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On Primitive Pythagorean Triples

César Aguilera.

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Abstract

We talk about different kinds of Primitive Pythagorean Triples. Sylvester's Sequence, Prime numbers, Fermat Primes and their relation with H. Lee Price Pythagorean Tree.

Introduction.

Back in 1966, C. Stanley Ogilvy and John T. Anderson, published a book called "Excursions in Number Theory" [1], at this book appears a triangular array composed by summations of squared terms, we found this triangle very interesting as he can be related to Primitive Pythagorean Triples, so let's explore this triangle and find where he takes us.

Table 1: Triangle.

n	2n+1	
1	3	$3^2 + 4^2 = 5^2$
2	5	$10^2 + 11^2 + 12^2 = 13^2 + 14^2$
3	7	$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$
4	9	$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$
5	11	$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$
6	13	$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$

Where **n** is the nth line and **2n+1** is the number of terms on the line. Notice that given a **n** line, with an initial term \mathbf{n}_i , then $\frac{\mathbf{n}_i}{\mathbf{n}} = \mathbf{2n} + \mathbf{1}$, and the last term $\frac{\mathbf{n}_k}{\mathbf{n}} + \mathbf{n}_k$ is equal to \mathbf{n}_i of the next line. Also we can notice that the term before the equal sign is equal to $\mathbf{2n}^2 + \mathbf{2n}$, the term after the equal sign is $\mathbf{2n}^2 + \mathbf{2n} + \mathbf{1}$ and they are part of a Primitive Pythagorean Triple of the form: $\mathbf{2n} + \mathbf{1}$, $\mathbf{2n}^2 + \mathbf{2n}$, $\mathbf{2n}^2 + \mathbf{2n} + \mathbf{1}$.

A representation that will be useful later is this:
 $2n + 1, 2 \cdot \int 2n + 1 \cdot dn, 2 \cdot \int 2n + 1 \cdot dn + 1.$

H. Lee Price. Fibonacci Boxes.

The H. Lee Price Fibonacci Boxes^[2] are representations of 2×2 matrices with numerators in the first row and denominators in the second row.

For our specific case, we use boxes of the form:

$$\begin{vmatrix} \frac{n-1}{2} & 1 \\ \frac{n+1}{2} & n \end{vmatrix} \text{ if we substitute } \mathbf{n} \text{ with } \mathbf{2n+1}, \text{ we have } \begin{vmatrix} \frac{2n+1-1}{2} & 1 \\ \frac{2n+1+1}{2} & 2n+1 \end{vmatrix}$$

and we get:

$$\begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix} \text{ Note that it follows the fibonacci property.}$$

Also follows the other properties:

**The hypotenuse is the sum of the diagonal products and it is also the difference of the two row products.

So we have:

$$\begin{aligned} (2n+1)(n) + (n+1)(1) &= 2n^2 + n + n + 1 = 2n^2 + 2n + 1 \\ (n+1)(2n+1) - (n)(1) &= 2n^2 + n + 2n + 1 - n = 2n^2 + 2n + 1 \end{aligned}$$

**There is a single even number and is always in the first column (if \mathbf{n} is even, then $\mathbf{n+1}$ is not and viceversa, and $\mathbf{2n+1}$ can not be even).

Another property is:

$$2(n+1)(n) = 2n^2 + 2n.$$

Also:

$$2n^2 + 2n - \text{Det} \begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix} = 2n + 1$$

Geometric representation.

We can represent these Pythagorean Triples as circles and find some properties. Given the Pythagorean Triple (3,4,5), we can intersect it's circle equation with the equations of the next Pythagorean Triples and find the intersection points and the slope.

The second term of all the Pythagorean Triples, divided by the second term of (3,4,5) is equal to the Triangular Numbers, this is:

$$\frac{2n^2+2n}{4} = \frac{n(n+1)}{2}$$

Table 2: Intersection points and slope respect to $x^2 - 6x + y^2 - 8y = 0$.

