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Universality for random permutations and some other groups[☆]

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Abstract

We present a Markovian approach to prove universality results for general statistics on the symmetric group. We prove, in particular, that the number of occurrences of a vincular pattern satisfies a CLT for conjugation invariant random permutations with few cycles and we improve the results already known for the longest increasing subsequence. We give also a generalization to other random permutations and other sets having a similar structure to the symmetric group.

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1. The ping-pong method

In probability, there are some limiting laws (Tracy–Widom, Airy point process, sine point process, semi-circular law, Gaussian distribution, etc.) that appear as a limit of different statistics with no obvious link between them. This phenomenon is known as universality. Let \mathfrak{S}_n be the group of permutations of $[n]$. For $\sigma \in \mathfrak{S}_n$, we denote by $\#(\sigma)$ the number of cycles of σ . In this section, we are interested in proving universality for conjugation invariant random permutations with few cycles. We give direct applications to known statistics in the second section. In the third section, we push the technique to improve some known results. In particular, we obtain a new lower bound for the expected value of the length of the longest increasing subsequence of a conjugation invariant permutation. In the fourth section, we generalize this result to other sets, including colored permutations and other distributions, including Mallows laws.

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A sequence of random permutations $(\sigma_n)_{n \geq 1}$ is said to be conjugation invariant if σ_n is supported on \mathfrak{S}_n and

$$\forall n \geq 1, \forall \sigma \in \mathfrak{S}_n, \sigma_n \stackrel{d}{=} \sigma^{-1} \sigma_n \sigma. \tag{H_{inv}}$$

For example, the uniform law on any conjugacy class is conjugation invariant. Many examples of conjugation invariant random permutation have been studied in the literature we refer for example to the Ewens permutations and their generalizations (the generalized Ewens permutations, the Ewens–Pitman permutations, the Kingman virtual permutations etc.). We refer to the [Appendix](#) for a brief introduction to Ewens permutations.

For $\alpha \geq 1$ and $p \in [1, \infty]$, we say that the sequence of random permutations $(\sigma_n)_{n \geq 1}$ satisfies $\mathcal{H}_{inv,\alpha}^{\mathbb{P}}$ if

$$(\sigma_n)_{n \geq 1} \text{ is conjugation invariant and } \frac{\#(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0, \tag{H_{inv,\alpha}^{\mathbb{P}}}$$

and we say that it satisfies $\mathcal{H}_{inv,\alpha}^{\mathbb{L}^p}$ if

$$(\sigma_n)_{n \geq 1} \text{ is conjugation invariant and } \frac{\#(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} 0. \tag{H_{inv,\alpha}^{\mathbb{L}^p}}$$

1.1. Rebound on the Ewens zero distribution

Given $n \geq 1$ and $E \subset \mathfrak{S}_n$, we define

$$\text{next}(E) := \{\rho \circ (i, j); \rho \in E, \#(\rho \circ (i, j)) = \#(\rho) - 1\} \cup \{\rho \in E; \#(\rho) = 1\}$$

and

$$\text{final}(\sigma) := \begin{cases} \text{next}^{\#(\sigma)-1}(\{\sigma\}) & \text{if } \#(\sigma) > 1 \\ \{\sigma\} & \text{otherwise.} \end{cases}$$

In other words, $\text{next}(E)$ is the set of permutations obtained by concatenating, if possible, two cycles of some $\sigma \in E$, and $\text{final}(\sigma)$ is the set of permutations obtained by concatenating all the cycles of σ . In particular,

$$\text{final}(\sigma) \subset \mathfrak{S}_n^0 := \{\sigma \in \mathfrak{S}_n; \#(\sigma) = 1\}.$$

Let $\mathcal{G}_{\mathfrak{S}_n}$ be the directed graph with vertices \mathfrak{S}_n and edges $\{(\sigma, \rho); \sigma \in \mathfrak{S}_n, \rho \in \text{next}(\{\sigma\})\}$. We represent $\mathcal{G}_{\mathfrak{S}_3}$ in [Fig. 1](#). $\mathcal{G}_{\mathfrak{S}_n}$ can be seen as a directed version of the Cayley graph of \mathfrak{S}_n generated by transpositions where the edges are oriented toward the permutations with fewer cycles (the further from the identity according to the graph distance), for which we added loops at the permutations of \mathfrak{S}_n^0 .

Let f be a function defined on $\mathfrak{S}_\infty := \cup_{i=1}^\infty \mathfrak{S}_i$ and taking its values in some metric space (F, d_F) , for example $\mathbb{Z}, \mathbb{R}, \mathbb{R}^d$ or $\mathcal{C}^0(\mathbb{R})$. It turns out that the uniform distribution on \mathfrak{S}_n^0 , also known as the Ewens distribution with parameter 0 (See the [Appendix](#) for more details.), is useful to obtain universality results for conjugation invariant permutations if f does not change too much by merging two cycles. More precisely, we define for $1 \leq k \leq n$,

$$\varepsilon'_{n,k}(f) := \max_{\sigma \in \mathfrak{S}_n, \#(\sigma)=k} \max_{\rho \in \text{final}(\sigma)} d_F(f(\sigma), f(\rho)).$$

We present now our main result.

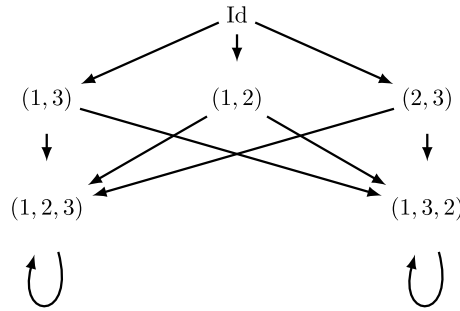


Fig. 1. The directed graph $\mathcal{G}_{\mathfrak{S}_3}$.

Theorem 1. Assume that $(\sigma_n)_{n \geq 1}$ and $(\sigma_{ref,n})_{n \geq 1}$ satisfy (\mathcal{H}_{inv}) . Suppose that there exists $x \in F$ such that

$$f(\sigma_{ref,n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x, \tag{1}$$

$$\varepsilon'_{n, \#(\sigma_{ref,n})}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \tag{2}$$

and that $\varepsilon'_{n, \#(\sigma_n)}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \tag{3}$

Then

$$f(\sigma_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x. \tag{4}$$

Moreover, if the assumptions (1)–(3) hold true for the \mathbb{L}^p convergence for some $p \geq 1$ instead of the convergence in probability, then so does (4).

When $F = \mathbb{R}^d$, we obtain also the convergence in distribution.

Theorem 2. Assume that $F = \mathbb{R}^d$ and that $(\sigma_n)_{n \geq 1}$ and $(\sigma_{ref,n})_{n \geq 1}$ satisfy (\mathcal{H}_{inv}) . Suppose that (2) and (3) hold true and that there exists a random variable X supported on F such that $f(\sigma_{ref,n}) \xrightarrow[n \rightarrow \infty]{d} X$. Then

$$f(\sigma_n) \xrightarrow[n \rightarrow \infty]{d} X. \tag{5}$$

Let $\sigma_{unif,n}$ and $\sigma_{Ew,0,n}$ be uniform random permutations respectively on \mathfrak{S}_n and \mathfrak{S}_n^0 . The idea of the proof is to compare both $f(\sigma_n)$ and $f(\sigma_{ref,n})$ with $f(\sigma_{Ew,0,n})$. In general, the choice $\sigma_{ref,n} \stackrel{d}{=} \sigma_{unif,n}$ is interesting since the convergence in (1) is known for many statistics. We chose to give a very simple version that can be checked easily for many statistics. For almost sure convergence, one can obtain similar results after defining the spaces properly. We will not discuss here this type of convergence. We will give many applications using the following observation.

Remark 3. By the triangle inequality, we have

$$\varepsilon'_{n,k}(f) \leq \sum_{i=2}^k \varepsilon_{n,i}(f) \leq (k - 1)\varepsilon_n(f),$$

where

$$\varepsilon_{n,k}(f) := \max_{\sigma \in \mathfrak{S}_n, \#(\sigma)=k} \max_{\rho \in \text{next}(\{\sigma\})} d_F(f(\sigma), f(\rho)) \quad \text{and} \quad \varepsilon_n(f) := \max_{1 \leq k < n} \varepsilon_{n,k}(f).$$

Consequently, if there exists some $\alpha \leq 1$ such that $\varepsilon_n(f) = O\left(n^{-\frac{1}{\alpha}}\right)$ then $(\mathcal{H}_{inv,\alpha}^{\mathbb{P}})$ implies (3) and $(\mathcal{H}_{inv,\alpha}^{\mathbb{L}^p})$ implies the equivalent hypothesis in \mathbb{L}^p . Moreover, if $\sigma_{ref,n} \stackrel{d}{=} \sigma_{unif,n}$, then Proposition 36 implies (2) and the \mathbb{L}^p version of (2) for every p . We will give some direct applications of this observation in the next subsection.

1.2. Some applications

In the next corollary, we will give some applications. The second column of Table 1 contains the function to study. We apologize to the reader because those statistics are not defined yet. One can check the corresponding result in the sixth column for more details.

Corollary 4. For the functions f the distribution X and the real α in Table 1, if $(\mathcal{H}_{inv,\alpha}^{\mathbb{P}})$ is satisfied, then

$$f(\sigma_n) \xrightarrow[n \rightarrow \infty]{d} X$$

except for the sixth example where the convergence holds in probability.¹ For the first and the fourth examples, the convergence holds also in \mathbb{L}^p under $(\mathcal{H}_{inv,\alpha}^{\mathbb{L}^p})$. For the fifth example please check the corresponding theorem for more details about the type of convergence.

Note that:

- We give in the fourth column the inequality we used to obtain our results. Except for the cases where we study the RSK image of the permutation, the longest alternating subsequence and the descent process, the inequality is trivial, but we will prove all the inequalities in the sequel.
- We want to emphasize that these results are just a direct application of Theorems 1 and 2. Using more sophisticated controls of the error, one could obtain larger classes of universality as we will detail in the sequel.
- For all our examples, the special case of the Ewens distribution satisfies the hypothesis.

1.3. Proof of Theorems 1 and 2

Let ρ_n be a conjugation invariant random permutation. To prove Theorems 1 and 2, the idea is to modify ρ_n to obtain a conjugation invariant random permutation supported on \mathfrak{S}_n^0 . We define the Markov operator T associated to the uniform random walk over $\mathcal{G}_{\mathfrak{S}_n}$. Another way to see it is the following:

¹ In the space of continual diagrams i.e. the set of 1-Lipschitz real functions f such that outside a compact, $f(x) = |x - a|$. One can see [21,27] for more details for continual diagrams. We will use as distance, $d_F(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$ which is finite since both functions are continuous and, outside a compact of \mathbb{R} , $f - g$ is constant.

Table 1
Some examples.

