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Domain of existence of the Laplace transform of negative multinomial distributions and simulations

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Abstract

This article provides the domain of existence of the Laplace transform of infinitely divisible negative multinomial distributions, defined by their probability generating functions. It makes it possible to construct all of these distributions. We give examples in dimensions 2 and 3. We give a construction and a simulation of negative multivariate distributions in dimension 2.

KEY WORDS: Laplace transform ; mixed multivariate Poisson distributions ; multivariate gamma distributions ; negative multinomial distributions ; probability generating function ; simulations.

1 Introduction

In this article, we consider the following definition, see references in (Bernardoff, 2003). We denote by \mathbb{N} the set of non-negative integers. We shall say that the probability distribution $\sum_{\alpha \in \mathbb{N}^n} p_\alpha \delta_\alpha$ on \mathbb{N}^n , where n is a positive integer, is a *negative multinomial distribution* if there exists an affine polynomial $P(z_1, \dots, z_n)$ and $\lambda > 0$ such that $P(0, \dots, 0) \neq 0$, $P(1, \dots, 1) = 1$, and

$$\sum_{\alpha \in \mathbb{N}^n} p_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} = (P(z_1, \dots, z_n))^{-\lambda}. \quad (1)$$

This corresponds to a polynomial which is affine with respect to each $z_j, j = 1, \dots, n$, or for which $\partial^2 / \partial z_j^2 P = 0$ for all $j = 1, \dots, n$. Moreover, $P(z_1, \dots, z_n) = \sum_{T \in \mathfrak{P}_n} a_T \mathbf{z}^T$, where \mathfrak{P}_n is the family of subsets of $\{1, 2, \dots, n\} = [n]$, and where $\mathbf{z}^T = \prod_{t \in T} z_t$, if $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$. For instance, for $n = 2$, such P has the form $P(z_1, z_2) = a_\emptyset + a_{\{1\}}z_1 + a_{\{2\}}z_2 + a_{\{1,2\}}z_1z_2$ with $a_\emptyset \neq 0$. However, finding exactly which pairs (P, λ) are compatible is an unsolved problem. Bernardoff (2003) explained why this distribution is a generalization of the usual negative multinomial distribution defined in (Johnson *et al.*, 1997).

Before giving the main result, let us make an observation. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, then we denote $\mathbf{z}^\alpha = \prod_{i=1}^n z_i^{\alpha_i} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Let A be any polynomial such that $A(0, \dots, 0) = 1$, and suppose that the Taylor expansion $(A(z_1, \dots, z_n))^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) \mathbf{z}^\alpha$ has non-negative coefficients $c_\alpha(\lambda)$. Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be positive numbers such that $\sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) \mathbf{a}_1^{\alpha_1} \cdots \mathbf{a}_n^{\alpha_n} < \infty$. With such a sequence $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ we associate the negative multinomial distribution $\sum_{\alpha \in \mathbb{N}^n} p_\alpha \delta_\alpha$ defined by

$$\sum_{\alpha \in \mathbb{N}^n} p_\alpha \mathbf{z}^\alpha = [A(\mathbf{a}_1 z_1, \dots, \mathbf{a}_n z_n) / A(\mathbf{a}_1, \dots, \mathbf{a}_n)]^{-\lambda}, \text{ thus}$$

$$P(z_1, \dots, z_n) = A(\mathbf{a}_1 z_1, \dots, \mathbf{a}_n z_n) / A(\mathbf{a}_1, \dots, \mathbf{a}_n) \text{ in the notation (1).}$$

Bernardoff (2003) defined the numbers $b_T = b_T(P)$ by:

Definition 1 Let $P(\mathbf{z}) = \sum_{T \in \mathfrak{P}_n} a_T \mathbf{z}^T$ be an affine polynomial $P(z_1, \dots, z_n)$ such that $P(0, \dots, 0) = 0$, and $A = 1 - P$. Let \mathfrak{P}_n^* the family of non-empty subsets of $[n]$. For $T \in \mathfrak{P}_n^*$, let us introduce the number b_T , in fact a polynomial with respect to the a_T , defined by $b_T = \partial^{|T|} / \partial z^T (\log(1 - P))|_{\mathbf{0}}$, where $|T|$ is the cardinality of T and $\partial z^T = \prod_{t \in T} \partial z_t$. Then

$$b_T = \sum_{l=1}^{|T|} (l-1)! \sum_{\mathcal{T} \in \Pi_T^l} a_{\mathcal{T}} \quad (2)$$

where Π_T is the set of partitions of T , and Π_T^l is the set of partitions of length l of T (if $\mathcal{T} = \{T_1, T_2, \dots, T_l\}$, the partition \mathcal{T} of T is of length l).

For instance, for $n = 3$, $b_{\{1\}} = a_{\{1\}}$, $b_{\{1,2\}} = a_{\{1,2\}} + a_{\{1\}}a_{\{2\}}$ and $b_{\{1,2,3\}} = a_{\{1,2,3\}} + a_{\{1\}}a_{\{2,3\}} + a_{\{2\}}a_{\{1,3\}} + a_{\{3\}}a_{\{1,2\}} + 2a_{\{1\}}a_{\{2\}}a_{\{3\}}$. Now, if there is no ambiguity, for simplicity we omit the braces.

Using the numbers b_T , Bernardoff (2003) proves the following theorem.

Theorem 2 *Let $P(z) = \sum_{T \in \mathfrak{P}_n^*} a_T z^T$, as before, and suppose that $(1 - P(z))^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) z^\alpha$. Then $c_\alpha(\lambda) \geq 0$ for all positive λ if and only if b_T , given by (2), is non-negative for all $T \in \mathfrak{P}_n^*$.*

See examples in dimension $n = 2, 3$ in (Bernardoff, 2003).

This article is preliminary to another one on the simulation of infinitely divisible negative multinomial distributions in dimension $n > 2$. Before writing such an article, knowing all these distributions is a prerequisite.

This article is organized as follows. Section 2 gives the main result. Section 3 applies the main result to bivariate and trivariate cases. Section 4 gives a construction of some of these negative multivariate distributions in dimension 2. Section 5 gives definition and construction of multifactor negative multinomial distributions. Section 6 gives simulations for the examples in Section 3.

