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# Vortices in $\mathcal{PT}$ -symmetric non-Hermitian two-component superfluid

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We discuss vortex solutions in the non-Hermitian parity-time-symmetric relativistic model with two interacting scalar complex fields. In the London limit, the vortex singularities in different condensates experience dissipative dynamics unless they overlap. At finite quartic couplings, the vortices appear in the  $\mathcal{PT}$ -symmetric regions with broken U(1) symmetry. We find the phase diagram of the interacting model where the  $\mathcal{PT}$ -symmetric and  $\mathcal{PT}$ -broken regions together with the U(1)-symmetric and U(1)-broken phases form complicated patterns.

## I. INTRODUCTION

Quantum-mechanical systems are traditionally described by Hermitian Hamiltonians which ensure the real-valuedness of the full energy spectrum and, therefore, the unitary evolution of the system as a whole. It turns out, however, that the Hermitian description can be extended with a large class of non-Hermitian terms which are invariant under combined parity-time ( $\mathcal{PT}$ ) transformations. The  $\mathcal{PT}$ -symmetric non-Hermitian systems are as meaningful as the conventional Hermitian quantum mechanics in the regions where the  $\mathcal{PT}$ -symmetry is not broken spontaneously [1, 2].

Mathematically, in the  $\mathcal{PT}$ -symmetric systems the familiar Hermiticity condition  $H^\dagger = H$  is replaced by the requirement of the  $\mathcal{PT}$ -symmetricity  $(\mathcal{PT})H(\mathcal{PT}) = H$ , which is equivalent to the commutation of the Hamiltonian with the combined parity  $\mathcal{P}$  and time-reversal inversion  $\mathcal{T}$  operation [2],  $[\mathcal{PT}, H] = 0$ . This combined symmetry leads to real-valued energy spectrum that ensures the stability of the system. One can show that all  $\mathcal{PT}$ -symmetric non-Hermitian Hamiltonians belong to the class of the so-called pseudo-Hermitian Hamiltonians  $\eta H \eta^{-1} = H^\dagger$  where  $\eta$  is a Hermitian linear automorphism [3]. The pseudo-Hermiticity is a generalization of the  $\mathcal{PT}$ -symmetry which, in turn, depends crucially on the fact that the time-reversal transformation  $\mathcal{T}$  is an anti-linear operation.

As it was shown in Ref. [4] and most recently emphasized in Ref. [5], it is the anti-linearity property rather than Hermiticity which is important for the self-consistent description of stable quantum-mechanical systems. The non-Hermitian  $\mathcal{PT}$ -invariant quantum systems can be mapped to their Hermitian counterparts via a non-Unitary transformation [6, 7] (the existence of the map is not guaranteed as there are known exceptions in quantum mechanics [8]). These extensions broaden the class of stable physical systems beyond the tight Hermiticity constraints and open new horizons for the research.

The non-Hermitian description has been extended to interacting relativistic field theories, including the systems of fundamental particle interactions. The  $\mathcal{PT}$ -symmetric interactions which explicitly break the non-

Hermiticity of the system can arise in fermionic theories [9], contribute to the mass gap generation in the NJL model and affect the phase structure of the model [10]. The non-Hermitian Dirac fermions allow for the realization of an anomalous equilibrium transport [11]. The ordinary Hermitian models can generate a new, non-Hermitian ground state which could potentially be formed, for example, in fireballs of quark-gluon plasma created after heavy-ion collisions [12]. In the context of the extensions of the Standard Model of particle interactions, anti-Hermitian Yukawa interactions may lead to an anomalous radiative mass-gap generation in a model of the right-handed sterile neutrinos [13, 14]. The concept of non-Hermitian quantum theory allows an extension via the gauge-gravity duality well beyond the scope of the field-theoretical models [15].

The  $\mathcal{PT}$ -symmetric non-Hermitian Hamiltonians arise in the description of various open quantum systems in optics and solid state physics where this symmetry can be interpreted as a result of a perfect balance between the gains and losses as the system interacts with the external environment [16, 17]. The recent works also include the studies of the effect of non-Hermitian terms in topological superconductivity which leads to nonlocal anomalous transport effects [18] as well as in the conventional superconductivity which gives rise to the unusual first-order phase transition between the phases [19]. The possibility of non-Hermitian superfluidity with a complex-valued, non-Hermitian interaction constant naturally arises from inelastic scattering between fermions [20]. The associated non-Hermitian BCS-BEC crossover of Dirac fermions in field-theoretical models of many-body systems reveals a nontrivial phase diagram as a function of the complex coupling [21].

In our paper, we work with vortex topological defects in a bosonic non-Hermitian model which involves a pair of scalar fields associated with interacting condensates. The topological solutions in the multicomponent scalar models are interesting because they appear in the models which have applications from condensed matter to high energy physics. Some of these models can serve as viable extensions of the Standard model of fundamental particle physics [23–26]. Similarly to the Grand Unification particle models and their close counterparts,

they host 't Hooft–Polyakov monopole configurations [27] along with complex skyrmions [28] and kink/anti-kink solutions [29] with real-valued energies. As in the Hermitian models, these classical solutions are associated with the saddle points of the corresponding partition functions.

At the condensed matter side, the many-condensate systems possess richer dynamics than their one-condensate counterparts. For example, the standard classification of superconductivity into types I and II fails to describe the phases of multiband condensates so that a proposal to adopt a new terminology, a type-1.5 superconductivity, appeared in the theoretical community [30]. Experimentally, the existence of the type-1.5 superconductivity has been demonstrated shortly afterwards [31]. The semi-Meissner state of a type-1.5 superconductor demonstrates non-pairwise interaction between the vortices which leads to formation of a multitude of complicated vortex states [32]. We discuss the non-Hermitian extension of the two-component model possessing a global, rather than local, continuous symmetry, appropriate for the two-component superfluidity. We concentrate on stability of the ground state, fate of the  $\mathcal{PT}$  symmetry in the interacting model, and the properties of the vortex configurations. In a different context, the vortices in the weakly interacting superfluid Bose-Einstein condensates with complex-valued  $\mathcal{PT}$ -symmetric potentials have been investigated in Ref. [33].

The plan of our paper is as follows. In Section II we briefly overview the Lagrangian and its symmetries, and discuss the ground state of the minimal non-Hermitian theory with two scalar fields. The special attention is paid to the extension of the analysis of the  $\mathcal{PT}$  symmetries to the case of interacting model. In Section III we consider the vortex solutions in the London limit of the theory and in Section IV we describe the examples of the vortex solutions at finite quartic couplings. The last section is devoted to our conclusions.

## II. (NON-)HERMITIAN SCALAR THEORY

### A. Lagrangians

We consider a simplest example of a scalar non-Hermitian theory which describes a  $\mathcal{PT}$ -symmetric dynamics of two complex scalar fields  $\phi_1$  and  $\phi_2$  conveniently grouped into the single doublet field,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (1)$$

The Lagrangian of the theory [7],

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - \Phi^\dagger \hat{M}^2 \Phi - V(\Phi), \quad (2)$$

includes the classical Hermitian self-interaction potential for the scalar fields:

$$V(\Phi) \equiv V(\phi_1, \phi_2) = \lambda_1 |\phi_1|^4 + \lambda_2 |\phi_2|^4. \quad (3)$$

The non-Hermiticity is encoded in the real-valued mass matrix  $\hat{M}^2$  of the Lagrangian (2):

$$\hat{M}^2 = \hat{M}_{\text{NH}}^2 = \begin{pmatrix} m_1^2 & m_5^2 \\ -m_5^2 & m_2^2 \end{pmatrix}, \quad (4)$$

provided the off-diagonal element<sup>1</sup> of this matrix is a nonzero,  $m_5^2 \neq 0$ . To see how the non-Hermiticity enters the theory, it is instructive to write the Lagrangian in terms of the individual fields  $\phi_1$  and  $\phi_2$ :

$$\begin{aligned} \mathcal{L}_{\text{NH}} = & \partial_\nu \phi_1^* \partial^\nu \phi_1 + \partial_\nu \phi_2^* \partial^\nu \phi_2 - m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 \\ & - m_5^2 (\phi_1^* \phi_2 - \phi_2^* \phi_1) - \lambda_1 |\phi_1|^4 - \lambda_2 |\phi_2|^4. \end{aligned} \quad (5)$$

The first term of the second line in Eq. (5) takes a purely complex value:  $-2im_5^2 \text{Im}(\phi_1^* \phi_2)$  if the off-diagonal component of the mass matrix (4) is a real-valued nonzero quantity. The complex valuedness of the Lagrangian (5) is consistent with the non-Hermiticity of the mass matrix in Eq. (2):  $\hat{M}_{\text{NH}}^{2,\dagger} \neq \hat{M}_{\text{NH}}^2$ . The off-diagonal mass with  $m_5^2 \neq 0$  sets up the non-Hermitian regime in (4), while the point  $m_5^2 = 0$  corresponds to a Hermitian theory (with  $\hat{M}_{\text{NH}}^{2,\dagger} = \hat{M}_{\text{NH}}^2$ ) which describes two non-interacting scalar fields  $\phi_1$  and  $\phi_2$ .

The model (2) describes two relativistic superfluids which interact with each other via the off-diagonal non-Hermitian coupling. We consider the potential (3) in the form which explicitly breaks the  $U(2)$  symmetry,  $\Phi \rightarrow \Omega \Phi$  with the  $2 \times 2$  matrix  $\Omega \in U(2)$ , down to its Cartan  $[U(1)]^2$  subgroup since the  $U(2)$  group is explicitly broken by the mass matrix (4) anyway provided  $\hat{M}_{\text{NH}}^2 \not\propto \mathbb{1}$ .

In order to highlight the features of non-Hermiticity, we briefly discuss the Hermitian version of the model with the following mass matrix in the Lagrangian (2):

$$\hat{M}_{\text{H}}^2 = \begin{pmatrix} m_1^2 & m_5^2 \\ m_5^2 & m_2^2 \end{pmatrix} \quad (6)$$

Notice that  $\hat{M}_{\text{H}}^{2,\dagger} = \hat{M}_{\text{H}}^2$  as expected and the Hermitian Lagrangian is a real-valued expression for all values of the parameter  $m_5$ :

$$\begin{aligned} \mathcal{L}_{\text{H}} = & \partial_\nu \phi_1^* \partial^\nu \phi_1 + \partial_\nu \phi_2^* \partial^\nu \phi_2 - m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 \\ & - m_5^2 (\phi_1^* \phi_2 + \phi_2^* \phi_1) - \lambda_1 |\phi_1|^4 - \lambda_2 |\phi_2|^4. \end{aligned} \quad (7)$$

Below, we will consider the classical equations of motion concentrating on the non-Hermitian theory described by Lagrangian (2). While we work with the classical solutions, we would like to notice that on quantum level, the non-Hermiticity propagates into the loops of perturbation theory. The quantum corrections could, therefore, induce a complex term in the interaction potential (3) and literally complexify the phase diagram of the theory. Leaving aside the quantum corrections, which were discussed in Ref. [25], in our paper we concentrate on classical properties of the Lagrangian (2).