Statistic	$f(\sigma)$	X	Error	Hypotheses	Theorem
Longest monotone subsequence: first order	$\frac{\text{LIS}(\sigma)}{\sqrt{n}}, \frac{\text{LDS}(\sigma)}{\sqrt{n}}$	2	$\varepsilon_n \leq \frac{2}{\sqrt{n}}$	$(\mathcal{H}_{inv,2}^{\mathbb{P}})$ $(\mathcal{H}_{inv,2}^{L^p})$	Theorem 7
Longest monotone circular subsequence: first order	$\frac{\text{LISC}(\sigma)}{\sqrt{n}}, \frac{\text{LDSC}(\sigma)}{\sqrt{n}}$	2	$\varepsilon_n \leq \frac{2}{\sqrt{n}}$	$(\mathcal{H}_{inv,2}^{\mathbb{P}})$	Corollary 10
Longest monotone subsequence: fluctuations	$\frac{\text{LIS}(\sigma)-2\sqrt{n}}{n^{\frac{6}{5}}}, \frac{\text{LDS}(\sigma)-2\sqrt{n}}{n^{\frac{6}{5}}}$	Tracy–Widom	$\varepsilon_n \leq \frac{2}{n^{\frac{6}{5}}}$	$(\mathcal{H}_{inv,6}^{\mathbb{P}})$	Theorem 7
Edge of RSK: first order	$\frac{\lambda_i(\sigma)}{\sqrt{n}}$	2	$\varepsilon_n \leq \frac{4}{\sqrt{n}}$	$(\mathcal{H}_{inv,2}^{\mathbb{P}})$ $(\mathcal{H}_{inv,2}^{L^p})$	Proposition 12
Edge of RSK: fluctuations	$\left(\frac{\lambda_i(\sigma)-2\sqrt{n}}{n^{\frac{1}{6}}}\right)_{1 \leq i \leq d}$	Airy ensemble	$\varepsilon_n \leq \frac{4}{n^{\frac{1}{6}}}$	$(\mathcal{H}_{inv,6}^{\mathbb{P}})$	Theorem 13
Shape of RSK	$s \rightarrow \frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}}$	Ω	$\varepsilon'_{n,k} \leq \frac{2\sqrt{k-1}}{\sqrt{n}}$	$(\mathcal{H}_{inv,1}^{\mathbb{P}})$	Theorem 14
j -cliques: first order	$\frac{\mathcal{K}_j(\sigma)}{n^j}$	$\frac{1}{j!^2}$	$\varepsilon_n \leq \frac{2j}{n}$	$(\mathcal{H}_{inv,1}^{\mathbb{P}})$	Corollary 17
j -cliques: fluctuations	$\frac{\mathcal{K}_j(\sigma) - \frac{n^j}{(j!)^2}}{\sqrt{n}}$	$\mathcal{N}\left(0, \frac{(4j-2) - 2(2j-1)^2}{2(2m-1)^2}\right)$	$\varepsilon_n \leq \frac{2j}{\sqrt{n}}$	$(\mathcal{H}_{inv,2}^{\mathbb{P}})$	Corollary 17
Exceedances: first order	$\frac{\mathcal{N}_{exc}(\sigma)}{n}$	$\frac{1}{2}$	$\varepsilon_n \leq \frac{4}{n}$	$(\mathcal{H}_{inv,1}^{\mathbb{P}})$	Corollary 17
Exceedances: fluctuations	$\frac{\mathcal{N}_{exc}(\sigma) - \frac{n}{2}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{1}{12})$	$\varepsilon_n \leq \frac{4}{\sqrt{n}}$	$(\mathcal{H}_{inv,2}^{\mathbb{P}})$	Corollary 17
Descent process	$\mathbb{1}_{D(\sigma) \subset A}$	$Ber(\det([k_0(j-i)]_A))$	Proposition 23	$(\mathcal{H}_{inv,1}^{\mathbb{P}})$	Corollary 25
Patterns: fluctuations	$\frac{\mathcal{N}_{(\tau,X)}(\sigma) - \frac{n^{p-q}}{p!(p-q)!}}{n^{p-q+\frac{1}{2}}}$	$\mathcal{N}(0, V_{\tau,X})$	$\varepsilon_n \leq \frac{C}{\sqrt{n}}$	$(\mathcal{H}_{inv,2}^{\mathbb{P}})$	Proposition 19
The results below are fully understood in the conjugation invariant case.					
Descents: first order	$\frac{\mathcal{N}_D(\sigma)}{n}$	$\frac{1}{2}$	$\varepsilon_n \leq \frac{4}{n}$	$(\mathcal{H}_{inv,1}^{\mathbb{P}})$	Corollary 17
Descents: fluctuations	$\frac{\mathcal{N}_D(\sigma) - \frac{n}{2}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{1}{12})$	$\varepsilon_n \leq \frac{4}{\sqrt{n}}$	$(\mathcal{H}_{inv,2}^{\mathbb{P}})$	Corollary 17
Peaks: first order	$\frac{\mathcal{N}_{peak}(\sigma)}{n}$	$\frac{1}{3}$	$\varepsilon_n \leq \frac{6}{n}$	$(\mathcal{H}_{inv,1}^{\mathbb{P}})$	Corollary 17
Peaks: fluctuations	$\frac{\mathcal{N}_{peak}(\sigma) - \frac{n}{3}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{2}{45})$	$\varepsilon_n \leq \frac{6}{\sqrt{n}}$	$(\mathcal{H}_{inv,2}^{\mathbb{P}})$	Corollary 17

(continued on next page)

Table 1 (continued).

Statistic	$f(\sigma)$	X	Error	Hypotheses	Theorem
Longest alternating subsequence: first order	$\frac{\text{LAS}(\sigma)}{n}$	$\frac{2}{3}$	$\varepsilon_n \leq \frac{6}{n}$	$(\mathcal{H}_{inv,1}^{\mathbb{P}})$	Corollary 17
Longest alternating subsequence: fluctuations	$\frac{\text{LAS}(\sigma) - \frac{2n}{3}}{\sqrt{n}}$	$\mathcal{N}(0, \frac{8}{45})$	$\varepsilon_n \leq \frac{6}{\sqrt{n}}$	$(\mathcal{H}_{inv,2}^{\mathbb{P}})$	Corollary 17

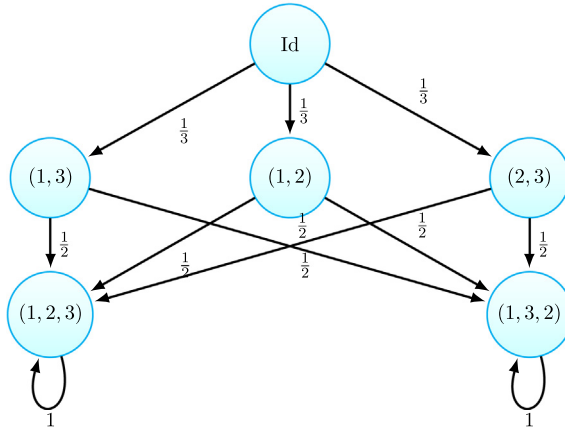


Fig. 2. The transition probabilities of T for $n = 3$.

- If the realization σ of ρ_n has one cycle, σ remains unchanged ($T(\sigma) = \sigma$).
- Otherwise, we choose a couple (i, j) uniformly from the nonempty set

$$\{(i, j) : j \notin \mathcal{C}_i(\sigma)\}$$

and we take $T(\sigma) = \sigma \circ (i, j)$. Here $\mathcal{C}_i(\sigma)$ is the cycle of σ containing i .

For example, for $n = 3$, transition probabilities of T are given in Fig. 2. Slightly different Markov operators have already been studied in [19,20], we modify a little the two operators presented in the cited papers to obtain a uniform random walk easy to generalize to other sets.

We denote by $T^k(\rho_n)$ the random permutation obtained after applying k times the operator T . It is the random permutation obtained after k steps of the uniform random walk on $\mathcal{G}_{\mathfrak{S}_n}$ with initial state ρ_n . Remark that the condition $j \notin \mathcal{C}_i(\sigma)$ guarantees that $\#(\sigma \circ (i, j)) = \#(\sigma) - 1$ since the cycles containing i and j are merged and the remaining of cycles are the same for σ and $\sigma \circ (i, j)$.

In particular,

$$\#(T^i(\rho_n)) \stackrel{a.s.}{=} \max(\#(\rho_n) - i, 1). \tag{6}$$

The invariant measure of this walk (for conjugation invariant permutations) is trivial.

Lemma 5. If ρ_n is a conjugation invariant random permutation of \mathfrak{S}_n then the law of $T^{n-1}(\rho_n)$ is the uniform distribution on \mathfrak{S}_n^0 i.e.

$$T^{n-1}(\rho_n) \stackrel{d}{=} \sigma_{Ew,0,n}.$$

Proof. First, by construction, if ρ_n is conjugation invariant then $T(\rho_n)$ is also conjugation invariant. Indeed, one can see that $T(\rho_n)$ is conjugation invariant since the construction depends only on the cycle structure of ρ_n and all the integers between 1 and n play a symmetric role. By iteration, $T^{n-1}(\rho_n)$ is conjugation invariant. Moreover, using (6),

$$\#(T^{n-1}(\sigma_n)) \stackrel{a.s}{=} 1. \tag{7}$$

Knowing that all the elements of \mathfrak{S}_n^0 belong to the same conjugacy class, they are equally distributed and Lemma 5 follows from (7). \square

We now prove Theorems 1 and 2.

Proof of Theorems 1 and 2. Equality (6) implies that

$$T^{n-1}(\rho_n) \stackrel{a.s}{=} T^{\#(\rho_n)-1}(\rho_n).$$

Therefore, almost surely,

$$d_F(f(T^{n-1}(\rho_n)), f(\rho_n)) = d_F(f(T^{\#(\rho_n)-1}(\rho_n)), f(\rho_n)) \leq \varepsilon'_{n,\#(\rho_n)}.$$

Thus, if $\varepsilon'_{n,\#(\rho_n)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$, then for any $\varepsilon > 0$,

$$\mathbb{P}(d_F(f(T^{n-1}(\rho_n)), f(\rho_n)) > \varepsilon) \xrightarrow[n \rightarrow \infty]{} 0.$$

According to Lemma 5, $T^{n-1}(\rho_n)$ does not depend on the law of ρ_n . By choosing at first $\rho_n = \sigma_{ref,n}$, (2) then yields

$$f(\sigma_{Ew,0,n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x.$$

By choosing at a second step $\rho_n = \sigma_n$, we obtain (4) for any σ_n satisfying the hypothesis of Theorem 1. One can prove Theorem 2 using the same argument. \square

2. Proof of Corollary 4

In the first subsection, we will mainly recover some already known results related to the longest increasing subsequence. Our main contribution, is the study of what we will call later local statistics is Section 2.2. Some of them are already studied for conjugation in the invariant case but not the general case. We will focus in particular in the spacial case of permutation patterns, which is to the best of our knowledge is not studied in the conjugation invariant case.