2 Domain of existence of the Laplace transform

Let A be an affine polynomial on \mathbb{R}^n and let $\lambda > 0$ be such that $A(0, \dots, 0) = 1$ and such that $(A(z_1, \dots, z_n))^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) z^\alpha$ satisfies $c_\alpha(\lambda) > 0$ for all α in \mathbb{N}^n . The problem of finding the affine polynomials $A = 1 - P$ such that for all $\lambda > 0$, $A^{-\lambda}$ has non-negative coefficients is solved by Theorem 2. This is equivalent to finding all negative multinomial distributions on \mathbb{N}^n which are infinitely divisible. We now only consider infinitely divisible negative multinomial distributions. Consider the discrete measure on \mathbb{N}^n , $\mu_\lambda = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) \delta_\alpha$. The present section aims to describe the convex set $D(\mu_\lambda) = \{\boldsymbol{\theta} \in \mathbb{R}^n, \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) \exp(\alpha_1 \theta_1 + \dots + \alpha_n \theta_n) < +\infty\}$, which is an important object in the study of the natural exponential family generated by μ_λ (Bar-Lev *et al.*, 1994; Letac, 1991). The answer is contained in the following result.

Theorem 3 *With the above notation, we denote $H = \{\mathbf{s} \in \mathbb{R}^n, s_1 + \dots + s_n = 0\}$. For $\mathbf{s} \in H$, we denote by $R_{\mathbf{s}}$ the smallest positive zero of the polynomial $P_{\mathbf{s}}(t) = A(\mathbf{t}e^{s_1}, \dots, \mathbf{t}e^{s_n})$. Then the map $\mathbf{s} \mapsto \mathbf{s} + \log R_{\mathbf{s}}(1, \dots, 1)$ is a parametrization by H of a hypersurface in \mathbb{R}^n which is the boundary of $D(\mu_\lambda)$. More specifically, if $\boldsymbol{\theta}$ is in \mathbb{R}^n , $\bar{\theta}_n = (\theta_1 + \dots + \theta_n)/n$ and $\mathbf{s} = \boldsymbol{\theta} - \bar{\theta}_n(1, \dots, 1)$, then $\boldsymbol{\theta}$ is in $D(\mu_\lambda)$ if and only if $\bar{\theta}_n < \log R_{\mathbf{s}}$. Finally, $D(\mu_\lambda)$ is an open set.*

Proof. We first prove that the radius of convergence R of the power series

$$P_{\mathbf{s}}^{-\lambda}(t) = \sum_{n \in \mathbb{N}} u_n(\lambda) t^n \tag{3}$$

is equal to $R_{\mathbf{s}}$. This comes from the following fact: since $u_n(\lambda) \geq 0$, a known result in the theory of analytic functions (Titchmarsh, 1939, **7.21**) implies that $t \mapsto P_{\mathbf{s}}^{-\lambda}(t)$ is not analytic at R . Since $P_{\mathbf{s}}(0) = 1$, $P_{\mathbf{s}}(t) > 0$ for $0 < t < R_{\mathbf{s}}$ and $P_{\mathbf{s}}(R_{\mathbf{s}}) = 0$, clearly $R = R_{\mathbf{s}}$.

We now observe that if $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ is such that

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} c_{\boldsymbol{\alpha}}(\lambda) \exp(\alpha_1 \theta_1 + \dots + \alpha_n \theta_n) < +\infty, \text{ then for all } p \geq 0 \text{ we have}$$

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} c_{\boldsymbol{\alpha}}(\lambda) \exp[\alpha_1 (\theta_1 - p) + \dots + \alpha_n (\theta_n - p)] < +\infty.$$

We write $\bar{\theta}_n = (\theta_1 + \dots + \theta_n)/n$. The orthogonal projection of $\boldsymbol{\theta}$ on H is $\mathbf{s} = \boldsymbol{\theta} - \bar{\theta}_n(1, \dots, 1) = (s_1, \dots, s_n)$. Thus for all $j = 1, \dots, n$, we have $\theta_j - s_j = \bar{\theta}_n$. We claim that $\bar{\theta}_n < \log R_{\mathbf{s}}$. If not, we have $t_0 = e^{\bar{\theta}_n} \geq R_{\mathbf{s}}$. But $A(\mathbf{e}^{\theta_1}, \dots, \mathbf{e}^{\theta_n})$ is $P_{\mathbf{s}}(t_0)$ and the previous remark shows that for all $p \geq 0$, $p \mapsto P_{\mathbf{s}}(e^{-p} t_0)$ is positive. This contradicts the fact that $t_0 \geq R_{\mathbf{s}}$.

Conversely, if $\boldsymbol{\theta}$ is such that $\bar{\theta}_n < \log R_{\mathbf{s}}$, similar reasoning shows that $\boldsymbol{\theta} \in D(\mu_{\lambda})$.

Finally, for $t = R_{\mathbf{s}}$ in (3) the series diverges. A short proof goes as follows:

For any k in \mathbb{N} and $\lambda > 0$, we define $(\lambda)_k = \lambda(\lambda + 1) \dots (\lambda + k - 1) = \Gamma(\lambda + k)/\Gamma(\lambda)$, where Γ is Euler's gamma function and $(\lambda)_0 = 1$. We write $P_{\mathbf{s}}(t) = \left(1 - \frac{t}{r_0}\right) \dots \left(1 - \frac{t}{r_k}\right)$ where $|r_j| \geq r_0 = R_{\mathbf{s}}$ by definition of $R_{\mathbf{s}}$. Thus $\sum_{n \in \mathbb{N}} u_n(\lambda) t^n$ is the product of the Newton Series $\sum_{n \in \mathbb{N}} \frac{1}{n!} (\lambda)_n \left(\frac{t}{r_k}\right)^n$ and the series corresponding to $k = 0$ diverges for $t = r_0$. That is, $\sum_{n \in \mathbb{N}} u_n(\lambda) r_0^n = +\infty$.

■

Remark 4 *With the notations of Theorem 3, if $\boldsymbol{\theta}$ is in the boundary of $D(\mu_{\lambda})$, $\exists \mathbf{s} \in H, \boldsymbol{\theta} = \mathbf{s} + \log R_{\mathbf{s}}(1, \dots, 1)$, then $z_i = e^{\theta_i} = R_{\mathbf{s}} e^{s_i}$ for $i \in [n]$, and $A(z_1, \dots, z_n) = A(R_{\mathbf{s}} e^{s_1}, \dots, R_{\mathbf{s}} e^{s_n}) = 0$, by the definition of $R_{\mathbf{s}}$.*

3 Examples in dimensions 2 and 3

Example 1. For $n = 2$, we take $a_1 = 1$, $a_2 = 1$ and $a_{1,2} = a \geq -1$ so that the conditions of Theorem 2 are satisfied. Hence $A(z_1, z_2) = 1 - z_1 - z_2 - a z_1 z_2$, and for $z_1 = t e^{s_1}$, $z_2 = t e^{s_2}$, with $s_1 + s_2 = 0$, we have $P_{\mathbf{s}}(t) = 1 - t(\mathbf{e}^{s_1} + \mathbf{e}^{-s_1}) - a t^2$. If $\mathbf{s} = (s_1, s_2)$, then $R_{\mathbf{s}} = \left[-\cosh s_1 + (1/2 + a + 1/2 \cosh 2s_1)^{1/2}\right]/a$, and we obtain the parametrization of the boundary of $D(u_{\lambda}) : x = \theta_1 = s_1 + \log R_{\mathbf{s}}$, $y = \theta_2 = -s_1 + \log R_{\mathbf{s}}$, whose graphic representation is given in Figure 1.