Pythagorean Triple	Equation	Intersection Points	Slope
(5,12,13)	$x^2 - 10x + y^2 - 24y = 0$	$(\frac{64}{17}, -(\frac{16}{17}))$	4
(7,24,25)	$x^2 - 14x + y^2 - 48y = 0$	$(\frac{59}{13}, -(\frac{11}{13}))$	5
(9,40,41)	$x^2 - 18x + y^2 - 80y = 0$	$(\frac{168}{37}, -(\frac{28}{37}))$	6
(11,60,61)	$x^2 - 22x + y^2 - 120y = 0$	$(\frac{119}{25}, -(\frac{17}{25}))$	7
(13,84,85)	$x^2 - 26x + y^2 - 168y = 0$	$(\frac{64}{13}, -(\frac{8}{13}))$	8

The function $y = x^3 - x$

Let n be the n th line of the triangular array and $2n+1$ the number of terms of such given line, a Pythagorean Triple (a,b,c) can be expressed as:

$$a, \sqrt{b}, \sqrt{b} + 1$$

where:

$$2n + 1 = a \quad b = \int_1^a x^3 - x \cdot dx$$

All these can take us to the Sylvester's Sequence.

The Sylvester's Sequence.

The Sylvester's sequence is an integer sequence where every number is the product of the previous numbers plus one.

The first terms are: $\{2, 3, 7, 43, 1807, 3263443, \dots\}$

We can obtain the Sylvester's sequence using the function $f(x) = x^3 - x$ as we can see next:

$$A_1 = \int_1^3 x^3 - x \cdot dx, \quad A_2 = \int_1^5 x^3 - x \cdot dx, \quad A_3 = \int_1^{13} x^3 - x \cdot dx$$

$$A_4 = \int_1^{85} x^3 - x \cdot dx, \quad A_5 = \int_1^{3613} x^3 - x \cdot dx, \quad A_6 = \int_1^{6526885} x^3 - x \cdot dx$$

$$\sqrt{A_1} = 4, \quad \sqrt{A_2} = 12, \quad \sqrt{A_3} = 84, \quad \sqrt{A_4} = 3612, \quad \sqrt{A_5} = 6526884,$$

$$\frac{\sqrt{A_1}}{2} = 2, \quad \frac{\sqrt{A_2}}{\sqrt{A_1}} = 3, \quad \frac{\sqrt{A_3}}{\sqrt{A_2}} = 7, \quad \frac{\sqrt{A_4}}{\sqrt{A_3}} = 43, \quad \frac{\sqrt{A_5}}{\sqrt{A_4}} = 1807,$$

And we got the sequence. The upper-bounds of the Integrals correspond to the hypotenuses of the Pythagorean Triples:
 $(3,4,5), (5,12,13), (13,84,85), (85,3612,3613)$, etc. (No need to say that it is a Pythagorean Spiral).

Notice that: $n_k(n_k - 1) = n_{k+1}$ and as $\int_1^{2n+1} x^3 - x \cdot dx = 2n^2 + 2n$, we can now define the Sylvester's Sequence as numbers of the form $n^2 - n + 1$ where the first two terms are $n^2 + n, n^2 + n + 1$.

Sylvester's Sequence and the H. Lee Price. Fibonacci Boxes.

Given the spiral formed by the abc triangles:

$(3,4,5), (5,12,13), (13,84,85), (85,3612,3613)$, ... we have:

From $\Delta(3, 4, 5)$:

$$\frac{b}{c+a} = \frac{4}{8} = \frac{1}{2}; \frac{a}{c+b} = \frac{3}{9} = \frac{1}{3} \rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

From $\Delta(5, 12, 13)$:

$$\frac{b}{c+a} = \frac{5}{25} = \frac{1}{5}; \frac{a}{c+b} = \frac{12}{18} = \frac{2}{3} \rightarrow \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

