



Domain of existence of the Laplace transform of infinitely divisible negative multinomial distributions

Philippe Bernardoff

► To cite this version:

Philippe Bernardoff. Domain of existence of the Laplace transform of infinitely divisible negative multinomial distributions. 2021. hal-03243831v1

HAL Id: hal-03243831

<https://hal.science/hal-03243831v1>

Preprint submitted on 31 May 2021 (v1), last revised 8 Jun 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Domain of existence of the Laplace transform of infinitely divisible negative multinomial distributions

Philippe Bernardoff

Université de Pau et des Pays de l'Adour

Laboratoire de Mathématiques et de leurs Applications de Pau,

UMR 5142, BP1155, 64013 PAU France

e.mail : philippe.bernardoff@univ-pau.fr

Mai 30, 2021

Abstract

This article provides the domain of existence Ω of the Laplace transform of infinitely divisible negative multinomial distributions. We define a negative multinomial distribution on \mathbb{N}^n , where \mathbb{N} is the set of nonnegative integers, by its probability generating function which will be of the form $(A(a_1 z_1, \dots, a_n z_n) / A(a_1, \dots, a_n))^{-\lambda}$ where $A(\mathbf{z}) = \sum_{T \subset \{1, 2, \dots, n\}} a_T \prod_{i \in T} z_i$, where $a_\emptyset \neq 0$, and where λ is a positive number. Finding couples (A, λ) for which we obtain a probability generating function is a difficult problem. Necessary and sufficient conditions on the coefficients a_T of A for which we obtain a probability generating function for any positive number λ are known by (Bernardoff, 2003). Thus we obtain necessary and sufficient conditions on $\mathbf{a} = (a_1, \dots, a_n)$ so that $\mathbf{a} = (\mathbf{e}^{t_1}, \dots, \mathbf{e}^{t_n})$ with $\mathbf{t} = (t_1, \dots, t_n)$ belonging to Ω . This makes it possible to construct all the infinitely divisible multinomial distributions on \mathbb{N}^n . We give examples of construction in dimensions 2 and 3.

KEY WORDS: discrete exponential families ; domain of existence ; infinitely divisible distribution ; natural exponential family ; Laplace transform ; negative multinomial distribution ; probability generating function.

1 Introduction

In this article, we consider the following definition, see references in Bernardoff (2003). We shall say that the probability distribution $\sum_{\alpha \in \mathbb{N}^n} p_\alpha \delta_\alpha$ on \mathbb{N}^n , where n is a non negative number, is a *negative multinomial distribution* if there exists an affine polynomial $P(z_1, \dots, z_n)$ and $\lambda > 0$ such that $P(0, \dots, 0) \neq 0$, and $P(1, \dots, 1) = 1$.

$$\sum_{\alpha \in \mathbb{N}^n} p_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} = (P(z_1, \dots, z_n))^{-\lambda}. \quad (1)$$

That means a polynomial which is affine with respect to each z_j , $j = 1, \dots, n$, or for which $\frac{\partial^2}{\partial z_j^2} P = 0$ for all $j = 1, \dots, n$. That is $P(z_1, \dots, z_n) = \sum_{T \in \mathfrak{P}_n} a_T \mathbf{z}^T$, where \mathfrak{P}_n is the set of the subset of $\{1, 2, \dots, n\} = [n]$, and where $\mathbf{z}^T = \prod_{t \in T} z_t$. if $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$. For instance, for $n = 2$, such as P has the form $P(z_1, z_2) = a_\emptyset + a_{\{1\}}z_1 + a_{\{2\}}z_2 + a_{\{1,2\}}z_1z_2$ with $a_\emptyset \neq 0$. However, finding exactly which pairs (P, λ) are compatible is an unsolved problem.

Before giving the main result, let us make an observation. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, then we denote

$$\mathbf{z}^\alpha = \prod_{i=1}^n z_i^{\alpha_i} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}. \quad (2)$$

Let A be any polynomial such that $A(0, \dots, 0) = 1$, and suppose that the Taylor expansion

$$(A(z_1, \dots, z_n))^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) \mathbf{z}^\alpha$$

has non-negative coefficients $c_\alpha(\lambda)$. Let a_1, \dots, a_n be positive numbers such that $\sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) a_1^{\alpha_1} \cdots a_n^{\alpha_n} < \infty$. With such a sequence $\mathbf{a} = (a_1, \dots, a_n)$ we associate the negative multinomial distribution $\sum_{\alpha \in \mathbb{N}^n} p_\alpha \delta_\alpha$ defined by

$$\sum_{\alpha \in \mathbb{N}^n} p_\alpha \mathbf{z}^\alpha = \left(\frac{A(a_1 z_1, \dots, a_n z_n)}{A(a_1, \dots, a_n)} \right)^{-\lambda}, \quad (3)$$

thus

$$P(z_1, \dots, z_n) = \frac{A(a_1 z_1, \dots, a_n z_n)}{A(a_1, \dots, a_n)} \quad (4)$$

in the notation (1).

