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# Dynamic substructuring for multilevel spent nuclear fuel containers with high modal density

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## Abstract

The vibration analysis of a multilevel spent nuclear fuel canister loaded with 68 fuel assemblies (FA) is considered. The multilevel aspect leads to large numbers of degrees of freedom (DOF) and vibration eigenmodes. The Craig-Bampton (CB) substructuring technique is well suited to perform the modal analysis in presence of few interface DOF between the components and the structural levels, as well as for tackling the high modal density. The Shift-Invert Lanczos solver is used for solving the large-scale CB generalized eigenvalue problem (GEP), whose large dimension results from the numerous component modes. A block factorization based on the Schur complement that exploits the sparsity of the CB matrices enables considerable speed-up. A CB model for the FA is also implemented as it exhibits two structural levels. Since many local vibration modes do not contribute much to the response, a strategy for ranking the importance of the component modes is developed, so that a significant portion of them are removed and the CB GEP lightened.

## 1 Introduction

This paper deals with the linear vibration analysis of a spent nuclear fuel (SNF) canister loaded with 68 identical fuel assemblies (FA). Each FA gathers 92 fuel rods that are wrapped together and maintained in parallel configuration by spacer grids. The loaded SNF canister being considered in vertical configuration, each FA is inserted in one of the basket cells of the canister, maintained by localized attachments at the top and simply supported by the bottom plate of the cylindrical canister. The detailed geometry of the FA requires a finer mesh in comparison to that of the canister and its basket. The structure is characterized by the presence of three distinct structural levels, namely (i) a main body made up of the canister and the basket as the upper level, (ii) the 68 FA as the intermediate level, and (iii) the  $68 \times 92 = 6256$  fuel rods as the lower level. There is the repetition of a very large number of interconnected components, each of which having its own resonant vibrations. As a result, the multilevel structure exhibits a very large number of modes, among which many consist of local vibrations of small components. The vibration analysis relies on computing nearly half a million modes using a finite element (FE, [1]) model with more than 130 million degrees of freedom (DOF). Fortunately, the localized attachments between the several components and scales allows an advantageous domain decomposition to be considered, with small interfaces. The vibration modes are the eigenvectors of the generalized eigenvalue problem (GEP) involving the mass and stiffness FE matrices [2, 3]. For several reasons, the Craig-Bampton (CB) substructuring technique [4, 5] is considered as an approximate solver for this GEP. First, the substructural modes of the FA need not be computed 68 times. Second, the small interfaces between the components lead to a small number of interface coupling DOF. Third, the iterative solvers such as Shift-Invert Lanczos (SIL, [6, 7, 8]) or the Subspace Iteration Method [9, 10, 11], although very efficient general-purpose solvers, turn out to be very time consuming in the present case.

The CB method consists in approximating the system modes as linear combinations of the component modes and interface static constraint modes through Galerkin projection of the FE GEP. The number of static constraint modes is given by the number of interface DOF, whereas the number of component modes is generally adjusted by considering a cutoff frequency. Even though the number of static constraint modes is small, the dimension of the CB GEP obtained through Galerkin projection is very large, as a result of the large number of component modes. Usually for the CB method, the CB GEP is solved directly for all the eigenpairs using for instance Householder tridiagonalization, with a negligible computational cost. Unfortunately, in the present case, the CB GEP is a large-scale GEP with sparse matrices, and for which a large portion of the eigenpairs are required. Although the SIL solver is better suited to solving a few eigenpairs of a sparse GEP, spectrum slicing with multiple SIL is a possible strategy for solving many eigenpairs. The strategy consists in dividing the frequency band into several slices and considering eigenvalue shifts given by the center of the slices [12]. Independent eigenvalue computations with the different shifts are then run in parallel, for which the eigenvectors are orthogonal by construction (upon convergence). For each shift, the matrix to be factored for use in the iterative process is different. However, the sparsity of the CB matrices allows an efficient factorization. With a small number of boundary (interface) DOF and a large number of substructure (modal) DOF, a block factorization based on the Schur complement (see [13, 14]) of the interface coordinates enables considerable gains. Thus, the CB GEP is solved by multiple SIL with block factorization by Schur complement.

The accuracy of the CB approximation of the system modes depends on the number of component modes kept. Several criteria can be used to quantify the error perpetrated by the CB approximation, such as convergence of the eigenvalues and modal error. However, presumably due to the high modal density of the multilevel structure, these criteria turned out to lack robustness. The objective being to calculate the vibration response, given by frequency response functions (FRF) between given input excitation DOF and output observation DOF, the CB error is quantified through an FRF error criterion. For one given FRF, it is defined as the root mean square error in decibel (RMS-dB) of the modulus of the response. It should be noted that the exact response is required for computing such an error measure, and that it is not available. Since the response can be written as modal superposition and since numerous modes have negligible contribution, it is proposed to substitute the true FRF with the FRF restrained to a subspace spanned by a small subset of dominant modes. Such FRF spanned by a subset of modes is referred to as “partial FRF”. The few dominant modes are recomputed exactly by using the SIL solver with block factorization by Schur complement according to the domain decomposition (this differs from the Schur complement factorization of the CB matrices). Then the partial FRF can be compared (CB approximation against FE approximation). It has been shown that this procedure allows to characterize the accuracy of the CB approximation with a good confidence.