2.1. First application: Longest increasing subsequence

Given $\sigma \in \mathfrak{S}_n$, a subsequence $(\sigma(i_1), \dots, \sigma(i_k))$ is an increasing (resp. decreasing) subsequence of σ of length k if $i_1 < \dots < i_k$ and $\sigma(i_1) < \dots < \sigma(i_k)$ (resp. $\sigma(i_1) > \dots > \sigma(i_k)$). We denote by $\text{LIS}(\sigma)$ (resp. $\text{LDS}(\sigma)$) the length of the longest increasing (resp. decreasing) subsequence of σ For example,

$$\text{if } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}, \text{LIS}(\sigma) = 2 \text{ and } \text{LDS}(\sigma) = 4.$$

The study of the limiting behavior of $\text{LIS}(\sigma_{\text{unif},n})$, here $\sigma_{\text{unif},n}$ is a uniform random permutation on \mathfrak{S}_n , is known as the Ulam’s problem (or the Ulam–Hammersley problem) In particular, the asymptotic fluctuations were studied by Baik, Deift and Johansson. They proved the following result:

Theorem 6 (Baik et al. [4]). For all $s \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{\text{LIS}(\sigma_{\text{unif},n}) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \xrightarrow[n \rightarrow \infty]{} F_2(s),$$

where F_2 is the cumulative distribution function (CDF) of the GUE Tracy–Widom distribution.

For historical details, full proofs and applications, we refer to Romik [26]. For our purpose, by remarking that for any permutation σ and for any transposition τ ,

$$|\text{LIS}(\sigma \circ \tau) - \text{LIS}(\sigma)| \leq 2, \quad |\text{LDS}(\sigma \circ \tau) - \text{LDS}(\sigma)| \leq 2,$$

we have immediately.

Theorem 7.

- Kammoun [19, Theorem 1.2] Under $(\mathcal{H}_{\text{inv},6}^{\mathbb{P}})$, for all $s \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{\text{LIS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \xrightarrow[n \rightarrow \infty]{} F_2(s) \text{ and } \mathbb{P}\left(\frac{\text{LDS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \xrightarrow[n \rightarrow \infty]{} F_2(s). \tag{TW}$$

- Under $(\mathcal{H}_{\text{inv},2}^{\mathbb{P}})$,

$$\frac{\text{LIS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2 \quad \text{and} \quad \frac{\text{LDS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2.$$

- For any $p \in [1, \infty)$, under $(\mathcal{H}_{\text{inv},2}^{\mathbb{L}^p})$,

$$\frac{\text{LIS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} 2 \quad \text{and} \quad \frac{\text{LDS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} 2.$$

This result is direct application of Theorems 1 and 2 with $\sigma_{\text{ref},n} = \sigma_{\text{unif},n}$ and by taking as functions to study

$$f_{\text{LIS1}}(\sigma) := \frac{\text{LIS}(\sigma)}{\sqrt{n}} \quad \text{and} \quad f_{\text{LIS2}}(\sigma) := \frac{\text{LIS}(\sigma) - 2\sqrt{n}}{n^{\frac{1}{6}}}.$$

For the uniform case, one can see [23,29] for the convergence of f_{LIS1} in probability, Baik et al. [5] for the convergence in \mathbb{L}^p of f_{LIS1} and Baik et al. [4] for the convergence of f_{LIS2} in probability. For the fluctuations, we prove a stronger result.

Theorem 8. Assume that $(\sigma_n)_{n \geq 1}$ is conjugation invariant and

$$\frac{1}{n^{\frac{1}{6}}} \min_{1 \leq i \leq n} \left(\left(\sum_{j=1}^i \#_j(\sigma_n) \right) + \frac{\sqrt{n}}{i} \sum_{j=i+1}^n \#_j(\sigma_n) \right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \tag{8}$$

Then (TW). Here, $\#_j(\sigma)$ is the number of cycles of σ of length j .

We prove this result in Section 3.2.

A similar application is the length of the longest increasing (resp. decreasing) circular subsequence.

Definition 9. Given $\sigma \in \mathfrak{S}_\infty$, a subsequence is said to be increasing (resp. decreasing) circular if it is increasing (resp. decreasing) up to a circular permutation. We refer to Albert et al. [1] for a rigorous definition and more details. We denote by $\text{LICS}(\sigma)$ (resp. $\text{LDCS}(\sigma)$) the length of the longest increasing (resp. decreasing) circular subsequence.

Corollary 10. If $(\mathcal{H}_{inv,2}^{\mathbb{P}})$ is satisfied then

$$\frac{\text{LICS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2 \quad \text{and} \quad \frac{\text{LDCS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2.$$

The uniform case is proved in [1, Theorem 1] for the LICS and the case of the LDCS can be obtained by composition by the permutation $i \mapsto n - i + 1$. and one conclude since, for any permutation σ and any transposition τ ,

$$|\text{LICS}(\sigma \circ \tau) - \text{LICS}(\sigma)| \leq 2 \quad \text{and} \quad |\text{LDCS}(\sigma \circ \tau) - \text{LDCS}(\sigma)| \leq 2.$$

We will give now a generalization for the universality for the LIS. Given $\sigma \in \mathfrak{S}_n$, let $(\lambda_i(\sigma))_{i \geq 1}$ and $(\lambda'_i(\sigma))_{i \geq 1}$ be respectively the shape of image of σ by the RSK correspondence and its transpose. One way to define it is the following. Let

$$\begin{aligned} \mathfrak{J}_1(\sigma) &:= \{s \subset \{1, 2, \dots, n\}; \forall i, j \in s, (i - j)(\sigma(i) - \sigma(j)) \geq 0\}, \\ \mathfrak{D}_1(\sigma) &:= \{s \subset \{1, 2, \dots, n\}; \forall i, j \in s, (i - j)(\sigma(i) - \sigma(j)) \leq 0\}, \\ \mathfrak{J}_{k+1}(\sigma) &:= \{s \cup s', s \in \mathfrak{J}_k, s' \in \mathfrak{J}_1\}, \\ \mathfrak{D}_{k+1}(\sigma) &:= \{s \cup s', s \in \mathfrak{D}_k, s' \in \mathfrak{D}_1\}. \end{aligned}$$

For example, for

$$\sigma_{ex,3} := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mathfrak{J}_1(\sigma_{ex,3}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}\}$$

and

$$\mathfrak{J}_2(\sigma_{ex,3}) = \mathfrak{D}_2(\sigma_{ex,3}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

The RSK image is defined as follows. For any permutation $\sigma \in \mathfrak{S}_n$,

$$\max_{s \in \mathfrak{J}_i(\sigma)} \text{card}(s) = \sum_{k=1}^i \lambda_k(\sigma), \quad \max_{s \in \mathfrak{D}_i(\sigma)} \text{card}(s) = \sum_{k=1}^i \lambda'_k(\sigma). \tag{9}$$

In particular,

$$\max_{s \in \mathfrak{J}_1(\sigma)} \text{card}(s) = \lambda_1(\sigma) = \text{LIS}(\sigma), \quad \max_{s \in \mathfrak{D}_1(\sigma)} \text{card}(s) = \lambda'_1(\sigma) = \text{LDS}(\sigma).$$

A more general version of Theorem 6 is the following.

Theorem 11 (Borodin et al. [9, Theorem 5] and Johansson [18, Theorem 1.4]). *If σ_n follows the uniform distribution, then for all positive integer k , for all real numbers s_1, s_2 ,*

$\dots, s_k,$

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\forall i \leq k, \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i \right) = \lim_{n \rightarrow \infty} \mathbb{P} \left(\forall i \leq k, \frac{\lambda'_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i \right) \tag{Ai}$$

$$= F_{2,k}(s_1, s_2, \dots, s_k),$$

where $F_{2,k}$ is the CDF of the first k particles of the Airy Ensemble.

A direct consequence of the construction above is that for any permutation σ and any transposition τ

$$|\lambda_i(\sigma) - \lambda_i(\sigma \circ \tau)| \leq 4 \quad \text{and} \quad |\lambda'_i(\sigma) - \lambda'_i(\sigma \circ \tau)| \leq 4. \tag{10}$$

Consequently, we have

Proposition 12.

- If $(\mathcal{H}_{inv,2}^{\mathbb{P}})$ is satisfied then for any $i \geq 1$

$$\frac{\lambda_i(\sigma_n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2 \quad \text{and} \quad \frac{\lambda'_i(\sigma_n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2.$$

- For any $p \in [1, \infty)$, under $(\mathcal{H}_{inv,2}^{\mathbb{L}^p})$,

$$\frac{\lambda_i(\sigma_n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} 2 \quad \text{and} \quad \frac{\lambda'_i(\sigma_n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} 2.$$

- Kammoun [19] If $(\mathcal{H}_{inv,6}^{\mathbb{P}})$ is satisfied, then (Ai) holds true.

For the permutations satisfying the same assumptions as in Theorem 8, we have the same asymptotics as in the uniform setting at the edge.

Theorem 13. Assume that $(\sigma_n)_{n \geq 1}$ is conjugation invariant and

$$\frac{1}{n^{\frac{1}{6}}} \min_{1 \leq i \leq n} \left(\left(\sum_{j=1}^i \#_j(\sigma_n) \right) + \frac{\sqrt{n}}{i} \sum_{j=i+1}^n \#_j(\sigma_n) \right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \tag{11}$$

Then (Ai) holds true.

The typical shape of $(\lambda_i(\sigma_{unif,n}))_{i \geq 1}$ seen as a Young diagram was studied separately by Logan and Shepp [23] and Vershik and Kerov [29]. Stronger results are proved by Vershik and Kerov [30]. In 1993, Kerov studied the limiting fluctuations but did not publish his results. See [16] for further details. Let $L_{\lambda(\sigma)}$ be the height function of $\lambda(\sigma) = (\lambda_i(\sigma))_{i \geq 1}$ rotated by $\frac{5\pi}{4}$ and extended by the function $x \mapsto |x|$ to obtain a function defined on \mathbb{R} . For example, if $\lambda(\sigma) = (7, 5, 2, 1, 1, 0)$ the associated function $L_{\lambda(\sigma)}$ is represented by Fig. 3. Our technique can easily extend Vershik, Kerov, Logan and Sheep’s result but fails to give any universality result for fluctuations.

Theorem 14 (Kammoun [19]). Under $(\mathcal{H}_{inv,1}^{\mathbb{P}})$,

$$\sup_{s \in \mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)} \left(s\sqrt{2n} \right) - \Omega(s) \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0, \tag{VKLS}$$

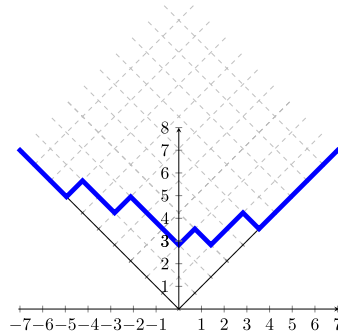


Fig. 3. $L_{(7,5,2,1,1,0)}$.

where

$$\Omega(s) := \begin{cases} \frac{2}{\pi}(s \arcsin(s) + \sqrt{1 - s^2}) & \text{if } |s| < 1 \\ |s| & \text{if } |s| \geq 1. \end{cases}$$

Proof. Let now F be the set of continal diagrams i.e. the set of 1-Lipschitz real functions g from \mathbb{R} to \mathbb{R}_+ such that $\exists a, b > 0$ s.t. $\forall x \notin [-b, b], g(x) = |x - a|$. For $g, h \in F$, we denote by $d_F(g, h) = \sup_{\mathbb{R}} |h - g|$. For $\sigma \in \mathfrak{S}_n, f(\sigma)$ is the function $s \rightarrow \frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}}$. f is a function from \mathfrak{S}_∞ taking values in the metric space (F, d_F) . If we choose $\sigma_{ref,n} = \sigma_{unif,n}$ and x to be the function Ω , the convergence

$$f(\sigma_{ref,n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x$$

is proven by Logan and Shepp [23] and Vershik and Kerov [29]. Using [19, Lemma 3.7.], for any $1 \leq k \leq n$,

$$\varepsilon'_{n,k}(f) \leq 2\sqrt{\frac{k-1}{n}}. \tag{12}$$

Theorem 1 gives the conclusion. \square

2.2. Local statistics

Definition 15. Given $k \geq 1$, we call a function f defined on \mathfrak{S}_∞ a local function of type k , and we write $f \in \mathcal{Loc}_k$, if there exist a positive integer $m \geq 1$, a Boolean function g defined on $\mathbb{N}^{(m+1)k}$ such that, for any $n \geq k + m - 1$ and any $\sigma \in \mathfrak{S}_n$,

$$f(\sigma) = \sum_{1 \leq i_1 < \dots < i_k \leq n} g(i_1, \dots, i_k, \sigma(i_1), \sigma(i_1 - 1), \dots, \sigma(i_1 - m + 1), \sigma(i_2), \dots, \sigma(i_k - m + 1)).$$

We used the convention $\sigma(i) = 0$ when $i \leq 0$.