<sup>1</sup> We use the notation  $m_5$  which is also appropriate for the non-Hermitian mass of a fermionic model [11].

## B. Symmetries

We consider the non-degenerate model (2) with a nonzero off-diagonal mass  $m_5 \neq 0$  which possesses a couple of continuous and discrete symmetries. Both Hermitian and non-Hermitian versions of the model are invariant under the global  $U(1)$  transformation

$$U(1) : \quad \Phi(t, \mathbf{x}) \rightarrow \Phi'(t, \mathbf{x}) = e^{i\omega} \Phi(t, \mathbf{x}), \quad (8)$$

in which the single phase factor (with a real-valued parameter  $\omega$ ) is shared by both complex scalar fields  $\phi_1$  and  $\phi_2$ . If  $m_5 = 0$ , both components of the doublet field  $\Phi$  can be transformed independently so that the symmetry is enlarged to the global  $[U(1)]^2$ . We do not consider this trivial case.

The non-Hermitian theory (4) is also invariant under a discrete transformation corresponding to the product of the parity inversion  $\mathcal{P}$  and the time conjugation  $\mathcal{T}$  operators, respectively:

$$\text{NH} : \quad \begin{cases} \mathcal{P} : & \Phi(t, \mathbf{x}) \rightarrow \Phi'(t, -\mathbf{x}) = \sigma_3 \Phi(t, \mathbf{x}), \\ \mathcal{T} : & \Phi(t, \mathbf{x}) \rightarrow \Phi'(-t, \mathbf{x}) = \Phi^*(t, \mathbf{x}). \end{cases} \quad (9)$$

Due to the presence of the Pauli matrix  $\sigma_3$ , the upper  $\phi_1$  component transforms under the parity inversion  $\mathcal{P}$  as a genuine scalar while the lower component  $\phi_2$  transforms as a pseudoscalar field.

The  $\mathcal{P}$  and  $\mathcal{T}$  operations whose product leaves invariant the Hermitian theory (6),

$$\text{H} : \quad \begin{cases} \mathcal{P} : & \Phi(t, \mathbf{x}) \rightarrow \Phi'(t, -\mathbf{x}) = \Phi(t, \mathbf{x}), \\ \mathcal{T} : & \Phi(t, \mathbf{x}) \rightarrow \Phi'(-t, \mathbf{x}) = \Phi^*(t, \mathbf{x}), \end{cases} \quad (10)$$

indicate that both fields  $\phi_1$  and  $\phi_2$  behave as true scalars under the parity inversion.

The model (2), along with its extensions, possesses interesting features of the Goldstone modes associated with the spontaneous breaking of the continuous symmetry (8) as well as the unusual properties of the conserved currents [7, 22, 23]. Below we consider the ground state and the vortex solutions of the model.

## C. Ground states

### 1. Non-Hermitian ground state

The analysis of the ground state of the two-field model (2) has already been done analytically in Ref. [22] for the special case of the potential (3) in which one of the fields was not self-interacting ( $\lambda_1 \neq 0$  and  $\lambda_2 = 0$ ). In our paper, we complement this work with the numerical analysis of the system in which both fields experience the self-interaction,  $\lambda_{1,2} \neq 0$ . We show that the simple extension (or, better to say, completion) of the model makes the analysis of the phase diagram very complicated.

Let us start from the simplest case when the quartic interaction is absent:  $\lambda_1 = \lambda_2 = 0$ . The Hermitian (6)

and non-Hermitian (4) mass-squared matrices have the following eigenvalues, respectively:

$$M_{\text{H},\pm}^2 = \frac{1}{2} \left( m_1^2 + m_2^2 \pm \sqrt{(m_1^2 - m_2^2)^2 + 4m_5^4} \right), \quad (11a)$$

$$M_{\text{NH},\pm}^2 = \frac{1}{2} \left( m_1^2 + m_2^2 \pm \sqrt{(m_1^2 - m_2^2)^2 - 4m_5^4} \right). \quad (11b)$$

The vacua in these models are stable provided the eigenmasses have no imaginary parts. For the Hermitian model with the mass matrix (6), this requirement implies  $M_{\text{H},-}^2 \geq 0$  or, naturally,

$$\text{H} : \quad \begin{cases} m_1^2 + m_2^2 > 0, \\ m_1^2 m_2^2 - m_5^4 \geq 0. \end{cases} \quad (12)$$

These equations determine the region in the parameter space where the instability due to a negative mode (or, modes) at the trivial minimum,  $\phi_1 = \phi_2 = 0$ , does not occur. In an interacting theory with  $\lambda_{1,2} \neq 0$ , this instability corresponds to the spontaneous symmetry breaking.

In the non-Hermitian model (4), the spectrum of the free (i.e., with  $\lambda_1 = \lambda_2 = 0$ ) theory does not contain complex energy eigenvalues provided

$$\text{NH} : \quad \begin{cases} m_1^2 + m_2^2 > 0, \\ m_1^2 m_2^2 + m_5^4 \geq 0, \\ (m_1^2 - m_2^2)^2 - 4m_5^4 \geq 0, \end{cases} \quad (13)$$

where we also took into account a possibility that the squared masses  $m_1^2$  and  $m_2^2$  can take negative values. The first two requirements in Eq. (13) correspond to the instability related to the spontaneous symmetry breaking rather than to the non-Hermiticity of the model. The last condition in Eq. (13) highlights the region of the parameter space where the  $\mathcal{PT}$  symmetry is said to be unbroken [7, 22, 23]. Together, these conditions guarantee the stability of the ground state.

The classical equations of motion of the model (2) are obtained by the variation of the Lagrangian (5) with respect to the independent fields  $\phi_1^*$  and  $\phi_2^*$ , respectively:

$$\square \phi_1 + m_1^2 \phi_1 + m_5^2 \phi_2 + \frac{\partial V}{\partial \phi_1^*} = 0, \quad (14a)$$

$$\square \phi_2 + m_2^2 \phi_2 - m_5^2 \phi_1 + \frac{\partial V}{\partial \phi_2^*} = 0. \quad (14b)$$

A variation with respect to the fields  $\phi_1$  and  $\phi_2$  gives us an inequivalent set of equations:

$$\square \phi_1^* + m_1^2 \phi_1^* - m_5^2 \phi_2^* + \frac{\partial V}{\partial \phi_1} = 0, \quad (15a)$$

$$\square \phi_2^* + m_2^2 \phi_2^* + m_5^2 \phi_1^* + \frac{\partial V}{\partial \phi_2} = 0, \quad (15b)$$

which differs from Eq. (14) by the sign flip of the off-diagonal mass term,  $m_5^2 \rightarrow -m_5^2$ .

The striking inconsistency of the two pairs of equations of motion, (14) and (15), poses the natural question of how to treat this non-Hermitian system at the classical level. One of the earlier proposals [7] suggests to use only one set of equations of motion, either (14) or (15), and omit the complementary set. Indeed, the choice of the set flips the sign of the off-diagonal mass,  $m_5^2 \rightarrow -m_5^2$ , which does not affect the physical spectrum neither in free model (11) nor in the interacting model, as we will see below. Moreover, the non-Hermitian system must be open via coupling to an external source which equilibrates non-vanishing surface terms in the variation of the action. When the off-diagonal mass vanishes, the system becomes Hermitian, and the single set of equations is enough to describe the solutions consistently.

Another proposal to resolve the apparent problem with the inconsistency of the classical equations of motion was put forward in Ref. [22] where a similarity transformation for the action has been used to achieve harmony between the two sets of equations of motion. This elegant idea initially required an extension of the field space using two complex components (with four degrees of freedom) for every original complex field (which hosts two degrees of freedom). The same strategy has later been adapted and extended to many-field theories in Ref. [23] where two real-valued fields were used to represent one complex field. However, this procedure, used for the mapping of the non-Hermitian theory to the Hermitian one via the similarity transformation, leads to the appearance of a negative kinetic energy term for one of the fields. While it was argued that this artifact does not change the signature of the appropriate Hilbert space [22], the appearance of the negative kinetic action leads to a negative contribution to the energy-momentum tensor so that the energy of a classical configuration (which is not necessarily a classical solution) would become unbounded from below. This pseudo-Hermitian method can be adapted, in specifying the appropriate physical regions, to give the classical solutions for non-Abelian monopoles in a spontaneously broken theory [27] (we refer the Reader to Refs. [36, 37] for complementary discussions of a non-Hermitian non-Abelian gauge theory).

In our paper we follow the approach of Ref. [7] where only one set of equations of motion – either (14) or (15), but not the both of them – is considered. At certain stage, we use a numerical method to find the classical solutions using the criteria of the energy minimization as a selection principle of the right, “minimal energy” solution among all other available solutions. In our approach, the energy of the classical configuration in the non-Hermitian theory is bounded from below so that the numerical approach is self-consistent in finding the correct configuration. If we would otherwise employ the pseudo-Hermitian procedure of mapping the non-Hermitian theory to the Hermitian one, then the classical energy becomes unbounded due to the negative sign in the kinetic terms, and the numerical procedure fails to converge to a reasonable solution.

In the ground state the condensates are coordinate-independent quantities, and Eqs. (14) reduce to the non-linear algebraic relations:

$$m_1^2 \phi_1 + m_5^2 \phi_2 + 2\lambda_1 \phi_1^2 \phi_1^* = 0, \quad (16a)$$

$$m_2^2 \phi_2 - m_5^2 \phi_1 + 2\lambda_2 \phi_2^2 \phi_2^* = 0. \quad (16b)$$

The use of the complementary set of equations (15) instead of Eqs. (14) would lead to an equivalent physical solutions. Indeed, the swap of equations leads to the sign flip in Eq. (16) which corresponds to a simple swap of the fields  $\phi_1$  and  $\phi_2$ .