Once its accuracy is quantified and deemed sufficient, the CB model can be considered as a reference. Then, it is possible to decrease the cutoff frequency without altering the accuracy. This is explained by the fact that the cutoff frequency required for obtaining rank correspondence between the exact and CB eigenvalues in the validation step above can be very high due to the small eigenvalue separation (caused by the very high modal density), which needs to remain larger than the relative eigenvalue error (otherwise, mode swap is susceptible to occur, in which case the exact and CB partial FRF may not coincide).

Since the structure is multilevel, another CB step (inner CB), nested in the first one, is well indicated. The FA exhibits numerous components (the fuel rods) attached to the main body by localized connections (through the spacer grids). The cost for computing the FA modes (component modes of the outer CB) is thus decreased.

The most computationally demanding operation is the large-scale CB GEP. Its large dimension is due to the numerous component modes, which correspond to local vibrations of small components. It has been observed that many of the system modes made up of such local vibrations do not contribute much to the response of the canister. In this research, the focus is on the prediction of the vibration response of the canister exterior. Therefore, the removal of component modes with local vibrations is investigated in order to decrease the dimension and cost of the CB GEP while keeping a satisfactory level of accuracy. For doing so, a substructural (component) modal importance measure is introduced, defined as the RMS-dB error re-

sulting from the sole removal of the substructural mode for which the error is to be characterized. However, the calculation of the RMS-dB error (with respect to the CB model) requires calculating the FRF with the CB model. It is proposed to do so with coarse frequency sampling and observation grid, using direct numerical simulation (frequency-by-frequency) involving a dynamic stiffness matrix that can again be factored by Schur complement. Then, it is shown that the approximate response resulting from the removal of one given mode can be calculated through a rank-1 modification of the Schur complement. Using the Sherman-Morrison-Woodbury matrix identity, the response can be computed very efficiently. This enables a precise yet affordable characterization of the substructural modal importance. Then, the least important modes can be removed and the induced error examined with respect to the cost reduction.

The paper is organized as follows. In Section 2 some well known concepts for modal analysis in computational structural dynamics are briefly described, including modal projection, domain decomposition, Craig-Bampton method, and Shift-Invert Lanczos approach with spectrum slicing. In Section 3 the methodology proposed for performing an efficient multiscale modal analysis is presented. Section 4 presents developments toward substructural mode selection that allow the overall cost to be significantly reduced. Finally, Section 5 presents numerical results and Section 6 draws conclusions. The theoretical developments and numerical results can be found with more details in [15, 16].

## 2 Modal analysis in computational structural dynamics

### 2.1 Modal projection

The structural dynamics is described in the frequency domain by the discretization  $\mathbb{U}(\omega)$  of the displacement field by using the FE method. For all circular frequency  $\omega$  belonging to a given frequency band of analysis  $\mathcal{B} = 2\pi \times ]0, f_u]$  ( $f_u$  is the upper bound in Hz), the complex vector  $\mathbb{U}(\omega)$  is given by

$$(-\omega^2[\mathbb{M}] + i\omega[\mathbb{D}] + [\mathbb{K}]) \mathbb{U}(\omega) = \mathbb{F}(\omega), \quad (1)$$

in which  $[\mathbb{M}]$ ,  $[\mathbb{D}]$ , and  $[\mathbb{K}]$  are respectively the mass, damping, and stiffness matrices and where vector  $\mathbb{F}(\omega)$  is the discretization of the external forces. A reduced representation of the response can be obtained through linear combination of the vibration eigenmodes  $\varphi$  such that

$$\mathbb{U}(\omega) = \sum_{\alpha=1}^{n_e} \varphi_{\alpha} q_{\alpha}(\omega), \quad (2)$$

in which  $\alpha$  denotes the mode number (the modes are sorting according to their eigenfrequency or eigenvalue, as described below) and  $n_e$  denotes the number of modes kept, while quantity  $\mathbf{q}_{\alpha}(\omega)$  constitutes a so-called ‘‘generalized’’ coordinate. In matrix form, one can write  $\mathbb{U}(\omega) = [\Phi]\mathbf{q}(\omega)$  with  $[\Phi]$  the modal matrix whose size is  $(N \times n_e)$  with  $N$  the number of DOF. The vibration modes  $\varphi$  are the eigenvectors satisfying the GEP

$$[\mathbb{K}]\varphi = \lambda[\mathbb{M}]\varphi, \quad (3)$$

with  $\lambda$  the associated eigenvalue and  $f = \sqrt{\lambda}/2\pi$  the eigenfrequency in Hz.