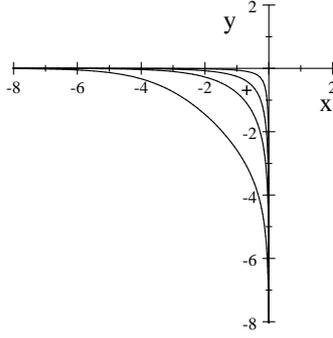


Figure 1 : the boundary of $D(u_\lambda)$ for $a = -\frac{9}{10}, -\frac{1}{2}, 1, 20$.

Remark 5 Using Remark 4 we obtain another parametrization of the boundary of $D(u_\lambda)$: $\theta_1 < 0, \theta_2 = -\log [1 + (a + 1) / (e^{-\theta_1} - 1)]$. In addition, because $D(\mu_\lambda)$ is a convex set, $\theta = (\theta_1, \theta_2) \in D(\mu_\lambda)$ is defined by

$$\theta_1 < 0, \theta_2 < -\log [1 + (a + 1) / (e^{-\theta_1} - 1)]. \quad (4)$$

In this case, for $a = -1/2$, $R_s = 2 \cosh s_1 - \sqrt{2} \cosh 2s_1$ and if we choose $s_1 = 0$, the condition $\bar{\theta}_2 < \log R_s$ becomes $\bar{\theta}_2 < -\log (1 + \sqrt{2}/2)$. As $\bar{\theta}_2 = -\log 2 < -\log (1 + \sqrt{2}/2)$, then $(-\log 2, -\log 2) \in D(\mu_\lambda)$. Then, the introduction proves that $[A(z_1/2, z_2/2) / A(1/2, 1/2)]^{-\lambda} = (8 - 4z_1 - 4z_2 + z_1z_2)^{-\lambda}$ is a generating function for all $\lambda > 0$.

Remark 6 For $a = -1/2$, the condition (4) gives for $\theta_1 = -\log 2$, $\theta_2 < -\log (3/2)$, and $\theta_2 = -\log 2$ is suitable. Hence $(-\log 2, -\log 2) \in D(\mu_\lambda)$.

Again with $a = -1/2$, if we choose $s_1 = \log 2$, $R_s = \frac{5}{2} - \frac{1}{2}\sqrt{17}$, then $\bar{\theta}_2 = -2\log 2 < \log R_s$ and $\theta = \mathbf{s} + \bar{\theta}_2(1, 1) = (-\log 2, -3\log 2)$. Then, the introduction proves that $[A(z_1/2, z_2/8) / A(1/2, 1/8)]^{-\lambda} = (32/13 - 16/13z_1 - 4/13z_2 + 1/13z_1z_2)^{-\lambda}$ is a generating function for all $\lambda > 0$.

Remark 7 For $a = -1/2$, the condition (4) gives for $\theta_1 = -\log 2$, $\theta_2 < -\log (3/2)$, and $\theta_2 = -3\log 2$ is suitable. Hence $(-\log 2, -3\log 2) \in D(\mu_\lambda)$.

Example 2. For $n = 3$, the conditions of Theorem 2 are for $i, j = 1, 2, 3$:

$$b_i = a_i \geq 0; a_{ij} \geq -a_i a_j; a_{123} \geq -(a_1 a_{23} + a_2 a_{13} + a_3 a_{12} + 2a_1 a_2 a_3)$$

We take $a_1 = a_2 = a_3 = 1$, $a_{12} = a_{13} = a_{23} = a$ and $a_{123} = b$, so that $A(z) = 1 - [(z_1 + z_2 + z_3) + a(z_1z_2 + z_1z_3 + z_2z_3) + b z_1z_2z_3]$.

The conditions of Theorem 2 are satisfied for $a \geq -1$ and $b \geq -3a - 2$. We take $a = 1$ and $b = 0$, hence

$A(z) = 1 - z_1 - z_2 - z_3 - z_1z_2 - z_1z_3 - z_2z_3$. Let $z_1 = te^{s_1}$, $z_2 = te^{s_2}$ and $z_3 = te^{s_3}$, with $s_1 + s_2 + s_3 = 0$.

If $\mathbf{s} = (s_1, s_2, s_3)$, then $P_s(t) = 1 - (e^{s_1} + e^{s_2} + e^{-s_1-s_2})t - (e^{-s_1} + e^{-s_2} + e^{s_1+s_2})t^2$, and we have

$$R_{\mathbf{s}} = \frac{-e^{s_1} - e^{s_2} - e^{-s_1 - s_2} + (e^{2s_1} + e^{2s_2} + 6e^{s_1 + s_2} + e^{-2(s_1 + s_2)} + 6e^{-s_1} + 6e^{-s_2})^{1/2}}{2(e^{-s_1} + e^{-s_2} + e^{s_1 + s_2})}.$$

Finally, the parametrization of the boundary of $D(u_\lambda)$ is $x = \theta_1 = s_1 + \log R_{\mathbf{s}}$, $y = \theta_2 = s_2 + \log R_{\mathbf{s}}$, $z = \theta_3 = -s_1 - s_2 + \log R_{\mathbf{s}}$ whose graphic representation is given in Figure 2.

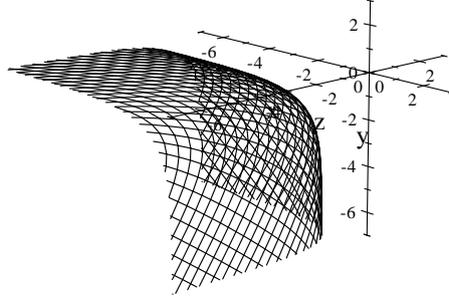


Figure 2 : The boundary of $D(\mu_\lambda)$

If we choose $s_1 = s_2 = s_3 = 0$, the condition $\bar{\theta}_3 < \log R_{\mathbf{s}}$ becomes $\bar{\theta}_3 < \log(\sqrt{21}/6 - 1/2)$. As $\bar{\theta}_3 = -\log 4 < \log(\sqrt{21}/6 - 1/2)$, then $(-\log 4, -\log 4, -\log 4) \in D(\mu_\lambda)$. Then, the introduction proves that $[A(z_1/4, z_2/4, z_3/4) / A(1/4, 1/4, 1/4)]^{-\lambda} = (16 - 4z_1 - 4z_2 - 4z_3 - z_1z_2 - z_1z_3 - z_2z_3)^{-\lambda}$ is a generating function for all $\lambda > 0$.

