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Unification with parameters in the implication fragment of Classical Propositional Logic

Philippe Balbiani^a and Mojtaba Mojtahedi^b

^aToulouse Institute of Computer Science Research
CNRS — Toulouse University, Toulouse, France

^bSchool of Mathematics, Statistics and Computer Science
College of Science
University of Tehran, Tehran, Iran

Abstract

In this paper, we show that the implication fragment of Classical Propositional Logic is finitary for unification with parameters.

Keywords: Classical Propositional Logic. Implication fragment. Unification with parameters. Unification type.

1 Introduction

Based on a definitionally complete set of connectives, Classical Propositional Logic (denoted **CPL**) seems to have all good properties a propositional logic can possess. For instance, its satisfiable formulas constitutes an **NP**-complete set of formulas [23, Chapter 9] and its admissible rules are all derivable [9, Chapter 1]. These properties, however, are lost when the syntax is enriched as, for example, in modal logic: the satisfiable formulas defined by **S4** constitute a **PSPACE**-complete set of formulas [6, Chapter 6] whereas there exists **S4**-admissible rules that are not derivable [25, Chapter 3]. Hence, it is natural to ask what happens to the good properties of **CPL** when its syntax is impoverished to a definitionally incomplete set of connectives.

According to Dunn and Hardegree [11, Chapter 9], the interesting fragments of **CPL** are the implication fragment and the positive fragment. In this paper, we only consider the implication fragment. It is algebraically characterizable in terms of semi-Boolean algebras [1], i.e. algebraic structures of the form (A, \triangleright) where $(a \triangleright b) \triangleright a = a$, $a \triangleright (b \triangleright c) = b \triangleright (a \triangleright c)$ and $(a \triangleright b) \triangleright b = (b \triangleright a) \triangleright a$. Its tautologies are axiomatizable by adding to the rule of modus ponens the axioms $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$, $\varphi \rightarrow (\psi \rightarrow \varphi)$ and $((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$, the Pierce formula. Moreover, in it, the Deduction Theorem — if ψ is derivable from φ and a set Γ of formulas then $\varphi \rightarrow \psi$ is derivable

from Γ — still holds. But what can we say about its satisfiable formulas and its admissible rules?

Concerning the satisfiable formulas, they constitute a trivial set: as the reader can prove by induction on the implicative formula φ , φ is satisfiable¹. Concerning the admissible rules, some of them are not derivable as soon as the language contains propositional parameters, i.e. atomic formulas that cannot be replaced by formulas when one applies substitutions. In order to understand why, consider the rule $\frac{x \rightarrow p}{p \rightarrow x}$ where x is a variable and p is a parameter. As the reader can prove by induction on the implicative formula φ , if $\varphi \rightarrow p$ is valid then $p \rightarrow \varphi$ is valid. Hence, the considered rule is admissible. Nevertheless, it is not derivable. Indeed, suppose the considered rule is derivable. Thus, by using the Deduction Theorem, the formula $(x \rightarrow p) \rightarrow (p \rightarrow x)$ is valid: a contradiction.

In a propositional logic \mathbf{L} , the admissibility problem is related to the unifiability problem [16, 17]: given a formula φ , determine whether it has an \mathbf{L} -unifier, i.e. a substitution σ such that $\sigma(\varphi)$ is in \mathbf{L} . A unifiable formula is finitary if it has a finite complete set of unifiers and unitary if this set is a singleton. It is well-known that, based on a definitionally complete set of connectives, \mathbf{CPL} is unitary [4, 13, 22]. We usually distinguish between elementary unification — when all atomic formulas are considered as variables — and unification with parameters — when some atomic formulas are considered as parameters.

It may happen that in a propositional logic, every parameter-free unifiable formula is unitary whereas some unifiable formulas with parameters are finitary. For instance, in [15], Ghilardi has shown that the implication-conjunction fragment of Intuitionistic Propositional Logic is unitary for elementary unification and finitary for unification with parameters. From [10, 24], we know that the implication fragment of every intermediate logic is unitary for elementary unification. In this paper, we show that the implication fragment of Classical Propositional Logic is finitary for unification with parameters.

2 Classical Propositional Logic and its implication fragment

Atoms and models Let \mathbf{VAR} be a countable set of *variables* (with typical members denoted x, y , etc) and \mathbf{PAR} be a countable set of *parameters* (with typical members denoted p, q , etc). *Atoms* (denoted α, β , etc) are variables or parameters. Let \mathbf{ATO} be the set of all atoms. A *model* is a function of the form

- $V : \mathbf{ATO} \longrightarrow \{0, 1\}$.

¹Concerning the valid formulas, they constitute a **coNP**-hard set. Why? Simply because for all formulas F in the definitionally complete set $\{\neg, \rightarrow\}$ of connectives and for all atomic formulas α not occurring in F , F is valid if and only if the implicative formula $(\varphi \rightarrow \alpha) \rightarrow \alpha$ is valid, φ being the implicative formula obtained from F after having inductively replaced each subformula of the form $\neg H$ by $H \rightarrow \alpha$.

Formulas *Formulas* (denoted φ, ψ , etc) are inductively defined as follows:

- $\varphi, \psi ::= \alpha \mid (\varphi \rightarrow \psi)$.

Let **FOR** be the set of all formulas. We adopt the standard rules for omission of the parentheses. The Boolean connective of *disjunction* is defined, at the level of formulas, as follows:

- $(\varphi \vee \psi) ::= ((\varphi \rightarrow \psi) \rightarrow \psi)$.