It is convenient to represent the fields  $\phi_a$  in the radial form  $\phi_a = v_a e^{i\theta_a}$  for  $a = 1, 2$ . Equations (16) can possess nontrivial solutions provided the phases  $\theta_a$  satisfy one of the following relations:

$$\theta_1 = \theta_2, \quad \theta_1 = \theta_2 + \pi. \quad (17)$$

One can cover both these cases assuming  $\theta_1 = \theta_2$  and, simultaneously, allowing the amplitudes  $v_a$  to take both positive and negative values,  $v_a \in \mathbb{R}$ . The classical equations (16) then determine the amplitudes:

$$m_1^2 v_1 + m_5^2 v_2 + 2\lambda_1 v_1^3 = 0, \quad (18a)$$

$$m_2^2 v_2 - m_5^2 v_1 + 2\lambda_2 v_2^3 = 0. \quad (18b)$$

The canonical energy-momentum of the model (2) can be calculated by endowing the model with the gravitational background  $g_{\mu\nu}$ , considering the variation of the theory action,  $S = \int d^4\mathcal{L}$  with respect to the metric,

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}(x)}, \quad (19)$$

and then setting the metric to its Minkowski values back again,  $g_{\mu\nu} \rightarrow \eta_{\mu\nu} \equiv \text{diag}(+1, -1, -1, -1)$ . The energy density of the model (5) is given by the  $\mu = \nu = 0$  component of the energy-momentum tensor (19):

$$\begin{aligned} T^{00} = & \partial_0 \phi_1^* \partial_0 \phi_1 + \nabla \phi_1^* \nabla \phi_1 + \partial_0 \phi_2^* \partial_0 \phi_2 + \nabla \phi_2^* \nabla \phi_2 \\ & + m_1^2 |\phi_1|^2 + m_2^2 |\phi_2|^2 + m_5^2 (\phi_1^* \phi_2 - \phi_2^* \phi_1) \\ & + \lambda_1 |\phi_1|^4 + \lambda_2 |\phi_2|^4. \end{aligned} \quad (20)$$

One gets the following simple expression for the energy density of a uniform time-independent (ground) state:

$$E_{\text{NH},0} \equiv T^{00} = m_1^2 v_1^2 + m_2^2 v_2^2 + \lambda_1 v_1^4 + \lambda_2 v_2^4. \quad (21)$$

Both solutions (17) for the phases  $\theta_a \equiv \arg \phi_a$  of the condensates  $\phi_a$  lead to the vanishing non-Hermitian term  $\delta\mathcal{L}_{\text{NH}} = -2im_5^2 \text{Im}(\phi_1^* \phi_2) \equiv -2im_5^2 v_1 v_2 \sin(\theta_1 - \theta_2)$  in the energy density (21). Thus, the inconvenient imaginary terms do not enter the ground state of the model (we will see below that any solution of the single set of the classical equations of motion (14) has a real-valued energy density). However, despite the non-Hermitian mass  $m_5$  does not enter explicitly the vacuum energy (21), it affects the ground state indirectly via the solutions of the equations of motion (18).

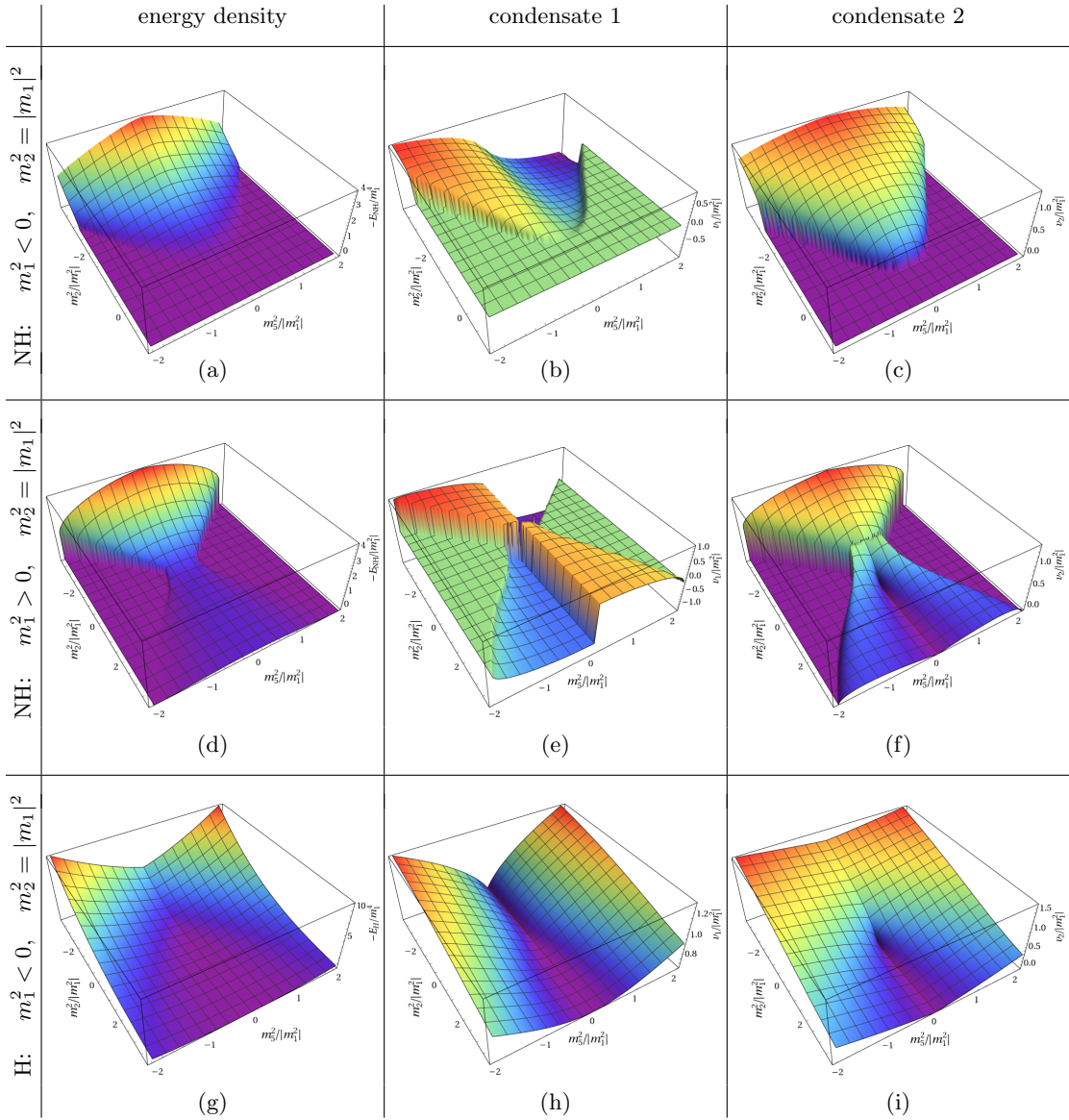


FIG. 1. The upper panel: (a) the (minus) energy density (21) and the condensates (b)  $v_1$  and (c)  $v_2$  in the ground state of the non-Hermitian model (5) are shown in the plane of the mass parameter squared  $m_2^2$  and the non-Hermitian mass squared  $m_5^2$ . The mass (squared) of the first field  $\phi_1$  is taken positive,  $m_1^2 > 0$ , and the quartic couplings for both scalar fields are fixed:  $\lambda_1 = \lambda_2 = 1$ . The middle panel, with the plots (d), (e) and (f), corresponds to the same quantities obtained for the negative diagonal mass  $m_1^2 < 0$ . The lower panel, with the plots (g), (h) and (i), depicts the Hermitian case (7). All dimensionful quantities are shown in the units of the absolute value of the first mass parameter,  $|m_1|$ .

## 2. Hermitian ground state

The equations of motion of the Hermitian model (7) take the following form:

$$\square\phi_1 + m_1^2\phi_1 + m_5^2\phi_2 + \frac{\partial V}{\partial\phi_1^*} = 0, \quad (22a)$$

$$\square\phi_2 + m_2^2\phi_2 + m_5^2\phi_1 + \frac{\partial V}{\partial\phi_2^*} = 0, \quad (22b)$$

which differ from Eq. (14) of the non-Hermitian model (5) as the both signs in front of the off-diagonal mass terms  $m_5^2$  are the same in the Hermitian case (22). This small

difference naturally propagates into the equations for the amplitudes in the ground state,

$$m_1^2 v_1 + m_5^2 v_2 + 2\lambda_1 v_1^3 = 0, \quad (23a)$$

$$m_2^2 v_2 + m_5^2 v_1 + 2\lambda_2 v_2^3 = 0, \quad (23b)$$

as compared to its non-Hermitian analogue (18). However, in the striking dissimilarity with non-Hermitian case, the variations over the fields and their complex conjugates are obviously consistent with each other in the Hermitian model. Moreover, the energy density of the

ground state of the Hermitian model,

$$E_{\text{H},0} = m_1^2 v_1^2 + m_2^2 v_2^2 + 2m_5^2 v_1 v_2 + \lambda_1 v_1^4 + \lambda_2 v_2^4, \quad (24)$$

explicitly includes the  $m_5^2$  coupling between the amplitudes of the different condensates. We remind that this coupling does not enter the non-Hermitian energy density (21).

### 3. Hermitian model vs. non-Hermitian model

In the Hermitian model, a negative diagonal mass can trigger a natural spontaneous symmetry breaking of the global U(1) symmetry (8) which leads to a non-zero expectation value of the doublet field,  $\Phi_0 \neq 0$ . The runaway of the scalar field from the symmetric state is balanced by quartic (self)-interactions so that the symmetry-broken ground state of the system is stable. In an interacting model, the stability of the symmetric,  $\Phi = 0$ , state is determined by Eq. (12).

The interacting non-Hermitian theory possesses two types of instabilities. In addition to the mentioned spontaneous symmetry breaking stipulated by the first two relations in Eq. (13), the symmetric ground state can also experience the U(1) symmetry breaking which could be caused, in turn, by the broken discrete  $\mathcal{PT}$  symmetry. This purely non-Hermitian effect is dictated by the third relation in Eq. (13), the violation of which leads to the complex mass spectrum and, consequently, to the instability. The  $\mathcal{PT}$ -induced instability could be well seen when both diagonal masses are positive so that the first two relations in Eq. (13) are satisfied. As in the purely Hermitian case, the symmetry-broken ground state of the non-Hermitian system is expected to be stabilized by the quartic (self)-interactions.

In Fig. 1 we compare the U(1) symmetry breaking patterns in the non-Hermitian model with  $m_1^2 > 0$  (the upper panel) and  $m_1^2 < 0$  (the middle panel) as well as the Hermitian model with  $m_1^2 < 0$  (the lower panel). We show the total energy density  $E$  determined by Eqs. (21) and (24), and the condensates  $v_1$  and  $v_2$ . Notice that the non-Hermitian (18) and Hermitian (23) equations on the ground state imply that the relative sign of the condensates is determined by the ground state while the overall sign is not fixed. To remove this arbitrariness, we require the condensate  $v_2$  to be positive.

