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Domain of existence of the Laplace transform of infinitely divisible negative multinomial distributions

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Abstract

This article provides the domain of existence Ω of the Laplace transform of infinitely divisible negative multinomial distributions. We define a negative multinomial distribution on \mathbb{N}^n , where \mathbb{N} is the set of nonnegative integers, by its probability generating function which will be of the form $(A(a_1 z_1, \dots, a_n z_n) / A(a_1, \dots, a_n))^{-\lambda}$ where $A(\mathbf{z}) = \sum_{T \subset \{1, 2, \dots, n\}} a_T \prod_{i \in T} z_i$, where $a_\emptyset \neq 0$, and where λ is a positive number. Finding couples (A, λ) for which we obtain a probability generating function is a difficult problem. Necessary and sufficient conditions on the coefficients a_T of A for which we obtain a probability generating function for any positive number λ are known by (Bernardoff, 2003). Thus we obtain necessary and sufficient conditions on $\mathbf{a} = (a_1, \dots, a_n)$ so that $\mathbf{a} = (\mathbf{e}^{t_1}, \dots, \mathbf{e}^{t_n})$ with $\mathbf{t} = (t_1, \dots, t_n)$ belonging to Ω . This makes it possible to construct all the infinitely divisible multinomial distributions on \mathbb{N}^n . We give examples of construction in dimensions 2 and 3.

KEY WORDS: discrete exponential families ; domain of existence ; infinitely divisible distribution ; natural exponential family ; Laplace transform ; negative multinomial distribution ; probability generating function.

1 Introduction

In this article, we consider the following definition, see references in Bernardoff (2003). We shall say that the probability distribution $\sum_{\alpha \in \mathbb{N}^n} p_\alpha \delta_\alpha$ on \mathbb{N}^n , where n is a non negative number, is a *negative multinomial distribution* if there exists an affine polynomial $P(z_1, \dots, z_n)$ and $\lambda > 0$ such that $P(0, \dots, 0) \neq 0$, and $P(1, \dots, 1) = 1$.

$$\sum_{\alpha \in \mathbb{N}^n} p_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} = (P(z_1, \dots, z_n))^{-\lambda}. \quad (1)$$

That means a polynomial which is affine with respect to each z_j , $j = 1, \dots, n$, or for which $\frac{\partial^2}{\partial z_j^2} P = 0$ for all $j = 1, \dots, n$. That is $P(z_1, \dots, z_n) = \sum_{T \in \mathfrak{P}_n} a_T \mathbf{z}^T$, where \mathfrak{P}_n is the set of the subset of $\{1, 2, \dots, n\} = [n]$, and where $\mathbf{z}^T = \prod_{t \in T} z_t$. if $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$. For instance, for $n = 2$, such as P has the form $P(z_1, z_2) = a_\emptyset + a_{\{1\}}z_1 + a_{\{2\}}z_2 + a_{\{1,2\}}z_1z_2$ with $a_\emptyset \neq 0$. However, finding exactly which pairs (P, λ) are compatible is an unsolved problem.

Before giving the main result, let us make an observation. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, then we denote

$$\mathbf{z}^\alpha = \prod_{i=1}^n z_i^{\alpha_i} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}. \quad (2)$$

Let A be any polynomial such that $A(0, \dots, 0) = 1$, and suppose that the Taylor expansion

$$(A(z_1, \dots, z_n))^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) \mathbf{z}^\alpha$$

has non-negative coefficients $c_\alpha(\lambda)$. Let a_1, \dots, a_n be positive numbers such that $\sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) a_1^{\alpha_1} \cdots a_n^{\alpha_n} < \infty$. With such a sequence $\mathbf{a} = (a_1, \dots, a_n)$ we associate the negative multinomial distribution $\sum_{\alpha \in \mathbb{N}^n} p_\alpha \delta_\alpha$ defined by

$$\sum_{\alpha \in \mathbb{N}^n} p_\alpha \mathbf{z}^\alpha = \left(\frac{A(a_1 z_1, \dots, a_n z_n)}{A(a_1, \dots, a_n)} \right)^{-\lambda}, \quad (3)$$

thus

$$P(z_1, \dots, z_n) = \frac{A(a_1 z_1, \dots, a_n z_n)}{A(a_1, \dots, a_n)} \quad (4)$$

in the notation (1).