### 2.2 Domain decomposition

The structural domain can be partitioned. The DOF that are connected to several subdomains or substructures are referred to as boundary DOF and denoted by letter  $B$ . The other DOF belong to the interior of the substructures are denoted by letter  $S$ . These substructure DOF are not coupled with any other DOF than those present in the same substructure. Therefore, the coupling blocks between substructures are zero and consequently, a DOF ordering according to the domain decomposition leads to block-diagonal submatrices

$[\mathbb{K}_{SS}]$ ,  $[\mathbb{M}_{SS}]$ , and  $[\mathbb{D}_{SS}]$ . Based on this domain decomposition, the linear system  $[\mathbb{K}]\mathbf{x} = \mathbf{f}$  can be parallelized over the substructure, as described hereinafter. Let  $\mathbf{x}$  (resp.  $\mathbf{f}$ ) be partitioned based on the subvectors  $\mathbf{x}_B$  and  $\mathbf{x}_S$  (resp.  $\mathbf{f}_B$  and  $\mathbf{f}_S$ ) associated with the boundary DOF and the substructure DOF. Then, it can be shown that introducing the Schur complement  $[\mathbb{S}_{BB}^{\mathbb{K}}] = [\mathbb{K}_{BB}] - [\mathbb{K}_{BS}][\mathbb{K}_{SS}]^{-1}[\mathbb{K}_{SB}]$  leads to the solution

$$\mathbf{x}_S = [\mathbb{K}_{SS}]^{-1} (\mathbf{f}_S - [\mathbb{K}_{SB}]\mathbf{x}_B), \quad \mathbf{x}_B = [\mathbb{S}_{BB}^{\mathbb{K}}]^{-1} (\mathbf{f}_B - [\mathbb{K}_{BS}][\mathbb{K}_{SS}]^{-1}\mathbf{f}_S). \quad (4)$$

### 2.3 Craig-Bampton method

The above equations can be obtained by projecting the linear equations onto a square matrix  $[\mathbb{B}]$  such that

$$[\mathbb{B}] = \begin{bmatrix} I_S & t_{SB} \\ 0 & I_B \end{bmatrix}, \quad (5)$$

in which  $[t_{SB}] = -[\mathbb{K}_{SS}]^{-1}[\mathbb{K}_{SB}]$  is the matrix of the static constraint modes that represent the static reactions of the interfaces between the substructures. The CB technique first introduces this change of coordinates and then, for dimension reduction, introduces a projection on the component modes  $[\Phi_S]$  gathered in a matrix  $[R]$  such that

$$[R] = \begin{bmatrix} \Phi_S & 0 \\ 0 & I_B \end{bmatrix}, \quad (6)$$

where matrix  $[\Phi_S]$  is block diagonal with each block containing the modal matrix of its corresponding substructure. The CB matrices are obtained by Galerkin projection over the matrix  $[V] = [\mathbb{B}][R]$  and yields the CB GEP

$$[K]\mathbf{q} = \lambda[M]\mathbf{q} \quad (7)$$

whose eigenvectors  $\mathbf{q}$  yield an approximation  $\varphi = [V]\mathbf{q}$  to the eigenvectors  $\varphi$  of the GEP of Eq. (3). The accuracy and the cost of the CB procedure are related to the modal truncations for the substructures. Denoting as  $n_J$  the number of modes kept for substructure number  $J$ , as  $\mathcal{N}$  the number of substructures, and as  $N_B$  the number of boundary DOF, the dimension  $\nu$  of the CB model is  $\nu = N_B + n_S$  with  $n_S = \sum_{J=1}^{\mathcal{N}} n_J$ .

### 2.4 Shift-Invert Lanczos approach

For solving a GEP of the type of Eq. (3) one possible strategy is to introduce an eigenvalue shift  $\sigma \geq 0$  and consider shifting as  $([\mathbb{K}] - \sigma[\mathbb{M}])\varphi = (\lambda - \sigma)[\mathbb{M}]\varphi$  and then inverting (useful for eigenvalue separation) as

$$([\mathbb{K}] - \sigma[\mathbb{M}])^{-1}[\mathbb{M}]\varphi = \left( \frac{1}{\lambda - \sigma} \right) \varphi. \quad (8)$$