For all formulas φ , let $\text{var}(\varphi)$ be the set of all variables occurring in φ and $\text{par}(\varphi)$ be the set of all parameters occurring in φ . We shall say that a formula φ is *variable-free* if $\text{var}(\varphi) = \emptyset$ and *parameter-free* if $\text{par}(\varphi) = \emptyset$. For all finite $S \subseteq \mathbf{FOR}$, let $\text{var}(S)$ be $\bigcup\{\text{var}(\varphi) : \varphi \in S\}$ and $\text{par}(S)$ be $\bigcup\{\text{par}(\varphi) : \varphi \in S\}$. Given a model $V : \mathbf{ATO} \rightarrow \{0, 1\}$, the function $\bar{V} : \mathbf{FOR} \rightarrow \{0, 1\}$ is inductively defined as follows:

- $\bar{V}(\alpha) = V(\alpha)$,
- $\bar{V}(\varphi \rightarrow \psi) = \max\{1 - \bar{V}(\varphi), \bar{V}(\psi)\}$.

As a result,

- $\bar{V}(\varphi \vee \psi) = \max\{\bar{V}(\varphi), \bar{V}(\psi)\}$.

A formula φ is *valid* (in symbols $\models \varphi$) if for all models V , $\bar{V}(\varphi) = 1$. A finite set S of formulas is *valid* (in symbols $\models S$) if for all $\varphi \in S$, $\models \varphi$.

Proposition 1 *For all formulas φ , there exists an atom α such that $\models \alpha \rightarrow \varphi$.*

Proof: By induction on φ . \dashv

Proposition 2 *Let α be an atom and φ be a formula. For all $n \geq 1$ and for all atoms β_1, \dots, β_n ,*

1. *if $\models \alpha \rightarrow \beta_1 \vee \dots \vee \beta_n$ then there exists $i \in \{1, \dots, n\}$ such that $\alpha = \beta_i$,*
2. *if $\models \varphi \rightarrow \beta_1 \vee \dots \vee \beta_n$ then there exists $i \in \{1, \dots, n\}$ such that $\models \beta_i \rightarrow \varphi$.*

Proof: Let $n \geq 1$ and β_1, \dots, β_n be atoms.

(1). Left to the reader.

(2). Suppose $\models \varphi \rightarrow \beta_1 \vee \dots \vee \beta_n$. By Proposition 1, let α be an atom such that $\models \alpha \rightarrow \varphi$. Since $\models \varphi \rightarrow \beta_1 \vee \dots \vee \beta_n$, $\models \alpha \rightarrow \beta_1 \vee \dots \vee \beta_n$. Hence, by Item (1), let $i \in \{1, \dots, n\}$ be such that $\alpha = \beta_i$. Since $\models \alpha \rightarrow \varphi$, $\models \beta_i \rightarrow \varphi$. \dashv

Extended formulas *Extended formulas* (denoted F, G , etc) are inductively defined as follows:

- $F, G ::= \alpha \mid (F \rightarrow G) \mid \neg F$.

Let **EXT** be the set of all extended formulas. We adopt the standard rules for omission of the parentheses. The Boolean connectives of *disjunction*, *conjunction* and *equivalence* are defined, at the level of extended formulas, as usual. For all extended formulas F , let $\text{var}(F)$ be the set of all variables occurring in F and $\text{par}(F)$ be the set of all parameters occurring in F . We shall say that an extended formula F is *variable-free* if $\text{var}(F) = \emptyset$ and *parameter-free* if $\text{par}(F) = \emptyset$. Given a model $V : \mathbf{ATO} \rightarrow \{0, 1\}$, the function $\bar{V} : \mathbf{EXT} \rightarrow \{0, 1\}$ is inductively defined as usual. An extended formula F is *valid* (in symbols $\models F$) if for all models V , $\bar{V}(F) = 1$. We shall say that the extended formula F is *logically equivalent* to the extended formula G if $\models F \leftrightarrow G$.

A useful result Obviously, every formula is an extended formula. We shall say that an extended formula is *implicative* if it is logically equivalent to a formula. For all atoms α , when we write α^0 and α^1 , we respectively mean $\neg\alpha$ and α .

Proposition 3 *Let α be an atom. For all $k \in \mathbb{N}$, for all atoms β_1, \dots, β_k and for all $a_1, \dots, a_k \in \{0, 1\}$, $\bigwedge\{\beta_j^{a_j} : 1 \leq j \leq k\} \vee \alpha$ is implicative.*

Proof: Let $k \in \mathbb{N}$, β_1, \dots, β_k be atoms and $a_1, \dots, a_k \in \{0, 1\}$. Obviously, $\models (\bigwedge\{\beta_j^{a_j} : 1 \leq j \leq k\} \vee \alpha) \leftrightarrow (\bigvee\{\beta_j^{1-a_j} : 1 \leq j \leq k\} \rightarrow \alpha)$. Hence, $\models (\bigwedge\{\beta_j^{a_j} : 1 \leq j \leq k\} \vee \alpha) \leftrightarrow (\bigvee\{\beta_j : 1 \leq j \leq k \text{ and } a_j = 0\} \vee \bigvee\{\neg\beta_j : 1 \leq j \leq k \text{ and } a_j = 1\} \rightarrow \alpha)$. Thus, $\models (\bigwedge\{\beta_j^{a_j} : 1 \leq j \leq k\} \vee \alpha) \leftrightarrow (\bigvee\{\beta_j : 1 \leq j \leq k \text{ and } a_j = 0\} \vee \bigvee\{\beta_j \rightarrow \alpha : 1 \leq j \leq k \text{ and } a_j = 1\} \rightarrow \alpha)$. Consequently, $\bigwedge\{\beta_j^{a_j} : 1 \leq j \leq k\} \vee \alpha$ is implicative². \dashv

Proposition 4 *For all extended formulas F , the following conditions are equivalent:*

1. F is implicative,
2. there exists an atom α such that $\models \alpha \rightarrow F$,
3. there exists an extended formula G and there exists an atom α such that $\models F \leftrightarrow (G \vee \alpha)$,
4. there exists an extended formula G and there exists an atom α such that $\models F \leftrightarrow (G \rightarrow \alpha)$.