Bernardoff (2003) define the polynomials b_T by

Definition 1 Let $P(\mathbf{z}) = \sum_{T \in \mathfrak{P}_n} a_T \mathbf{z}^T$ be an affine polynomial $P(z_1, \dots, z_n)$ such that $P(0, \dots, 0) = 0$, and $A = 1 - P$. Let T be in \mathfrak{P}_n^* the set of the nonempty subset of $[n]$ let us denote by b_T the number defined by

$$b_T = \left. \frac{\partial^{|T|}}{\partial z^T} (\log(1 - P)) \right|_{\mathbf{0}},$$

where then $|T|$ is the cardinal of T and $\partial z^T = \prod_{t \in T} \partial z_t$, then

$$b_T = \sum_{l=1}^{|T|} (l-1)! \sum_{\mathcal{T} \in \Pi_l^T} a_{\mathcal{T}} \quad (5)$$

where Π_T is the set of the partition of T , and Π_T^l is the set of the partition of length l of T (if $\mathcal{T} = \{T_1, T_2, \dots, T_l\}$, the partition \mathcal{T} of T is of length l).

For instance, for $n = 3$, $b_{\{1\}} = a_{\{1\}}$, $b_{\{1,2\}} = a_{\{1,2\}} + a_{\{1\}}a_{\{2\}}$ and $b_{\{1,2,3\}} = a_{\{1,2,3\}} + a_{\{1\}}a_{\{2,3\}} + a_{\{2\}}a_{\{1,3\}} + a_{\{3\}}a_{\{1,2\}} + 2a_{\{1\}}a_{\{2\}}a_{\{3\}}$. Now, if there is no ambiguity, for simplicity we omit the braces.

Using the numbers b_T , Bernardoff (2003) proves the following theorem.

Theorem 2 Let $P(z) = \sum_{T \in \mathfrak{P}_n^*} a_T z^T$, as before, and suppose that $(1 - P(z))^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) z^\alpha$. Then $c_\alpha(\lambda) \geq 0$ for all positive λ if and only if b_T , given by (5), is non negative for all $T \in \mathfrak{P}_n^*$.

See examples in dimension $n = 2, 3$ in Bernardoff (2003).

This article is organized as follows. Section 2 gives the main result. Section 3 applies the main result to bivariate and trivariate cases.

2 Domain of existence of the Laplace transform

Let A be an affine polynomial on \mathbb{R}^n and let $\lambda > 0$ such that $A(0, \dots, 0) = 1$ and such that $(A(z_1, \dots, z_n))^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) z^\alpha$ satisfies $c_\alpha(\lambda) > 0$ for all α in \mathbb{N}^n . Consider the discrete measure on \mathbb{N}^n , $\mu_\lambda = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) \delta_\alpha$.

The present section aims to describe the convex set

$$D(\mu_\lambda) = \left\{ \boldsymbol{\theta} \in \mathbb{R}^n, \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) e^{\theta_1 \alpha_1 + \dots + \theta_n \alpha_n} < +\infty \right\},$$

which is an important object in order to study the natural exponential family generated by μ_λ (see Letac, 1991 and Bar-Lev *et al.*, 1994). The answer is contained in the following Proposition.

Theorem 3 With the above notation, denote $H = \{\mathbf{s} \in \mathbb{R}^n, s_1 + \dots + s_n = 0\}$. For $\mathbf{s} \in H$, we denote by $R_{\mathbf{s}}$ the smallest positive zero of the polynomial $P_{\mathbf{s}}(t) = A(te^{s_1}, \dots, te^{s_n})$. Then the map $\mathbf{s} \mapsto \mathbf{s} + \log R_{\mathbf{s}}(1, \dots, 1)$ is a parametrization by H of a hypersurface in \mathbb{R}^n which is the boundary of $D(\mu_\lambda)$. More specifically if $\boldsymbol{\theta}$ is in \mathbb{R}^n and if $\bar{\theta}_n = \frac{1}{n}(\theta_1 + \dots + \theta_n)$ and if $\mathbf{s} = \boldsymbol{\theta} - \bar{\theta}_n(1, \dots, 1)$, then $\boldsymbol{\theta}$ is in $D(\mu_\lambda)$ if and only if $\bar{\theta}_n < \log R_{\mathbf{s}}$.