Bernardoff (2003) define the polynomials b_T by

Definition 1 Let $P(\mathbf{z}) = \sum_{T \in \mathcal{P}_n} a_T \mathbf{z}^T$ be an affine polynomial $P(z_1, \dots, z_n)$ such that $P(0, \dots, 0) = 0$, and $A = 1 - P$. Let T be in \mathfrak{P}_n^* the set of the nonempty subset of $[n]$ let us denote by b_T the number defined by

$$b_T = \left. \frac{\partial^{|T|}}{\partial \mathbf{z}^T} (\log(1 - P)) \right|_{\mathbf{0}},$$

where then $|T|$ is the cardinal of T and $\partial \mathbf{z}^T = \prod_{t \in T} \partial z_t$, then

$$b_T = \sum_{l=1}^{|T|} (l-1)! \sum_{\tau \in \Pi_T^l} a_\tau \quad (5)$$

where Π_T is the set of the partition of T , and Π_T^l is the set of the partition of lenght l of T (if $\mathcal{T} = \{T_1, T_2, \dots, T_l\}$, the partition \mathcal{T} of T is of length l).

For instance, for $n = 3$, $b_{\{1\}} = a_{\{1\}}$, $b_{\{1,2\}} = a_{\{1,2\}} + a_{\{1\}}a_{\{2\}}$ and $b_{\{1,2,3\}} = a_{\{1,2,3\}} + a_{\{1\}}a_{\{2,3\}} + a_{\{2\}}a_{\{1,3\}} + a_{\{3\}}a_{\{1,2\}} + 2a_{\{1\}}a_{\{2\}}a_{\{3\}}$. Now, if there is no ambiguity, for simplicity we omit the braces.

Using the numbers b_T , Bernardoff (2003) proves the following theorem.

Theorem 2 Let $P(z) = \sum_{T \in \mathfrak{P}_n^*} a_T z^T$, as before, and suppose that $(1 - P(z))^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) z^\alpha$. Then $c_\alpha(\lambda) \geq 0$ for all positive λ if and only if b_T , given by (5), is non negative for all $T \in \mathfrak{P}_n^*$.

See examples in dimension $n = 2, 3$ in Bernardoff (2003).

This article is organized as follows. Section 2 gives the main result. Section 3 applies the main result to bivariate and trivariate cases.

2 Domain of existence of the Laplace transform

Let A be an affine polynomial on \mathbb{R}^n and let $\lambda > 0$ such that $A(0, \dots, 0) = 1$ and such that $(A(z_1, \dots, z_n))^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) z^\alpha$ satisfies $c_\alpha(\lambda) > 0$ for all α in \mathbb{N}^n . Consider the discrete measure on \mathbb{N}^n , $\mu_\lambda = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) \delta_\alpha$.

The present section aims to describe the convex set

$$D(\mu_\lambda) = \left\{ \theta \in \mathbb{R}^n, \sum_{\alpha \in \mathbb{N}^n} c_\alpha(\lambda) e^{\theta_1 \alpha_1 + \dots + \theta_n \alpha_n} < +\infty \right\},$$

which is an important object in order to study the natural exponential family generated by μ_λ (see Letac, 1991 and Bar-Lev *et al.*, 1994). The answer is contained in the following Proposition.