If both diagonal squared masses are positive,  $m_1^2 > 0$  and  $m_2^2 > 0$ , and the off-diagonal mass is zero,  $m_5 = 0$ , then the ground state resides in the symmetric phase with vanishing condensates. As the off-diagonal mass increases in its absolute value, the theory should experience the  $\mathcal{PT}$ -symmetric instability and we expect the appearance of the condensates in the ( $m_1^2 > 0, m_2^2 > 0$ ) plane. However, this does not happen, as one can see in Figs. (1)(b) and (c). The presence of the “ $\mathcal{PT}$ -induced” condensates is forbidden by the requirement of the minimization of the energy density (21) as these condensates would have made the energy density positive while the

trivial ground state with  $v_1 = v_2 = 0$  has zero energy. This conclusion appears to be the consequence of the fact that the expression for the non-Hermitian energy density (21) does not contain a “compensating”  $m_5^2$  term which is present, on the contrary, in the expression of the Hermitian energy density (24). Therefore, the U(1) symmetric ground state with vanishing energy is preferred over the U(1) broken state with the nonvanishing condensates. We will see below that some of these symmetric states belong to the  $\mathcal{PT}$ -broken phases of the interacting model and, therefore, are physically meaningless.

In the symmetry-broken phase with  $m_2^2 < 0$  (while we still keep  $m_1^2 > 0$ ), the conventional U(1) symmetry breaking does occur. The upper panel of Fig. 1 shows that the non-Hermitian mass influences the conventional symmetry breaking in a somewhat controversial way: at sufficiently large  $m_5^4$ , the third requirement of Eq. (13) is violated, the  $\mathcal{PT}$  symmetry gets broken and the U(1) symmetry gets restored because the ground-state condensates  $v_1$  and  $v_2$  vanish. In the symmetry-broken region, the flip of the sign in the  $m_5^2$  mass flips the sign of the  $v_1$  condensate (we remind that we always keep the  $v_2$  condensate non-negative).

A rather similar picture occurs when one of the diagonal mass squared is taken to be negative,  $m_1^2 < 0$ , as shown in the middle panel of Fig. 1. The spontaneous symmetry breaking occurs at both signs of the remaining mass squared,  $m_2^2$ , while the increase of the absolute value of the off-diagonal mass  $m_5^2$  leads to the restoration of the U(1) symmetry. We would like to stress again that this particular picture is enforced by the requirement of minimization of the non-Hermitian energy density (21) which allows us to choose the correct ground state from the multitude of the solutions of the non-Hermitian equations of motion (18).

Finally, the ground state of the Hermitian two-scalar model with  $m_1^2 < 0$  is shown in the lower panel of Fig. 1. The condensates appear almost at every point of the phase diagram except for the line  $m_5 = 0$ , where the condensates decouple and the  $v_2$  condensate ceases to exist at  $m_2^2 > 0$ . At this semi-infinite line, the condensate  $v_1$  takes its minimum.

### 4. Stability of the ground state and the $\mathcal{PT}$ symmetry in the non-Hermitian model

Before proceeding to the discussion of the vortex solutions, let us address the formal stability issues of the ground state. Usually, the local stability of a classical configuration is probed by expanding the scalar fields in the vicinity of the configuration,  $\phi_a = v_a + \hat{\phi}_a$  with  $|\hat{\phi}_a| \ll |v_a|$ . The configuration is unstable if the fluctuation matrix corresponding to the variation of the action with respect to the fluctuation of the fields contains negative modes.

In the Hermitian theory, the global minimum in the total energy of the solution corresponds to an absolutely

stable state. In the non-Hermitian theory, this criterion may not work since even in the classical theory, the ground state is determined by a single set of classical equations of motion out of the existing two sets. Therefore, mathematically, one could expect the emergence of negative directions in the space of fields. This expectation is coherent with our physical intuition since the non-Hermitian system resides in the steady state which is generally not a thermodynamic equilibrium. In the remaining part of this section, we argue that the formal stability criteria can still be applied to the interacting non-Hermitian systems. These criteria give us the generalization of the  $\mathcal{PT}$ -symmetric and  $\mathcal{PT}$ -broken regions for the interacting theory.

From the classical equations of motion (16), we obtain that the field fluctuations around the condensates are governed by the following equation:

$$\square \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_1^* \\ \hat{\phi}_2 \\ \hat{\phi}_2^* \end{pmatrix} + \mathcal{M}^2 \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_1^* \\ \hat{\phi}_2 \\ \hat{\phi}_2^* \end{pmatrix} = 0, \quad (25)$$

where the fluctuation matrix  $\mathcal{M}_{\text{NH}}^2$  of the non-Hermitian model is as follows:

$$\mathcal{M}_{\text{NH}}^2 = \begin{pmatrix} 4\lambda_1 v_1^2 + m_1^2 & 2v_1^2 \lambda_1 & m_5^2 & 0 \\ 2v_1^2 \lambda_1 & 4\lambda_1 v_1^2 + m_1^2 & 0 & m_5^2 \\ -m_5^2 & 0 & 4\lambda_2 v_2^2 + m_2^2 & 2v_2^2 \lambda_2 \\ 0 & -m_5^2 & 2v_2^2 \lambda_2 & 4\lambda_2 v_2^2 + m_2^2 \end{pmatrix}.$$

In the U(1) broken phase, this matrix has one zero eigenvalue which corresponds to the Goldstone mode. In the symmetric U(1) phase, all eigenvalues are generally nonzero.

For the ground state to be stable, one expects that the all eigenvalues of the fluctuation matrix are non-negative. In the non-Hermitian model, this requirement is not always satisfied. In Fig. 2 we show the (in)stability phases for various quartic couplings  $\lambda_1$  and  $\lambda_2$ .

The (in)stability phase diagram has a rather curious form. One notices that the borderlines between the stable and unstable areas involve both U(1) symmetric and broken regions which possess, on the contrary, rather featureless, smooth behaviour of the condensates as shown in Fig. 1. In order to understand the appearance of the negative modes, let us consider any unstable point in the symmetry-restored region with masses  $m_1^2 > 0$ ,  $m_2^2 > 0$  (these regions are marked by the white color in the upper panel of Fig. 2). Since the both condensates are zero and both diagonal masses as well as the couplings are positive, the energy density (21) takes an absolute minimum. On the other hand, in these regions, the third criterion of Eq. (13) is not satisfied, indicating that the symmetric state resides in the  $\mathcal{PT}$ -broken region and is thus unstable. The instability is not captured by the minimization of the energy density (21) since this particular expression is derived for the classical solutions while the instability

can drive the configuration out of the classical subspace of configurations. The non-classical configuration can acquire even a complex value thus invalidating the criterion of the energy minimization outside of the classical subspace of field configurations.

Thus, we arrive to the conclusion that the ground state of the non-Hermitian model is formally not stable in the in certain regions of the parameter space. One can easily check that these unstable regions become the  $\mathcal{PT}$ -broken regions when the quartic interaction couplings are set to zero,  $\lambda_{1,2} \rightarrow 0$ . Therefore, similarly to the free non-interacting model, the unstable regions indicated in Fig. 2 can be interpreted as the  $\mathcal{PT}$ -broken regions in the interacting case where the model cannot be used as a invalid prescription of any steady state in a physical system. The  $\mathcal{PT}$ -symmetric regions, on the contrary, are valid stable zones where the steady-state physics can be realized.

The stability, now associated with the  $\mathcal{PT}$ -symmetric regions, is thus determined by the standard requirement that the quadratic fluctuation matrix,

$$\mathcal{M}_{ab}^2 = \frac{\delta^2 S_{\text{NH}}[\chi]}{\delta\chi_a(x)\delta\chi_b(x)}, \quad (26)$$

does not possess negative eigenvalues. Here the vector  $\vec{\chi} = (\phi_1, \phi_2, \phi_1^*, \phi_2^*)$  denotes the original fields and their conjugates.

For the sake of completeness, we show the fluctuation matrix of the Hermitian model (7),

$$\mathcal{M}_{\text{H}}^2 = \begin{pmatrix} 4\lambda_1 v_1^2 + m_1^2 & 2v_1^2 \lambda_1 & m_5^2 & 0 \\ 2v_1^2 \lambda_1 & 4\lambda_1 v_1^2 + m_1^2 & 0 & m_5^2 \\ m_5^2 & 0 & 4\lambda_2 v_2^2 + m_2^2 & 2v_2^2 \lambda_2 \\ 0 & m_5^2 & 2v_2^2 \lambda_2 & 4\lambda_2 v_2^2 + m_2^2 \end{pmatrix},$$

which has only non-negative eigenmodes, as expected. The Hermitian ground state does not have any ambiguities in the stability criteria.

### III. VORTICES IN THE LONDON LIMIT

#### A. Vortices in one-component superfluids: brief review

The notion of the London limit appears originally in the Ginzburg-Landau models of superconductivity. This limit corresponds to the type II regime in which the penetration depth of the magnetic field is much longer than the correlation length of the superconducting condensate. Equivalently, this limit poses a constraint on the modulus of the superconducting condensate which can be considered as a rigidly fixed, non-fluctuating quantity. The gauge invariance implies that the phase of the superconducting condensate is not constrained by this limit. However, the phase is absorbed by the electromagnetic gauge field, which becomes – in consistency with the Meissner effect – a massive vector field via the Anderson-Higgs mechanism. [34]