From $\Delta(13, 84, 85)$:

$$\frac{b}{c+a} = \frac{13}{169} = \frac{1}{13}; \frac{a}{c+b} = \frac{84}{98} = \frac{6}{7} \rightarrow \begin{bmatrix} 6 & 1 \\ 7 & 13 \end{bmatrix}$$

From $\Delta(85, 3612, 3613)$:

$$\frac{b}{c+a} = \frac{85}{7225} = \frac{1}{85}; \frac{a}{c+b} = \frac{3612}{3698} = \frac{42}{43} \rightarrow \begin{bmatrix} 42 & 1 \\ 43 & 85 \end{bmatrix}$$

This is:

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}; \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}; \begin{bmatrix} 6 & 1 \\ 7 & 13 \end{bmatrix}; \begin{bmatrix} 42 & 1 \\ 43 & 85 \end{bmatrix}, \dots$$

From:

$$\begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix}$$

We have:

$$\left\{ \begin{array}{l} S_k = (2n+1) - n = n+1 \\ S_{k+1} = n(n+1) + 1 = n^2 + n + 1 \end{array} \right\}$$

Variations of the Sylvester's Sequence.

We can now construct the variations of the Sylvester's Sequence for different values of 'n'.

Table 3: Sylvester's Sequence for n=1 and initial Pythagorean Triple (3,4,5).

2	3	7	43	1807	3263443	10650056950807	113423713055421844361000443
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Table 4: Sylvester's Sequence for n=2 and initial Pythagorean Triple (5,12,13).

6	7	43	1807	3263443	10650056950807	113423713055421844361000443
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Table 5: Sylvester's Sequence for n=3 and initial Pythagorean Triple (7,24,25).

12	13	157	24493	599882557	359859081592975693	129498558604939936868397356895854557
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Table 6: Sylvester's Sequence for n=4 and initial Pythagorean Triple (9,40,41).

20	21	421	176821	31265489221	977530816197201697621
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Table 7: Sylvester's Sequence for n=5 and initial Pythagorean Triple (11,60,61).

30	31	931	865831	749662454731	561993796032558961827631
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Table 8: Sylvester's Sequence for n=6 and initial Pythagorean Triple (13,84,85).

42	43	1807	3263443	10650056950807	113423713055421844361000443
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(Tables: 3,4,8, are the same sequence.)

Data taken from the original by Artur Jasinski. Under permission of OEIS, (Neil Sloane and Friends).

The Jasinski Sequence.

The Jasinski's sequence is formed by the terms: $\frac{(p_k-1)(p_k)(p_k+1)}{24}$, being p_k any prime number larger or equal to 3.[4]

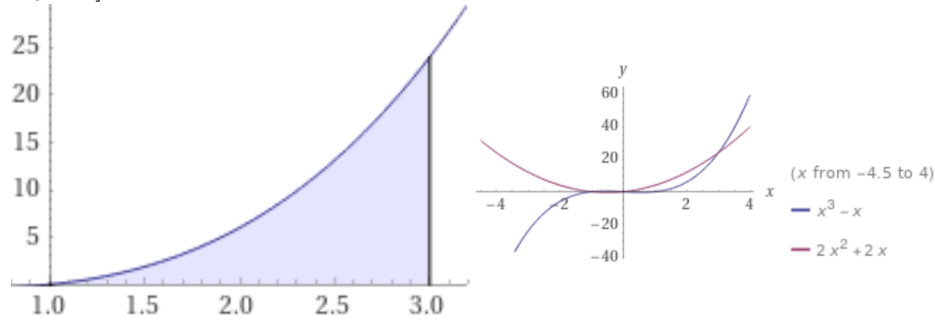
We later found that:

Given any prime number p_k then: $\frac{(p_k-1)(p_k)(p_k+1)}{24} = \sum_1^{\infty} \frac{1}{3^{2x+1}} \div \sum_1^{\infty} \frac{1}{p_k^{2x+1}}$.