Theorem 3 With the above notation, denote $H = \{\mathbf{s} \in \mathbb{R}^n, s_1 + \dots + s_n = 0\}$. For $\mathbf{s} \in H$, we denote by $R_{\mathbf{s}}$ the smallest positive zero of the polynomial $P_{\mathbf{s}}(t) = A(te^{s_1}, \dots, te^{s_n})$. Then the map $\mathbf{s} \mapsto \mathbf{s} + \log R_{\mathbf{s}}(1, \dots, 1)$ is a parametrization by H of a hypersurface in \mathbb{R}^n which is the boundary of $D(\mu_\lambda)$. More specifically if θ is in \mathbb{R}^n and if $\bar{\theta}_n = \frac{1}{n}(\theta_1 + \dots + \theta_n)$ and if $\mathbf{s} = \theta - \bar{\theta}_n(1, \dots, 1)$, then θ is in $D(\mu_\lambda)$ if and only if $\bar{\theta}_n < \log R_{\mathbf{s}}$. Finally $D(\mu_\lambda)$ is an open set.

Proof We first prove that the radius of convergence R of the power series

$$P_{\mathbf{s}}^{-\lambda}(t) = \sum_{n \in \mathbb{N}} u_n(\lambda) t^n \quad (6)$$

is equal to $R_{\mathbf{s}}$. This comes from the following fact: since $u_n(\lambda) \geq 0$, a known result in theory of

analytic functions (see Titchmarsh (1939) **7.21**) implies that $t \mapsto P_{\mathbf{s}}^{-\lambda}(t)$ is not analytic at R . Since $P_{\mathbf{s}}(0) = 1$, $P_{\mathbf{s}}(t) > 0$ for $0 < t < R_{\mathbf{s}}$ and $P_{\mathbf{s}}(R_{\mathbf{s}}) = 0$ clearly $R = R_{\mathbf{s}}$.

We now observe that if $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ is such that

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} c_{\boldsymbol{\alpha}}(\lambda) \mathbf{e}^{\alpha_1 \theta_1 + \dots + \alpha_n \theta_n} < +\infty$$

then for all $p \geq 0$ we have

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} c_{\boldsymbol{\alpha}}(\lambda) \mathbf{e}^{\alpha_1(\theta_1 - p) + \dots + \alpha_n(\theta_n - p)} < +\infty.$$

We now fix $\boldsymbol{\theta}$ in $D(\mu_{\lambda})$. Write $\bar{\theta}_n = (\theta_1 + \dots + \theta_n)/n$. The orthogonal projection of $\boldsymbol{\theta}$ on H is $\mathbf{s} = \boldsymbol{\theta} - \bar{\theta}_n(1, \dots, 1) = (s_1, \dots, s_n)$. Thus for all $j = 1, \dots, n$ we have $\theta_j - s_j = \bar{\theta}_n$. We claim that $\bar{\theta}_n < \log R_{\mathbf{s}}$. If not, we have $t_0 = e^{\bar{\theta}_n} \geq R_{\mathbf{s}}$. But $A(\mathbf{e}^{\theta_1}, \dots, \mathbf{e}^{\theta_n})$ is $P_{\mathbf{s}}(t_0)$ and the previous remark shows that for all $p \geq 0$, $p \mapsto P_{\mathbf{s}}(\mathbf{e}^{-p} t_0)$ is always positive. This contradicts the fact that $t_0 \geq R_{\mathbf{s}}$. Conversely if $\boldsymbol{\theta}$ is such that $\bar{\theta}_n < \log R_{\mathbf{s}}$ with $\mathbf{s} = \boldsymbol{\theta} - \bar{\theta}_n(1, \dots, 1)$ a similar reasoning shows that $\boldsymbol{\theta} \in D(\mu_{\lambda})$.

Finally for $t = R_{\mathbf{s}}$ in (6) the series diverges. A short proof goes as follows:

Write $P_{\mathbf{s}}(t) = \left(1 - \frac{t}{r_0}\right) \left(1 - \frac{t}{r_1}\right) \dots \left(1 - \frac{t}{r_k}\right)$ where $|r_j| \geq r_0 = R_{\mathbf{s}}$ by definition of $R_{\mathbf{s}}$.

Thus $\sum_{n \in \mathbb{N}} u_n(\lambda) t^n$ is the product of Newton Series $\sum_{n \in \mathbb{N}} \frac{1}{n!} \langle \lambda \rangle_n \left(\frac{t}{r_k}\right)^n$ and the series corresponding to $k = 0$ diverges for $t = r_0$. Thus $\sum_{n \in \mathbb{N}} u_n(\lambda) r_0^n = +\infty$. ■

Remark 4 With the notations of Theorem 3, if $\boldsymbol{\theta}$ is in the boundary of $D(\mu_{\lambda})$, $\exists \mathbf{s} \in H, \boldsymbol{\theta} = \mathbf{s} + \log R_{\mathbf{s}}(1, \dots, 1)$, then $z_i = e^{\theta_i} = R_{\mathbf{s}} e^{s_i}$ for $i \in [n]$, and

$$A(z_1, \dots, z_n) = A(R_{\mathbf{s}} \mathbf{e}^{s_1}, \dots, R_{\mathbf{s}} \mathbf{e}^{s_n}) = 0,$$

by the definition of $R_{\mathbf{s}}$.