²In this argument, we assume $k \geq 1$. The truth is that if $k = 0$ then $\bigwedge\{\beta_j^{a_j} : 1 \leq j \leq k\} \vee \alpha$ is implicative, seeing that $\models (\bigwedge\{\beta_j^{a_j} : 1 \leq j \leq k\} \vee \alpha) \leftrightarrow (\alpha \rightarrow \alpha)$.

Proof: Let F be an extended formula.

(1 \Rightarrow 2). Suppose F is implicative. Let φ be a formula such that $\models F \leftrightarrow \varphi$. By Proposition 1, let α be an atom such that $\models \alpha \rightarrow \varphi$. Since $\models F \leftrightarrow \varphi$, $\models \alpha \rightarrow F$.

(2 \Rightarrow 3). Suppose there exists an atom α such that $\models \alpha \rightarrow F$. Hence, $\models F \leftrightarrow (F \vee \alpha)$.

(3 \Rightarrow 4). Suppose there exists an extended formula G and there exists an atom α such that $\models F \leftrightarrow (G \vee \alpha)$. Hence, $\models F \leftrightarrow (\neg G \rightarrow \alpha)$.

(4 \Rightarrow 1). Suppose there exists an extended formula G and there exists an atom α such that $\models F \leftrightarrow (G \rightarrow \alpha)$. Let H be a conjunctive normal form of G , i.e. let $n \in \mathbb{N}$, $k_1, \dots, k_n \in \mathbb{N}$, $\beta_{1,1}, \dots, \beta_{1,k_1}, \dots, \beta_{n,1}, \dots, \beta_{n,k_n} \in \mathbf{ATO}$ and $a_{1,1}, \dots, a_{1,k_1}, \dots, a_{n,1}, \dots, a_{n,k_n} \in \{0, 1\}$ be such that $H = \bigwedge \{ \bigvee \{ \beta_{i,j}^{a_{i,j}} : 1 \leq j \leq k_i \} : 1 \leq i \leq n \}$ and $\models G \leftrightarrow H$. Since $\models F \leftrightarrow (G \rightarrow \alpha)$, $\models F \leftrightarrow (\bigwedge \{ \bigvee \{ \beta_{i,j}^{a_{i,j}} : 1 \leq j \leq k_i \} : 1 \leq i \leq n \} \rightarrow \alpha)$. Hence, $\models F \leftrightarrow (\bigvee \{ \bigwedge \{ \beta_{i,j}^{1-a_{i,j}} : 1 \leq j \leq k_i \} : 1 \leq i \leq n \} \vee \alpha)$. Thus, $\models F \leftrightarrow \bigvee \{ \bigwedge \{ \beta_{i,j}^{1-a_{i,j}} : 1 \leq j \leq k_i \} \vee \alpha : 1 \leq i \leq n \}$. Consequently, by Proposition 3, F is implicative³.
 \dashv

3 Unification with parameters

Substitutions A *substitution* is a function of the form

- $\sigma : \mathbf{VAR} \rightarrow \mathbf{FOR}$.

We shall say that a substitution σ *moves* a variable x if $\sigma(x) \neq x$. Following the standard assumption considered in the literature about unification [2, 5], we will always assume that substitutions move at most finitely many variables. Let **SUB** be the set of all substitutions. For all finite $X \subseteq \mathbf{VAR}$, an *X-substitution* is a substitution that does not move the variables in $\mathbf{VAR} \setminus X$. For all formulas $\varphi(x_1, \dots, x_n)$ and for all substitutions σ , let $\sigma(\varphi(x_1, \dots, x_n))$ be $\varphi(\sigma(x_1), \dots, \sigma(x_n))$. For all finite $S \subseteq \mathbf{FOR}$ and for all substitutions σ , let $\sigma(S)$ be $\{\sigma(\varphi) : \varphi \in S\}$. For all extended formulas $F(x_1, \dots, x_n)$ and for all substitutions σ , let $\sigma(F(x_1, \dots, x_n))$ be $F(\sigma(x_1), \dots, \sigma(x_n))$. The *composition* $\sigma \circ \tau$ of the substitutions σ and τ is the substitution associating to each variable x the formula $\tau(\sigma(x))$. The *identity substitution* (denoted **id**) is the substitution associating to each variable x the formula x . For all $n \in \mathbb{N}$ and for all substitutions σ , the substitution σ^n is inductively defined as follows:

- $\sigma^0 = \mathbf{id}$,
- $\sigma^{n+1} = \sigma^n \circ \sigma$.

³In this argument, we assume $n \geq 1$. The truth is that if $n = 0$ then F is implicative, seeing that $\models F \leftrightarrow \alpha$.

Proposition 5 For all $n \in \mathbb{N}$ and for all substitutions σ , $\sigma \circ \sigma^n = \sigma^n \circ \sigma$.

Proof: By induction on n . \dashv

For all finite $X \subseteq \mathbf{VAR}$, the equivalence relation \simeq_X on \mathbf{SUB} is defined by

- $\sigma \simeq_X \tau$ if and only if for all $x \in X$, $\models \sigma(x) \leftrightarrow \tau(x)$.

For all finite $X \subseteq \mathbf{VAR}$, the quasi-order \preceq_X on \mathbf{SUB} is defined by

- $\sigma \preceq_X \tau$ if and only if there exists $v \in \mathbf{SUB}$ such that for all $x \in X$, $\models v(\sigma(x)) \leftrightarrow \tau(x)$.