Remark 8 In this case, using Remark 4 we obtain that another definition of $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) \in$ the boundary of $D(u_\lambda)$ is : $\theta_1 < 0, \theta_2 < -\log\left(\frac{1-e^{\theta_1}}{1+e^{\theta_1}}\right), \theta_3 = \log\left(\frac{1-e^{\theta_1}-e^{\theta_2}-e^{\theta_1+\theta_2}}{1+e^{\theta_1}+e^{\theta_2}}\right)$, and another parametrization of the boundary of $D(u_\lambda)$ is $x = \theta_1 = u, u < 0; y = \theta_2 = v - \log\left(\frac{1-e^u}{1+e^u}\right), v < 0; z = \theta_3 = \log\left[\frac{(1-e^{2u})(1-e^v)}{1+2e^u+e^v-e^ue^v+e^{2u}}\right]$. In addition, because $D(\mu_\lambda)$ is a convex set, $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) \in D(\mu_\lambda)$ is defined by $\theta_1 < 0, \theta_2 < -\log\left(\frac{1+e^{\theta_1}}{1-e^{\theta_1}}\right), \theta_3 < \log\left(\frac{1-e^{\theta_1}-e^{\theta_2}-e^{\theta_1+\theta_2}}{1+e^{\theta_1}+e^{\theta_2}}\right)$. Hence $(-\log 4, -\log 4, -\log 4) \in D(\mu_\lambda)$ because $-\log 4 < 0, -\log 4 < -\log(5/3)$ and $-\log 4 < -\log(4 - 4/7)$.

4 Construction of bivariate negative multinomial distributions

Before giving the construction and simulation of infinitely divisible negative multinomial distributions, we recall some definitions, notations and results. We denote by \mathbb{E} the expectation. For a random vector $\mathbf{X} = (X_1, \dots, X_n)$ with pd $\mu_{\mathbf{X}}$, we write $\mathbf{X} \sim \mu_{\mathbf{X}}$.

Definition 9 We fix $\lambda > 0$, let P be an affine polynomial. If a random vector $\mathbf{X} = (X_1, \dots, X_n)$ on \mathbb{R}^n with pd $\mu_{\mathbf{X}}$ is such that its Laplace transform is

$$L_{\mu_{\mathbf{X}}}(\boldsymbol{\theta}) = \mathbb{E}\{\exp[-(\theta_1 X_1 + \dots + \theta_n X_n)]\} = [P(\boldsymbol{\theta})]^{-\lambda}, \quad (5)$$

for a set of θ with non empty interior. We denote $\mu_{\mathbf{X}} = \gamma_{(P,\lambda)}$, and $\gamma_{(P,\lambda)}$ will be called the multivariate gamma distribution associated with (P, λ) (Bar-Lev et al., 1994). We say also that $\gamma_{(P,\lambda)}$ is the multivariate gamma distribution with scale parameter P and shape parameter λ .

We remark that $L_{\mu_{X_i}}(\theta_i) = (1 + p_i\theta_i)^{-\lambda}$. Hence $X_i \sim \gamma_{(p_i,\lambda)}$ where $\gamma_{(p_i,\lambda)}$ is the ordinary gamma distribution with scale parameter p_i and shape parameter λ such that $\gamma_{(p_i,\lambda)}(dx) = x_i^{\lambda-1} p_i^{-\lambda} / \Gamma(\lambda) \exp(-x/p_i) \mathbf{1}_{(0,\infty)}(x) dx$ with $\mathbf{1}_{(0,\infty)}(x) = 1$ if $x \in (0, \infty)$ and $\mathbf{1}_{(0,\infty)}(x) = 0$ if $x \notin (0, \infty)$.

We give the notation of *negative multinomial distribution*.

Notation 10 We fix $\lambda > 0$, let P be an affine polynomial. If a random vector $\mathbf{X} = (X_1, \dots, X_n)$ on \mathbb{N}^n with pd $\mu_{\mathbf{X}}$ is such that its pgf $g_{\mathbf{X}}$ is

$$g_{\mathbf{X}}(\mathbf{z}) = \mathbb{E}(\mathbf{z}^{\mathbf{X}}) = [P(\mathbf{z})]^{-\lambda}. \quad (6)$$

We denote $\mu_{\mathbf{X}} = \mathcal{NM}_{(P,\lambda)}$, and $\mathcal{NM}_{(P,\lambda)}$ will be called the *negative multinomial distribution associated with (P, λ)* .

Let us now recall the definitions and results of Ferrari et al. (2004). An MMPD (Mixed Multivariate Poisson Distribution) is defined by assuming that the random variables N_i , $i = 1, \dots, n$ are independent and distributed according to Poisson distributions with means $(\lambda_1, \dots, \lambda_n)$, conditioned upon the vector of intensities $\lambda = (\lambda_1, \dots, \lambda_n)$. In this case, the probability masses of $\mathbf{N} = (N_1, \dots, N_n)$ are defined as $Pr(\mathbf{N} = \mathbf{k}) = \int_{(0,\infty)^n} \prod_{\ell} \frac{(\lambda_{\ell})^{k_{\ell}}}{k_{\ell}!} \exp(-\lambda_{\ell}) \mu(d\lambda)$ where $\mu(d\lambda)$ is the probability of λ defined on $(0, \infty)^n$. The MMPD defined above is fully characterized by the measure $\mu(d\lambda)$ and will be denoted by $\text{MMPD}(\mu)$. The pgf of \mathbf{N} expresses as: $g_{\mathbf{N}}(\mathbf{z}) = L_{\mu}(\mathbf{1} - \mathbf{z})$, where $\mathbf{1} = (1, \dots, 1)$. Let A be an affine polynomial such that $A(0, \dots, 0) = 1$, and $P(\mathbf{z}) = A(\mathbf{a}_1 z_1, \dots, \mathbf{a}_n z_n) / A(\mathbf{a}_1, \dots, \mathbf{a}_n)$ for any $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_n) = (\exp(\theta_1), \dots, \exp(\theta_n))$, with $\theta \in D(\mu_{\lambda})$, and $Q(\mathbf{z}) = P(\mathbf{1} - \mathbf{z})$, if $\mathbf{Y} = (Y_1, \dots, Y_n) \sim \gamma_{(Q,\lambda)}$, then we have $\mathbf{N} \sim \text{MPPD}(\gamma_{(Q,\lambda)}) = \mathcal{NM}_{(P,\lambda)}$.

Let us denote $\mathcal{P}(\lambda)$ the Poisson distribution with mean λ . Finally, if $N_i \sim \mathcal{P}(Y_i)$, $i = 1, \dots, n$ are independent, then we have $\mathbf{N} \sim \mathcal{NM}_{(P,\lambda)}$. This allows us to construct and simulate $\mathbf{N} \sim \mathcal{NM}_{(P,\lambda)}$ if we can construct and simulate $\mathbf{Y} \sim \gamma_{(Q,\lambda)}$.