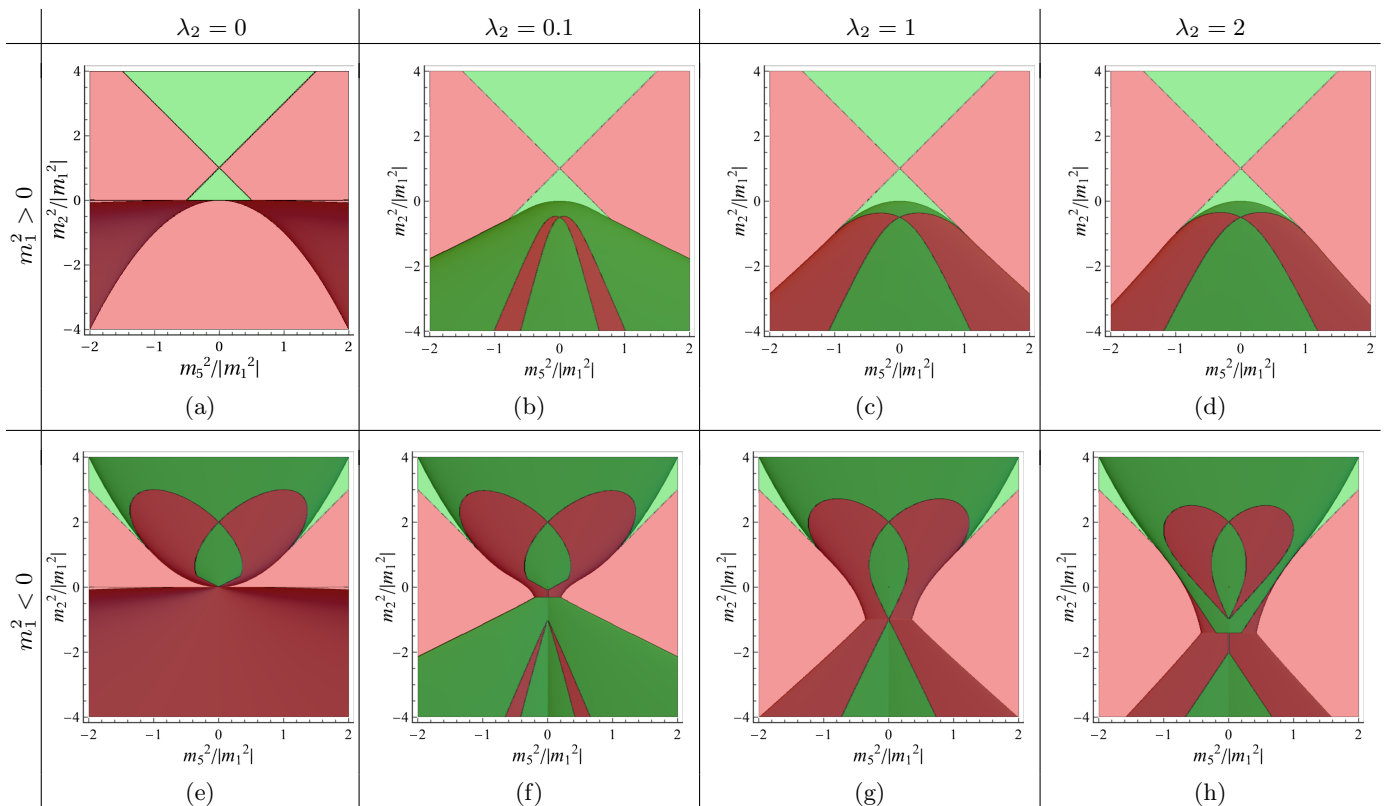


FIG. 2. The stable ( $\mathcal{PT}$ -symmetric, marked by the green color) region and the unstable (spontaneously  $\mathcal{PT}$ -broken, marked by the red color) regions in the interacting non-Hermitian theory in the  $(m_5^2, m_2^2)$  plane. The mass squared of the first scalar field takes positive values ( $m_1^2 > 0$ ) at the upper panel and negative values ( $m_1^2 < 0$ ) at the lower panel. The dark green and light green colors denote regions with stable  $\mathcal{PT}$ -symmetric ground states with non-vanishing and vanishing condensates, respectively. The dark red and light red areas are unstable  $\mathcal{PT}$ -broken regions with, respectively, non-trivial and vanishing solutions of the classical equations of motion (22).

In the background of sufficiently strong magnetic field, the type-II superconductors enter the mixed (Abrikosov) phase where the magnetic flux penetrates the condensate in a form of parallel vortices which form the regular Abrikosov lattice. In the London limit, the vortex is characterized by a singularity in the phase of the condensate which reflects itself in a singular behavior of the gauge field close to the vortex core. [35]

In our paper, we consider the model with the global, rather than local/gauge, U(1) symmetry. In the global case, the condensate is associated with the electrically neutral complex scalar field which corresponds to the phenomenon of superfluidity rather than superconductivity. The Anderson-Higgs mechanism is evidently absent and the massless mode in the phase of the condensate appears to be the Goldstone boson associated with the global U(1) symmetry breaking. In the case of a non-relativistic Bose gas, one identifies the Goldstone mode with a phonon. Instead of the Abrikosov strings with magnetic flux, the superfluid condensates host the vortices characterized by singularities in the phases of the condensate. These vortices are also called the “global vortices” since they appear in theories possessing a global symmetry. Below, we will briefly review the theory of

vortices for a non-relativistic one-component superfluid following Ref. [35] and then we proceed to the generalization of our approach to the London limit of the relativistic non-Hermitian two-field model.

Consider a superfluid condensate of a non-relativistic Bose gas of particles with the mass  $M$ . It is convenient to describe by the wave function in the radial coordinates  $\psi = |\psi|e^{i\theta}$ . The London limit corresponds to the fixed radial degree of freedom  $|\psi|$  which is a good approximation at very low temperature. Then the energy of the fluctuations in the superfluid can be written in the following form (we work in units  $c = \hbar = 1$ ):

$$E = \int d^3x \frac{\rho_s}{2} \mathbf{v}_s^2 \equiv \frac{|\psi|^2}{2M} \int d^3x (\nabla\theta)^2, \quad (27)$$

where the radial part  $|\psi|$  of the condensate determines the superfluid density  $\rho_s = M|\psi|^2$  while its phase  $\theta$  provides us with the velocity of the superfluid condensate,  $\mathbf{v}_s = (\nabla\theta)/M$ . We neglected here qualitatively inessential contributions coming from the inhomogeneity of the radial part of the condensate. An extremization of the energy (27) gives us the Poisson’s wave equation,  $\Delta\theta = 0$  which corresponds to the incompressibility condition of the superfluid,  $\nabla \cdot \mathbf{v}_s = 0$ .

The superfluid theory (27) also incorporates singular configurations of the vortex fluid which correspond to the superfluid currents that wind around a line (called the vortex line) in three-dimensional space. Precisely at the vortex line, the superfluid condensate vanishes,  $\phi = 0$ , while the phase of the superfluid condensate is equal – in the vicinity to the vortex core – to a geometrical angle (times an integer) in the plane transverse to the vortex line. The integer is a topological quantity which characterizes the vortex winding number.

The vortices can be accounted in the model (27) despite the fact that this model describes a long-range macroscopic physics with a globally uniform, constant condensate. Consider, for simplicity, a straight static vortex line along the direction  $x_3$  located at the center  $x_1 = x_2 = 0$  in the transverse plane. The phase of the vortex is  $\theta(x_1, x_2) = n\varphi$  where  $n$  is the winding number of the vortex and  $\varphi = \arg(x_1 + ix_2)$  is the azimuthal angle in the two-dimensional  $(x_1, x_2)$  plane. The velocity of the superfluid,

$$\mathbf{v}_s = \frac{n}{M} \nabla \arg(x_1 + ix_2) = \frac{n}{Mr} \mathbf{e}_\varphi, \quad (28)$$

is directed along the unit vector in the azimuthal  $\mathbf{e}_\varphi$ , so that the fluid “winds” around the vortex center at  $x_1 = x_2 = 0$ . Alternatively, the location of the vortex singularity corresponds to the point in the two-dimensional plane, where the derivatives do not commute,

$$(\partial_1 \partial_2 - \partial_2 \partial_1) \theta(x_1, x_2) = 2\pi n \delta(x_1) \delta(x_2). \quad (29)$$

The total energy (27) evaluated at this static straight vortex configuration is:

$$E = L \frac{|\psi|^2 n^2}{2M} 2\pi \int_\xi^R r dr \frac{1}{r^2} = L \frac{\pi |\psi|^2 n^2}{M} \ln \frac{R}{\xi}, \quad (30)$$

where  $L$  is the length of the vortex. The infrared cutoff of this integral is the size of the system  $R$  while the size  $\xi$  of the vortex core – typically, of the interatomic distance – serves as an ultraviolet cutoff. The important message of Eq. (30) is that the vortex energy (mass) is proportional to the length of the vortex  $L$ . One can show that the vortex curvature provides a subleading correction. The energy density per unit length is a finite quantity since the logarithmic divergence in Eq. (30) is very mild.

Consider now the relativistic one-component model with the action:

$$S = \int d^4x \partial_\mu \phi^* \partial^\mu \phi \equiv \kappa^2 \int d^4x \partial_\mu \theta \partial^\mu \theta, \quad (31)$$

where we adopted the London limit with the constant radial condensate  $\kappa = |\phi|$ . The variation of the action with respect to the phase  $\theta$  gives us the equation for the propagation of the massless Goldstone particle,  $\square \theta = 0$ , where  $\square = \partial_t^2 - \nabla^2$  is the d’Alembert operator.

In the non-relativistic limit, the model (31) possesses the energy (27) for static vortex configurations. It is convenient to parameterize the coordinates  $\tilde{x}^\mu = \tilde{x}^\mu(\vec{\sigma})$  of

the two-dimensional vortex singularities by the two component vector  $\vec{\sigma} = (\sigma_1, \sigma_2)$ . We split the phase  $\theta$  into the regular and singular parts,  $\theta = \theta^r + \theta^s$ . The regular part  $\theta^r$  of the phase corresponds to the perturbative fluctuations while the singular part  $\theta^s$  encodes the position of the vortex (29). In the relativistic notations,

$$\partial_{[\mu} \partial_{\nu]} \theta^s(x, \tilde{x}) = 2\pi \cdot \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \Sigma_{\alpha\beta}(x, \tilde{x}), \quad (32)$$

where the singularity itself is given by the tensor current:

$$\Sigma_{\alpha\beta}(x, \tilde{x}) = \int_\Sigma d^2 \sigma_{\alpha\beta}(\tilde{x}) \delta^{(4)}[x - \tilde{x}(\vec{\sigma})], \quad (33)$$

which is expressed via the differential measure at the vortex world-sheet  $\Sigma$ :

$$d^2 \sigma_{\alpha\beta}(\tilde{x}) = \epsilon^{ab} \frac{\partial x_\alpha}{\partial \sigma_a} \frac{\partial x_\beta}{\partial \sigma_b} d^2 \sigma. \quad (34)$$

Here  $\epsilon^{ab}$  is the fully anti-symmetrized tensor in two dimensions,  $\epsilon^{12} = -\epsilon^{21} = +1$  and  $\epsilon^{11} = \epsilon^{22} = 0$ . The vortex tensor (33) is a two-dimensional delta-function at the surface of the vortex, with the orientation at the vortex world-sheet. For example, for a straight vortex mentioned above, one uses the parameterization  $\tilde{x}_0 = \sigma_1$ ,  $\tilde{x}_1 = \tilde{x}_2 = 0$ ,  $\tilde{x}_3 = \sigma_2$ , and obtains

$$\Sigma_{\alpha\beta}(x) = n (\delta_{\alpha,0} \delta_{\beta,3} - \delta_{\alpha,3} \delta_{\beta,0}) \delta(x_1) \delta(x_2), \quad (35)$$

where  $n \in \mathbb{Z}$  is the vorticity. Note that the regular part of the phase does not contain any singularity by definition,  $[\partial_\mu, \partial_\nu] \theta^r \equiv 0$ .