3 Examples in dimensions 2 and 3

Example 1. For $n = 2$, we take $a_1 = 1$, $a_2 = 1$ and $a_{1,2} = a \geq -1$ so that the conditions of Theorem 2 are satisfied. Hence $A(z_1, z_2) = 1 - z_1 - z_2 - a z_1 z_2$. For $z_1 = t e^{s_1}$, $z_2 = t e^{s_2}$, with $s_1 + s_2 = 0$ we have $A(z_1, z_2) = 1 - t(\mathbf{e}^{s_1} + \mathbf{e}^{-s_1}) - a t^2$. If $s = (s_1, -s_1)$, $R_{\mathbf{s}} = \frac{1}{a} \left(-\cosh s_1 + \sqrt{\frac{1}{2} + a + \frac{1}{2} \cosh 2s_1} \right)$, we obtain the parametrization of the boundary of $D(u_{\lambda})$:

$$\begin{cases} x = \theta_1 = s_1 + \log \frac{1}{a} \left(-\cosh s_1 + \sqrt{\frac{1}{2} + a + \frac{1}{2} \cosh 2s_1} \right) \\ y = \theta_2 = -s_1 + \log \frac{1}{a} \left(-\cosh s_1 + \sqrt{\frac{1}{2} + a + \frac{1}{2} \cosh 2s_1} \right) \end{cases},$$

whose graphic representation is

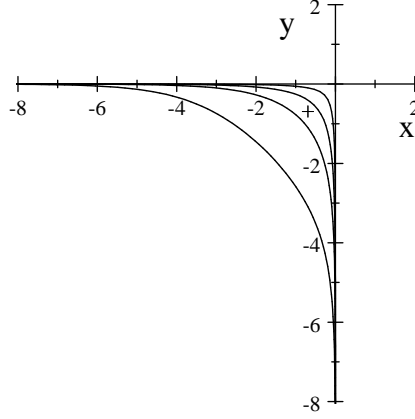


figure 1 : the boundary of $D(u_\lambda)$ for $a = -\frac{9}{10}, -\frac{1}{2}, 1, 20$.

Remark 5 Using the Remark 4 we obtain an other parametrization of the boundary of $D(u_\lambda)$:

$$\theta_1 < 0, \theta_2 = -\log(1 + (a+1) / (e^{-\theta_1} - 1)).$$

In addition, because $D(\mu_\lambda)$ is a convex set, $\theta = (\theta_1, \theta_2) \in D(\mu_\lambda)$ is defined by

$$\theta_1 < 0, \theta_2 < -\log(1 + (a+1) / (e^{-\theta_1} - 1)). \quad (7)$$

In this case, for $a = -1/2$, $R_s = 2 \cosh s_1 - \sqrt{2 \cosh 2s_1}$ and if we choose $s_1 = 0$, the condition $\bar{\theta}_2 < \log R_s$ becomes $\bar{\theta}_2 < -\log(1 + \sqrt{2}/2)$. As $\bar{\theta}_2 = -\log 2 < -\log(1 + \sqrt{2}/2)$, then $(-\log 2, -\log 2) \in D(\mu_\lambda)$. Then, the introduction proves that $(A(\frac{1}{2}z_1, \frac{1}{2}z_2) / A(\frac{1}{2}, \frac{1}{2}))^{-\lambda} = (8 - 4z_1 - 4z_2 + z_1z_2)^{-\lambda}$ is a generating function for all $\lambda > 0$.

Remark 6 For $a = -1/2$, the condition 7 gives for $\theta_1 = -\log 2$, $\theta_2 < -\log(3/2)$, and $\theta_2 = -\log 2$ is suitable. Hence $(-\log 2, -\log 2) \in D(\mu_\lambda)$.