Here are some examples of local statistics.

- The number of fixed points:
By choosing $k = m = 1$ and $g(x, y) = \mathbb{1}_{x=y}$, we obtain that $\text{tr} \in \mathcal{Loc}_1$.

- $\#_k \in \mathcal{Loc}_k$ and $\sigma \mapsto \text{tr}(\sigma^k) \in \mathcal{Loc}_k$.
- The number of j -exceedances²:
 For $j \in \mathbb{N}$ fixed, we define for $\sigma \in \mathfrak{S}_n$ and, we define $\mathcal{N}_{exc_j}(\sigma) := \text{card}(\{\sigma_i \geq i + j\})$. We choose again $k = m = 1$ and $g(x, y) = \mathbb{1}_{x+j \leq y}$ and we obtain again $\mathcal{N}_{exc_j} \in \mathcal{Loc}_1$.
- Longest alternating subsequence (LAS):
 Given $\sigma \in \mathfrak{S}_n$, $(\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k))$ is said to be an alternating subsequence of σ of length k if $i_1 < i_2 < \dots < i_k$ and $\sigma(i_1) > \sigma(i_2) \leq \sigma(i_3) \geq \dots \sigma(i_k)$. We denote by $\text{LAS}(\sigma)$ the length of the longest alternating subsequence of σ . In particular, Romik [25, Corollary 2] states that $\text{LAS}(\sigma) = 1 + \sum_{i=1}^{n-1} M_k(\sigma)$, where $M_1(\sigma) = \mathbb{1}_{\sigma(1) > \sigma(2)}$ and for $1 < k < n$,

$$M_k(\sigma) = \mathbb{1}_{\sigma(k-1) > \sigma(k) < \sigma(k+1)} + \mathbb{1}_{\sigma(k-1) < \sigma(k) > \sigma(k+1)}.$$

This implies in particular that $\text{LAS} \in \mathcal{Loc}_1$. Here, $k = 1, m = 3$ and

$$g(i, y_1, y_2, y_3) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ \mathbb{1}_{y_2 > y_1} & \text{if } i = 2 \\ \mathbb{1}_{i < k > j} + \mathbb{1}_{y_3 > y_2 < y_1} & \text{if } i > 2. \end{cases}$$

- Number of peaks: For $\sigma \in \mathfrak{S}_n$, we define $\mathcal{N}_{peak}(\sigma) := \text{card}(\{1 < i < n, \sigma(i - 1) < \sigma(i) > \sigma(i + 1)\})$. We choose again $k = 1, m = 3$ and $g(x, y_1, y_2, y_3) = \mathbb{1}_{x \geq 3} \mathbb{1}_{y_1 < y_2 > y_3}$ and we obtain again $\mathcal{N}_{peak} \in \mathcal{Loc}_1$.
- Number of j -descents:
 For $j \geq 1, \sigma \in \mathfrak{S}_n$, we define $\mathcal{N}_{D_j}(\sigma) := \text{card}\{1 \leq i \leq n - 1, \sigma(i + 1) + j \leq \sigma(i)\}$. We choose $k = 1, m = 2$ and $g(x, y_1, y_2) = \mathbb{1}_{x \geq 2} \mathbb{1}_{y_2 \geq y_1 + j}$ and we obtain again $\mathcal{N}_{D_j} \in \mathcal{Loc}_1$. When $j = 1$, the 1-descents are known as the descents. We also set

$$\mathcal{N}_D(\sigma) := \text{card}\{1 \leq i \leq n - 1, \sigma(i + 1) < \sigma(i)\} = \mathcal{N}_{D_1}(\sigma).$$

- Number of inversions and j -cliques of the permutation graph:
 Let $\sigma \in \mathfrak{S}_n$. Let $\mathfrak{G}(\sigma) = (V_{\mathfrak{G}(\sigma)}, E_{\mathfrak{G}(\sigma)})$ be the permutation graph of σ defined by

$$V_{\mathfrak{G}(\sigma)} = \{1, \dots, n\} \text{ and } E_{\mathfrak{G}(\sigma)} = \{(i, j) \in \{1, 2, \dots, n\}; (\sigma(i) - \sigma(j))(i - j) < 0\}.$$

For example, $E_{\mathfrak{G}(\sigma)} = \emptyset$ if and only if $\sigma = Id_n$ and for the permutation $\sigma : i \mapsto n - i + 1$, $\mathfrak{G}(\sigma)$ is the complete graph with n vertices.

Given $j \geq 2$, we denote by

$$\mathcal{K}_j(\sigma) := \text{card}(\{(i_1, i_2, \dots, i_j); 1 \leq i_1 < \dots < i_j \leq n, \sigma(i_1) > \dots > \sigma(i_j)\})$$

the number of j -cliques of $\mathfrak{G}(\sigma)$. In particular, $\mathcal{K}_2(\sigma)$ is the number of inversions of σ . One can easily check that with $\mathcal{K}_j \in \mathcal{Loc}_j$. Here,

$$g(x_1, \dots, x_j, y_1, \dots, y_j) = \mathbb{1}_{y_1 > y_2 > \dots > y_j}.$$

- Let $d_k(\sigma) := \text{card}(\{(i; (i, k) \in E_{\mathfrak{G}(\sigma)})\})$ be the degree of the vertex k in $\mathfrak{G}(\sigma)$. We have $d_k(\sigma) \in \mathcal{Loc}_2$.

² In the literature, j -exceedances is sometimes defined by the condition $\sigma_i \geq i + j$ and othertimes by $\sigma_i = i + j$. In both cases, the number j -exceedances is a local statistic but only the first case is in interest for our purpose.

Proposition 16. Given $k \geq 1$, $f \in \mathcal{L}oc_k$, a random real variable X , $k - 1 < \gamma \leq k$ and $(a_n)_{n \geq 0} \in \mathbb{R}^{\mathbb{N}}$ such that

$$\frac{f(\sigma_{unif,n}) - a_n}{n^\gamma} \xrightarrow[n \rightarrow \infty]{d} X,$$

if $(\mathcal{H}_{inv, \frac{1}{\gamma-k+1}}^{\mathbb{P}})$ holds then

$$\frac{f(\sigma_n) - a_n}{n^\gamma} \xrightarrow[n \rightarrow \infty]{d} X.$$

Proof. By counting the number of possible choices of $1 \leq i_1 < i_2, \dots < i_k \leq n$ such that $\{i, j\} \cap \{i_1, \dots, i_1 - m + 1, i_2, \dots, i_k - m + 1\} \neq \emptyset$, it is easy to see that for any permutation $\sigma \in \mathfrak{S}_n$ and any transposition (i, j) we have

$$|f(\sigma(i, j)) - f(\sigma)| \leq \frac{2km(n-1)!}{(k-1)!(n-k)!} \leq 2kmn^{k-1}.$$

Consequently for $h = \frac{f-a_n}{n^\gamma}$, $\varepsilon_n(h) \leq 2kn^{k-\gamma-1}m$ and one can conclude using Remark 3. \square

One can then easily apply this result.

Corollary 17. Under $(\mathcal{H}_{inv,1}^{\mathbb{P}})$, we have for any $j \geq 2$,

$$\begin{aligned} \frac{\mathcal{N}_{D_j}(\sigma_n)}{n} &\xrightarrow[n \rightarrow \infty]{\mathbb{L}^1} \frac{1}{2}, & \frac{\mathcal{N}_D(\sigma_n)}{n} &\xrightarrow[n \rightarrow \infty]{\mathbb{L}^1} \frac{1}{2}, & \frac{\mathcal{K}_j(\sigma_n)}{n^m} &\xrightarrow[n \rightarrow \infty]{\mathbb{L}^1} \frac{1}{(m!)^2}, \\ \frac{\mathcal{N}_{exc_j}(\sigma_n)}{n} &\xrightarrow[n \rightarrow \infty]{\mathbb{L}^1} \frac{1}{2}, & \frac{\mathcal{N}_{peak}(\sigma_n)}{n} &\xrightarrow[n \rightarrow \infty]{\mathbb{L}^1} \frac{1}{3}, & \frac{\text{LAS}(\sigma_n)}{n} &\xrightarrow[n \rightarrow \infty]{\mathbb{L}^1} \frac{2}{3}. \end{aligned}$$

Moreover, under $(\mathcal{H}_{inv,2}^{\mathbb{P}})$, we have for any $j \geq 2$,

$$\begin{aligned} \frac{\mathcal{N}_{D_j}(\sigma_n) - \frac{n}{2}}{\sqrt{n}} &\xrightarrow[n \rightarrow \infty]{d} \mathcal{N}\left(0, \frac{1}{12}\right) & \frac{\mathcal{N}_D(\sigma_n) - \frac{n}{2}}{\sqrt{n}} &\xrightarrow[n \rightarrow \infty]{d} \mathcal{N}\left(0, \frac{1}{12}\right), \\ \frac{\mathcal{K}_j(\sigma_n) - \frac{n^j}{(j!)^2}}{n^{j-\frac{1}{2}}} &\xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, v_j), & \frac{\mathcal{N}_{exc_j}(\sigma_n) - \frac{n}{2}}{\sqrt{n}} &\xrightarrow[n \rightarrow \infty]{d} \mathcal{N}\left(0, \frac{1}{12}\right), \\ \frac{\mathcal{N}_{peak}(\sigma_n) - \frac{n}{2}}{\sqrt{n}} &\xrightarrow[n \rightarrow \infty]{d} \mathcal{N}\left(0, \frac{2}{45}\right), & \frac{\text{LAS}(\sigma_n) - \frac{2}{3}n}{\sqrt{n}} &\xrightarrow[n \rightarrow \infty]{d} \mathcal{N}\left(0, \frac{8}{45}\right). \end{aligned}$$

where

$$v_j = \frac{\binom{4j-2}{2j-1} - 2\binom{2j-1}{j}^2}{2((2m-1)!)^2}.$$

The uniform case for \mathcal{N}_D , \mathcal{N}_{peak} , \mathcal{K}_j , \mathcal{N}_{exc_1} and LAS has already been studied. One can find a proof respectively in [22], [13], [14], [12] and [25,28]. For the conjugation invariant case, as we explained before, \mathcal{N}_D and \mathcal{N}_{peak} are fully understood but, to the best knowledge of the author, it is not the case for \mathcal{K}_j and \mathcal{N}_{exc_1} . For \mathcal{N}_{exc_1} , the special case of the Ewens distribution is studied in [12]. Moreover, the results for \mathcal{N}_{D_j} and \mathcal{N}_{exc_j} are direct consequences of respectively \mathcal{N}_D and \mathcal{N}_{exc_1} since for any conjugation invariant random permutation σ_n ,

$$0 \leq \mathbb{E}(\mathcal{N}_D(\sigma_n) - \mathcal{N}_{D_j}(\sigma_n)) = \frac{(j-1)(n-j-1)(1 - \mathbb{P}(\sigma_n(1) = 1))}{n-1} \leq j-1$$

and

$$0 \leq \mathbb{E}(\mathcal{N}_{exc_1}(\sigma_n) - \mathcal{N}_{exc_j}(\sigma_n)) \leq j - 1.$$

2.3. Number of occurrences of a vincular permutation pattern

Vincular Patterns also known as dashed patterns are introduced by Babson and Steingrímsson [3]. We use the same definition as in [12].