The largest eigenvalues of the standard eigenvalue problem  $[\mathbb{A}]\varphi = \mu\varphi$  with  $[\mathbb{A}] = ([\mathbb{K}] - \sigma[\mathbb{M}])^{-1}[\mathbb{M}]$  and  $\mu^{-1} = \lambda - \sigma$  correspond to the eigenvalues  $\lambda$  that are closest to the shift  $\sigma$ . Due to the orthogonality of the eigenvectors  $\varphi$ , two sets of eigenvectors (modes) computed from two different shifts are mutually orthogonal. This is verified as long as the eigensolutions are converged. This way, a large number of modes can be computed by considering several shifts. Thus, the eigenvalue calculation can be parallelized and the complexity is kept linear with respect to the number of modes sought. For each eigenvalue analysis of the type of Eq. (8), the matrix  $[\mathbb{H}] = [\mathbb{K}] - \sigma[\mathbb{M}]$  has to be factored for solving linear systems in the iterative Lanczos process. In this work, an LDL factorization (see [17])  $[\mathbb{P}]^T[\mathbb{H}][\mathbb{P}] = [\mathbb{L}][\mathbb{D}][\mathbb{L}]^T$  is considered, with  $[\mathbb{P}]$  a permutation matrix,  $[\mathbb{L}]$  a lower-triangular matrix, and  $[\mathbb{D}]$  a block-diagonal matrix with block size of 1 or 2. Dividing the frequency band (spectrum slicing) to perform the eigenvalue analysis of Eq. (3) requires a good knowledge of the modal density, to balance the number of modes per shift as well as to avoid computing the same modes twice. In this work, the matrix size makes the LDL factorization costly in terms of memory and CPU usage. Therefore, the domain decomposition of Section 2 is considered and the LDL factorization is performed blockwise based on the Schur complement (see Section 3.1).

### 3 Multiscale modal analysis

#### 3.1 Block factorization based on Schur complement

Using domain decomposition, the symmetric indefinite matrix  $[\mathbb{H}]$  can be factored as  $[\mathbb{P}]^T[\mathbb{H}][\mathbb{P}] = [\mathbb{L}][\mathbb{D}][\mathbb{L}]^T$  with the following construction:

$$[\mathbb{L}] = \begin{bmatrix} \mathbb{L}_S & 0 \\ \mathbb{P}_B^T \mathbb{H}_{BS} \mathbb{H}_{SS}^{-1} \mathbb{P}_S \mathbb{L}_S & \mathbb{L}_B \end{bmatrix}, \quad [\mathbb{D}] = \begin{bmatrix} \mathbb{D}_S & 0 \\ 0 & \mathbb{D}_B \end{bmatrix}, \quad [\mathbb{P}] = \begin{bmatrix} \mathbb{P}_S & 0 \\ 0 & \mathbb{P}_B \end{bmatrix}, \quad (9)$$

in which the matrices with index  $S$  satisfy the block LDL factorization  $[\mathbb{P}_S]^T[\mathbb{H}_{SS}][\mathbb{P}_S] = [\mathbb{L}_S][\mathbb{D}_S][\mathbb{L}_S]^T$  (this factorization can be parallelized over the substructures according to the block-diagonal structure of  $[\mathbb{H}_{SS}]$ ) and where the matrices with index  $B$  satisfy the LDL factorization  $[\mathbb{P}_B]^T[\mathbb{S}_{BB}^H][\mathbb{P}_B] = [\mathbb{L}_B][\mathbb{D}_B][\mathbb{L}_B]^T$  of the Schur complement. Such block LDL factorization as described in (9) will be used in several instances of the proposed methodology. It should be noted that the inertia count  $\mathcal{I}([\mathbb{H}])$  (number of negative eigenvalues of matrix  $[\mathbb{H}])$  associated with a shift  $\sigma$  can be obtained as

$$\mathcal{I}([\mathbb{H}]) = \mathcal{I}([\mathbb{D}]) = \mathcal{I}([\mathbb{D}_B]) + \mathcal{I}([\mathbb{D}_S]) = \mathcal{I}([\mathbb{D}_B]) + \sum_{J=1}^{\mathcal{N}} \mathcal{I}([\mathbb{D}_J]). \quad (10)$$

#### 3.2 Large-scale Craig-Bampton generalized eigenvalue problem

For performing the eigenvalue analysis of Eq. (3) in an affordable manner, the CB GEP of Eq. (7) is considered instead. In order to obtain an accurate CB model, a high frequency truncation is considered that leads to a large value for the number  $n_S$  of substructural (or component) modes. The small interfaces between components yield a small number  $N_B$  of boundary DOFs. The CB matrices are written as

$$[K] = \begin{bmatrix} \Lambda_S & 0 \\ 0 & K_{BB} \end{bmatrix}, \quad [M] = \begin{bmatrix} I_S & M_{BS}^T \\ M_{BS} & M_{BB} \end{bmatrix}, \quad (11)$$