We now consider the case $n = 2$. Let F_m^p be the generalized hypergeometric function (Slater, 1966) defined by

$$F_m^p(\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_m; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_m)_k} \frac{z^k}{k!}. \quad (7)$$

For simplification, we denote F_m^0 by F_m .

Bernardoff (2018) gave the following proposition:

Proposition 11 Let $P(\theta_1, \theta_2) = 1 + p_1\theta_1 + p_2\theta_2 + p_{1,2}\theta_1\theta_2$ be an affine polynomial where $p_1, p_2 > 0$ and $p_{1,2} > 0$. Let $\gamma_{(P,\lambda)}$ be the gamma distribution associated with (P, λ) . The measure $\gamma_{(P,\lambda)}$ exists if and only if $c = (p_1p_2 - p_{1,2})/p_{1,2}^2 > 0$. Then, we have

$$\begin{aligned} \gamma_{(P,\lambda)}(dx_1, dx_2) &= \frac{p_{1,2}^{-\lambda}}{[\Gamma(\lambda)]^2} \exp\left(-\frac{p_2}{p_{1,2}}x_1 - \frac{p_1}{p_{1,2}}x_2\right) (x_1x_2)^{\lambda-1} \\ &\times F_1(\lambda, cx_1x_2) \mathbf{1}_{(0,\infty)^2}(x_1, x_2) dx_1 dx_2. \end{aligned} \quad (8)$$

From (Watson, 1966) we also have $F_1(\lambda, z) = \Gamma(\lambda) I_{\lambda-1}(2\sqrt{z}) z^{-(\lambda-1)/2}$ where I_λ is the modified Bessel function of order λ . So that we have

$$\begin{aligned} \gamma_{(P,\lambda)}(dx_1, dx_2) &= c^{-(\lambda-1)/2} \frac{p_{1,2}^{-\lambda}}{\Gamma(\lambda)} \exp\left(-\frac{p_2}{p_{1,2}}x_1 - \frac{p_1}{p_{1,2}}x_2\right) (x_1x_2)^{(\lambda-1)/2} \\ &\times I_{\lambda-1}(2\sqrt{cx_1x_2}) \mathbf{1}_{(0,\infty)^2}(x_1, x_2) dx_1 dx_2. \end{aligned} \quad (9)$$

This last formula will be used to graphically represent the distribution $\gamma_{(P,\lambda)}$ in order to compare it to the simulation.

We can simplify these constructions by using the following proposition:

Proposition 12 If $\mathbf{X} \sim \gamma_{(P,\lambda)}$, $P(z_1, \dots, z_n) = \sum_{T \in \mathfrak{P}_n} p_T \mathbf{z}^T$, then for $\mathbf{b} = (b_1, \dots, b_n)$ with $b_1 > 0, \dots, b_n > 0$, the random vector $\mathbf{Y} = (Y_1, \dots, Y_n) = (b_1 X_1, \dots, b_n X_n) \sim \gamma_{(Q,\lambda)}$, with $Q(z_1, \dots, z_n) = \sum_{T \in \mathfrak{P}_n} q_T \mathbf{z}^T$ such that $q_T = \mathbf{b}^T p_T$. In particular, for $\mathbf{b} = (1/p_1, \dots, 1/p_n)$, if $\mathbf{p} = (p_1, \dots, p_n)$, with $p_1 > 0, \dots, p_n > 0$, then we have $q_i = 1$, for $i = 1, \dots, n$ and $q_T = p_T / \mathbf{p}^T$. As a result, we can construct $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_n) = (X_1/p_1, \dots, X_n/p_n)$, and $\mathbf{X} = (X_1, \dots, X_n) = (p_1 Y_1, \dots, p_n Y_n)$. Moreover, if we define $\tilde{b}_T(P) = b_T(\tilde{P})$, where \tilde{P} is such that $\tilde{P}_T = -p_{[n] \setminus T} / p_{[n]}$, then we have $\tilde{b}_T(P) = \mathbf{p}^{[n]} \tilde{b}_T(Q)$. This proves that the conditions for infinite divisibility in Bernardoff (2006) are equivalent for $\gamma_{(P,\lambda)}$ and $\gamma_{(Q,\lambda)}$.

Walker (2021) gave the following Theorem

Theorem 13 Let $Q_2(\theta_1, \theta_2) = 1 + \theta_1 + \theta_2 + q_{1,2}\theta_1\theta_2$, an affine polynomial with $q_{1,2} > 0$, we define the random variables (Y_1, V_1, Y_2) as

$$Y_1 \sim \gamma_{(1,\lambda)}, V_1 \sim \mathcal{P}(rY_1) \text{ and } Y_2 = \gamma_{(\frac{1}{1+r}, \lambda + V_1)},$$

where $r > 0$ and $q_{1,2} = 1/(1+r)$, moreover $r = 1/q_{1,2} - 1$, then $\mathbf{Y} = (Y_1, Y_2) \sim \gamma_{(Q_2,\lambda)}$.

We give the construction of $\mathbf{X} \sim \gamma_{(P,\lambda)}$, an infinitely divisible bivariate gamma distribution. We modify the construction given by Walker (2021) by using Proposition (12) and Theorem (13), and we give the following theorem.

Theorem 14 Let $P(\theta_1, \theta_2) = 1 + p_1\theta_1 + p_2\theta_2 + p_{1,2}\theta_1\theta_2$ an affine polynomial, where $p_i > 0$ $i = 1, 2$, $p_{1,2} > 0$, and $\tilde{b}_{1,2} = p_1p_2/p_{1,2}^2 - 1/p_{1,2} \geq 0$. Let $X_1 \sim \gamma_{(p_1, \lambda)}$, $V_1 \sim \mathcal{P}(cX_1)$ and $X_2 \sim \gamma_{(p_{1,2}/p_1, \lambda + V_1)}$ with $c = p_{1,2}\tilde{b}_{1,2}/p_1$, then we have $\mathbf{X} = (X_1, X_2) \sim \gamma_{(P, \lambda)}$.

Proof. Let us define Q by $Q(\theta_1, \theta_2) = P(\theta_1/p_1, \theta_2/p_2)$, hence $P(\theta_1, \theta_2) = 1 + \theta_1 + \theta_2 + p_{1,2}/(p_1p_2)\theta_1\theta_2$, so that $q_{1,2} = p_{1,2}/(p_1p_2)$. Using Theorem(13) we have $Y_1 \sim \gamma_{(1, \lambda)}$, $V_1 \sim \mathcal{P}(rY_1)$ and $Y_2 \sim \gamma_{(1/(1+r), \lambda + V_1)}$, where $r > 0$ and $q_{1,2} = 1/(1+r)$, so that $r = p_{1,2}\tilde{b}_{1,2}$, and $p_2/(1+r) = p_{1,2}/p_1$, hence $\mathbf{Y} = (Y_1, Y_2) \sim \gamma_{(Q, \lambda)}$.