Definition 18. A vincular pattern of size p is a couple (τ, X) such that $\tau \in \mathfrak{S}_p$ and $X \subset [p-1]$. Given $\sigma \in \mathfrak{S}_\infty$, an occurrence of (τ, X) is a list $i_1 < \dots < i_p$ such that

- $i_{x+1} = i_x + 1$ for any $x \in X$.
- $(\sigma(i_1), \dots, \sigma(i_p))$ is in the same relative order as $(\tau(i_1), \dots, \tau(i_p))$.

We denote by $\mathcal{N}_{(\tau, X)}(\sigma)$ the number of occurrences of (τ, X) in σ .

When $X = \emptyset$, (τ, X) is said to be a classic pattern. Here is some examples of vincular patterns:

- $\mathcal{N}_{(21, \emptyset)} = \mathcal{N}_{inv}$
- $\mathcal{N}_{(21, \{1\})} = \mathcal{N}_D$
- $\mathcal{N}_{(j \dots 21, \emptyset)} = \mathcal{K}_j$
- $\mathcal{N}_{(132, \{1, 2\})} + \mathcal{N}_{(231, \{1, 2\})} = \mathcal{N}_{peak}$.

Remark that for any (τ, X) , $\mathcal{N}_{(\tau, X)} \in \mathcal{L}oc_{p-\text{card}(X)}$.

For the uniform case, Bóna [6], Janson et al. [17] and Hofer [15] proved a CLT for monotone, classic and vincular patterns respectively. Féray [12] gives a generalization for the Ewens distribution. In particular, Hofer [15] proved that for any $\tau \in \mathfrak{S}_p$ and any $X \subset [p-1]$,

$$\frac{\mathcal{N}_{(\tau, X)}(\sigma_{unif, n}) - \frac{n^{p-q}}{p!(p-q)!}}{n^{p-q-\frac{1}{2}}} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, V_{\tau, X}).$$

Here, $q = \text{card}(X)$ and $V_{\tau, X} > 0$. Using Proposition 16, we have immediately the following.

Proposition 19. Under $(\mathcal{H}_{inv, 2}^{\mathbb{P}})$, for any $\tau \in \mathfrak{S}_p$ and any $X \subset [p-1]$

$$\frac{\mathcal{N}_{(\tau, X)}(\sigma_n) - \frac{n^{p-q}}{p!(p-q)!}}{n^{p-q-\frac{1}{2}}} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, V_{\tau, X}).$$

Here, $q = \text{card}(X)$ and $V_{\tau, X} > 0$.

3. Further discussion and improved bounds

In this section, we want to show that beyond Theorems 1 and 2, the technique of proof can be useful to have improved bounds with minor adaptations.

3.1. A lower bound for fluctuations

For some statistics, one can obtain a better lower bound by using a different way to go from $\sigma_{Ew, 0, n}$ to σ_n . Unlike the previous examples, the control of the error may depend on the statistic. Our first example is the longest increasing subsequence. We give a lower bound for

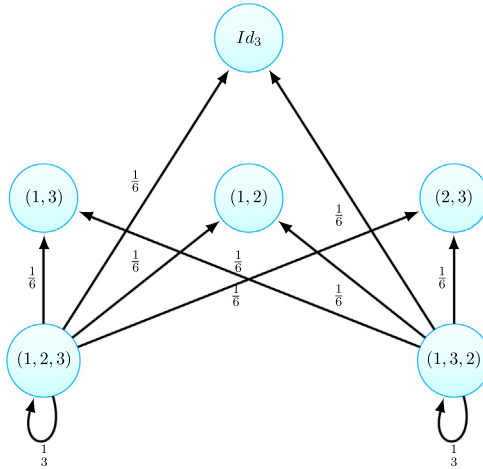


Fig. 4. The transition probabilities of $T_{\sigma_{unif,3}}$.

the fluctuations for a conjugation invariant random permutation. Using this inverse walk one can obtain the following results.

Proposition 20. *If $(\mathcal{H}_{inv, \frac{3}{2}}^{\mathbb{P}})$ is satisfied, then for any $k \geq 1$, for any $s_1, \dots, s_k \in \mathbb{R}$,*

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(\forall i \leq k', \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i \right) \leq F_2(s_1, s_2, \dots, s_k).$$

In particular,

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{\text{LIS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s \right) \leq F_2(s).$$

To do so, we define a new Markov operator. Let $\sigma \in \mathfrak{S}_n^0, \lambda \in \mathbb{Y}_n$ and $i \in \{1, \dots, n\}$.³ We define $\mathfrak{T}_{i,\lambda}(\sigma) := (\sigma^{\lambda_1+1}(i), \dots, \sigma^{\lambda_1+\lambda_2}(i)) \dots \left(\sigma^{\sum_{j=1}^{\ell(\lambda)-1} \lambda_j}(i), \dots, \sigma^n(i) \right)$. Now let σ_n be a conjugation invariant random permutation and let T_{σ_n} be the Markov operator defined on \mathfrak{S}_n^0 as follows. Starting from $\sigma \in \mathfrak{S}_n^0$, choose i uniformly in $\{1, \dots, n\}$ and λ randomly according to the distribution of $\hat{\lambda}(\sigma_n)$ ⁴ and then $T_{\sigma_n}(\sigma)$ returns $\mathfrak{T}_{i,\lambda}(\sigma)$. Here we define a different Markov operator for every distribution. For example, the transition probabilities of $T_{\sigma_{unif,3}}$ are shown in Fig. 4.

By construction, $\hat{\lambda}(T_{\sigma_n}(\sigma)) = \lambda$ and thus, for any cyclic permutation $\sigma \in \mathfrak{S}_n^0, \hat{\lambda}(T_{\sigma_n}(\sigma)) \stackrel{d}{=} \hat{\lambda}(\sigma_n)$. This yields, $\hat{\lambda}(T_{\sigma_n}(\sigma_{Ew,0,n})) \stackrel{d}{=} \hat{\lambda}(\sigma_n)$. Finally, since the construction depends only on the cycle structure, $T_{\sigma_n}(\sigma_{Ew,0,n})$ is conjugation invariant and

$$T_{\sigma_n}(\sigma_{Ew,0,n}) \stackrel{d}{=} \sigma_n. \tag{13}$$

Our main argument is the following lemma.

³ We recall that \mathfrak{S}_n^0 is the set of cyclic permutations.

⁴ $\hat{\lambda}(\sigma)$ is the cycle structure of σ .

Lemma 21. For any permutation $\rho \in \mathfrak{S}_n^0$, for any conjugation invariant random permutation σ_n , for any positive integer k , almost surely

$$\mathbb{E} \left(\left(\sum_{i=1}^k \lambda_j(T_{\sigma_n}(\rho)) - \lambda_j(\rho) \right) \Big|_{\#(T_{\sigma_n}(\rho))} \right) \leq \frac{\#(T_{\sigma_n}(\rho))}{n} \sum_{j=1}^k \lambda_j(\rho) \stackrel{d}{=} \frac{\#(\sigma_n)}{n} \sum_{i=1}^k \lambda_i(\rho).$$

Proof. Let $i_1 < i_2 < \dots < i_{\sum_{i=1}^k \lambda_i(\rho)}$ such that $\{i_1, i_2, \dots, i_{\sum_{i=1}^k \lambda_i(\rho)}\} \subset \mathfrak{I}_k(\rho)$. We have then for any permutation ρ' ,

$$\{i_1, i_2, \dots, i_{\sum_{i=1}^k \lambda_i(\rho)}\} \cap \{i, \rho'(i) = \rho(i)\} \subset \mathfrak{I}_k(\rho')$$

and then

$$\left(\sum_{j=1}^k \lambda_j(\rho') - \lambda_j(\rho) \right) \leq \text{card} \left\{ j \leq \sum_{i=1}^k \lambda_i(\rho); \rho(i_j) \neq \rho'(i_j) \right\}. \tag{14}$$

Consequently, almost surely

$$\begin{aligned} \mathbb{E} \left(\left(\sum_{j=1}^k \lambda_j(T_{\sigma_n}(\rho)) - \lambda_j(\rho) \right) \Big|_{\#(T_{\sigma_n}(\rho))} \right) &\leq \sum_{j=1}^{\sum_{i=1}^k \lambda_i(\rho)} \mathbb{E}(\mathbb{1}_{\rho(j) \neq \rho'(j)} \#(T_{\sigma_n}(\rho))) \\ &= \sum_{i=1}^k \lambda_i(\rho) \frac{\#(T_{\sigma_n}(\rho))}{n}. \quad \square \end{aligned}$$

Proof of Proposition 20. For any $\varepsilon > 0$ there exists n_0 such that for any $n \geq n_0$,

$$\mathbb{P} \left(\sum_{i=1}^k \lambda_i(\sigma_{Ew,0,n}) < 9k\sqrt{n} \right) \geq \sqrt{1 - \varepsilon}$$

and by hypothesis for any $\varepsilon' > 0$ there exist $n_1 >$ such that for any $n > n_1$

$$\mathbb{P} \left(\#(\sigma_n) \leq \varepsilon' \frac{n^{\frac{2}{3}}}{9k} \right) \geq \sqrt{1 - \varepsilon}.$$

Consequently,

$$\mathbb{P} \left(\frac{\mathbb{E} \left(\left(\sum_{j=1}^k \lambda_j(T_{\sigma_n}(\sigma_{Ew,0,n})) - \lambda_j(\sigma_{Ew,0,n}) \right) \Big|_{\#(T_{\sigma_n}(\sigma_{Ew,0,n}))} \right)}{n^{\frac{1}{6}}} \leq \varepsilon' \right) \geq 1 - \varepsilon.$$

This yields

$$\frac{\left(\sum_{j=1}^k \lambda_j(T_{\sigma_n}(\sigma_{Ew,0,n})) - \lambda_j(\sigma_{Ew,0,n}) \right) \Big|_{\#(T_{\sigma_n}(\sigma_{Ew,0,n}))}}{n^{\frac{1}{6}}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

which concludes the proof since $T_{\sigma_n}(\sigma_{Ew,0,n}) \stackrel{d}{=} \sigma_n$. \square