in which  $[I_S]$  is the identity matrix of dimension  $n_S$  and  $[\Lambda_S]$  is the diagonal matrix of the eigenvalues of the substructures, and where the submatrices  $[K_{BB}]$  and  $[M_{BB}]$  are rather dense and the submatrix  $[M_{BS}]$  is sparse with dense blocks (their expression can be deduced from the equality  $[K] = [V]^T[\mathbb{K}][V]$  or  $[M] = [V]^T[\mathbb{M}][V]$ ). Due to the large value of  $n_S$  related to the high modal density arising from the local vibrations of the numerous components, the dimension  $\nu$  of the CB matrices  $[K]$  and  $[M]$  can be several million. In this context, the GEP of Eq. (7) cannot be solved by the usual direct solvers by transformation methods such as the Householder tridiagonalization. Instead, the only tractable possibility is to consider the solvers for large-scale sparse GEP such as SIL or SIM. This is possible because the CB matrices exhibit quite a high sparsity, that is amplified by the large ratio  $n_S/N_B$  (large number of substructural modes and small number of boundary DOF). In addition, the block structure allows an LDL factorization just like that of Eq. (9) and, given that  $[\Lambda_S]$  and  $[I_S]$  are diagonal, significant gains can be obtained by doing so. For following the SIL approach of Section 2.4, the LDL factorization  $[P]^T[H][P] = [L][D][L]^T$  of  $[H] = [K] - \sigma[M]$  is considered:

$$[L] = \begin{bmatrix} I_S & 0 \\ -\sigma P_B^T M_{BS} (\Lambda_S - \sigma I_S)^{-1} & L_B \end{bmatrix}, \quad [D] = \begin{bmatrix} \Lambda_S - \sigma I_S & 0 \\ 0 & D_B \end{bmatrix}, \quad [P] = \begin{bmatrix} I_S & 0 \\ 0 & P_B \end{bmatrix}, \quad (12)$$

in which  $[P_B]$ ,  $[L_B]$ , and  $[D_B]$  satisfy the LDL factorization  $[P_B]^T[S_{BB}^H][P_B] = [L_B][D_B][L_B]^T$  of the Schur complement  $[S_{BB}^H]$ . It should be noted that the inertia count  $\mathcal{I}([H])$  (number of negative eigenvalues of matrix  $[H])$  associated with a shift  $\sigma$  can be obtained as

$$\mathcal{I}([H]) = \mathcal{I}([D]) = \mathcal{I}([D_B]) + \mathcal{I}([\Lambda_S] - \sigma[I_S]) = \mathcal{I}([D_B]) + \sum_{J=1}^{\mathcal{N}} \mathcal{I}([\Lambda_J] - \sigma[I_J]). \quad (13)$$

### 3.3 Accuracy verification

The modal density can be estimated using several shifts  $\sigma$  and computing, for each of them, the inertia count  $\mathcal{I}([D_B])$  through the LDL factorization of the Schur complement  $[S_{BB}^H]$ . Since the number  $N_B$  of boundary DOF is small, this is quite efficient. Then, several slices are introduced and parallel SIL is undertaken. For each GEP, the matrix factorization requires the LDL factorization of the Schur complement  $[S_{BB}^H]$ . Then, the linear solves involve thin matrix products involving matrices with similar structure as  $[M_{BS}]$ . The CB GEP can thus be solved very efficiently for a very large number of eigenpairs. Very large truncations are first considered, to obtain a CB model that can be considered as error-free.

Concerning the error quantification, the usual error measures based on eigenvalue convergence or the modal error did not turn out to be adequate for the multilevel structure considered. With the modal error, too large values were obtained for some modes while considering very high truncations. Furthermore, because of the very high modal density, the relative eigenvalue error could remain as high as the relative difference between two distinct eigenvalues. In fact, the eigenvalues are susceptible to swap even though the CB model is already very accurate. Since the purpose of the computational model is to calculate the FRF, the error measure is defined through the FRF error. Let  $U_{ij}(\omega)$  denote the entry  $i, j$  of the true FRF matrix and let  $\tilde{U}_{ij}(\omega)$  denote one approximation of it, for instance obtained with the CB model. The associated error measure  $\epsilon_{ij}$  for this particular FRF is defined as

$$\epsilon_{ij} = \sqrt{\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} (u_{ij}(\omega) - \tilde{u}_{ij}(\omega))^2 d\omega}, \quad (14)$$

in which  $u_{ij}(\omega) = 20 \log_{10}|U_{ij}(\omega)|$  and  $\tilde{u}_{ij}(\omega) = 20 \log_{10}|\tilde{U}_{ij}(\omega)|$  are the moduli in decibel (dB) scale, and where  $|\mathcal{B}|$  denotes the length of the frequency band of analysis. Error  $\epsilon_{ij}$  is the RMS-dB error. The global error measure  $\epsilon$  is then given by

$$\epsilon = \mu_\epsilon + k_\epsilon \sigma_\epsilon, \quad \mu_\epsilon = \frac{1}{N_I^2} \sum_{i=1}^{N_I} \sum_{j=1}^{N_I} \epsilon_{ij}, \quad k_\epsilon > 0, \quad \sigma_\epsilon^2 = \frac{1}{N_I^2 - 1} \sum_{i=1}^{N_I} \sum_{j=1}^{N_I} (\epsilon_{ij} - \mu_\epsilon)^2, \quad (15)$$

in which  $k_\epsilon$  is a parameter that indirectly controls the probability of obtaining an error greater than  $\epsilon$  when randomly considering one FRF with error  $\epsilon_{ij}$ .