Integrating out or, equivalently, solving the equations of motion for the regular component the phase,  $\theta^r$ , allows us to rewrite the action (31) in terms of the singular part of the phase  $\theta^s$ , which, in turn, depends only on the vortex world-sheet (33):

$$S[\Sigma] = 4\pi^2 \kappa^2 \int_\Sigma d^2 \sigma(\tilde{x}) \int_\Sigma d^2 \sigma(\tilde{x}') D(\tilde{x} - \tilde{x}'). \quad (36)$$

This action is a nonlocal functional which features two integrals that are taken over the same vortex worldsheet. In the case of many vortices, the worldsheet  $\Sigma$  includes all their worldsheets:  $\Sigma_{\mu\nu} = \Sigma_{1,\mu\nu} + \Sigma_{2,\mu\nu} + \dots$

The nonlocal action (36) represents the self-interaction of the vortex line as well as the interactions of the distinct vortex segments via propagation of a massless Goldstone particle between the vortex segments. In Eq. (36), this long-range interaction is represented by the advanced Green’s function  $D(x)$  of the d’Alembert operator:

$$\square D(x) = -\delta^{(4)}(x). \quad (37)$$

In the case of a static straight vortex line of large length  $L$ , the action (36) calculated for the time interval  $\delta t$  gives us  $S = E \delta t$ , where  $E$  is, up to parameter redefinitions, the known vortex energy (30). In order to demonstrate this fact, it is convenient to make a Wick transformation

in the integral (36) to the Euclidean spacetime. In the Euclidean space, the massless propagator (37) is:

$$D(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ipx}}{p^2} = \frac{1}{8\pi^2} \frac{1}{|x|^2}, \quad (38)$$

where  $x$  is the 4-distance. For a single straight static vortex with the surface (35), the longitudinal (along the vortex) and temporal coordinates in the action (36) can be integrated out, and we get the following formal expression for the vortex energy:

$$E = 4\pi^2 \kappa^2 L D_{2D}(0), \quad (39)$$

where

$$D_{2D}(\rho) = -\frac{1}{2\pi} \ln \frac{\rho}{\rho_0}, \quad (40)$$

is the two-dimensional massless propagator [a solution of Eq. (37) in two Euclidean dimensions] as the function of the two-dimensional distance  $\rho$ . The parameter  $\rho_0$ , which has the dimension ‘‘length’’, is introduced for the consistency reasons. The argument ‘‘0’’ in Eq. (39) highlights the fact that the formal expression for the energy in the London limit is a logarithmically divergent quantity similarly to the non-relativistic expression (30). A more accurate derivation in a finite cylindrical box of the radius  $R_0$  leads us to

$$E = 2\pi \kappa^2 L \log \frac{R_0}{\xi}, \quad (41)$$

where  $\xi$  is the size of the vortex core.

The general expression (36) also gives us the interaction energy of the two straight static vortices with vorticities  $n_1$  and  $n_2$  separated by the distance  $R_{12}$ :

$$\begin{aligned} V(R_{12}) &= 8\pi^2 \kappa^2 L D_{2D}(R_{12}) \\ &\equiv -4\pi n_1 n_2 \kappa^2 L \log \frac{R_{12}}{\xi}, \end{aligned} \quad (42)$$

so that the like-charged vortices repel each other while the vortices with opposite vorticities attract each other. Finishing this section, we notice that in order to get Eq. (42) it is sufficient to take in the action (36), instead of the single-vortex current (35), the following expression:

$$\begin{aligned} \Sigma_{\alpha\beta}(x) &= (\delta_{\alpha,0}\delta_{\beta,3} - \delta_{\alpha,3}\delta_{\beta,0})\delta(x_1) \\ &\quad \times [n_1\delta(x_2 - R_{12}/2) + n_2\delta(x_2 - R_{12}/2)]. \end{aligned} \quad (43)$$

## B. Two-component superfluids in London limit

### 1. Lagrangians in the London limit

The London limit of both Hermitian (7) and non-Hermitian (5) two-component models is obtained by expressing the diagonal masses  $m_a^2 = -2\lambda_a \kappa_a^2$  via the parameters  $\kappa_a^2 > 0$  and  $a = 1, 2$  and then making the quartic interaction large,  $\lambda_1 = \lambda_2 \rightarrow \infty$ . The parameters  $\kappa_a > 0$  fix the radial amplitudes for each field,

$\phi_a = \kappa_a e^{i\theta_a}$ , while leaving the phases  $\theta_a$  as the only dynamical variables. One gets in the London limit for the Hermitian and non-Hermitian theories, respectively:

$$\begin{aligned} \mathcal{L}_H &= \kappa_1^2 \partial_\nu \theta_1 \partial^\nu \theta_1 + \kappa_2^2 \partial_\nu \theta_2 \partial^\nu \theta_2 \\ &\quad - 2m_5^2 \kappa_1 \kappa_2 \cos(\theta_1 - \theta_2), \end{aligned} \quad (44)$$

$$\begin{aligned} \mathcal{L}_{NH} &= \kappa_1^2 \partial_\nu \theta_1 \partial^\nu \theta_1 + \kappa_2^2 \partial_\nu \theta_2 \partial^\nu \theta_2 \\ &\quad + 2im_5^2 \kappa_1 \kappa_2 \sin(\theta_1 - \theta_2). \end{aligned} \quad (45)$$

Both these models have similar Lagrangians. The only difference between them appears in the interaction between the phases of different condensates: instead of the cosine function in the Hermitian model, its non-Hermitian model has a sine function preceded by a purely imaginary coupling.

According to the third criterion of Eq. (13), the non-Hermitian theory in the London limit at  $\kappa_1 \neq \kappa_2$  would correspond to the  $\mathcal{PT}$  unbroken phase if the theory were non-interacting. Below, we will see that in the interacting theory this criterion, unsurprisingly, does not work. Still, Eq. (45) represents a meaningful theory even if  $m_5 \neq 0$ .

### 2. Hermitian two-condensate model

From the classical equations of motion of the Hermitian model,

$$\kappa_1 \square \theta_1 - m_5^2 \kappa_2 \sin(\theta_1 - \theta_2) = 0, \quad (46)$$

$$\kappa_2 \square \theta_2 + m_5^2 \kappa_1 \sin(\theta_1 - \theta_2) = 0, \quad (47)$$

one immediately determines the presence of the Goldstone massless mode

$$\chi = \theta_1 \sin^2 \beta + \theta_2 \cos^2 \beta, \quad (48)$$

and the massive excitation,

$$\gamma = \theta_1 - \theta_2. \quad (49)$$

These degrees of freedom satisfy, respectively, the following equations:

$$\square \chi = 0, \quad (50)$$

$$\square \gamma - M^2 \sin \gamma = 0. \quad (51)$$

Here

$$M^2 = \frac{2m_5^2}{\sin 2\beta} \equiv \frac{\kappa_1^2 + \kappa_2^2}{\kappa_1 \kappa_2} m_5^2, \quad (52)$$

is the mass of the mode (49),

$$\tan \beta = \frac{\kappa_1}{\kappa_2}, \quad (53)$$

is the angle which determines the relative strength of the condensates, and

$$\kappa = \sqrt{\kappa_1^2 + \kappa_2^2}. \quad (54)$$

Since  $\kappa_a > 0$ , the mass (52) can always be chosen as a positive quantity. The Hermitian Lagrangian (44) can

therefore be rewritten as a sum of independent contributions coming from massless (48) and massive (49) fields:

$$\mathcal{L}_H = \kappa^2 \partial_\nu \chi \partial^\nu \chi + \mathcal{L}_H^{(M)}(\gamma). \quad (55)$$

The Goldstone mode  $\chi$  corresponds to the massless excitation of the one-component Bose gas superfluid (31). The massive mode  $\gamma$  is described by the sine-Gordon Lagrangian:

$$\mathcal{L}_{\text{NH}}^{(M)}(\gamma) = \tilde{\kappa}^2 (\partial_\nu \gamma \partial^\nu \gamma - 2M^2 \cos \gamma), \quad (56)$$

where the parameter

$$\tilde{\kappa} = \frac{\kappa \sin 2\beta}{2}, \quad (57)$$

plays a role of the amplitude corresponding to the massive condensate. In Eq. (56), the term  $\cos \gamma$  appears naturally instead of the usual  $\gamma^2$  mass term in agreement with the  $\gamma \rightarrow \gamma + 2\pi$  periodicity coming from the symmetry of the original fields  $\theta_1$  and  $\theta_2$ .

Following the analogy of the one-component model, we conclude the two-component model should contain two types of superfluid vortices associated with singularities (winding) in the phases of the fields  $\theta_1$  and  $\theta_2$ . Using the decompositions (48) and (49), we identify the massless  $\Sigma^{(0)}$  and massive  $\Sigma^{(M)}$  combinations of the vortex worldsheets which appear as the singularities in  $\chi$  and  $\gamma$  fields, respectively:

$$\Sigma_{\mu\nu}^{(0)} = \Sigma_{\mu\nu}^{(1)} \sin^2 \beta + \Sigma_{\mu\nu}^{(2)} \cos^2 \beta, \quad (58)$$

$$\Sigma_{\mu\nu}^{(M)} = \Sigma_{\mu\nu}^{(1)} - \Sigma_{\mu\nu}^{(2)}. \quad (59)$$

Here the individual phase windings are defined according to Eq. (32):

$$\partial_{[\mu, \partial_\nu]} \theta_i^s(x, \tilde{x}) = 2\pi \cdot \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \Sigma_{\alpha\beta}^{(i)}(x, \tilde{x}), \quad i = 1, 2. \quad (60)$$

Therefore, the effective theory of vortices is written as follows:

$$S[\Sigma] = 4\pi^2 \kappa^2 \int_{\Sigma^{(0)}} d^2\sigma(\tilde{x}) \int_{\Sigma^{(0)}} d^2\sigma(\tilde{x}') D(\tilde{x} - \tilde{x}') \quad (61)$$

$$+ 4\pi^2 \tilde{\kappa}^2 \int_{\Sigma^{(M)}} d^2\sigma(\tilde{x}) \int_{\Sigma^{(M)}} d^2\sigma(\tilde{x}') D_M(\tilde{x} - \tilde{x}').$$

where  $D_M(x)$  is the advanced Green's function corresponding to the propagator of the massive particle,  $(\square + M^2) D_M(x) = -\delta^{(4)}(x)$ .