3.2. Proof of Theorem 8 and of Theorem 13

Since Theorem 8 is the particular case $k = 1$ of Theorem 13, we will prove only Theorem 13. Moreover $(\mathcal{H}_{inv, \frac{3}{2}}^{\mathbb{P}})$ implies clearly (11) and consequently, the first bound of Theorem 13 is a direct application of Proposition 20. So it is sufficient to prove that under (\mathcal{H}_{inv}) and (11), we have

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left(\forall i \leq k', \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i \right) \geq F_2(s_1, s_2, \dots, s_k). \tag{15}$$

Sketch of proof. We will not go through all the details since we have already presented similar techniques many times. The idea is to modify the random walk associated to T as following. Given $1 \leq j \leq n - 1$, we define \hat{T}_j the Markov operator as following. $\hat{T}_j(\sigma)$ is a permutation chosen uniformly at random among the permutations obtained by merging all cycles of length less than j to (one of) the biggest cycles of σ to obtain a permutation with cycles of length more than j . Since this construction depends only on the cycle structure, under (\mathcal{H}_{inv}) , $\hat{T}_j(\sigma_n)$ is conjugation invariant. Therefore $T^n(\hat{T}_j(\sigma_n))$ is distributed according to $Ew(0)$. Similarly to the previous proofs, we have

$$\mathbb{E} \left(\left(\sum_{i=1}^k \lambda_j(T^n(\hat{T}_j(\sigma_n))) - \lambda_j(\hat{T}_j(\sigma_n)) \right) \Big| \#(\hat{T}_j(\sigma_n)) \right) \leq \frac{\#(\hat{T}_j(\sigma_n))}{j} \sum_{i=1}^k \lambda_i(\hat{T}_j(\sigma_n)).$$

Let $(j_n)_{n>1}$ be such that

$$\frac{1}{n^{\frac{1}{6}}} \left(\left(\sum_{k=1}^{j_n} \#_k(\sigma_n) \right) + \frac{\sqrt{n}}{j_n} \sum_{k=j_n+1}^n \#_k(\sigma_n) \right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

We have then $(T_{j_n}(\sigma_n))_{n \geq 1}$ satisfies $(\mathcal{H}_{inv,6}^{\mathbb{P}})$,

$$\frac{\sum_{i=1}^k \lambda_i(\hat{T}_{j_n}(\sigma_n))}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2k$$

and

$$\frac{\mathbb{E} \left(\left(\sum_{i=1}^k \lambda_j(T^n(\hat{T}_j(\sigma_n))) - \lambda_j(\hat{T}_j(\sigma_n)) \right) \Big| \#(\hat{T}_j(\sigma_n)) \right)}{n^{\frac{1}{6}}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

This yields (15). \square

3.3. Lower bound for the longest increasing subsequence

Proposition 22. *If $(\sigma_n)_{n \geq 1}$ is conjugation invariant then for any $\varepsilon > 0$,*

$$\mathbb{P} \left(\text{LIS}(\sigma_n) > (2\sqrt{13} - 6 - \varepsilon)\sqrt{n} \right) \xrightarrow[n \rightarrow \infty]{} 1.$$

This yields the following lower bound

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(\text{LIS}(\sigma_n))}{\sqrt{n}} \geq 2\sqrt{13} - 6 \simeq 1.21 \dots$$

Motivated by a conjecture of Bukh and Zhou [10], the author tried in a previous work to prove an asymptotic lower bound on the expectation of the longest increasing subsequence of a conjugation invariant random permutation without cycle conditions. In particular, under the same hypothesis, it was proved in [20] that

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(\text{LIS}(\sigma_n))}{\sqrt{n}} \geq 0.564\dots, \tag{16}$$

Sketch of the proof of Proposition 22. The proof is an adaptation of the proof of [20, Thm 1]. Before we start, let

$$\theta' := 4 - \sqrt{13} \text{ and } \theta'' := 2(1 - \theta') = 2\sqrt{6\theta' - 2} = 2\sqrt{13} - 6.$$

In this proof, we use the following convention. Let $A, B \subset \mathfrak{S}_n$ and $f : \mathfrak{S}_n \rightarrow \mathbb{R}$. If $\mathbb{P}(\sigma_n \in A) = 0$, we assign $\mathbb{P}(\sigma_n \in B | \sigma_n \in A) = 0$ and $\mathbb{E}(f(\sigma_n) | \sigma_n \in A) = 0$.

We have,

$$\begin{aligned} \mathbb{E}(\text{LIS}(\sigma_n)) &= \mathbb{E} \left(\text{LIS}(\sigma_n) \middle| \#_1(\sigma_n) < \theta'' \sqrt{n} \right) \mathbb{P}(\#_1(\sigma_n) < \theta'' \sqrt{n}) \\ &\quad + \mathbb{E} \left(\text{LIS}(\sigma_n) \middle| \#_1(\sigma_n) \geq \theta'' \sqrt{n} \right) \mathbb{P}(\#_1(\sigma_n) \geq \theta'' \sqrt{n}) \\ &\geq E \left(\text{LIS}(\sigma_n) \middle| \#_1(\sigma_n) < \theta'' \sqrt{n} \right) \mathbb{P}(\#_1(\sigma_n) < \theta'' \sqrt{n}) \\ &\quad + \theta'' \sqrt{n} \mathbb{P}(\#_1(\sigma_n) \geq \theta'' \sqrt{n}). \end{aligned}$$

Since the condition on the fixed points is conjugation invariant, it is sufficient to prove this result in the case where almost surely $\#_1(\sigma_n) < \theta'' \sqrt{n}$. Using the same argument and since the condition on the number of cycles is conjugation invariant, it is sufficient to prove this result in the two particular cases.

- If almost surely $\#(\sigma_n) > n\theta'$.

We recall that

$$\#_1(\sigma^2) \geq 6\#(\sigma) - 3\#_1(\sigma) - 2n.$$

Consequently, under the condition $\#_1(\sigma_n) < \theta'' \sqrt{n}$, almost surely,

$$\#_1(\sigma_n^2) > n(6\theta' - 2) - 3\theta' \sqrt{n}.$$

We can then conclude using [20, Proposition 15] that

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(\text{LIS}(\sigma_n))}{\sqrt{n(6\theta' - 2) - 3\theta' \sqrt{n}}} \geq 2.$$

Thus,

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(\text{LIS}(\sigma_n))}{\sqrt{n}} \geq 2\sqrt{6\theta' - 2} = \theta''.$$

- If almost surely $\#(\sigma_n) \leq n\theta'$. Using Lemma 21 for $k = 1$, we obtain that for any $\varepsilon, \varepsilon' > 0$ there exists n_0 such that for any $n > n_0$, for any conjugation invariant random permutation σ_n such that almost surely $\#(\sigma_n) \leq n\theta'$,

$$\mathbb{P}(\text{LIS}(\sigma_n) > 2\sqrt{n}(1 - \theta' - \varepsilon)) > 1 - \varepsilon'.$$

Consequently,

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(\text{LIS}(\sigma_n))}{\sqrt{n}} \geq 2(1 - \theta') = \theta''.$$

This concludes the proof. \square

3.4. Universality for $\widetilde{\mathcal{L}oc}$

We denote by $\widetilde{\mathcal{L}oc}$ the set of local functions $f \in \mathcal{L}oc_k$ for some k that depends only on a finite number of entries of σ . For this class, it is simple to obtain the convergence of the expectation. It can be seen as a macroscopic universality result.

Let $A \subset \mathbb{N}^*$ be finite, $n > \max(A)$ and $(\sigma_n)_{n \geq 1}$ satisfying (\mathcal{H}_{inv}) . Using again the random walk associated to T and seeing that

$$\mathbb{P}(\exists i \in \{i_1 - i_2; i_1 \in A, 0 \leq i_2 < m - 1\}, (T^{n-1}(\sigma_n))(i) \neq \sigma_n(i)) \leq \frac{2\#(\sigma_n)\text{card}(A)m}{n},$$

we obtain the following.

Proposition 23. *Given $f \in \widetilde{\mathcal{L}oc}$ and assuming that $(\sigma_n)_{n \geq 1}$ and $(\sigma_{ref,n})_{n \geq 1}$ satisfy $(\mathcal{H}_{inv,1}^P)$ we have*

$$\mathbb{E}(f(\sigma_n)) - \mathbb{E}(f(\sigma_{ref,n})) \xrightarrow[n \rightarrow \infty]{} 0.$$

Moreover, if $f(\sigma_{ref,n})$ converges in distribution then $f(\sigma_n)$ does also converge to the same limit.

We give now an application: Let n be a positive integer and $\sigma \in \mathfrak{S}_n$, we define

$$D(\sigma) := \{i \in \{1, \dots, n - 1\}; \sigma(i + 1) < \sigma(i)\}.$$

When σ is random, $D(\sigma)$ is known as a descent process.

Given $A \subset \mathbb{N}^*$ finite, if we introduce $D^A(\sigma) := \mathbb{1}_{A \subset D(\sigma)}$, then $D^A \in \mathcal{L}oc_{|A|} \cap \widetilde{\mathcal{L}oc}$. Here,

$$g(x_1, x_2, \dots, x_{|A|}, y_1, y'_1, y_2, \dots, y_{|A|}, y'_{|A|}) = \mathbb{1}_{A=\{x_i-1, 1 \leq i \leq |A|\}} \prod_{i=1}^{|A|} \mathbb{1}_{y_i < y'_i}.$$

We further investigate the descent process. First, the descent process is well understood in the uniform case.

Theorem 24 (Borodin et al. [8, Theorem 5.1]). *For any positive integer n and any $A \subset \{1, 2, \dots, n - 1\}$,*

$$\mathbb{P}(A \subset D(\sigma_{unif,n})) = \det([k_0(j - i)]_{i,j \in A}),$$

where,

$$\sum_{i \in \mathbb{Z}} k_0(i)z^i = \frac{1}{1 - e^z}.$$

We say that the descent process is determinantal with kernel $K_0(i, j) := k_0(j - i)$.

In the non-uniform setting, the descent process is already studied for the Mallows law with Kendall tau metric: it is also determinantal with different kernels, see [8, Proposition 5.2]. We showed in [19] that for a large class of random permutations, the limiting descent process is determinantal with the same kernel as the uniform setting. We obtain a weaker result than [19].

Corollary 25. Under $(\mathcal{H}_{inv,1}^{\mathbb{P}})$, for any finite set $A \subset \mathbb{N}^*$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(A \subset D(\sigma_n)) = \det([k_0(j - i)]_{i,j \in A}). \tag{DPP}$$

Proof. Just apply Proposition 23 for the statistic D^A . \square

The same argument can be applied for other local statistics but not necessarily in $\widetilde{\mathcal{L}oc}$. For example, we have similar results for the degree of vertices of the permutation graph.

Proposition 26. Under $(\mathcal{H}_{inv,1}^{\mathbb{P}})$,

$$\frac{d_k(\sigma_n)}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{1}{2}, \quad \frac{d_{\frac{n}{2}}(\sigma_n)}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{1}{2}, \quad \frac{d_n(\sigma_n)}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{1}{2}.$$

Moreover, under $(\mathcal{H}_{inv,2}^{\mathbb{P}})$,

$$\begin{aligned} \frac{d_{\frac{n}{2}}(\sigma_n) - \frac{n}{2}}{2\sqrt{n}} &\xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(U, 1 - U), & \frac{d_n(\sigma_n) - \frac{n}{2}}{\sqrt{n}} &\xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 6), \\ \frac{d_k(\sigma_n) - \frac{n}{2}}{\sqrt{n}} &\xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 6), \end{aligned}$$

where U is a uniform random variable on $[0, 1]$.