The exact FRF  $U_{ij}(\omega)$  are not available and consequently, this error measure  $\epsilon$  cannot directly be used to quantify the accuracy of the CB model. Since the response can be written as modal superposition and since numerous modes have negligible contribution, it is proposed to substitute the true FRF with the FRF restrained to a subspace spanned by a small subset of dominant modes. The dominant modes are defined as those for which the error induced by their sole removal is the greatest. Such FRF spanned by a subset of modes is referred to as ‘‘partial FRF’’. This way, it is possible to calculate the true ‘‘partial FRF’’ by computing the subset of dominant modes by using a solver as accurate as machine precision. To do so, the SIL approach of Section 2.4 is used along with domain decomposition for the linear solver as described in Section 2.2. The number  $n_d$  of dominant modes should be such that  $n_d \ll n_e$  and adjusted such that the eigenvalue computation of Eq. (3) be affordable. Then, the use of the error measure defined in Eqs. (14) and (15) allows for controlling the accuracy of the  $n_d$  dominant modes calculated through the CB GEP of Eq. (7). It should be noted that these dominant modes are very much likely to cover the whole frequency band. These dominant modes do not have any reason to have better accuracy than all the other modes that have not been verified by such direct comparison. To make sure of it, one possibility is to examine the other error measures (modal error and eigenvalue convergence) and verify that the dominant modes do not behave differently than the others.

### 3.4 Loose truncation and multilevel Craig-Bampton

The FRF error described in Section 3.3 required that the modes recomputed with SIL correspond to the same modes computed by the CB approach. Otherwise the error due to the mismatch between the FRF would not

be due to the inaccuracy of the CB model, and the accuracy verification would be biased and most likely exhibiting too large errors. The mode correspondence was thus necessary for the accuracy certification described in Section 3.3 and required large truncations (large value for  $n_S$ ). Nevertheless, the CB model does not necessarily need that large a value for  $n_S$ . The CB model for which the accuracy has been verified in Section 3.3 is used as a reference and then the FRF error (not that of the partial FRF but that of the true FRF) is used to study the convergence of the CB model with respect to the truncation of the substructural (or component) modes.

In fact, the accuracy verification of Section 3.3 can be seen as an offline step necessary for preparing an efficient procedure to perform the dynamic analysis for the same computational model but altered with few changes such as damage or randomness. Beside the loose truncation compared to that used in the accuracy verification step of Section 3.3, several other approximations are examined to decrease the cost of the computational model while keeping the same level of accuracy.

Since the multilevel structure presents three structural levels, it is also beneficial to consider a second CB approximation (referred to as “inner CB”), nested in the first one (referred to as “outer CB”), that is to say devoted to computing the component modes of the outer CB. The component modes of the outer CB are the FA modes and the modes of the so-called frame substructure constituted of the canister and its basket. The inner CB only considers the CB model of a FA (the 68 FA are nominally identical) as it is assumed that the frame substructure is not subject to change. Just like the FA are connected to the upper scale through localized attachments with the basket and the canister bottom plate, the fuel rods are connected to the FA skeleton through localized attachments with the spacer grids. The same methodology is used for performing this inner CB step.

## 4 Substructural mode selection

For one different instance of the computational model, that is to say e.g., either for one choice of the set of mechanical parameters describing the computational model or for one damage scenario, the FA modes and static constraint modes need to be recomputed (no change is applied to the canister and its basket). As a consequence, the CB model is changed and the CB GEP of Eq. (7) needs to be solved again. The main cost is that of the CB GEP and for decreasing it, the strategy proposed is to remove the FA modes that do not contribute much to the CB model. From one instance of the computational model to the other, the importance or dominance of the FA modes varies, in general. Therefore, it needs to be calculated “online”, that is to say for each instance of the computational model. Following the same definition as in Section 3.3, the dominant FA modes are defined as those for which the error induced by their sole removal is the greatest. Nevertheless, in this case, such calculation must be less time consuming than the time that is gained by dealing with a CB GEP with a reduced dimension resulting from the removal of FA modes.