When we integrate:

$$\int_1^3 x^3 - x \cdot dx = 16; [x = 3, y = 24]$$

The intersection point of the functions: $f(x) = x^3 - x$ and $f(x) = 2x^2 + 2x$ are $[x=3,y=24]$.



We can find some data in the next table.

Table 9: $\int_1^{2n+1} x^3 - x \cdot dx$

x	y	Area	Area square root
3	24	16	4
5	120	144	12
7	336	576	24
9	720	1600	40
11	1320	3600	60

From this data, we can get some properties.

$\frac{y}{x}$ **spiral.**

$\frac{y}{x}$ is the classic spiral of the (eight \times Triangular Numbers): $(4 \cdot n \cdot (n + 1))$.

64	65	66	67	68	69	70	71	72
63	36	37	38	39	40	41	42	73
62	35	16	17	18	19	20	43	74
61	34	15	4	5	6	21	44	75
60	33	14	3	0	7	22	45	76
59	32	13	2	1	8	23	46	77
58	31	12	11	10	9	24	47	78
57	30	29	28	27	26	25	48	79
56	55	54	53	52	51	50	49	80

Where

$$\sqrt{\frac{y}{x} + 1} = 2n + 1; \int 2n + 1 \cdot dn = n^2 + n + c$$

$(n^2 + n)$ are the Oblong numbers.

so: $\frac{4 \cdot n \cdot (n+1)}{4} = n(n + 1) = n^2 + n$

Areas and Triangular Numbers.

Given any Area A_k then: $\sqrt{\frac{A_k}{16}}$ is a Triangular Number. (see Table 2).

This can be applied to the Sylvester's Sequence. For example:

$$A_k = \int_1^{3613} x^3 - x \cdot dx$$

$$\sqrt{A_k} = 6526884$$

$$\frac{6526884}{4} = 1631721 = \frac{1806 \cdot 1807}{2}$$

$$\frac{1806}{2} = 903 = \frac{42 \cdot 43}{2}$$

$$\frac{42}{2} = 21 = \frac{6 \cdot 7}{2}$$

$$\frac{6}{2} = 3 = \frac{2 \cdot 3}{2}$$

Rook Chess moves and the average value of $x^3 - x$.

Given the average value of a function:

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) \cdot dx$$

We have:

$$f_{avg} = \frac{1}{(2n+1)-1} \int_1^{2n+1} x^3 - x \cdot dx = \frac{1}{2n} \int_1^{2n+1} x^3 - x \cdot dx$$

f_{avg} is the number of moves that a Rook Chess can perform on a $(d \times d)$ board, where $d = n + 1$.

Example:

$n = 1; 2n + 1 = 3; \text{board} = 2 \times 2$

$$\frac{1}{2} \int_1^3 x^3 - x \cdot dx = \frac{1}{2} \cdot 16 = 8$$

In general: Rook moves = $2n^3 + 4n^2 + 2n$.

Prime numbers and Fibonacci Boxes.

Given a Fibonacci Box $\left| \begin{array}{cc} n & 1 \\ n+1 & 2n+1 \end{array} \right|$,

the area of a given Pythagorean Triangle is given by:

$$1 \cdot n \cdot (n+1) \cdot (2n+1) = 2n^3 + 3n^2 + n.$$

For example, (5,12,13) have a Box $\left| \begin{array}{cc} 2 & 1 \\ 3 & 5 \end{array} \right|$ so,

$$\text{Area} = \frac{5 \cdot 12}{2} = 1 \cdot 2 \cdot 3 \cdot 5 = 2 \cdot 2^3 + 3 \cdot 2^2 + 2 = 30$$

In general, the Areas of this type of triangles, where the difference between the hypotenuse and the long leg is 1, is given by $\frac{y}{4}$ (see Table 9).