Useful results Let $k \in \mathbb{N}$. Let $\mathcal{S} = \{0, 1\}^k$ (with typical members denoted (a_1, \dots, a_k) , (b_1, \dots, b_k) , etc) and \ll be the binary relation on \mathcal{S} defined as follows:

- $(a_1, \dots, a_k) \ll (b_1, \dots, b_k)$ iff for all $i \in \{1, \dots, k\}$, $a_i \leq b_i$.

Obviously, (\mathcal{S}, \ll) is a finite lattice in which the length of an arbitrary chain is at most equal to k .

Proposition 6 Let $\{y_1, \dots, y_k\}$ be a finite set of variables and σ be a substitution such that for all $i \in \{1, \dots, k\}$, $\mathbf{var}(\sigma(y_i)) \subseteq \{y_1, \dots, y_k\}$. Let φ be a variable-free formula such that for all $i \in \{1, \dots, k\}$, $\models \varphi \rightarrow (y_i \rightarrow \sigma(y_i))$. For all $n \in \mathbb{N}$ and for all $j \in \{1, \dots, k\}$, if $n \geq k$ then $\models \varphi \rightarrow (\sigma(\sigma^n(y_j)) \leftrightarrow \sigma^n(y_j))$.

Proof: Let $n \in \mathbb{N}$ and $j \in \{1, \dots, k\}$ be such that $n \geq k$. For the sake of the contradiction, suppose $\not\models \varphi \rightarrow (\sigma(\sigma^n(y_j)) \leftrightarrow \sigma^n(y_j))$. Let V be a model such that $\bar{V}(\varphi) = 1$ and $\bar{V}(\sigma(\sigma^n(y_j)))$ and $\bar{V}(\sigma^n(y_j))$ are not equal. Since for all $i \in \{1, \dots, k\}$, $\mathbf{var}(\sigma(y_i)) \subseteq \{y_1, \dots, y_k\}$, for all models V', V'' , if $V'_{\mathbf{PAR}} = V''_{\mathbf{PAR}}$ and for all $i \in \{1, \dots, k\}$, $V'(y_i) = V''(y_i)$ then for all $i \in \{1, \dots, k\}$, $\bar{V}'(\sigma(y_i)) = \bar{V}''(\sigma(y_i))$. For all $(a_1, \dots, a_k) \in \mathcal{S}$, let $f((a_1, \dots, a_k))$ be the unique $(b_1, \dots, b_k) \in \mathcal{S}$ such that for all models V' , if $V'_{\mathbf{PAR}} = V_{\mathbf{PAR}}$ and for all $i \in \{1, \dots, k\}$, $V'(y_i) = a_i$ then for all $i \in \{1, \dots, k\}$, $\bar{V}'(\sigma(y_i)) = b_i$. Since φ is variable-free, for all $i \in \{1, \dots, k\}$, $\models \varphi \rightarrow (y_i \rightarrow \sigma(y_i))$ and $\bar{V}(\varphi) = 1$, for all $(a_1, \dots, a_k) \in \mathcal{S}$, $(a_1, \dots, a_k) \ll f((a_1, \dots, a_k))$. Since the length of an arbitrary chain in (\mathcal{S}, \ll) is at most equal to k and $n \geq k$, $f(f^n((V(y_1), \dots, V(y_k)))) = f^n((V(y_1), \dots, V(y_k)))$. Hence, $(\bar{V}(\sigma(\sigma^n(y_1))), \dots, \bar{V}(\sigma(\sigma^n(y_k)))) = (\bar{V}(\sigma^n(y_1)), \dots, \bar{V}(\sigma^n(y_k)))$. Thus, $\bar{V}(\sigma(\sigma^n(y_j)))$ and $\bar{V}(\sigma^n(y_j))$ are equal: a contradiction. \dashv

Proposition 7 Let $\{y_1, \dots, y_k\}$ be a finite set of variables and σ be a substitution such that for all $i \in \{1, \dots, k\}$, $\mathbf{var}(\sigma(y_i)) \subseteq \{y_1, \dots, y_k\}$. Let φ be a variable-free formula such that for all $i \in \{1, \dots, k\}$, $\models \varphi \vee (\sigma(y_i) \rightarrow y_i)$. For all $n \in \mathbb{N}$ and for all $j \in \{1, \dots, k\}$, if $n \geq k$ then $\models \varphi \vee (\sigma(\sigma^n(y_j)) \leftrightarrow \sigma^n(y_j))$.

Proof: Let $n \in \mathbb{N}$ and $j \in \{1, \dots, k\}$ be such that $n \geq k$. For the sake of the contradiction, suppose $\not\models \varphi \vee (\sigma(\sigma^n(y_j)) \leftrightarrow \sigma^n(y_j))$. Let V be a model such that $\bar{V}(\varphi) = 0$ and $\bar{V}(\sigma(\sigma^n(y_j)))$ and $\bar{V}(\sigma^n(y_j))$ are not equal. Since for all $i \in \{1, \dots, k\}$, $\text{var}(\sigma(y_i)) \subseteq \{y_1, \dots, y_k\}$, for all models V', V'' , if $V'_{|\text{PAR}} = V''_{|\text{PAR}}$ and for all $i \in \{1, \dots, k\}$, $V'(y_i) = V''(y_i)$ then for all $i \in \{1, \dots, k\}$, $\bar{V}'(\sigma(y_i)) = \bar{V}''(\sigma(y_i))$. For all $(a_1, \dots, a_k) \in \mathcal{S}$, let $f((a_1, \dots, a_k))$ be the unique $(b_1, \dots, b_k) \in \mathcal{S}$ such that for all models V' , if $V'_{|\text{PAR}} = V_{|\text{PAR}}$ and for all $i \in \{1, \dots, k\}$, $V'(y_i) = a_i$ then for all $i \in \{1, \dots, k\}$, $\bar{V}'(\sigma(y_i)) = b_i$. Since φ is variable-free, for all $i \in \{1, \dots, k\}$, $\models \varphi \vee (\sigma(y_i) \rightarrow y_i)$ and $\bar{V}(\varphi) = 0$, for all $(a_1, \dots, a_k) \in \mathcal{S}$, $f((a_1, \dots, a_k)) \ll (a_1, \dots, a_k)$. Since the length of an arbitrary chain in (\mathcal{S}, \ll) is at most equal to k and $n \geq k$, $f(f^n((V(y_1), \dots, V(y_k)))) = f^n((V(y_1), \dots, V(y_k)))$. Hence, $(\bar{V}(\sigma(\sigma^n(y_1))), \dots, \bar{V}(\sigma(\sigma^n(y_k)))) = (\bar{V}(\sigma^n(y_1)), \dots, \bar{V}(\sigma^n(y_k)))$. Thus, $\bar{V}(\sigma(\sigma^n(y_j)))$ and $\bar{V}(\sigma^n(y_j))$ are equal: a contradiction. \dashv