Again with $a = -1/2$, if we choose $s_1 = \log 2$, $R_s = \frac{5}{2} - \frac{1}{2}\sqrt{17}$, then $\bar{\theta}_2 = -2\log 2 < \log R_s$ and $\theta = s + \bar{\theta}_2(1, 1) = (-\log 2, -3\log 2)$. Then, the introduction proves that $(A(\frac{1}{2}z_1, \frac{1}{8}z_2) / A(\frac{1}{2}, \frac{1}{8}))^{-\lambda} = (\frac{32}{13} - \frac{16}{13}z_1 - \frac{4}{13}z_2 + \frac{1}{13}z_1z_2)^{-\lambda}$ is a generating function for all $\lambda > 0$.

Remark 7 For $a = -1/2$, the condition 7 gives for $\theta_1 = -\log 2$, $\theta_2 < -\log(3/2)$, and $\theta_2 = -3\log 2$ is suitable. Hence $(-\log 2, -3\log 2) \in D(\mu_\lambda)$. This method is easier to use.

Example 2. For $n = 3$, the conditions of Theorem 2 are for $i, j = 1, 2, 3$:

$$b_i = a_i \geq 0 ; a_{ij} \geq -a_i a_j ; a_{123} \geq -(a_1 a_{23} + a_2 a_{13} + a_3 a_{12} + 2a_1 a_2 a_3)$$

We take $a_1 = a_2 = a_3 = 1$, $a_{12} = a_{13} = a_{23} = a$ and $a_{123} = b$, so that

$$A(z) = 1 - ((z_1 + z_2 + z_3) + a(z_1z_2 + z_1z_3 + z_2z_3) + bz_1z_2z_3).$$

The conditions of Theorem 2 are satisfied for $a \geq -1$ and $b \geq -3a - 2$. We take $a = 1$ and $b = 0$, hence

$$A(z) = 1 - z_1 - z_2 - z_3 - z_1 z_2 - z_1 z_3 - z_2 z_3.$$

Let $z_1 = te^{s_1}$, $z_2 = te^{s_2}$ and $z_3 = te^{s_3}$, with $s_1 + s_2 + s_3 = 0$, then $P_s(t) = 1 - (e^{s_1} + e^{s_2} + e^{-s_1-s_2})t - (e^{-s_1} + e^{-s_2} + e^{s_1+s_2})t^2 = 0$, and we have

$$R_s = \frac{-e^{s_1} - e^{s_2} - e^{-s_1-s_2} + \sqrt{(e^{2s_1} + e^{2s_2} + 6e^{s_1+s_2} + e^{-2(s_1+s_2)} + 6e^{-s_1} + 6e^{-s_2})}}{2(e^{-s_1} + e^{-s_2} + e^{s_1+s_2})}.$$

Finally, the parametrization of the boundary of $D(u_\lambda)$ is:

$$\begin{cases} x = \theta_1 = s_1 + \log \frac{-e^{s_1} - e^{s_2} - e^{-s_1-s_2} + \sqrt{(e^{2s_1} + e^{2s_2} + 6e^{s_1+s_2} + e^{-2(s_1+s_2)} + 6e^{-s_1} + 6e^{-s_2})}}{2(e^{-s_1} + e^{-s_2} + e^{s_1+s_2})} \\ y = \theta_2 = s_2 + \log \frac{-e^{s_1} - e^{s_2} - e^{-s_1-s_2} + \sqrt{(e^{2s_1} + e^{2s_2} + 6e^{s_1+s_2} + e^{-2(s_1+s_2)} + 6e^{-s_1} + 6e^{-s_2})}}{2(e^{-s_1} + e^{-s_2} + e^{s_1+s_2})} \\ z = \theta_3 = -s_1 - s_2 + \log \frac{-e^{s_1} - e^{s_2} - e^{-s_1-s_2} + \sqrt{(e^{2s_1} + e^{2s_2} + 6e^{s_1+s_2} + e^{-2(s_1+s_2)} + 6e^{-s_1} + 6e^{-s_2})}}{2(e^{-s_1} + e^{-s_2} + e^{s_1+s_2})} \end{cases},$$