Given a Fibonacci Box $\left| \begin{array}{cc} 1 & 2n+1 \\ 2n+2 & 2n+3 \end{array} \right|$,

the area of a given Pythagorean Triangle is given by:

$$(2n+1) \cdot (1) \cdot (2n+2) \cdot (2n+3) = 8n^3 + 24n^2 + 22n + 6.$$

For example, (16,63,65) have a Box $\left| \begin{array}{cc} 1 & 7 \\ 8 & 9 \end{array} \right|$ so,

$$\text{Area} = \frac{16 \cdot 63}{2} = 7 \cdot 1 \cdot 8 \cdot 9 = 8 \cdot 3^3 + 24 \cdot 3^2 + 22 \cdot 3 + 6 = 504$$

Given a Fibonacci Box $\begin{vmatrix} b & p_k \\ p_k + b & p_{(k+1)} \end{vmatrix}$,

the area of a given Pythagorean Triangle is given by:

$$(p_k) \cdot (4b) \cdot (p_k + b) \cdot (p_{(k+1)})$$

For example, (168,874,890) have a Box $\begin{vmatrix} 2 & 19 \\ 21 & 23 \end{vmatrix}$ so,

$$\text{Area} = \frac{168 \cdot 874}{2} = 19 \cdot 8 \cdot 21 \cdot 23 = 73416$$

This is, given a Pythagorean Triple (A,B,C), then: $\frac{A}{p_k + b} = 4b$.

Area of Pythagorean Triangles related to Prime Numbers,

Here we have some Non Primitive Pythagorean Triples related to Prime Numbers and their respective Fibonacci Boxes.

$$\begin{array}{cccccccccc} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} & \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} & \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} & \begin{bmatrix} 2 & 7 \\ 9 & 11 \end{bmatrix} & \begin{bmatrix} 1 & 11 \\ 12 & 13 \end{bmatrix} & \begin{bmatrix} 2 & 13 \\ 15 & 17 \end{bmatrix} & \begin{bmatrix} 1 & 17 \\ 18 & 19 \end{bmatrix} & \begin{bmatrix} 2 & 19 \\ 21 & 23 \end{bmatrix} & \begin{bmatrix} 3 & 23 \\ 26 & 29 \end{bmatrix} \\ (5,12,13) & (16,30,34) & (24,70,74) & (72,154,170) & (48,286,290) & (120,442,458) & (72,646,650) & (168,874,890) & (312,1334,1370) \end{array}$$

They are represented by the Triple:

$$(p_{(k+1)})^2 - (p_k)^2, (2 \cdot p_k)(p_{(k+1)}), (p_{(k+1)})^2 + (p_k)^2 [4]$$

where $\frac{(p_{(k+1)})^2 - (p_k)^2}{24}$ is integer. And since every Fibonacci Box have the form:

$$\begin{vmatrix} \frac{p_{(k+1)} - p_k}{2} & p_k \\ \frac{p_{(k+1)} + p_k}{2} & p_{(k+1)} \end{vmatrix} \text{ then from: } (p_k) \cdot (4b) \cdot (p_k + b) \cdot (p_{(k+1)})$$

$$(p_k) \left(\frac{p_{(k+1)} - p_k}{2} \right) \left(\frac{p_{(k+1)} + p_k}{2} \right) (p_{(k+1)}) (4)$$

$$= \left(\frac{1}{4} \right) (p_k \cdot p_{(k+1)}) (p_{(k+1)} - p_k) (p_{(k+1)} + p_k) (4)$$

$$= (p_{(k+1)})^3 (p_k) - (p_k)^3 (p_{(k+1)}) = \text{Area.}$$

Variation of Euclid's Formula for Primitive Pythagorean Triples.