Unifiers Let $S \subseteq \mathbf{FOR}$ be finite. A *unifier* of S is a $\text{var}(S)$ -substitution σ such that $\models \sigma(S)$. S is *unifiable* if there exists a unifier of S .

Proposition 8 For all unifiers σ of S , there exists a unifier τ of S such that $\tau \preceq_{\text{var}(S)} \sigma$ and $\text{par}(\tau(S)) \subseteq \text{par}(S)$.

Proof: Let σ be a unifier of S . Hence, $\models \sigma(S)$. Let q_1, \dots, q_l be the list of the parameters q such that $q \in \text{par}(\sigma(S)) \setminus \text{par}(S)$. Let z_1, \dots, z_l be new variables. Let τ be the substitution obtained from σ by respectively replacing in $\sigma(x)$ each occurrence of q_1, \dots, q_l by z_1, \dots, z_l for each variable x in S . Obviously, $\tau \preceq_{\text{var}(S)} \sigma$. Moreover, since $\models \sigma(S)$, $\models \tau(S)$. Thus, τ is a unifier of S . In other respect, obviously, $\text{par}(\tau(S)) \subseteq \text{par}(S)$. \dashv

Unification types Let $S \subseteq \mathbf{FOR}$ be finite. If S is unifiable then a set Σ of unifiers of S is *complete* if for all unifiers σ of S , there exists $\tau \in \Sigma$ such that $\tau \preceq_{\text{var}(S)} \sigma$. As is well-known, if S is unifiable then for all minimal complete sets Σ, Δ of unifiers of S , $\text{Card}(\Sigma) = \text{Card}(\Delta)$ ⁴. As a result, if S is unifiable then

- S is *unitary* if there exists a minimal complete set of unifiers of S with cardinality 1,
- S is *finitary* if there exists a minimal complete set of unifiers of S with finite cardinality ≥ 2 ,
- S is *infinitary* if there exists a minimal complete set of unifiers of S with infinite cardinality,

⁴This is a standard result, although we have not been able to find a reference proving it. Suppose S is unifiable. Let Σ, Δ be minimal complete sets of unifiers of S . By the completeness of Σ and Δ , one can readily define functions $f : \Sigma \rightarrow \Delta$ and $g : \Delta \rightarrow \Sigma$ such that $f(\sigma) \preceq_{\text{var}(S)} \sigma$ for each $\sigma \in \Sigma$ and $g(\delta) \preceq_{\text{var}(S)} \delta$ for each $\delta \in \Delta$. By the minimality of Σ and Δ , it easily follows that f and g are injective. Thus, Σ and Δ have the same cardinality.

- S is *nullary* if there exists no minimal complete set of unifiers of S .

4 About the type of the implication fragment

Let $S \subseteq \mathbf{FOR}$ be finite. Assume $\text{par}(S) \neq \emptyset$ and S is unifiable. Let σ be a unifier of S . By Proposition 8, let τ be a unifier of S such that $\tau \preceq_{\text{var}(S)} \sigma$ and $\text{par}(\tau(S)) \subseteq \text{par}(S)$. Since $\text{par}(S) \neq \emptyset$, let p_0 be a parameter in S . Let μ be the substitution defined as follows:

- for all variables x , if there exists $y \in \text{var}(S)$ such that $x \in \text{var}(\tau(y))$ then $\mu(x) = p_0 \rightarrow p_0$,
- for all variables x , if for all $y \in \text{var}(S)$, $x \notin \text{var}(\tau(y))$ then $\mu(x) = x$.

Note that $\text{var}(\mu(\tau(S))) = \emptyset$. For all $x \in \text{var}(S)$, by Proposition 4, there exists an atom α_x in $\tau(x)$ such that $\models \alpha_x \rightarrow \tau(x)$. For all $x \in \text{var}(S)$, let F_x be the extended formula defined as follows:

- if α_x is a variable then let $F_x = \neg(p_0 \rightarrow p_0)$,
- if α_x is a parameter then let $F_x = \alpha_x$.

Note that for all $x \in \text{var}(S)$, F_x is variable-free.

Lemma 1 For all $x \in \text{var}(S)$,

1. $\models F_x \rightarrow \tau(x)$,
2. $\models F_x \rightarrow \mu(\tau(x))$.

Proof: Let $x \in \text{var}(S)$.

(1). **Case α_x is a variable:** Hence, $F_x = \neg(p_0 \rightarrow p_0)$. Thus, $\models F_x \rightarrow \tau(x)$.

Case α_x is a parameter: Hence, $F_x = \alpha_x$. Since $\models \alpha_x \rightarrow \tau(x)$, $\models F_x \rightarrow \tau(x)$.