Note that d_k is a local statistic for fixed k but it is not the case for d_n . The uniform case is already studied by Güerker et al. [14]. The problem for d_n is that for any $2 < k < n$, $\varepsilon_n(d_n) = n - 1$ since $d_n(Id_n) = 0$ and $d_n((n, 1)) = n - 1$ and thus we cannot apply directly our previous approach. The idea of the proof is the following. If we condition on the event

$$E_n = \{T^1, T^2, \dots, T^n \text{ do not change } \sigma_n(n)\},$$

then d_n changes at most by 2 every time we apply T and one concludes easily since

$$\mathbb{P}(E_n) \geq 1 - 2 \frac{\mathbb{E}(\#(\sigma_n))}{n}.$$

4. Other groups

4.1. General idea and main results

The same technique of proof we presented in Section 1 can be applied to other sets having a similar structure to the symmetric group. We will give applications in the next subsection. In general, one can apply the same techniques when there exists a “nice” sequence of undirected graphs $G := (G_n = (V_n, E_n))_{n \geq 1}$ ⁵ such that.⁶

$$\forall n \geq 1, G_n \text{ is locally finite.} \tag{17}$$

$$\forall n \geq 1, \text{ there exist a countable set } I_n \text{ and finite sets } (V_n^i)_{i \in I_n} \text{ such that } V_n = \sqcup_{i \in I_n} V_n^i. \tag{18}$$

⁵ We use the usual notations i.e. V_n is the set of vertices and E_n is the set of edges.

⁶ We use \sqcup to denote disjoint union.

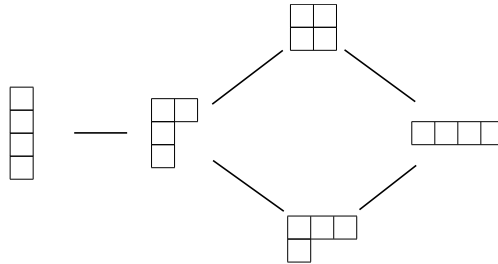


Fig. 5. The classes graph for the Cayley graph of \mathfrak{S}_n generated by transpositions for $n = 4$.

For any $n \geq 1$, for any $i, j \in I_n$, for any $\sigma_1, \sigma_2 \in V_n^i$,

$$\text{card}(\{\sigma' \in V_n^j; (\sigma', \sigma_1) \in E_n\}) = \text{card}(\{\sigma' \in V_n^j; (\sigma', \sigma_2) \in E_n\}) =: e_{j,i}. \tag{19}$$

i.e. the number of neighbors in V_n^j of any element of V_n^i only depends on (i, j) ; we denote it by $e_{i,j}$. We denote by $\widetilde{E}_n := \{(i, j) \in I_n^2; e_{i,j} > 0\}$ and by $\widetilde{G}_n := (I_n, \widetilde{E}_n)$ the classes graph. We need moreover in the sequel of this Section 1 that

$$\forall n \geq 1, \text{ the classes graph } \widetilde{G}_n \text{ is connected.} \tag{20}$$

In the sequel of this section, we assume (17)–(20).

For example, if G_n is the Cayley graph of the symmetric group generated by transpositions we have

- $V_n = \mathfrak{S}_n$
- $E_n = \{(\sigma, \sigma \circ (i, j)); \sigma \in \mathfrak{S}_n, i \neq j\}$
- $I_n = \mathbb{Y}_n$ (the set of Young diagrams of size n).
- $V_n^i = \{\sigma \in \mathfrak{S}_n; \hat{\lambda}(\sigma) = i\}$,
- \widetilde{E}_n the set of couples of Young diagrams such that one can obtain one from the other by concatenating two arrows. For example, for $n = 4$, we obtain the classes graph in Fig. 5.

With analogy with Section 1, we will now construct a new directed graph for which we will consider the uniform random walk. Let d_{G_n} be the usual graph distance and for $\sigma \in V_n$, we denote by $Class(\sigma)$ the unique $i \in I_n$ such that $\sigma \in V_n^i$. Let $(i_n^*)_{n \geq 1} \in \prod_{n \geq 1} I_n$ be a “nice” sequence of classes. We denote by $\underline{d}(\sigma) := \min_{\rho \in V_n^{i_n^*}} d_{G_n}(\sigma, \rho)$. The random walk we use to prove universality will be the uniform random walk on the directed graph $G'_n := (V_n, E'_n)$ where

$$E'_n = \{(\sigma_1, \sigma_2) \in E_n; \underline{d}(\sigma_2) = \underline{d}(\sigma_1) - 1\} \cup \{(\sigma, \sigma), \sigma \in V_n^{i_n^*}\}.$$

Back to the example of the Cayley graph of the symmetric group generated by transpositions we have

- $Class(\sigma) = \hat{\lambda}(\sigma)$,
- $i_n^* = (n, \underline{0})$ is the Young diagram with a unique row of length n ,
- $(V_n, E'_n) = \mathcal{G}_{\mathfrak{S}_n}$
- $\underline{d}(\sigma) = \#(\sigma) - 1$.

With analogy with the symmetric group, let $T_{G'_n}$ be the Markov operator associated to the uniform random walk on G'_n , $V_\infty := \cup_{n \geq 1} V_n$ and f be a function defined on V_∞ and having values on some metric space (F, d_F) . With analogy with Section 1, for $S \subset V_n$ and $\sigma \in V_n$, let

$$\text{next}(S) := \{\sigma_2; \sigma_1 \in S \text{ and } (\sigma_1, \sigma_2) \in E'_n\}, \quad \text{final}(\sigma) := \begin{cases} \text{next}^{d(\sigma)}(\{\sigma\}) & \text{if } \underline{d}(\sigma) > 1 \\ \{\sigma\} & \text{otherwise} \end{cases}$$

and for $i \in I_n$ and $p \geq 1$, we define

$$\begin{aligned} \varepsilon_{n,i,p}(f) &:= \left(\sum_{\sigma \in V_n^i} \sum_{\rho \in \text{next}(\{\sigma\})} \frac{(d_F(f(\sigma), f(\rho)))^p}{\text{card}(V_n^i)\text{card}(\text{next}(\{\sigma\}))} \right)^{\frac{1}{p}}, & \varepsilon_{n,p}(f) &:= \sup_{i \in I_n} \varepsilon_{n,i,p}(f), \\ \varepsilon_{n,i,\infty}(f) &:= \max_{\sigma \in V_n^i} \max_{\rho \in \text{next}(\{\sigma\})} d_F(f(\sigma), f(\rho)), & \varepsilon_{n,\infty}(f) &:= \sup_{i \in I_n} \varepsilon_{n,i,\infty}(f), \\ \varepsilon'_{n,i,p}(f) &:= \left(\sum_{\sigma \in V_n^i} \sum_{\rho \in \text{final}(\sigma)} \frac{(d_F(f(\sigma), f(\rho)))^p}{\text{card}(V_n^i)\text{card}(\text{final}(\sigma))} \right)^{\frac{1}{p}}, & \varepsilon'_{n,i,\infty}(f) &:= \max_{\sigma \in V_n^i} \max_{\rho \in \text{final}(\sigma)} d_F(f(\sigma), f(\rho)). \end{aligned}$$

Finally, let $(\sigma_n)_{n \geq 1}$ be a sequence of random variables such that σ_n is supported on V_n . We say that σ_n is G_n invariant (with respect to the partition $\{V_n^i\}_{i \in I_n}$) if for any $i \in I_n$ and any $\sigma, \rho \in V_n^i$

$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho),$$

and we say that $(\sigma_n)_{n \geq 1}$ is G -invariant if σ_n is G_n -invariant $\forall n \geq 1$.

Definition 27. For $\alpha > 0$ and $p \in [1, \infty]$, we say that $(\sigma_n)_{n \geq 1}$ satisfies $\mathcal{H}_{G\text{-inv},\alpha}^{\mathbb{P}}$ if

$$(\sigma_n)_{n \geq 1} \text{ is } G\text{-invariant and } \frac{d(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0, \tag{\mathcal{H}_{G\text{-inv},\alpha}^{\mathbb{P}}}$$

we say that it satisfies $\mathcal{H}_{G\text{-inv},\alpha}^{\mathbb{L}^p}$ if

$$(\sigma_n)_{n \geq 1} \text{ is } G\text{-invariant and } \frac{d(\sigma_n)}{n^{\frac{1}{\alpha}}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} 0. \tag{\mathcal{H}_{G\text{-inv},\alpha}^{\mathbb{L}^p}}$$

Interesting results can be obtained if the graph satisfies an additional symmetry property:

For any $\sigma_1 \in V_n$, for any $\sigma_2, \sigma_3 \in \text{final}(\sigma_1)$, the number of paths in G'_n of length $\underline{d}(\sigma)$ from σ_1 to σ_2 is equal to that from σ_1 to σ_3 i.e. $A_{G'_n}$ the adjacency matrix of G'_n satisfies the following:

$$\forall \sigma_1 \in \mathfrak{S}_n, \exists c_{\sigma_1} \in \mathbb{N} \text{ such that } \forall \rho \in \mathfrak{S}_n, A_{G'_n}^{d(\sigma)}(\sigma_1, \rho) = c_{\sigma_1} \mathbb{1}_{\rho \in \text{final}(\sigma_1)}. \tag{21}$$

In particular, we have the following:

Lemma 28. Under (17)–(21), for any $(\sigma_n)_{n \geq 1}$ G -invariant, for any $\mathbb{P} \in [1, \infty]$,

$$\mathbb{E} \left(\left(d_F \left(f(\sigma_n), f \left(T_{G'_n}^{d(\sigma_n)}(\sigma_n) \right) \right) \right)^p \right) = \mathbb{E}((\varepsilon'_{n,Class(\sigma_n),p})^p).$$

Proof. For any random variable σ_n , we have

$$\begin{aligned} \mathbb{E} \left(\left(d_F \left(f(\sigma_n), f \left(T_{G'_n}^{d(\sigma_n)}(\sigma_n) \right) \right) \right)^p \right) &= \mathbb{E} \left(\mathbb{E} \left(\left(d_F \left(f(\sigma_n), f \left(T_{G'_n}^{d(\sigma_n)}(\sigma_n) \right) \right) \right)^p \middle| \sigma_n \right) \right) \\ &= \sum_{i \in I_n} \sum_{\sigma \in V_n^i} \mathbb{P}(\sigma_n = \sigma) \\ &\quad \times \mathbb{E} \left(\left(d_F \left(f(\sigma_n), f \left(T_{G'_n}^{d(\sigma_n)}(\sigma_n) \right) \right) \right)^p \middle| \sigma_n = \sigma \right). \end{aligned}$$

If $(\sigma_n)_{n \geq 1}$ is G -invariant, then $\mathbb{P}(\sigma_n = \sigma) = \frac{1}{\text{card}(\text{Class}(\sigma))} \mathbb{P}(\text{Class}(\sigma_n) = \text{Class}(\sigma))$. Moreover, under (21),

$$\mathbb{E} \left(\left(d_F \left(f(\sigma_n), f \left(T_{G'_n}^{d(\sigma_n)}(\sigma_n) \right) \right) \right)^p \middle| \sigma_n = \sigma \right) = \frac{\sum_{\rho \in \text{final}(\sigma)} (d_F(f(\sigma), f(\rho)))^p}{\text{card}(\text{final}(\sigma))}.$$

Consequently, one can conclude since

$$\begin{aligned} \mathbb{E}((\underline{\varepsilon}'_{n, \text{Class}(\sigma_n), p})^p) &= \mathbb{E} \left(\mathbb{E}((\underline{\varepsilon}'_{n, \text{Class}(\sigma_n), p})^p) \middle| \text{Class}(\sigma_n) \right) \\ &= \sum_{i \in I_n} \mathbb{P}(\text{Class}(\sigma_n) = i) (\underline{\varepsilon}'_{n, i, p})^p \\ &= \sum_{i \in I_n} \mathbb{P}(\text{Class}(\sigma_n) = i) \sum_{\sigma \in V_n^i} \sum_{\rho \in \text{final}(\sigma)} \frac{(d_F(f(\sigma), f(\rho)))^p}{\text{card}(V_n^i) \text{card}(\text{final}(\sigma))}. \quad \square \end{aligned}$$

Similarly, one can prove the following.