Here we have some Primitive Pythagorean Triples related to Prime Numbers and their respective Fibonacci Boxes.

$$\begin{array}{cccccccccc} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} & \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} & \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} & \begin{bmatrix} 2 & 7 \\ 9 & 11 \end{bmatrix} & \begin{bmatrix} 1 & 11 \\ 12 & 13 \end{bmatrix} & \begin{bmatrix} 2 & 13 \\ 15 & 17 \end{bmatrix} & \begin{bmatrix} 1 & 17 \\ 18 & 19 \end{bmatrix} & \begin{bmatrix} 2 & 19 \\ 21 & 23 \end{bmatrix} & \begin{bmatrix} 3 & 23 \\ 26 & 29 \end{bmatrix} \\ (5,12,13) & (8,15,17) & (12,35,37) & (36,77,85) & (24,143,145) & (60,221,229) & (36,323,325) & (84,437,445) & (156,667,685) \end{array}$$

$$\text{From: } (p_{(k+1)})^2 - (p_k)^2, (2 \cdot p_k)(p_{(k+1)}), (p_{(k+1)})^2 + (p_k)^2$$

$$\text{We have: } \frac{(p_{(k+1)})^2 - (p_k)^2}{2}, \frac{(2 \cdot p_k)(p_{(k+1)})}{2}, \frac{(p_{(k+1)})^2 + (p_k)^2}{2} = \frac{(p_{(k+1)})^2 - (p_k)^2}{2}, (p_k)(p_{(k+1)}), \frac{(p_{(k+1)})^2 + (p_k)^2}{2}$$

This can be used for any pair of prime numbers.

Given two prime numbers p_1 and p_2 such that $p_2 > p_1 > 2$, this is:

$$b \cdot (p_1 + p_2), (p_1 \cdot p_2), \left(\frac{(p_1)^2 + (p_2)^2}{2} \right)$$

$$\text{Since: } b = \frac{p_{(k+1)} - p_k}{2} = \frac{p_2 - p_1}{2}$$

We have:

$$\left(\frac{(p_2)^2 - (p_1)^2}{2} \right), (p_1 \cdot p_2), \left(\frac{(p_1)^2 + (p_2)^2}{2} \right)$$

In general, to know if two numbers $\{a, b | a > b\}$ can perform a Primitive Pythagorean Triple, we test if $\frac{a^2 - b^2}{24}$ is integer, all of them run on the function $y = \frac{x^3 - x}{24x}$.

Prime numbers, Angle between Vectors and a "Tree" of Primitive Pythagorean Triples.

Given any two odd prime numbers, we can perform a Primitive Pythagorean Triple, if we take the primes in ascending order we have the next one's:

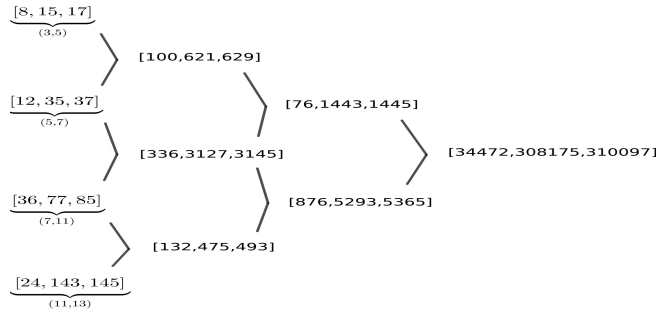
Prime numbers	Pythagorean Triple
(3,5)	[8,15,17]
(5,7)	[12,35,37]
(7,11)	[36,77,85]
(11,13)	[24,143,145]
(13,17)	[60,221,229]
⋮	⋮

Given a PPT $(a_1)^2 + (b_1)^2 = (c_1)^2$ and a vector $A = [a_1, a_2]$ where $a_1 = a_1$ and $b_1 = a_2$ and given a PPT $(a_2)^2 + (b_2)^2 = (c_2)^2$ and a vector $B = [b_1, b_2]$ where $a_2 = b_1$ and $b_2 = b_2$ then $\cos \theta = \frac{A \cdot B}{|A||B|} = \frac{b'}{c'}$ and $a' = \sqrt{(c')^2 - (b')^2}$ where $(a')^2 + (b')^2 = (c')^2$ is a PPT.