Inversely, we have $P(\theta_1, \theta_2) = Q(p_1\theta_1, p_2\theta_2)$ and $\mathbf{X} = (X_1, X_2) = (p_1Y_1, p_2Y_2) \sim \gamma_{(P, \lambda)}$. Because $X_1 = p_1Y_1$ and $X_2 = p_2Y_2$, we have $X_1 \sim \gamma_{(p_1, \lambda)}$, $V_1 \sim \mathcal{P}(r/p_1X_1)$ and $X_2 \sim \gamma_{(p_{1,2}/p_1, \lambda + V_1)}$, we have well $\mathbf{X} \sim \gamma_{(P, \lambda)}$.

■

For $n = 2$, we can now construct and simulate some of the random vectors $\mathbf{X} \sim \mathcal{NM}_{(P, \lambda)}$. Let A_2 be an affine polynomial such that $A_2(z_1, z_2) = 1 - a_1z_1 - a_2z_2 - a_{1,2}z_1z_2$. The conditions for infinite divisibility of $\mathcal{NM}_{(P, \lambda)}$ are $a_1 \geq 0$, $a_2 \geq 0$, $b_{1,2} = a_{1,2} + a_1a_2 \geq 0$. We consider the case $a_1 > 0$, $a_2 > 0$, $b_{1,2} = a_{1,2} + a_1a_2 > 0$. The conditions for $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2)$ are $0 < \mathbf{a}_1 < 1/a_1$, $0 < \mathbf{a}_2 < (1 - a_1\mathbf{a}_1)/(a_2 + a_{1,2}\mathbf{a}_1) = (a_2 + b_{1,2}\mathbf{a}_1/(1 - a_1\mathbf{a}_1))^{-1} < 1/a_2$, and we have $1 - a_1\mathbf{a}_1 > 0$, $1 - a_2\mathbf{a}_2 > 0$, $1 - a_1\mathbf{a}_1 - a_2\mathbf{a}_2 - a_{1,2}\mathbf{a}_1\mathbf{a}_2 > 0$, $a_2 + a_{1,2}\mathbf{a}_1 = a_2(1 - a_1\mathbf{a}_1) + b_{1,2}\mathbf{a}_1 > 0$ because $b_{1,2} \geq 0$ and $a_1 + a_{1,2}\mathbf{a}_2 = a_1(1 - a_2\mathbf{a}_2) + b_{1,2}\mathbf{a}_2 > 0$ because $b_{1,2} \geq 0$.

For $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2)$, we have $P_2(\mathbf{z}) = \frac{1 - a_1\mathbf{a}_1z_1 - a_2\mathbf{a}_2z_2 - a_{1,2}\mathbf{a}_1\mathbf{a}_2z_1z_2}{1 - a_1\mathbf{a}_1 - a_2\mathbf{a}_2 - a_{1,2}\mathbf{a}_1\mathbf{a}_2}$ and if $Q_2(\mathbf{z}) = P_2(\mathbf{1} - \mathbf{z}) = 1 + q_1z_1 + q_2z_2 + q_{1,2}z_1z_2$, we have $q_1 = \frac{(a_1 + a_{1,2}\mathbf{a}_2)\mathbf{a}_1}{1 - a_1\mathbf{a}_1 - a_2\mathbf{a}_2 - a_{1,2}\mathbf{a}_1\mathbf{a}_2}$, $q_2 = \frac{(a_2 + a_{1,2}\mathbf{a}_1)\mathbf{a}_2}{1 - a_1\mathbf{a}_1 - a_2\mathbf{a}_2 - a_{1,2}\mathbf{a}_1\mathbf{a}_2}$, and $q_{1,2} = \frac{-a_{1,2}\mathbf{a}_1\mathbf{a}_2}{1 - a_1\mathbf{a}_1 - a_2\mathbf{a}_2 - a_{1,2}\mathbf{a}_1\mathbf{a}_2}$. We remark that $q_1 > 0$, $q_2 > 0$. In this case $q_{1,2}$ must check $q_{1,2} > 0$ (Bernardoff, 2006), hence $a_{1,2}$ must check $a_{1,2} < 0$. Obviously, we have $Q_2(\mathbf{1} - \mathbf{z}) = P_2(\mathbf{z})$.

As a result, if $\mathbf{Y} = (Y_1, Y_2) \sim \gamma_{(Q_2, \lambda)}$, then $\mathbf{X} = (X_1, X_2)$ such that $X_1 \sim \mathcal{P}(Y_1)$, $X_2 \sim \mathcal{P}(Y_2)$, and X_1 and X_2 are independent, conditioned upon the vector \mathbf{Y} , then $\mathbf{X} \sim \mathcal{NM}_{(P_2, \lambda)}$.

To simulate \mathbf{X} , first we simulate $\mathbf{Y} = (Y_1, Y_2) \sim \gamma_{(Q_2, \lambda)}$, second we simulate independently X_1 such that $X_1 \sim \mathcal{P}(Y_1)$ and X_2 such that $X_2 \sim \mathcal{P}(Y_2)$, then $\mathbf{X} = (X_1, X_2) \sim \mathcal{NM}_{(P_2, \lambda)}$.

In order to compare the masses of bivariate negative multinomial distributions, we recall the following result from (Bernardoff *et al.*, 2013).

The Taylor expansions of $[A(z)]^{-\lambda}$ and $[P(z)]^{-\lambda}$ in the neighborhood of $(0, \dots, 0)$ will be denoted as follows: $[A(z)]^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(A, \lambda) \mathbf{z}^\alpha$, $[P(z)]^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(P, \lambda) \mathbf{z}^\alpha$. We have $c_\alpha(P, \lambda) = c_\alpha(A, \lambda) \mathbf{a}^\alpha [A(\mathbf{z}^\alpha)]^\lambda$.