### 3. Non-Hermitian two-condensate model

Exactly the same field combinations (48) and (49) can be used to rewrite the non-Hermitian theory (45) in terms of the massless and massive field combinations:

$$\mathcal{L}_{\text{NH}} = \kappa^2 \partial_\nu \chi \partial^\nu \chi + \mathcal{L}_{\text{NH}}^{(M)}(\gamma). \quad (62)$$

This non-Hermitian model possesses the usual Hermitian Goldstone mode  $\chi$  which leads to the long-range interactions between the combinations of the worldsheets (58).

However, the would-be massive excitation  $\gamma$  exhibits a non-Hermitian behaviour described by the Lagrangian:

$$\mathcal{L}_{\text{NH}}^{(M)}(\gamma) = \tilde{\kappa}^2 (\partial_\nu \gamma \partial^\nu \gamma + 2iM^2 \sin \gamma), \quad (63)$$

and therefore the interaction between the would-be massive components of the vortex sheets (59) is not evident.

The model (63) is nothing but a non-Hermitian version of the Sine-Gordon model in (3+1) dimensions. As a side remark, we notice that the Lagrangian (63) appears as a bosonic dual of the non-Hermitian massive Thirring Model in (1+1) dimensions [6]. According to Ref. [6], this model with the purely imaginary coupling in front of the sine-term resides in the  $\mathcal{PT}$ -broken domain and, therefore, should be characterized by complex energy dispersions which correspond to dissipation or instability, or the both.

The second term in the Lagrangian (63) implies that the  $\gamma = 0$  point is not a local extremum of the corresponding action. The stable minima could appear around the values  $\gamma_\pm = \pm\pi/2$ . Defining  $\gamma = \pm\pi/2 + \delta\gamma$ , we get the following equations of motion:

$$\square \delta\gamma \pm iM^2 \sin \delta\gamma = 0, \quad (64)$$

which differ from the classical equations of motion of the Hermitian model (51) by the purely complex coefficient in front of the sine term. For small fluctuations around the minimum,  $\delta\gamma = 0$ , the solutions of Eq. (64) give us the dispersions for the energy:  $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 \pm iM^2}$ . We find that the particle-like, positive-energy solutions with  $\text{Re } \omega > 0$  lead to an explosive behavior near the  $\gamma = -\pi/2$  minimum which appears to be unstable. For example, the amplitude of any zero-momentum solution ( $\mathbf{k} = 0$ ) diverges with time  $t$  as  $\delta\gamma \sim \exp(\Gamma t)$  where

$$\Gamma = \frac{M}{\sqrt{2}} \equiv \frac{\kappa_1^2 + \kappa_2^2}{\kappa_1 \kappa_2} \frac{m_5^2}{\sqrt{2}} > 0. \quad (65)$$

In the language of a Hermitian theory, the point  $\gamma = -\pi/2$  would correspond to an extremum.

However, the minimum  $\gamma = +\pi/2$  is stable so that all particle excitations around it behave as dissipative solutions  $\delta\gamma \sim \exp(-\Gamma t)$  that approach the minimum point  $\gamma = +\pi/2$  should the field  $\gamma$  deviate from it. Since the angle  $\gamma$  takes a constant value in the stable minimum, it evidently means that the field  $\gamma$  contains no vortex singularities in this ground state. According to Eq. (59), the vortex singularities in the both phases  $\theta_1$  and  $\theta_2$  should coincide with each other:  $\Sigma_{\mu\nu}^{(M)} = \Sigma_{\mu\nu}^{(1)} - \Sigma_{\mu\nu}^{(2)} = 0$ .

Thus, the vortices in the  $\phi_1$  and  $\phi_2$  condensate can only exist provided they coincide with each other,  $\Sigma_{\mu\nu}^{(0)} = \Sigma_{\mu\nu}^{(1)} = \Sigma_{\mu\nu}^{(2)}$  thus forming a single double-vortex sheet  $\Sigma_{\mu\nu}^{(0)}$  according to Eq. (58). Any fluctuation that separates the vortices leads to the energy dissipation with the dissipation rate (65) which returns the vortices back to their common stable non-dissipative minimum.

The common vortex line is described by action of the vortex in the one-component condensate (36), where the

coupling  $\kappa$  is given in Eq. (54). The energy per unit length of the joint vortex is thus given by Eq. (41). In this state, the common vortex segments interact with other segments via a long-range interaction mediated by massless particles. For the straight static vortices separated by the distance  $R_{12}$ , the interaction is given by Eq. (42).

#### IV. VORTICES AT FINITE COUPLINGS

The non-Hermitian two-field model possesses vortex solutions not only in the London limit but also at finite values of the quartic couplings  $\lambda_1$  and  $\lambda_2$ . In this section we consider examples of the static straight vortex solutions of the classical equations of motion (14) assuming the standard axial ansatz for the scalar fields

$$\phi_a(r, \theta) = v_a f_a(r) e^{in\theta}, \quad a = 1, 2, \quad (66)$$

where  $r$  and  $\theta$  are the radial coordinates in the  $(x_1, x_2)$  plane and  $n \in \mathbb{Z}$  is the vorticity of the solution. These static and straight field configurations do not depend on time and  $x_3$  coordinates. The vacuum values of the condensates,  $v_1$  and  $v_2$ , are the solutions of Eqs. (18).

The consistency of the coupled solutions at  $m_5 \neq 0$  imply that the vortices in  $\phi_1$  and  $\phi_2$  condensates should be superimposed on each other and they should have the same winding numbers  $n_1 = n_2 = n$  in Eq. (66). Below we concentrate on the non-Hermitian model which is the subject of our paper (the analysis of the Hermitian counterpart can also be performed in the same way).

The radial profiles of the vortices are described by the functions  $f_a$  with the following asymptotics:

$$\lim_{r \rightarrow \infty} f_a(r) = 1, \quad \lim_{r \rightarrow 0} f_a(r) = 0, \quad a = 1, 2, \quad (67)$$

which guarantee that the total energy of the vortex solution is converging both at the spatial infinity and at the origin, respectively.

The classical equations of motion (14) lead to the following system of equations for the profile functions:

$$f_1''(r) + \frac{f_1'(r)}{r} - \frac{n^2}{r^2} f_1(r) - m_1^2 f_1(r) - m_5^2 \frac{v_2}{v_1} f_2(r) - 2\lambda_1 v_1^2 f_1^3 = 0, \quad (68a)$$

$$f_2''(r) + \frac{f_2'(r)}{r} - \frac{n^2}{r^2} f_2(r) - m_2^2 f_2(r) + m_5^2 \frac{v_1}{v_2} f_1(r) - 2\lambda_2 v_2^2 f_2^3 = 0, \quad (68b)$$

which, evidently, do not possess straightforward analytical solutions.

Close to the origin, the last non-linear terms in both equations (68) can be neglected and the differential equations can be linearized. The solutions can be represented in the form of the polynomials,

$$f_a(r) = \sum_{k=0}^{\infty} A_a^{(2k)} r^{n+2k}, \quad (69)$$

which involve only the positive even powers of the radius starting from the power  $r^n$  determined by the vorticity number  $n = 1, 2, \dots$ . The ansatz (69) is thus consistent with the asymptotics (67). One gets for the first three coefficients:

$$\begin{aligned} A_1^{(0)} &= A_1, & A_1^{(2)} &= \frac{A_1 v_1 m_1^2 + A_2 v_2 m_5^2}{4(n+1)v_1}, \\ A_1^{(4)} &= \frac{A_1 v_1 (m_1^4 - m_5^4) + A_2 v_2 m_5^2 (m_1^2 + m_2^2)}{32(n+1)(n+2)v_1}, & (70a) \\ A_2^{(0)} &= A_2, & A_2^{(2)} &= \frac{A_2 v_2 m_1^2 - A_1 v_1 m_5^2}{4(n+1)v_2}, \\ A_2^{(4)} &= \frac{A_2 v_2 (m_1^4 - m_5^4) - A_1 v_1 m_5^2 (m_1^2 + m_2^2)}{32(n+1)(n+2)v_2}, \end{aligned}$$

where  $A_1$  and  $A_2$  are free parameters of the solution which cannot be fixed at this stage. The series (70a) of the  $f_1$  and  $f_2$  profile functions are related to each other by the flip of the sign in front of the off-diagonal mass term  $m_5^2$ .

In the large-distance region,  $r \rightarrow \infty$ , the asymptotics (67) imply  $f_a = 1 - h_a$  where  $|h_a| \ll 1$  at sufficiently large distances. The linearized equations of motion,

$$h_1''(r) + \frac{h_1'(r)}{r} - \frac{n^2}{r^2} h_1(r) - m_1^2 h_1(r) - m_5^2 \frac{v_2}{v_1} h_2(r) - 6\lambda_1 v_1^2 h_1(r) = 0, \quad (71)$$

$$h_2''(r) + \frac{h_2'(r)}{r} - \frac{n^2}{r^2} h_2(r) - m_2^2 h_2(r) + m_5^2 \frac{v_1}{v_2} h_1(r) - 6\lambda_2 v_2^2 h_2(r) = 0, \quad (72)$$

suggest that their solutions can be represented in the following form:

$$h_a(r) = B_a^{(0)} r^s e^{-\mu r}. \quad (73)$$

The self-consistency of the solutions provides us with the power  $s = -1/2$  of the algebraic prefactor  $r^s$  and also imposes two simultaneous constraints:

$$\mu^2 - m_1^2 - m_5^2 \frac{B_2^{(0)} v_2}{B_1^{(0)} v_1} - 6\lambda_1 v_1^2 = 0, \quad (74a)$$

$$\mu^2 - m_2^2 + m_5^2 \frac{B_1^{(0)} v_1}{B_2^{(0)} v_2} - 6\lambda_2 v_2^2 = 0. \quad (74b)$$

These equations give us two possible solutions for the ratio of the coefficients  $B_a^{(0)}$  from Eq. (73):

$$\frac{B_1^{(0)}}{B_2^{(0)}} = \frac{v_2}{v_1} \left( \alpha \pm \sqrt{\alpha^2 - 1} \right), \quad (75)$$

and also determines the common exponent:

$$\mu = \sqrt{6\lambda_1 v_1^2 + m_1^2 + m_5^2 \left( \alpha \pm \sqrt{\alpha^2 - 1} \right)}. \quad (76)$$

We denoted for brevity:

$$\alpha = \frac{6(\lambda_2 v_2^2 - \lambda_1 v_1^2) + m_2^2 - m_1^2}{2m_5^2}. \quad (77)$$