(2). By item 1, $\models F_x \rightarrow \tau(x)$. Hence, $\models F_x \rightarrow \mu(\tau(x))$. \dashv

For all $x \in \text{var}(S)$, let $G_x = (\bigwedge S \wedge x) \vee (\neg \bigwedge S \wedge \mu(\tau(x))) \vee F_x$ ⁵.

Lemma 2 For all $x \in \text{var}(S)$, G_x is implicative.

Proof: Let $x \in \text{var}(S)$.

Case α_x is a variable: Hence, $F_x = \neg(p_0 \rightarrow p_0)$. Thus, $\models G_x \leftrightarrow (\bigwedge S \wedge x) \vee (\neg \bigwedge S \wedge \mu(\tau(x)))$. Since α_x is an atom in $\tau(x)$, $\mu(\alpha_x) = p_0 \rightarrow p_0$. Since $\models \alpha_x \rightarrow \tau(x)$, $\models \mu(\alpha_x) \rightarrow \mu(\tau(x))$. Since $\mu(\alpha_x) = p_0 \rightarrow p_0$, $\models \mu(\tau(x))$. Since $\models G_x \leftrightarrow (\bigwedge S \wedge x) \vee (\neg \bigwedge S \wedge \mu(\tau(x)))$, $\models G_x \leftrightarrow (\bigwedge S \rightarrow x)$. Consequently, by

⁵Notice that for all $x \in \text{var}(S)$, if α_x is a variable then G_x is logically equivalent to the variant $(\bigwedge S \wedge x) \vee (\neg \bigwedge S \wedge \mu(\tau(x)))$ of the so-called Löwenheim formula [4, 13, 22].

Proposition 4, G_x is implicative.

Case α_x is a parameter: Hence, $F_x = \alpha_x$. Thus, by Proposition 4, G_x is implicative. \dashv

Let ϵ be the substitution defined as follows:

- for all variables x , if $x \in \text{var}(S)$ then $\epsilon(x) = G_x$,
- for all variables x , if $x \notin \text{var}(S)$ then $\epsilon(x) = x$.

Note that for all variables x , if $x \in \text{var}(S)$ then $\text{var}(\epsilon(x)) \subseteq \text{var}(S)$ and $\text{par}(\epsilon(x)) \subseteq \text{par}(S)$ else $\epsilon(x) = x$.

Lemma 3 For all variables x , $\models \tau(\epsilon(x)) \leftrightarrow \tau(x)$.

Proof: Let x be a variable.

Case $x \in \text{var}(S)$: Hence, $\epsilon(x) = (\bigwedge S \wedge x) \vee (\neg \bigwedge S \wedge \mu(\tau(x))) \vee F_x$. Since τ is a unifier of S , $\models \tau(S)$. Since $\epsilon(x) = (\bigwedge S \wedge x) \vee (\neg \bigwedge S \wedge \mu(\tau(x))) \vee F_x$, $\models \tau(\epsilon(x)) \leftrightarrow \tau(x) \vee F_x$. Thus, by Lemma 1, $\models \tau(\epsilon(x)) \leftrightarrow \tau(x)$.

Case $x \notin \text{var}(S)$: Hence, $\epsilon(x) = x$. Thus, $\models \tau(\epsilon(x)) \leftrightarrow \tau(x)$. \dashv

Lemma 4 For all $n \in \mathbb{N}$ and for all variables x , $\models \tau(\epsilon^n(x)) \leftrightarrow \tau(x)$.

Proof: By induction on n . \dashv

Lemma 5 For all $n \in \mathbb{N}$, $\epsilon^n \preceq_{\text{var}(S)} \sigma$.

Proof: Let $n \in \mathbb{N}$. By Lemma 4, $\epsilon^n \preceq_{\text{var}(S)} \tau$. Since $\tau \preceq_{\text{var}(S)} \sigma$, $\epsilon^n \preceq_{\text{var}(S)} \sigma$. \dashv

Lemma 6 For all variables x , if $x \in \text{var}(S)$ then $\models \bigwedge S \rightarrow (\epsilon(x) \leftrightarrow (x \vee F_x))$ and $\models \neg \bigwedge S \rightarrow (\epsilon(x) \leftrightarrow \mu(\tau(x)))$.

Proof: Let x be a variable such that $x \in \text{var}(S)$. Hence, $\epsilon(x) = (\bigwedge S \wedge x) \vee (\neg \bigwedge S \wedge \mu(\tau(x))) \vee F_x$. Thus, $\models \bigwedge S \rightarrow (\epsilon(x) \leftrightarrow (x \vee F_x))$ and $\models \neg \bigwedge S \rightarrow (\epsilon(x) \leftrightarrow (\mu(\tau(x)) \vee F_x))$. Consequently, by Lemma 1, $\models \neg \bigwedge S \rightarrow (\epsilon(x) \leftrightarrow \mu(\tau(x)))$. \dashv

Lemma 7 For all formulas φ , if $\text{var}(\varphi) \subseteq \text{var}(S)$ then $\models \neg \bigwedge S \rightarrow (\epsilon(\varphi) \leftrightarrow \mu(\tau(\varphi)))$.

Proof: By induction on φ . \dashv

Lemma 8 For all variables x , if $x \in \text{var}(S)$ then $\models (x \wedge \mu(\tau(x))) \rightarrow \epsilon(x)$ and $\models \epsilon(x) \rightarrow (x \vee \mu(\tau(x)))$.