Finally $D(\mu_\lambda)$ is an open set.

Proof We first prove that the radius of convergence R of the power series

$$P_{\mathbf{s}}^{-\lambda}(t) = \sum_{n \in \mathbb{N}} u_n(\lambda) t^n \tag{6}$$

is equal to $R_{\mathbf{s}}$. This comes from the following fact: since $u_n(\lambda) \geq 0$, a known result in theory of

analytic functions (see Titchmarsh (1939) **7.21**) implies that $t \mapsto P_{\mathbf{s}}^{-\lambda}(t)$ is not analytic at R . Since $P_{\mathbf{s}}(0) = 1$, $P_{\mathbf{s}}(t) > 0$ for $0 < t < R_{\mathbf{s}}$ and $P_{\mathbf{s}}(R_{\mathbf{s}}) = 0$ clearly $R = R_{\mathbf{s}}$.

We now observe that if $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ is such that

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} c_{\boldsymbol{\alpha}}(\lambda) \mathbf{e}^{\alpha_1 \theta_1 + \dots + \alpha_n \theta_n} < +\infty$$

then for all $p \geq 0$ we have

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} c_{\boldsymbol{\alpha}}(\lambda) \mathbf{e}^{\alpha_1(\theta_1 - p) + \dots + \alpha_n(\theta_n - p)} < +\infty.$$

We now fix $\boldsymbol{\theta}$ in $D(\mu_{\lambda})$. Write $\bar{\theta}_n = (\theta_1 + \dots + \theta_n)/n$. The orthogonal projection of $\boldsymbol{\theta}$ on H is $\mathbf{s} = \boldsymbol{\theta} - \bar{\theta}_n(1, \dots, 1) = (s_1, \dots, s_n)$. Thus for all $j = 1, \dots, n$ we have $\theta_j - s_j = \bar{\theta}_n$. We claim that $\bar{\theta}_n < \log R_{\mathbf{s}}$. If not, we have $t_0 = e^{\bar{\theta}_n} \geq R_{\mathbf{s}}$. But $A(\mathbf{e}^{\theta_1}, \dots, \mathbf{e}^{\theta_n})$ is $P_{\mathbf{s}}(t_0)$ and the previous remark shows that for all $p \geq 0$, $p \mapsto P_{\mathbf{s}}(\mathbf{e}^{-p} t_0)$ is always positive. This contradicts the fact that $t_0 \geq R_{\mathbf{s}}$. Conversely if $\boldsymbol{\theta}$ is such that $\bar{\theta}_n < \log R_{\mathbf{s}}$ with $\mathbf{s} = \boldsymbol{\theta} - \bar{\theta}_n(1, \dots, 1)$ a similar reasoning shows that $\boldsymbol{\theta} \in D(\mu_{\lambda})$.

Finally for $t = R_{\mathbf{s}}$ in (6) the series diverges. A short proof goes as follows:

Write $P_{\mathbf{s}}(t) = \left(1 - \frac{t}{r_0}\right) \left(1 - \frac{t}{r_1}\right) \dots \left(1 - \frac{t}{r_k}\right)$ where $|r_j| \geq r_0 = R_{\mathbf{s}}$ by definition of $R_{\mathbf{s}}$.

Thus $\sum_{n \in \mathbb{N}} u_n(\lambda) t^n$ is the product of Newton Series $\sum_{n \in \mathbb{N}} \frac{1}{n!} \langle \lambda \rangle_n \left(\frac{t}{r_k}\right)^n$ and the series corresponding to $k = 0$ diverges for $t = r_0$. Thus $\sum_{n \in \mathbb{N}} u_n(\lambda) r_0^n = +\infty$. ■

Remark 4 *With the notations of Theorem 3, if $\boldsymbol{\theta}$ is in the boundary of $D(\mu_{\lambda})$, $\exists \mathbf{s} \in H, \boldsymbol{\theta} = \mathbf{s} + \log R_{\mathbf{s}}(1, \dots, 1)$, then $z_i = e^{\theta_i} = R_{\mathbf{s}} e^{s_i}$ for $i \in [n]$, and*

$$A(z_1, \dots, z_n) = A(R_{\mathbf{s}} \mathbf{e}^{s_1}, \dots, R_{\mathbf{s}} \mathbf{e}^{s_n}) = 0,$$

by the definition of $R_{\mathbf{s}}$.