Example:

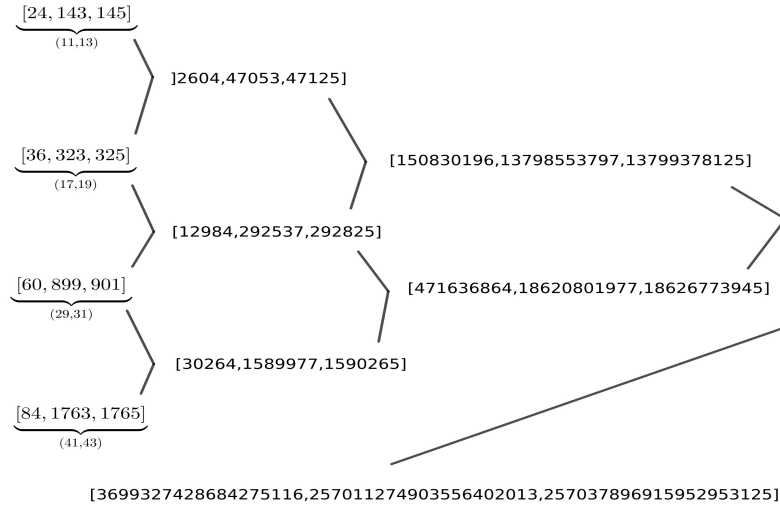
Given the first two PPT of the table we have: $A = [8, 15]$ and $B = [12, 35]$ then $\cos \theta = \frac{A \cdot B}{|A||B|} = \frac{621}{629} \rightarrow c' = 629, b' = 621, a' = \sqrt{629^2 - 621^2} = 100$.

Example of a "Tree" of the first 4 PPT related to prime numbers.



Note that if we make combinations at the same level, we will obtain repeated one's on the next, for example, the vectors [8,15] and [36,77] have the same angle of [100,621] and [336,3127].

We can perform "Trees" of PPT with 'n' primes and they don't need to be consecutive, the next "Tree" was built with four "Twin Primes".



Searching for the next prime.

We have the prime number (31) and we want to find the next prime.

We have the Box $\begin{vmatrix} 2 & 29 \\ 30 & 31 \end{vmatrix}$ and the next Box will be $\begin{vmatrix} b & 31 \\ 31 + b & p_{(k+1)} \end{vmatrix}$

$$\frac{(p_{(k+1)})^2 - (p_k)^2}{24} = \frac{(33)^2 - (31)^2}{24} = \frac{16}{3} \text{ (Is not integer, so (33) is discarded).}$$

$$\frac{(p_{(k+1)})^2 - (p_k)^2}{24} = \frac{(37)^2 - (31)^2}{24} = 17 \text{ and we have the Box } \begin{vmatrix} 3 & 31 \\ 34 & 37 \end{vmatrix}$$

We check: $\frac{A}{p_k + b} = 4b$, then: $\frac{37^2 - 31^2}{34} = 4 \cdot 3$

We ignore the number (35) but anyway we can check if he is Pentagonal:

$$\sqrt{(24)(35) + 1} = 29 \text{ (is Pentagonal).}$$

Fermat Primes.

The Fermat Primes, are prime numbers of the form $2^{2^n} + 1$ for $n \geq 0$, there are only five known numbers of this type (3,5,17,257,65537), all the Fermat Primes run on the function $y = 2^n + 1$ and (quoting Charles R. Grathouse IV.) "Frénicle de Bessy (1657) proved that 9 is the only square in this sequence".

We can use Pythagorean Triples and Fibonacci Boxes to obtain the Fermat Primes.

H. Lee Price gives three matrices:

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

We use the first one to obtain Pythagorean Triples whose first term runs on the function $y = 2^n + 1$.