For quantifying the error due to removing one given FA mode, the FRF error is again used as in Section 3.3. The true FRF is that obtained with the CB model without FA mode removal and the approximate FRF is that with single FA mode removal. Again, the true FRF is not available. The strategy thus proposed is to calculate the true FRF using direct numerical simulation using the dynamic stiffness matrix  $[H(\omega)] = -\omega^2[M] + i\omega[D] + [K]$ , for which the damping model is hysteretic such that  $[D] = \frac{2\xi(\omega)}{\omega}[K]$  with  $\xi(\omega)$  a frequency-dependent damping ratio. This way, the dynamic stiffness matrix exhibits the same sparsity as the shifted matrix  $[H]$  of the CB GEP of Eq. (7). The approximate FRF then considers the same damping model and is obtained in considering the removal of one FA mode (thus decreasing the dimension of the CB model by one unit). Let  $[M_{\beta A}]$  be the matrix similar to  $[M_{BS}]$  but associated with the FA substructure and for which the rows are restricted to its  $N_\beta$  boundary DOF. The matrix  $[s_{\beta\beta}^{H,A}(\omega)] = [M_{\beta A}][H_{AA}(\omega)]^{-1}[M_{\beta A}]^T$  with  $[H_{AA}(\omega)] = -\omega^2[I_A] + (1 + 2i\xi(\omega))[\Lambda_A]$  (dynamic stiffness matrix of the FA) includes the contribution of the FA modes to the Schur complement  $[S_{\beta\beta}^H(\omega)]$  of the dynamic stiffness matrix  $[H(\omega)]$  of the complete system. Let  $\mathbf{v}_\beta$  denote the column vector of matrix  $[M_{\beta A}]$  corresponding to the FA mode to be removed.



Then it can be shown that the contribution  $[\tilde{s}_{\beta\beta}^{H,A}(\omega)]$  of the remaining FA modes to the Schur complement  $[S_{BB}^H(\omega)]$  is given by

$$[\tilde{s}_{\beta\beta}^{H,A}(\omega)] = [s_{\beta\beta}^{H,A}(\omega)] - \frac{1}{h(\omega)} \mathbf{v}_\beta \mathbf{v}_\beta^T, \quad (16)$$

in which  $h(\omega) = -\omega^2 + (1 + 2i\xi(\omega)) \lambda_A$  with  $\lambda_A$  the eigenvalue of the removed FA mode. It can be deduced that the modified Schur complement  $[\tilde{S}_{BB}^H(\omega)]$  resulting from the single FA mode removal can be written as

$$[\tilde{S}_{BB}^H(\omega)] = [S_{BB}^H(\omega)] + \frac{\omega^4}{h(\omega)} \mathbf{v}_B \mathbf{v}_B^T, \quad (17)$$

in which vector  $\mathbf{v}_B$  is the extension of vector  $\mathbf{v}_\beta$  with zeros in the entries corresponding to boundary DOF of other substructures. It can be seen that the single mode removal comes down to a rank-1 modification of the Schur complement. The Sherman-Morrison-Woodbury formula is then used to efficiently deduce the inverse of the modified Schur complement based on the inverse of the original Schur complement. This is quite useful, considered that there are as many modified Schur complements to consider as the number of FA modes (they are each removed one at a time). It can be shown that then inverse  $[\tilde{S}_{BB}^H(\omega)]^{-1}$  of the modified Schur complement can be written as

$$[\tilde{S}_{BB}^H(\omega)]^{-1} = [S_{BB}^H(\omega)]^{-1} - \frac{1}{a(\omega)} \mathbf{p}_B(\omega) \mathbf{p}_B^T(\omega), \quad (18)$$

in which the frequency-dependent quantities  $a(\omega) = \frac{h(\omega)}{\omega^4} + \mathbf{v}_B^T \mathbf{p}_B(\omega)$  and  $\mathbf{p}_B(\omega) = [S_{BB}^H(\omega)]^{-1} \mathbf{v}_B$  then allow the modified FRF to be obtained.

For each mode of each FA and for a given sampling of frequencies, the approximate FRF are computed and used along the true FRF to calculate the error measure defined in Eqs. (14) and (15). The importance of a given FA mode is defined by the value of the error measure obtained through the sole removal of this substructural mode. Thus, the FA modes can be sorted according to their importance. Then, a given percentage of the most dominant FA mode is kept and the CB GEP solved. A convergence analysis is carried out to determine an adequate percentage of substructural mode filtering, with respect to both accuracy and efficiency.