3 Examples in dimensions 2 and 3

Example 1. For $n = 2$, we take $a_1 = 1$, $a_2 = 1$ and $a_{1,2} = a \geq -1$ so that the conditions of Theorem 2 are satisfied. Hence $A(z_1, z_2) = 1 - z_1 - z_2 - a z_1 z_2$. For $z_1 = t \mathbf{e}^{s_1}$, $z_2 = t \mathbf{e}^{s_2}$, with $s_1 + s_2 = 0$ we have $A(z_1, z_2) = 1 - t(\mathbf{e}^{s_1} + \mathbf{e}^{-s_1}) - a t^2$. If $s = (s_1, -s_1)$, $R_{\mathbf{s}} = \frac{1}{a} \left(-\cosh s_1 + \sqrt{\frac{1}{2} + a + \frac{1}{2} \cosh 2s_1} \right)$, we obtain the parametrization of the boundary of $D(u_{\lambda})$:

$$\begin{cases} x = \theta_1 = s_1 + \log \frac{1}{a} \left(-\cosh s_1 + \sqrt{\frac{1}{2} + a + \frac{1}{2} \cosh 2s_1} \right) \\ y = \theta_2 = -s_1 + \log \frac{1}{a} \left(-\cosh s_1 + \sqrt{\frac{1}{2} + a + \frac{1}{2} \cosh 2s_1} \right) \end{cases},$$

whose graphic representation is

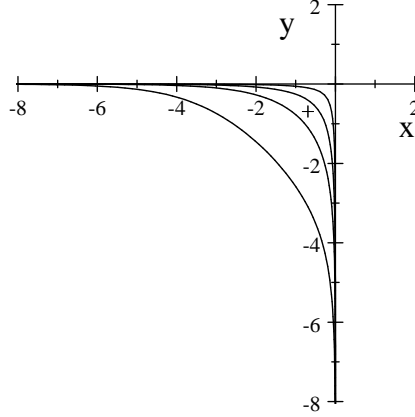


figure 1 : the boundary of $D(u_\lambda)$ for $a = -\frac{9}{10}, -\frac{1}{2}, 1, 20$.

Remark 5 Using the Remark 4 we obtain an other parametrization of the boundary of $D(u_\lambda)$:

$$\theta_1 < 0, \theta_2 = -\log(1 + (a + 1) / (e^{-\theta_1} - 1)).$$

In addition, because $D(\mu_\lambda)$ is a convex set, $\theta = (\theta_1, \theta_2) \in D(\mu_\lambda)$ is defined by

$$\theta_1 < 0, \theta_2 < -\log(1 + (a + 1) / (e^{-\theta_1} - 1)). \quad (7)$$

In this case, for $a = -1/2$, $R_s = 2 \cosh s_1 - \sqrt{2 \cosh 2s_1}$ and if we choose $s_1 = 0$, the condition $\bar{\theta}_2 < \log R_s$ becomes $\bar{\theta}_2 < -\log(1 + \sqrt{2}/2)$. As $\bar{\theta}_2 = -\log 2 < -\log(1 + \sqrt{2}/2)$, then $(-\log 2, -\log 2) \in D(\mu_\lambda)$. Then, the introduction proves that $(A(\frac{1}{2}z_1, \frac{1}{2}z_2) / A(\frac{1}{2}, \frac{1}{2}))^{-\lambda} = (8 - 4z_1 - 4z_2 + z_1z_2)^{-\lambda}$ is a generating function for all $\lambda > 0$.