Proof: Let x be a variable such that $x \in \mathbf{var}(S)$. By Lemma 6, $\models \bigwedge S \rightarrow (\epsilon(x) \leftrightarrow (x \vee F_x))$ and $\models \neg \bigwedge S \rightarrow (\epsilon(x) \leftrightarrow \mu(\tau(x)))$. Hence, $\models \bigwedge S \rightarrow ((x \wedge \mu(\tau(x))) \rightarrow \epsilon(x))$ and $\models \neg \bigwedge S \rightarrow ((x \wedge \mu(\tau(x))) \rightarrow \epsilon(x))$. Thus, $\models (x \wedge \mu(\tau(x))) \rightarrow \epsilon(x)$. Since $\models \bigwedge S \rightarrow (\epsilon(x) \leftrightarrow (x \vee F_x))$ and $\models \neg \bigwedge S \rightarrow (\epsilon(x) \leftrightarrow \mu(\tau(x)))$, $\models \bigwedge S \rightarrow (\epsilon(x) \rightarrow (x \vee F_x \vee \mu(\tau(x))))$ and $\models \neg \bigwedge S \rightarrow (\epsilon(x) \rightarrow (x \vee F_x \vee \mu(\tau(x))))$. Consequently, $\models \epsilon(x) \rightarrow (x \vee F_x \vee \mu(\tau(x)))$. Hence, by Lemma 1, $\models \epsilon(x) \rightarrow (x \vee \mu(\tau(x)))$. \dashv

Lemma 9 $\models \neg \bigwedge S \rightarrow \epsilon(\bigwedge S)$.

Proof: By Lemma 7, $\models \neg \bigwedge S \rightarrow (\epsilon(\bigwedge S) \leftrightarrow \mu(\tau(\bigwedge S)))$. Since τ is a unifier of S , $\models \tau(S)$. Hence, $\models \mu(\tau(S))$. Since $\models \neg \bigwedge S \rightarrow (\epsilon(\bigwedge S) \leftrightarrow \mu(\tau(\bigwedge S)))$, $\models \neg \bigwedge S \rightarrow \epsilon(\bigwedge S)$. \dashv

Lemma 10 For all $n \in \mathbb{N}$ and for all variables x , if $n \geq \|\mathbf{var}(S)\|$ and $x \in \mathbf{var}(S)$ then $\models \epsilon(\epsilon^n(x)) \leftrightarrow \epsilon^n(x)$.

Proof: Let $n \in \mathbb{N}$ and x be a variable such that $n \geq \|\mathbf{var}(S)\|$ and $x \in \mathbf{var}(S)$. By Lemma 8, $\models (x \wedge \mu(\tau(x))) \rightarrow \epsilon(x)$ and $\models \epsilon(x) \rightarrow (x \vee \mu(\tau(x)))$. Hence, $\models \mu(\tau(x)) \rightarrow (x \rightarrow \epsilon(x))$ and $\models \mu(\tau(x)) \vee (\epsilon(x) \rightarrow x)$. Since $n \geq \|\mathbf{var}(S)\|$, by Propositions 6 and 7, $\models \mu(\tau(x)) \rightarrow (\epsilon(\epsilon^n(x)) \leftrightarrow \epsilon^n(x))$ and $\models \mu(\tau(x)) \vee (\epsilon(\epsilon^n(x)) \leftrightarrow \epsilon^n(x))$. Thus, $\models \epsilon(\epsilon^n(x)) \leftrightarrow \epsilon^n(x)$. \dashv

Lemma 11 For all $n \in \mathbb{N}$, if $n \geq \|\mathbf{var}(S)\|$ then ϵ^n is a unifier of S .

Proof: Let $n \in \mathbb{N}$ be such that $n \geq \|\mathbf{var}(S)\|$. By Lemma 9, $\models \neg \bigwedge S \rightarrow \epsilon(\bigwedge S)$. Hence, $\models \bigwedge S \vee \epsilon(\bigwedge S)$. Thus, $\models \epsilon^n(\bigwedge S) \vee \epsilon^n(\epsilon(\bigwedge S))$. Consequently, by Proposition 5, $\models \epsilon^n(\bigwedge S) \vee \epsilon(\epsilon^n(\bigwedge S))$. Since $n \geq \|\mathbf{var}(S)\|$, by Lemma 10, $\models \epsilon(\epsilon^n(\bigwedge S)) \leftrightarrow \epsilon^n(\bigwedge S)$. Since $\models \epsilon^n(\bigwedge S) \vee \epsilon(\epsilon^n(\bigwedge S))$, $\models \epsilon^n(S)$. \dashv

Proposition 9 Let $S \subseteq \mathbf{FOR}$ be finite. If S is unifiable then either S is unitary, or S is finitary.

Proof: Suppose S is unifiable.

Case $\mathbf{par}(S) = \emptyset$: Let ϵ be the substitution defined by

- for all variables x , if $x \in \mathbf{var}(S)$ then $\epsilon(x) = \bigwedge S \rightarrow x$,
- for all variables x , if $x \notin \mathbf{var}(S)$ then $\epsilon(x) = x$.

The reader may easily verify that ϵ constitutes by itself a minimal complete set of unifiers of S [10, 24]. Hence, S is unitary.

Case $\mathbf{par}(S) \neq \emptyset$: Let p_0 be a parameter in S . Let Σ be the set of all unifiers of S . Note that Σ is complete. Let Σ' be the set of unifiers of S obtained from Σ by keeping only the unifiers σ of S such that $\mathbf{var}(\sigma(S)) \subseteq \mathbf{var}(S)$ and $\mathbf{par}(\sigma(S)) \subseteq \mathbf{par}(S)$. Since Σ is complete, by Lemmas 5 and 11, Σ' is complete.