## 5 Numerical results

The frame substructure (canister and its basket) has  $N_F = 2,209,084$  DOF, one FA has  $N_A = 1,935,837$  DOF, and there is a total of  $N_B = 1140$  boundary DOF. Therefore, the FE model is of dimension  $N = N_B + N_F + 68 \times N_A = 133,847,140$  DOF. Preliminary calculations pertaining to the offline stage are carried out. Among them, the first  $n_F = 211,057$  modes of the frame substructure are calculated, as well as the first  $n_A = 46,383$  FA modes. These numbers correspond to a frequency truncation of 20 kHz, given that the modal analysis is to be performed up to 1.2 kHz. Such a high truncation is considered for obtaining a reference computational model. The static constraint modes for both these structures are quite cheap to compute, as there are only  $N_B = 1140$  of them for the frame substructures and  $N_\beta = 27$  for the FA. Then, the CB GEP of Eq. (7) is solved for all the eigenpairs below 1.2 kHz: there are  $n_e = 458,910$  of them. The dimension of the CB GEP is  $\nu = N_B + n_F + 68 \times n_A = 3,366,241$ . Even though it exhibits higher numbers than for the frame substructure, the calculation is faster, thanks to the efficient block factorization.

The  $n_e = 458,910$  CB system modes are removed one at a time and the error measure defined in Eqs. (14) and (15) is calculated for each of them. This allows for the modes to be ranked according to their dominance. For accuracy certification as described in Section 3.3, the  $n_d = 200$  most dominant modes are considered. For each of them, the approximate eigenvalue given by the CB model with 20 kHz truncation is available. For each of these eigenvalues, the corresponding eigenpair is recomputed up to machine precision using the

SIL approach with domain decomposition as described in Sections 2.2 and 2.4, by using a frequency shift equal to the approximate eigenvalue. Such a precise shift is necessary because of the very high modal density. The matrix factorization necessary for the Lanczos iterative process also gives the inertia count as a byproduct, which allows the rank correspondence to be verified. Then the “partial FRF” restrained to these  $n_d = 200$  dominant modes are computed with both the exact modes and the CB modes, which then allows to calculate the FRF error defined in Eqs. (14) and (15). Using  $k_\epsilon = 4$  (the choice for this value is fixed for the rest of this paper), the decibel error is of 0.46 dB, which is very low. Therefore, the CB model with 20 kHz truncation can be considered as a reference.

Then, a convergence analysis with respect to the truncation frequency is carried out, which shows that a 4 kHz truncation is sufficient to reach a decibel error of 0.38 dB with respect to the reference computational model (CB model with 20 kHz truncation frequency). There are only  $n_F = 16,623$  frame modes and  $n_A = 14,064$  FA modes up to 4 kHz and consequently, the dimension of the CB model is decreased to  $\nu = 974,115$ .

Then, the FA modes are approximated by introducing an inner CB model for the FA. Using a 6 kHz inner truncation allows for obtaining a decibel error of 0.41 dB, that is to say no noticeable error is introduced. Due to this CB approximation, the number of FA modes slightly decreases to  $n_A = 14,047$ . Applying the methodology presented in Section 4, these modes are sorted according to their importance. For that, a coarse frequency sampling of 1000 frequency points instead of 3000 frequency points as used everywhere else in this paper is considered. Moreover, the number of FRF considered is decreased, such that the size of several of the frequency-dependent matrices involved in the substructural mode selection are reduced. Despite these approximations, the methodology delivers satisfying results, as a removal of 75% of the least important FA modes leads to a decibel error of 0.45 dB, which is still very low. Thanks to this filtering, the size of the CB model is decreased to  $\nu = 256,579$  with  $n_A = 3512$ . As a remark, the sole removal of the one most dominant FA mode gives an error of 7.43 dB, which is quite significant.

Regarding the computational cost, for the FA mode selection, the elapsed time for the true FRF is  $\sim 2$  minutes on one compute node devoted to one FA, and the elapsed time for computing the  $68 \times 14,047 = 955,196$  FRF with FA mode removal is  $\sim 12$  minutes (therefore, it is five orders of magnitude faster). The mode filtering allows the CB GEP to be solved in  $\sim 21$  minutes with dimension  $\nu = 256,579$  instead of being solved in  $\sim 118$  minutes with dimension  $\nu = 974,115$ . It should also be noted that the computational cost per mode is four orders of magnitude less for the reference CB GEP with 20 kHz truncation than for the SIL approach with domain decomposition as described in Sections 2.2 and 2.4. The proposed methodology allows the vibration analysis to be performed in about  $\sim 40$  minutes whereas it is nearly intractable with a standard approach.

## 6 Conclusions

For the vibration analysis of a multilevel structure characterized by pseudo-periodicity arising from the repetition of numerous identical components, an efficient solver combining Craig-Bampton substructuring and Shift-Invert Lanczos solver with spectrum slicing allowed to tackle the high computational cost induced by the large number of DOF and the very large modal density characterized by nearly half a million modes. The proposed methodology took advantage of the small number of interface DOF between components, which allowed efficient matrix factorizations. In addition, capturing the relevant physics by removing 75% of the least significant substructural modes allowed the computational cost to be further reduced. The proposed methodology enables the practical simulation of the complex multilevel structure for studying various instances of damage or variation of the mechanical properties.

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