(i_2, j_2)$. It is not hard to check that $i'_1 = i'_2$ and $j'_2 = 0$, from which we obtain $(i'_2, j'_2) \leq (i'_1, j'_1)$. Hence, setting $v_1 = v_2 = (i'_2, j'_2)$ gives us the desired witnesses.

By letting n vary, we see that no \mathcal{L}_{\square} -formula can be equivalent to $\diamond p$. \square

7 Conclusions

In this paper we have investigated on ITL^e , an intuitionistic analogue of LTL based on expanding domain models from modal logic. We have shown that, as happens in other modal intuitionistic logics or modal intermediate logics, modal operators are not interdefinable.

Many open questions remain regarding intuitionistic temporal logics. We know that ITL^e is decidable, but the proposed decision procedure is non-elementary. However, there seems to be little reason to assume that this is optimal, raising the following question:

Question 7.1. *Are the satisfiability and validity problems for ITL^e elementary?*

Meanwhile, we saw in Theorems 2.2 and 2.4 that ITL^e has the strong finite model property, while ITL^P does not have the finite model property at all. However, it may yet be that ITL^P is decidable despite this.

Question 7.2. *Is ITL^P decidable?*

Regarding expressive completeness, it is known that LTL is expressively complete [18, 29, 12, 16]; there exists a one-to-one correspondence (over \mathbb{N}) between the temporal language and the monadic first-order logic equipped with a linear order and ‘next’ relation [12]. It is not known whether the same property holds between ITL^e and first-order intuitionistic logic.

Question 7.3. *Is \mathcal{L} equally expressive to monadic first-order logic over the class of dynamic or persistent models?*

Finally, a sound and complete axiomatization for ITL^e remains to be found. The results we have presented here could be a first step in this direction, and we conclude with the following:

Question 7.4. *Are the ITL^e -valid formulas listed in this work, together with the intuitionistic tautologies and standard inference rules, complete for the class of dynamic posets? Is the logic augmented with $(\circ p \rightarrow \circ q) \rightarrow \circ(p \rightarrow q)$ complete for the class of persistent models?*

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