whose graphic representation is

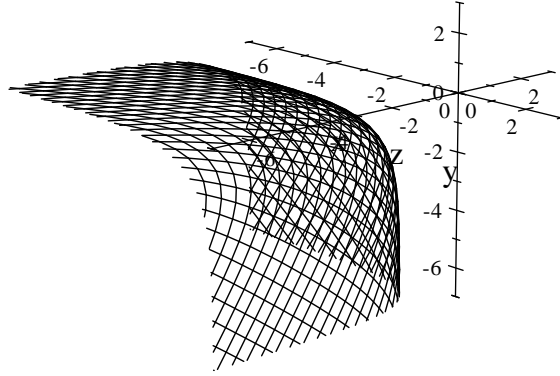


figure 2 : The boundary of $D(\mu_\lambda)$

If we choose $s_1 = s_2 = s_3 = 0$, the condition $\bar{\theta}_3 < \log R_s$ becomes $\bar{\theta}_3 < \log(\frac{1}{6}\sqrt{21} - \frac{1}{2})$. As $\bar{\theta}_3 = -\log 4 < \log(\frac{1}{6}\sqrt{21} - \frac{1}{2})$, then $(-\log 4, -\log 4, -\log 4) \in D(\mu_\lambda)$.

Then, the introduction proves that

$$\left(\frac{A(\frac{1}{4}z_1, \frac{1}{4}z_2, \frac{1}{4}z_3)}{A(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})} \right)^{-\lambda} = (16 - 4z_1 - 4z_2 - 4z_3 - z_1 z_2 - z_1 z_3 - z_2 z_3)^{-\lambda}$$

is a generating function for all $\lambda > 0$.

Remark 8 In this case, using the Remark 4 we obtain that an other definition of $\theta = (\theta_1, \theta_2, \theta_3) \in$ the boundary of $D(u_\lambda)$ is :

$$\theta_1 < 0, \theta_2 < -\log\left(\frac{1 - e^{\theta_1}}{1 + e^{\theta_1}}\right), \theta_3 = \log\left(\frac{1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_1+\theta_2}}{1 + e^{\theta_1} + e^{\theta_2}}\right),$$

and an other parametrization of the boundary of $D(u_\lambda)$ is

$$\begin{cases} x = \theta_1 = u, u < 0, \\ y = \theta_2 = v - \log\left(\frac{1 - e^{\theta_1}}{1 + e^{\theta_1}}\right), v < 0 \\ z = \theta_3 = \log\left(\frac{1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_1 + \theta_2}}{1 + e^{\theta_1} + e^{\theta_2}}\right) \end{cases},$$

i.e.

$$\begin{cases} x = \theta_1 = u, u < 0, \\ y = \theta_2 = v - \log\left(\frac{1 - e^u}{1 + e^u}\right), v < 0 \\ z = \theta_3 = \log\left(\frac{(1 - e^{2u})(1 - e^v)}{1 + 2e^u + e^v - e^u e^v + e^{2u}}\right) \end{cases}.$$

In addition, because $D(\mu_\lambda)$ is a convex set, $\theta = (\theta_1, \theta_2, \theta_3) \in D(\mu_\lambda)$ is defined by

$$\theta_1 < 0, \theta_2 < -\log\left(\frac{1 + e^{\theta_1}}{1 - e^{\theta_1}}\right), \theta_3 < \log\left(\frac{1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_1 + \theta_2}}{1 + e^{\theta_1} + e^{\theta_2}}\right).$$

Hence $(-\log 4, -\log 4, -\log 4) \in D(\mu_\lambda)$ because $-\log 4 < 0, -\log 4 < -\log\left(\frac{1 + e^{-\log 4}}{1 - e^{-\log 4}}\right) = -\ln \frac{5}{3}$ and $-\log 4 < \log\left(\frac{1 - e^{-\log 4} - e^{-\log 4} - e^{-\log 4 - \log 4}}{1 + e^{-\log 4} + e^{-\log 4}}\right) = -\log(4 - 4/7)$.

4 Acknowledgment

I thank Gérard Letac and Evelyne Bernadac for many helpful conversations.

References

- [1] Bar-Lev, S.K., Bshouty, D., Enis, P., Letac, G., Li Lu, I. and Richards, D. (1994). The Diagonal Multivariate Natural Exponential Families and Their Classification, *Journal of Theoretical Probability*, **7**, 883–929.
- [2] Bernardoff, P. (2003) Which negative multinomial distributions are infinitely divisible? *Bernoulli*, **9**, 877–893.
- [3] Letac, G. (1991). *Lectures on Natural Exponential Families and their Variance Functions*, Instituto de Matemática Pura e Aplicada, Rio de Janeiro.
- [4] Titchmarsh, E.C. (1939). *The Theory of Functions*, (2nd ed.), Oxford University Press.