For the bivariate negative multinomial distributions, let $A_2(\mathbf{z}) = 1 - a_1z_1 - a_2z_2 - a_{1,2}z_1z_2$, and $P_2(\mathbf{z}) = (A_2(\mathbf{az})/A_2(\mathbf{a}))$ for a suitable $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2)$, we have the following theorem:

Theorem 15 Consider the affine polynomial of order 2 with variables $\mathbf{z} = (z_1, z_2)$ defined by $A_2(\mathbf{z}) =$

$1 - a_1 z_1 - a_2 z_2 - a_{1,2} z_1 z_2$, and $P_2(\mathbf{z}) = A_2(\mathbf{az}) / A_2(\mathbf{a})$ for a suitable $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2)$. With $b_{1,2} = a_{1,2} + a_1 a_2$, the coefficient of \mathbf{z}^α in the Taylor expansion of $[A_2(\mathbf{z})]^{-\lambda}$ and $[P_2(\mathbf{z})]^{-\lambda}$, can be computed as follows

$$c_\alpha(A_2, \lambda) = (\lambda)_{\max(\alpha_1, \alpha_2)} \quad (10)$$

$$\times \sum_{\ell=0}^{\min(\alpha_1, \alpha_2)} \frac{(\lambda + \ell)_{\min(\alpha_1, \alpha_2) - \ell}}{(\alpha_1 - \ell)! (\alpha_2 - \ell)! \ell!} a_1^{\alpha_1 - \ell} a_2^{\alpha_2 - \ell} b_{1,2}^\ell$$

and

$$c_\alpha(P_2, \lambda) = [A_2(\mathbf{a}_1, \mathbf{a}_2)]^\lambda a_1^{\alpha_1} a_2^{\alpha_2} (\lambda)_{\max(\alpha_1, \alpha_2)} \quad (11)$$

$$\times \sum_{\ell=0}^{\min(\alpha_1, \alpha_2)} \frac{(\lambda + \ell)_{\min(\alpha_1, \alpha_2) - \ell}}{(\alpha_1 - \ell)! (\alpha_2 - \ell)! \ell!} a_1^{\alpha_1 - \ell} a_2^{\alpha_2 - \ell} b_{1,2}^\ell$$

5 Definition and construction of multifactor negative multinomial distributions

Bernardoff (2018) extended the definition of multivariate gamma distributions to *multifactor gamma distributions* by the following definition

Definition 16 We fix $\mathbf{\Lambda} = (\lambda_1, \dots, \lambda_n, \lambda)$ such that $\lambda_i \geq \lambda > 0$ for all $i = 1, \dots, n$, let P be an affine polynomial. If a random vector $\mathbf{X} = (X_1, \dots, X_n)$ on \mathbb{R}^n with pd $\mu_{\mathbf{X}}$ is such that its Laplace transform

$$L_{\mu_{\mathbf{X}}}(\theta) = [P(\theta)]^{-\lambda} \prod_{i=1}^n (1 + p_i \theta_i)^{-(\lambda_i - \lambda)}, \quad (12)$$

for a set of θ with non empty interior. We denote $\mu_{\mathbf{X}} = \gamma_{(P, \mathbf{\Lambda})}$, and $\gamma_{(P, \mathbf{\Lambda})}$ will be called the multifactor gamma distribution associated with $(P, \mathbf{\Lambda})$. We say also that $\gamma_{(P, \mathbf{\Lambda})}$ is the multifactor gamma distribution with scale parameter P and shape parameter $\mathbf{\Lambda}$.

We have $L_{\mu_{X_i}}(\theta_i) = (1 + p_i \theta_i)^{-\lambda_i}$, hence $X_i \sim \gamma_{(p_i, \lambda_i)}$. Bernardoff (2018) also gave the construction of $\gamma_{(P, \mathbf{\Lambda})}$ as follows. If $\mathbf{Y} = (Y_1, \dots, Y_n)$ is a vector of independent random variables such that $L_{\mu_{Y_i}}(\theta_i) = (1 + p_i \theta_i)^{-(\lambda_i - \lambda)}$, then if $\mathbf{Z} \sim \gamma_{(P, \lambda)}$ and \mathbf{Y} are independent, then $\mathbf{X} = \mathbf{Y} + \mathbf{Z} \sim \gamma_{(P, \mathbf{\Lambda})}$. This construction allows us to simulate $\gamma_{(P, \mathbf{\Lambda})}$ by simulating $\mathbf{Z} \sim \gamma_{(P, \lambda)}$ and \mathbf{Y} .

By analogy with the definition of multifactor gamma distributions, let us now define the multifactor negative multinomial distributions. This is a generalization of negative multinomial distributions.

Definition 17 We fix $\mathbf{\Lambda} = (\lambda_1, \dots, \lambda_n, \lambda)$ such that $\lambda_i \geq \lambda > 0$ for all $i = 1, \dots, n$, let P be an affine polynomial with respect to the variables z_1, \dots, z_n , so that $P(\mathbf{z}) = \sum_{S \in [n]} p_S \mathbf{z}^S$, and \mathbf{N} a random vector such that $\mu_{\mathbf{N}} = \mathcal{NM}_{(P, \lambda)}$. If a random vector $\mathbf{T} = (T_1, \dots, T_n)$ on \mathbb{N}^n with pd $\mu_{\mathbf{T}}$ is such that its pgf is

$$g_{\mathbf{T}}(\mathbf{z}) = \prod_{i=1}^n [P(1, \dots, 1, z_i, 1, \dots, 1)]^{-(\lambda_i - \lambda)} [P(\mathbf{z})]^{-\lambda}, \quad (13)$$

then we denote $\mu_{\mathbf{T}} = \mathcal{MNM}_{(P,\Lambda)}$, and $\mathcal{MNM}_{(P,\Lambda)}$ will be called the multifactor negative multinomial distribution associated with (P, Λ) .

Let us recall the definition of the negative binomial distribution on \mathbb{N} , of parameters $p, \lambda, p \in]0, 1[, \lambda > 0$ denoted by $\mathcal{NB}(p, \lambda)$: $\mathcal{NB}(p, \lambda) = \sum_{k \in \mathbb{N}} \frac{(\lambda)_k}{k!} p^\lambda (1-p)^k \delta_k$. Let $N \sim \mathcal{NB}(p, \lambda)$, then its pgf is $\mathbb{E}(z^N) = [1/p - q/pz]^{-\lambda}$, with $q = 1 - p$. We remark that $g_{T_i}(z_i) = [P(1, \dots, 1, z_i, 1, \dots, 1)]^{-\lambda_i} = (1/\mathbf{p}_i - (1 - \mathbf{p}_i)/\mathbf{p}_i z_i)^{-\lambda_i}$, with $P(1, \dots, 1, 0, 1, \dots, 1) = \sum_{S \subset [n] \setminus \{i\}} p_S = 1/\mathbf{p}_i$. This proves that $T_i \sim \mathcal{NB}(\mathbf{p}_i, \lambda_i)$. We also give the construction of $\mathbf{T} \sim \mathcal{MNM}_{(P,\Lambda)}$ in this way:

Proposition 18 *If $\mathbf{M} = (M_1, \dots, M_n)$ is a vector of independent random variables such that $\mathbb{E}(x_i^{M_i}) = \prod_{i=1}^n [P(1, \dots, 1, z_i, 1, \dots, 1)]^{-(\lambda_i - \lambda)}$, then if $\mathbf{N} \sim \mathcal{NM}_{(P,\lambda)}$ and \mathbf{M} are independent, then $\mathbf{T} = \mathbf{N} + \mathbf{M} \sim \mathcal{MNM}_{(P,\Lambda)}$. Clearly, we have $\mathcal{MNM}_{(P,\Lambda)} \sim \text{MMPD}(\gamma_{(Q,\Lambda)})$, with $Q(\mathbf{z}) = P(\mathbf{1} - \mathbf{z})$ and if we can simulate $\gamma_{(Q,\lambda)}$, we can simulate $\mathcal{MNM}_{(P,\Lambda)}$.*

6 Simulations

We now apply these results to the examples in Section 3 for $A_2(z_1, z_2) = 1 - z_1 - z_2 + 1/2 z_1 z_2$, $\mathbf{a} = (1/2, 1/2)$ and $\mathbf{a} = (1/2, 1/8)$. We respectively have $P_2(z_1, z_2) = 8 - 4z_1 - 4z_2 + z_1 z_2$ and $P_2(z_1, z_2) = 32/13 - 16/13 z_1 - 4/13 z_2 + 1/13 z_1 z_2$; $Q_2(z_1, z_2) = 1 + 3z_1 + 3z_2 + z_1 z_2$ and $Q_2(z_1, z_2) = 1 + 15/13 z_1 + 3/13 z_2 + 1/13 z_1 z_2$. Simulations are performed using R software.

The simulations for a sample of size 1,000 of $\gamma_{(Q_2, 2)}$ are illustrated by the graphical representations given in Figure 3.

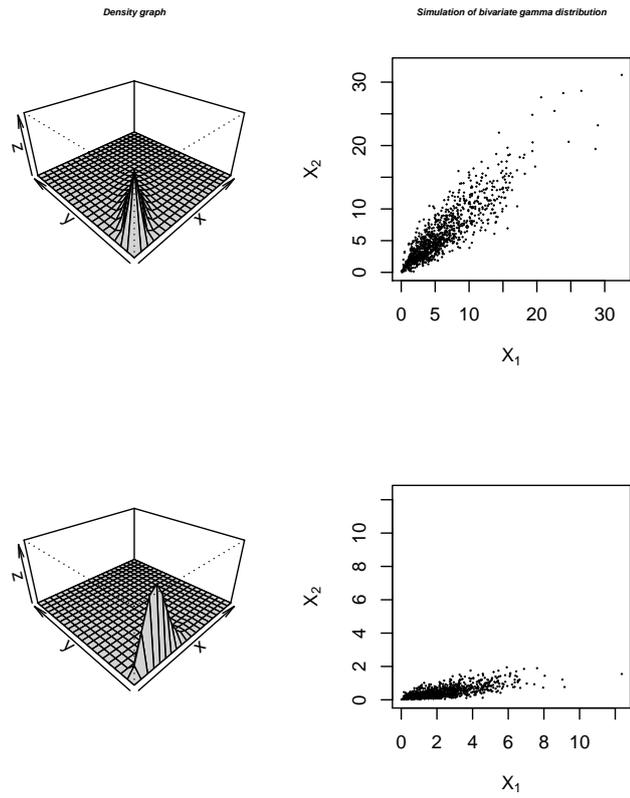


Figure 3 : Distribution and simulation

The simulations for a sample of size 1,000 of $\gamma_{(Q_2, (3,4,2))}$ are illustrated by the graphical representations given in Figure 4.

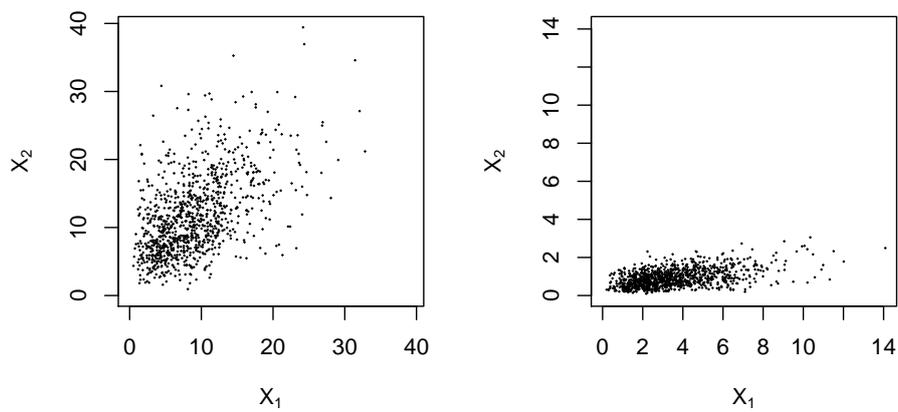


Figure 4 : Simulations of bifactor gamma distributions

The simulations for samples of size 100,000 of $\mathcal{NM}_{(P_2,2)}$ are presented in Tables 1 and 2.

Table 1	$A_2(z_1, z_2)$	$P_2(z_1, z_2)$	λ	\mathbf{a}	n
	$1 - z_1 - z_2 + 1/2z_1z_2$	$8 - 4z_1 - 4z_2 + z_1z_2$	2	(1/2, 1/2)	100000
$c_{(\alpha_1, \alpha_2)}(P_2)$			relative frequency		
$\alpha_1 \setminus \alpha_2$	0	1	$\alpha_1 \setminus \alpha_2$	0	1
0	0.015625	0.015625	0	0.01553	0.01631
1	0.015625	0.01953125	1	0.01517	0.01931

Table 2	$A_2(z_1, z_2)$	$P_2(z_1, z_2)$	λ	\mathbf{a}	n
	$1 - z_1 - z_2 + 1/2z_1z_2$	$\frac{32}{13} - \frac{16}{13}z_1 - \frac{4}{13}z_2 + \frac{1}{13}z_1z_2$	2	(1/2, 1/8)	100000
$c_{(\alpha_1, \alpha_2)}(P_2)$			relative frequency		
$\alpha_1 \setminus \alpha_2$	0	1	$\alpha_1 \setminus \alpha_2$	0	1
0	0.1650391	0.04125977	0	0.16552	0.04048
1	0.1650391	0.05157471	1	0.16407	0.05320

The simulations for samples of size 100,000 of $\mathcal{MM}_{(P_2, (3,4,2))}$ are presented in Tables 3 and 4.

Table 3	$P_2(z_1, z_2)$		Table 4	$P_2(z_1, z_2)$	
	$8 - 4z_1 - 4z_2 + z_1z_2$			$\frac{32}{13} - \frac{16}{13}z_1 - \frac{4}{13}z_2 + \frac{1}{13}z_1z_2$	
	relative frequency			relative frequency	
$\alpha_1 \setminus \alpha_2$	0	1	$\alpha_1 \setminus \alpha_2$	0	1
0	0.00021	0.00052	0	0.05157	0.03208
1	0.00040	0.00115	1	0.07845	0.05114

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