Remark 6 For $a = -1/2$, the condition 7 gives for $\theta_1 = -\log 2$, $\theta_2 < -\log(3/2)$, and $\theta_2 = -\log 2$ is suitable. Hence $(-\log 2, -\log 2) \in D(\mu_\lambda)$.

Again with $a = -1/2$, if we choose $s_1 = \log 2$, $R_s = \frac{5}{2} - \frac{1}{2}\sqrt{17}$, then $\bar{\theta}_2 = -2\log 2 < \log R_s$ and $\theta = \mathbf{s} + \bar{\theta}_2(1, 1) = (-\log 2, -3\log 2)$. Then, the introduction proves that $(A(\frac{1}{2}z_1, \frac{1}{8}z_2) / A(\frac{1}{2}, \frac{1}{8}))^{-\lambda} = (\frac{32}{13} - \frac{16}{13}z_1 - \frac{4}{13}z_2 + \frac{1}{13}z_1z_2)^{-\lambda}$ is a generating function for all $\lambda > 0$.

Remark 7 For $a = -1/2$, the condition 7 gives for $\theta_1 = -\log 2$, $\theta_2 < -\log(3/2)$, and $\theta_2 = -3\log 2$ is suitable. Hence $(-\log 2, -3\log 2) \in D(\mu_\lambda)$. This method is easier to use.

Example 2. For $n = 3$, the conditions of Theorem 2 are for $i, j = 1, 2, 3$:

$$b_i = a_i \geq 0 ; a_{ij} \geq -a_i a_j ; a_{123} \geq -(a_1 a_{23} + a_2 a_{13} + a_3 a_{12} + 2a_1 a_2 a_3)$$

We take $a_1 = a_2 = a_3 = 1$, $a_{12} = a_{13} = a_{23} = a$ and $a_{123} = b$, so that

$$A(z) = 1 - ((z_1 + z_2 + z_3) + a(z_1z_2 + z_1z_3 + z_2z_3) + bz_1z_2z_3).$$

The conditions of Theorem 2 are satisfied for $a \geq -1$ and $b \geq -3a - 2$. We take $a = 1$ and $b = 0$, hence

$$A(z) = 1 - z_1 - z_2 - z_3 - z_1z_2 - z_1z_3 - z_2z_3.$$

Let $z_1 = te^{s_1}$, $z_2 = te^{s_2}$ and $z_3 = te^{s_3}$, with $s_1 + s_2 + s_3 = 0$, then $P_s(t) = 1 - (e^{s_1} + e^{s_2} + e^{-s_1-s_2})t - (e^{-s_1} + e^{-s_2} + e^{s_1+s_2})t^2 = 0$, and we have

$$R_s = \frac{-e^{s_1} - e^{s_2} - e^{-s_1-s_2} + \sqrt{(e^{2s_1} + e^{2s_2} + 6e^{s_1+s_2} + e^{-2(s_1+s_2)} + 6e^{-s_1} + 6e^{-s_2})}}{2(e^{-s_1} + e^{-s_2} + e^{s_1+s_2})}.$$

Finally, the parametrization of the boundary of $D(u_\lambda)$ is:

$$\begin{cases} x = \theta_1 = s_1 + \log \frac{-e^{s_1} - e^{s_2} - e^{-s_1-s_2} + \sqrt{(e^{2s_1} + e^{2s_2} + 6e^{s_1+s_2} + e^{-2(s_1+s_2)} + 6e^{-s_1} + 6e^{-s_2})}}{2(e^{-s_1} + e^{-s_2} + e^{s_1+s_2})} \\ y = \theta_2 = s_2 + \log \frac{-e^{s_1} - e^{s_2} - e^{-s_1-s_2} + \sqrt{(e^{2s_1} + e^{2s_2} + 6e^{s_1+s_2} + e^{-2(s_1+s_2)} + 6e^{-s_1} + 6e^{-s_2})}}{2(e^{-s_1} + e^{-s_2} + e^{s_1+s_2})} \\ z = \theta_3 = -s_1 - s_2 + \log \frac{-e^{s_1} - e^{s_2} - e^{-s_1-s_2} + \sqrt{(e^{2s_1} + e^{2s_2} + 6e^{s_1+s_2} + e^{-2(s_1+s_2)} + 6e^{-s_1} + 6e^{-s_2})}}{2(e^{-s_1} + e^{-s_2} + e^{s_1+s_2})} \end{cases},$$