Let Σ'' be the set of unifiers of S obtained from Σ' by keeping only one representative of each equivalence class modulo $\simeq_{\text{var}(S)}$. Since Σ' is complete, Σ'' is complete. Moreover, Σ'' is finite. Thus, either S is unitary, or S is finitary. \dashv

Proposition 10 *Let $n \geq 1$ and $S = \{x \rightarrow p_1 \vee \dots \vee p_n\}$.*

1. *S is unifiable,*
2. *if for all $i, j \in \{1, \dots, n\}$, $p_i = p_j$ then S is unitary,*
3. *if there exists $i, j \in \{1, \dots, n\}$ such that $p_i \neq p_j$ then S is finitary.*

Proof:

(1). For all $i \in \{1, \dots, n\}$, let σ_i be the substitution defined as follows:

- $\sigma_i(x) = p_i$,
- for all variables y , if $y \neq x$ then $\sigma_i(y) = y$.

Obviously, for all $i \in \{1, \dots, n\}$, σ_i is a unifier of S . Hence, S is unifiable.

(2). Suppose for all $i, j \in \{1, \dots, n\}$, $p_i = p_j$. Let p be the parameter such that for all $i \in \{1, \dots, n\}$, $p = p_i$ and σ be the substitution defined as follows:

- $\sigma(x) = p$,
- for all variables y , if $y \neq x$ then $\sigma(y) = y$.

Obviously, σ is a unifier of S . Moreover, for all unifiers v of S , $\models v(x) \rightarrow p$ and by Proposition 2, $\models p \rightarrow v(x)$. Thus, for all unifiers v of S , $\models v(x) \leftrightarrow \sigma(x)$. Consequently, for all unifiers v of S , $v \simeq_{\text{var}(S)} \sigma$. Hence, S is unitary.

(3). Suppose there exists $i, j \in \{1, \dots, n\}$ such that $p_i \neq p_j$. For the sake of the contradiction, suppose S is not finitary. Thus, by Proposition 9, S is unitary. Let v be a unifier of S such that for all $k \in \{1, \dots, n\}$, $v \preceq_{\text{var}(S)} \sigma_k$. Thus, $\models v(x) \rightarrow p_1 \vee \dots \vee p_n$ and for all $k \in \{1, \dots, n\}$, there exists a substitution θ_k such that $\models \theta_k(v(x)) \leftrightarrow p_k$. Consequently, by Proposition 2, let $l \in \{1, \dots, n\}$ be such that $\models p_l \rightarrow v(x)$. Hence, $\models p_l \rightarrow \theta_i(v(x))$ and $\models p_l \rightarrow \theta_j(v(x))$. Since for all $k \in \{1, \dots, n\}$, $\models \theta_k(v(x)) \leftrightarrow p_k$, $\models p_l \rightarrow p_i$ and $\models p_l \rightarrow p_j$. Thus, $p_l = p_i$ and $p_l = p_j$. Consequently, $p_i = p_j$: a contradiction. Hence, S is finitary. \dashv

From Propositions 9 and 10, it follows that

Theorem 1 *The implication fragment of Classical Propositional Logic is finitary for unification with parameters, i.e.*

- *for all finite $S \subseteq \mathbf{FOR}$, if S is unifiable then either S is unitary, or S is finitary,*
- *there exists a finite $S \subseteq \mathbf{FOR}$ such that S is unifiable and S is finitary,*

5 Conclusion

In this paper, we have seen that, in contrast to the full language of **CPL** which is unitary both for elementary unification and for unification with parameters [4, 13, 22], the implication fragment of **CPL** is unitary for elementary unification and finitary for unification with parameters. Below, we propose numerous conjectures and varied problems, with the hope of setting goals for future research in unification theory.

It is known that when **CPL** is based on a definitionally complete set of connectives, elementary unification is **NP**-complete and unification with parameters is Π_2^P -complete [3]. From Section 4, it follows that elementary unification in the implication fragment of **CPL** is trivial and unification with parameters in the implication fragment of **CPL** is decidable. But what is the complexity of unification with parameters in the implication fragment of **CPL**?

As noted in the introduction, the rule $\frac{x \rightarrow p}{p \rightarrow x}$ is admissible in the implication fragment of **CPL**. Nevertheless, it is not derivable. Hence, the question arises: for the implication fragment of **CPL**, does the set of all admissible rules admit a finite basis, i.e. a finite set of rules that added to the axiomatization of the implication fragment of **CPL** mentioned in the introduction exactly produces the set of all admissible rules⁶?

For a modal language based on the connectives \rightarrow , \neg and \Box , it is known, for example, that **S5** is unitary [12], **S4** is finitary [17, 20] and **K** is nullary [21]. This immediately leads us to ask what is the unification type of **S5**, **S4** and **K** when \rightarrow and \Box are the sole connectives? This question is of course related to the question of the unification type of some of the modal algebras discussed by Celani and Montangie [7, 8]. It is not a trivial question. For instance, the formula $x \rightarrow \Box x$ used by Jeřábek [21] to show that **K** is nullary becomes unitary when \rightarrow and \Box are the sole connectives⁷.

Remind that in a modal language based on the connectives \rightarrow , \neg and \Box , the connective of strict implication is defined as follows: $(\varphi \Rightarrow \psi) ::= \Box(\varphi \rightarrow \psi)$ [18, Chapter 2]. Hence, there is also the question of the unification type of the frag-

⁶See [19] for a definition of what a basis of admissible rules is within the context of Intuitionistic Propositional Logic.

⁷In order to understand why, it suffices to notice that

- the substitution σ defined as follows:

- $\sigma(x) = x \rightarrow x$,
- for all variables y , if $x \neq y$ then $\sigma(y) = y$,

is a **K**-unifier of $x \rightarrow \Box x$,

- for all modal formulas φ with \rightarrow and \Box as its sole connectives, if $\varphi \rightarrow \Box \varphi$ is in **K** then φ is in **K**.

ments of the above-mentioned modal logics when \Rightarrow is the sole connective. For those who interest in such fragments which have been extensively studied over the years, we recommend the reading of [14].

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