The next-order correction to Eq. (73) can also be easily obtained,

$$h_a(r) = \left( \frac{B_a^{(0)}}{r^{\frac{1}{2}}} + \frac{B_a^{(1)}}{r^{\frac{3}{2}}} + O\left(r^{-\frac{5}{2}}\right) \right) e^{-\mu r}, \quad (78)$$

where

$$B_a^{(1)} = \frac{1}{2\mu} \left( n^2 - \frac{1}{4} \right) B_a^{(0)}. \quad (79)$$

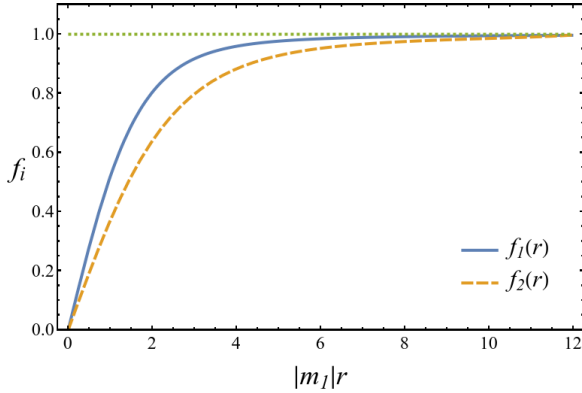


FIG. 3. The profile functions of the elementary  $n = 1$  vortex solution at the mass parameters  $m_2^2 = |m_1^2|$  and  $m_5^2 = 0.1|m_1^2|$  with  $m_1^2 < 0$  and the equal quartic couplings  $\lambda_1 = \lambda_2 = 1$ .

If the off-diagonal mass vanishes,  $m_5^2 = 0$ , the non-Hermitian two-scalar model reduces to two non-interacting scalar models  $\mathcal{L}(\phi_1, \phi_2) = \mathcal{L}_1(\phi_1) + \mathcal{L}_2(\phi_2)$  with

$$\mathcal{L}_a(\phi_a) = \partial_\nu \phi_a^* \partial^\nu \phi_a - m_a^2 |\phi_a|^2 - \lambda_a \phi_a^4. \quad (80)$$

This single-field model possesses the asymptotic solutions of the form (we omit the index  $a$  for simplicity):

$$f(r) = A \left[ r^n + \frac{m^2}{4(n+1)} r^{n+2} + \frac{m^4}{32(n+1)(n+2)} r^{n+4} + O(r^{n+6}) \right], \quad (81)$$

$$h(r) = B \left[ \frac{1}{\sqrt{r}} + \frac{1}{2\mu r^{3/2}} \left( n^2 - \frac{1}{4} \right) + O\left(r^{-5/2}\right) \right] e^{-\mu r},$$

where the asymptotic behavior is controlled by the mass of the single scalar field:

$$\mu = \sqrt{m^2 + 6\lambda v^2} = 2\sqrt{\lambda}v. \quad (82)$$

In this limit, the two equations (74) decouple, and the mass parameters reduce to Eq. (82) for each field. The asymptotics of the non-Hermitian solution (69), (70),

(78), and (79) are consistent with the single-field solution (81) as well.

Our numerical analysis confirms the existence of the stable vortex solutions in the regions of the phase diagram with non-zero condensates. An example of the profile functions for a set of coupling constants is shown in Fig. 3. All the radial  $n = 1$  profiles of the vortices exhibit the same qualitative features, the linear rise close to the origin and the exponentially slow approach of the corresponding vacuum expectation values at large distances. These properties reveal the generic behavior of all solutions that we have analyzed.

The energy density of the non-Hermitian vortex (calculated per unit vortex length),

$$E_{\text{NH}} = 2\pi \int_0^\infty r dr (|\nabla \phi_1|^2 + |\nabla \phi_2|^2 + m_1^2 \phi_1^2 + m_2^2 \phi_2^2 + \lambda_1 \phi_1^4 + \lambda_2 \phi_2^4), \quad (83)$$

can be simplified with the use of the corresponding equations of motion (14). It can be expressed via the profile functions  $f_a$  as follows:

$$E = 2\pi \int_0^\infty r dr \{ \lambda_1 v_1^4 [1 - f_1^4(r)] + \lambda_2 v_2^4 [1 - f_2^4(r)] \}, \quad (84)$$

where the energy is normalized in such a way that  $E = 0$  in the absence of the vortex. The very same expression (84) also gives us the energy of the vortex in the counterpart Hermitian theory (7),

$$E_{\text{H}} = 2\pi \int_0^\infty r dr (|\nabla \phi_1|^2 + |\nabla \phi_2|^2 + m_1^2 \phi_1^2 + m_2^2 \phi_2^2 + 2m_5^2 \phi_1 \phi_2 + \lambda_1 \phi_1^4 + \lambda_2 \phi_2^4), \quad (85)$$

after the use of the corresponding classical equations of motion (22).

Finally, despite of the mundane similarity of the numerically obtained vortex configurations in various regions, one notices rather unusual difference of the evolution of the vortex energy as the function of the off-diagonal mass parameter  $m_5$  in Hermitian and non-Hermitian regions. We show the examples of vortex energies in Hermitian and non-Hermitian theories in different stability areas, with  $m_1^2 < 0$  and  $m_2^2 > 0$  in Fig. 4(a) and with  $m_1^2, m_2^2 < 0$  in Fig. 4(b) at the same values of quartic couplings. The vortex energies in the Hermitian and non-Hermitian versions trivially coincide at  $m_5 = 0$  and then they tend to separate as the off-diagonal mass  $m_5$  increases. One notices non-monotonic behaviour of energies in the completely broken ( $m_1^2, m_2^2 < 0$ ) part of the phase diagram.

## V. CONCLUSIONS

We studied the properties of vortices in the non-Hermitian model  $\mathcal{PT}$  symmetric model with two interacting complex scalar fields.

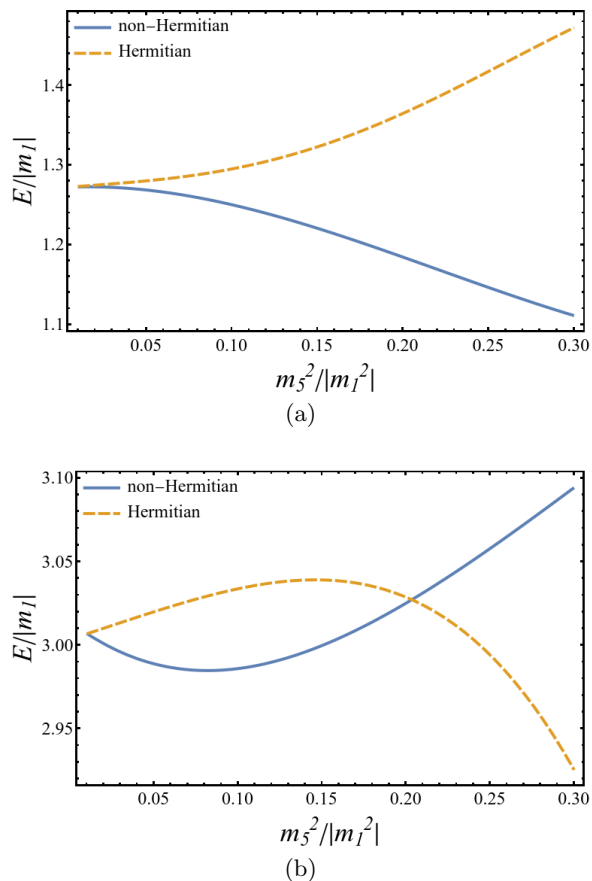


FIG. 4. Hermitian and non-Hermitian vortex energies vs. the off-diagonal mass squared  $m_5^2$  in different stability areas (a)  $m_5^2 = -m_1^2 > 0$  corresponding to the border of the stable and unstable regions and (b)  $m_5^2 = 2.5m_1^2 < 0$  residing within the stable region, at  $\lambda_1 = \lambda_2 = 1$ , Fig. 2(g).

Firstly, we considered the model in the London limit where the lengths of the condensates are frozen and the analytical analysis is possible. To reveal the vortex properties, we used a set of transformations that did not involve the explicit solution of the equations of motion. Noticing that the London limit of the model resides in the  $\mathcal{PT}$ -broken phase, we show that the superfluid vortices can propagate non-dissipatively if and only if the vortex singularities in different condensates have the same vorticity (winding number) and, in addition, they overlap. The joint vortex segments interact via a long-ranged exchange of a massless excitation, similarly to the vortices in a Hermitian one-condensate model. The dissipation rate of the individual (separated) vortex segments is controlled by the off-diagonal mass, which, in turn, determines the interaction between the condensates.

Secondly, we also studied the classical vortex configurations at finite quartic couplings. In order to identify the classical configurations, we used a single set of classical equations of motion, which is obtained by the variation of the action with respect to the original fields. We omitted the equivalent but the incompatible, complementary set of equations that correspond to the variation of the action with respect to the conjugated fields. This procedure, which follows Refs. [7, 36], seems appropriate in application to the open systems residing in a steady-state regime, which does not necessarily coincide with the (thermal) equilibrium. Moreover, in this approach, the classical solutions possess a real-valued energy spectrum bounded below. The latter property is essential on the practical level as we search for the classical states using a (numerical) procedure based on the energy minimization as a criterion for the true (ground) state.

An alternative approach, based on the similarity transformation, does not possess the incompatibility of the two sets of equations of motion. This property makes the analytical procedure of finding the classical solutions more elegant [22, 23]. However, the same approach makes the kinetic term (of, at least, one of the fields) negative, leading to the emergence of a negative quadratic mode for the classical solutions. Therefore, the mentioned class alternative solutions correspond to different, sphaleron-type saddle-point configurations which can be important for thermal properties of the system.

We analyzed the ground state of the model, and we have shown that the  $\mathcal{PT}$ -unbroken regions of the interacting system can be identified using, as a practical criterion, the absence of the negative modes in fluctuation matrix  $\mathcal{M}_{\text{NH}}^2$  that describes the quadratic fluctuations of the fields (25) over the ground-state condensates. On the contrary, in the spontaneously  $\mathcal{PT}$ -broken regions, the quadratic fluctuation matrix contains at least a single negative eigenvalue.

We found that the  $\mathcal{PT}$ -symmetric two-component model admits the vortex solutions inside and at the border of the  $\mathcal{PT}$ -broken regions. These two-condensate vortex solutions share similar behavior with the vortices in the one-component relativistic superfluids. For consistency of the classical solution, the vortices of different condensates should have the same position in space-time and possess the same vorticity (winding number).

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