whose graphic representation is

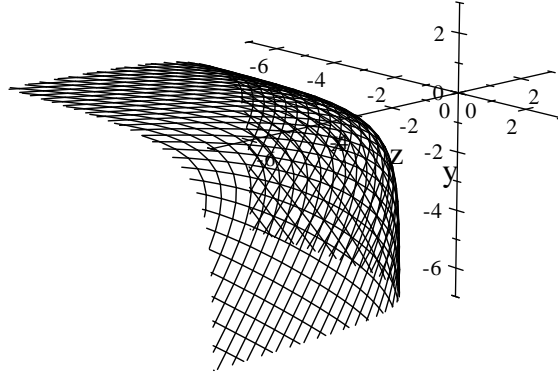


figure 2 : The boundary of $D(\mu_\lambda)$

If we choose $s_1 = s_2 = s_3 = 0$, the condition $\bar{\theta}_3 < \log R_s$ becomes $\bar{\theta}_3 < \log(\frac{1}{6}\sqrt{21} - \frac{1}{2})$. As $\bar{\theta}_3 = -\log 4 < \log(\frac{1}{6}\sqrt{21} - \frac{1}{2})$, then $(-\log 4, -\log 4, -\log 4) \in D(\mu_\lambda)$.

Then, the introduction proves that

$$\left(\frac{A(\frac{1}{4}z_1, \frac{1}{4}z_2, \frac{1}{4}z_3)}{A(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})} \right)^{-\lambda} = (16 - 4z_1 - 4z_2 - 4z_3 - z_1z_2 - z_1z_3 - z_2z_3)^{-\lambda}$$

is a generating function for all $\lambda > 0$.

Remark 8 In this case, using the Remark 4 we obtain that an other definition of $\theta = (\theta_1, \theta_2, \theta_3) \in$ the boundary of $D(u_\lambda)$ is :

$$\theta_1 < 0, \theta_2 < -\log\left(\frac{1 - e^{\theta_1}}{1 + e^{\theta_1}}\right), \theta_3 = \log\left(\frac{1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_1+\theta_2}}{1 + e^{\theta_1} + e^{\theta_2}}\right),$$

and an other parametrization of the boundary of $D(u_\lambda)$ is

$$\begin{cases} x = \theta_1 = u, u < 0, \\ y = \theta_2 = v - \log\left(\frac{1 - e^{\theta_1}}{1 + e^{\theta_1}}\right), v < 0 \\ z = \theta_3 = \log\left(\frac{1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_1 + \theta_2}}{1 + e^{\theta_1} + e^{\theta_2}}\right) \end{cases} ,$$

i.e.

$$\begin{cases} x = \theta_1 = u, u < 0, \\ y = \theta_2 = v - \log\left(\frac{1 - e^u}{1 + e^u}\right), v < 0 \\ z = \theta_3 = \log\left(\frac{(1 - e^{2u})(1 - e^v)}{1 + 2e^u + e^v - e^u e^v + e^{2u}}\right) \end{cases} .$$

In addition, because $D(\mu_\lambda)$ is a convex set, $\theta = (\theta_1, \theta_2, \theta_3) \in D(\mu_\lambda)$ is defined by

$$\theta_1 < 0, \theta_2 < -\log\left(\frac{1 + e^{\theta_1}}{1 - e^{\theta_1}}\right), \theta_3 < \log\left(\frac{1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_1 + \theta_2}}{1 + e^{\theta_1} + e^{\theta_2}}\right).$$

Hence $(-\log 4, -\log 4, -\log 4) \in D(\mu_\lambda)$ because $-\log 4 < 0, -\log 4 < -\log\left(\frac{1 + e^{-\log 4}}{1 - e^{-\log 4}}\right) = -\ln \frac{5}{3}$ and $-\log 4 < \log\left(\frac{1 - e^{-\log 4} - e^{-\log 4} - e^{-\log 4 - \log 4}}{1 + e^{-\log 4} + e^{-\log 4}}\right) = -\log(4 - 4/7)$.

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