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = [5 \quad 12 \quad 13]; \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix} = [9 \quad 40 \quad 41]$$

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 40 \\ 41 \end{bmatrix} = [17 \quad 144 \quad 145]; \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 17 \\ 144 \\ 145 \end{bmatrix} = [33 \quad 544 \quad 545]$$

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 33 \\ 544 \\ 545 \end{bmatrix} = [65 \quad 2112 \quad 2113]; \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 65 \\ 2112 \\ 2113 \end{bmatrix} = [129 \quad 8320 \quad 8321]$$

All these Pythagorean Triples have the form: $\begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix}$ of a Fibonacci

Box. So:

$$\text{If } A = \begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix} = \begin{vmatrix} n & 1 \\ n+1 & F_k \end{vmatrix} \rightarrow \begin{vmatrix} |A|+1 & 1 \\ |A|+2 & F_{(k+1)} \end{vmatrix}$$

$$(3, 4, 5) = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \text{ Determinant}=1.$$

$$(5, 12, 13) = \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} \text{ Determinant}=7.$$

$$(17, 144, 145) = \begin{vmatrix} 8 & 1 \\ 9 & 17 \end{vmatrix} \text{ Determinant}=127.$$

$$(257, 33024, 33025) = \begin{vmatrix} 128 & 1 \\ 129 & 257 \end{vmatrix} \text{ Determinant}=32767.$$

$$(65537, 214759184, 214759185) = \begin{vmatrix} 32768 & 1 \\ 32769 & 65537 \end{vmatrix} \text{ Determinant}=2147483647.$$

Note that n is solution for $2n^2 - 1 = |A|$. This also happens with the Mersenne Primes.

Appendix

$$A = \begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix} \text{ such that } |\mathbf{A}| \text{ is a Mersenne Prime } (M_p).$$

$$\begin{array}{cccccc} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} & \begin{bmatrix} 4 & 1 \\ 5 & 9 \end{bmatrix} & \begin{bmatrix} 8 & 1 \\ 9 & 17 \end{bmatrix} & \begin{bmatrix} 64 & 1 \\ 65 & 129 \end{bmatrix} & \begin{bmatrix} 256 & 1 \\ 257 & 513 \end{bmatrix} & \begin{bmatrix} 512 & 1 \\ 513 & 1025 \end{bmatrix} \\ \hline (5,12,13) & (9,40,41) & (17,144,145) & (129,8320,8321) & (513,131584,131585) & (1025,525312,525313) \end{array}$$

\mathbf{n} is a solution of: $2\mathbf{n}^2 - 1 = M_p$ and $\frac{\mathbf{n}_k}{\mathbf{n}_{(k-1)}}$ are the Relations between Perfect Numbers $2^{\frac{M_e-3}{2}}$ (M_e means Mersenne Exponent.)

$$\begin{vmatrix} 2 \cdot 2^{\frac{M_e-3}{2}} & 1 \\ 2 \cdot 2^{\frac{M_e-3}{2}} + 1 & 4 \cdot 2 \cdot 2^{\frac{M_e-3}{2}} \end{vmatrix} \text{ Relation between Perfect Numbers.}$$

$$\begin{vmatrix} 2^{2^{(n-1)}-1} & 1 \\ 2^{2^{(n-1)}-1} + 1 & 2^{2^{n-1}} + 1 \end{vmatrix} \text{ Fermat Primes. For } n > 0.$$

References

- [1] *C. Stanley Ogilvy and John T. Anderson, Excursions in Number Theory, Oxford University Press, NY, 1966, pp. 90-92.*
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- [3] *Two Prime Number Objects and The Velucchi Numbers. HAL Id : hal-02909691, version 1[math.NT], 2020.*
- [4] *Reinhard Zumkeller. OEIS Foundation Inc. (2021), The On-Line Encyclopedia of Integer Sequences, <http://oeis.org/A069487>*

Further Reading

Irreducibility Criteria for Polynomials with non-negative Coefficients.

<https://www.cambridge.org/core/journals/canadian-journal-of-mathematics/article/irreducibility-criteria-for-polynomials-with-nonnegative-coefficients/>

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