Lemma 29. Under (17)–(20), $(\sigma_n)_{n \geq 1}$ is G -invariant, for $n \geq 1$, for any $\mathbb{P} \in [1, \infty[$,

$$\mathbb{E} \left(\left(d_F \left(f(\sigma_n), f \left(T_{G'_n}(\sigma_n) \right) \right) \right)^p \right) = \mathbb{E}((\underline{\varepsilon}_{n, \text{Class}(\sigma_n), p})^p).$$

This gives us a universality result.

Theorem 30. Assume that (17)–(20) and that $(\sigma_n)_{n \geq 1}$ and $(\sigma_{ref, n})_{n \geq 1}$ are G -invariant. Suppose that there exists some deterministic $x \in F$ and $p \in [1, \infty[$ such that

$$f(\sigma_{ref, n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x \quad (\text{resp. } f(\sigma_{ref, n}) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} x),$$

$$\underline{\varepsilon}_{n, \text{Class}(\sigma_{ref, n}), \infty}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \quad (\text{resp. } \underline{\varepsilon}_{n, \text{Class}(\sigma_{ref, n}), \infty}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} 0) \tag{22}$$

and

$$\underline{\varepsilon}_{n, \text{Class}(\sigma_n), \infty}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \quad (\text{resp. } \underline{\varepsilon}_{n, \text{Class}(\sigma_n), \infty}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} 0). \tag{23}$$

Then

$$f(\sigma_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x \quad (\text{resp. } f(\sigma_n) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} x).$$

Moreover, under (21)–(23) can be replaced by

$$\underline{\varepsilon}'_{n, \text{Class}(\sigma_{ref, n}), 1}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \quad (\text{resp. } \underline{\varepsilon}'_{n, \text{Class}(\sigma_{ref, n}), p}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} 0)$$

and

$$\underline{\varepsilon}'_{n,Class(\sigma_n),1}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \quad (\text{resp. } \underline{\varepsilon}'_{n,Class(\sigma_n),p}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} 0).$$

Idea of the proof. The proof is identical to that of [Theorems 1 and 2](#). Indeed, (19) guarantees that under the G -invariance, for any $n \geq 1$, $T_{G'_n}(\sigma_n)$ is G_n invariant and by construction almost surely

$$\underline{d}(T_{G'_n}(\sigma_n)) = \max(0, \underline{d}(\sigma_n) - 1).$$

Consequently, by induction, $T_{G'_n}^{\underline{d}(\sigma_n)}(\sigma_n)$ is distributed according to the uniform distribution on $V_n^{i_n^*}$ and almost surely

$$d_F(f(T_{G'_n}^{\underline{d}(\sigma_n)}(\sigma_n)), f(\sigma_n)) \leq \underline{\varepsilon}_{n,class(\sigma_n),\infty}(f). \quad \square$$

Similarly to [Remark 3](#), by the triangle inequality and using that the arithmetic mean is smaller than the p -mean, we have⁷

$$(\underline{\varepsilon}_{n,k,p}(f))^p \leq \sum_{i=1}^{\underline{d}(k)} \max_{j:\underline{d}(j)=i} (\underline{\varepsilon}_{n,j,p}(f))^p \leq \underline{d}(k) \underline{\varepsilon}_{n,p}^p(f).$$

Consequently, if there exists $\alpha > 0$ such that

$$\underline{\varepsilon}_{n,p}^p(f) = O\left(\frac{1}{n^{\frac{1}{\alpha}}}\right),$$

then one can obtain (22) and (23) for the equivalent classes of $(\mathcal{H}_{G-inv,\alpha}^{\mathbb{P}})$ (resp. $(\mathcal{H}_{G-inv,\alpha}^{\mathbb{L}^p})$).

4.2. Some examples of finite graphs

In general, Cayley graphs are good candidates. An interesting case is when there exists $(i_n^*)_{n \geq 1} \in \prod_{n \geq 1} I_n$ such that

$$\frac{1}{\text{card}(V_n)} \sum_{\sigma \in V_n} \min_{\sigma' \in V_n^{i_n^*}} d_{G_n}(\sigma, \sigma') = o\left(\max_{\sigma_1, \sigma_2 \in V_n} d_{G_n}(\sigma_1, \sigma_2)\right),$$

in this case, the comparison with the uniform distribution can be done for reasonable statistics. The first four examples we give are different ways to apply our results to the symmetric group. The other four examples are different graphs. Our eight examples satisfy (17)–(21).

- The Cayley graph of symmetric group generated by transpositions: We recall that

- $(V_n, E'_n) = \mathcal{G}_{\mathfrak{S}_n}$,
- $I_n = \mathbb{Y}_n$,
- $V_n^i = \{\sigma \in \mathfrak{S}_n; \hat{\lambda}(\sigma) = i\}$,
- $Class(\sigma) = \hat{\lambda}(\sigma)$,
- $i_n^* = (n, \underline{0})$ the Young diagram with a unique row of length n ,
- $\underline{d}(\sigma) = \#(\sigma) - 1$.

⁷ There is here a notation abuse. Since \underline{d} is constant in any class, we denote by $\underline{d}(k)$, $\underline{d}(\sigma)$ for some $\sigma \in k$.

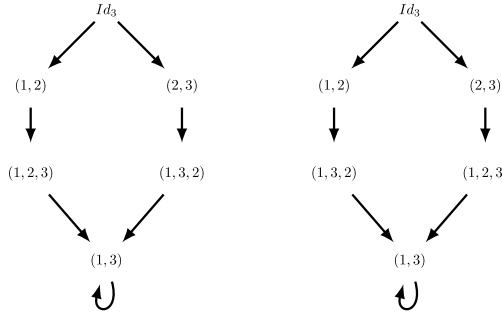


Fig. 6. G'_3 obtained by the transpositions (1, 2) and (2, 3).

- Even permutations: A permutation $\sigma \in \mathfrak{S}_n$ is said to be even if $n - \#(\sigma)$ is even. Cycles of length 3 are a generator of \mathfrak{S}_n . When n is odd, \mathfrak{S}_n^0 is a subset of the set of even permutations. One can choose for example.

- G_n the Cayley graph of \mathfrak{S}_{2n+1} generated by cycles of length 3
- $I_n = \{\lambda \in \mathbb{Y}_{2n+1}; \ell(\lambda) \equiv 1 \pmod{2}\}$,
- $V_n^i = \{\sigma \in \mathfrak{S}_{2n+1}, \hat{\lambda}(\sigma) = i\}$,
- $Class(\sigma) = \hat{\lambda}(\sigma)$,
- $i_n^* = (2n + 1, \underline{0})$,
- $\underline{d}(\sigma) = \frac{\#(\sigma)+1}{2}$.

- \mathfrak{S}_n seen as a Coxeter group: Here we take the right (or the left) Cayley graph generated by transpositions of type $(i, i + 1)$. Depending on the choose of the right or the left composition, one can obtain a different universality theorem. The classes are the same but the graph (and consequently error controls) are different. In this case we have:

- G_n the right (or the left) Cayley graph of \mathfrak{S}_n generated by $\{(i, i + 1); 1 \leq i \leq n - 1\}$.
- $I_n = \{0, 1, \dots, \frac{n(n-1)}{2}\}$,
- $V_n^i = \{\sigma; \mathcal{K}_2(\sigma) = i\}$, where we recall that $\mathcal{K}_2(\sigma)$ is the number of inversions of σ .
- $Class(\sigma) = \mathcal{K}_2(\sigma)$,
- $i_n^* = \lceil \frac{n^2}{4} \rceil$,
- $\underline{d}(\sigma) = |\lceil \frac{n^2}{4} \rceil - \mathcal{K}_2(\sigma)|$.

For example, G'_3 is represented in Fig. 6. Corollary 17 guarantees that $i_n^* = \lceil \frac{n^2}{4} \rceil$ is a good candidate if we want to compare with the uniform distribution. But also it is possible to choose $i_n^* = 0$ when looking for universality results for random permutations close to the identity. For this graph, the Mallows law with Kendall tau distance is G_n -invariant and one can obtain a first order universality for all local statistics we already studied in the previous sections and for the limiting shape

- Using the same previous graph (same G_n) but with only two classes even and odd permutations⁸ i.e. $I_n = \{\text{even}, \text{odd}\}$ we obtain that, if $f(\sigma_n)$ converges in probability (or \mathbb{L}^1) when σ_n follows one of these three distributions

⁸ Here, the choice of i_n^* is not important but the reader can take i_n^* = even.

- Uniform law of \mathfrak{S}_n
- Uniform law of even permutations
- Uniform law of odd permutations

it converges also for the two others as soon as

$$\frac{\min(\sum_{\sigma \in \mathfrak{S}_{n-1} \leq i < n} d_F(f(\sigma \circ (i, i+1)), f(\sigma)); \sum_{\sigma \in \mathfrak{S}_{n-1} \leq i < n} d_F(f((i, i+1) \circ \sigma), f(\sigma)))}{n!(n-1)} = o(1).$$

- Another possible application is the hypercube $(\mathbb{Z}/2\mathbb{Z})^{2n}$. In this case, we set
 - $G_n = (\mathbb{Z}/2\mathbb{Z})^{2n}$
 - $I_n = \{0, 1, \dots, 2n\}$
 - V_n^i is the set of edges of the graph such that the graph distance from $(0, \dots, 0)$ is i .
 - $i_n^* = n$.
- $(\mathbb{Z}/d\mathbb{Z})^{nd}$: Let \mathcal{R}_n be the equivalent relation defined as follows: For any

$$x = (x_i)_{1 \leq i \leq nd}, y = (y_i)_{1 \leq i \leq nd} \in (\mathbb{Z}/d\mathbb{Z})^{nd}, x \mathcal{R}_n y \Leftrightarrow \exists \sigma \in \mathfrak{S}_{nd}, y = (x_{\sigma(i)})_{1 \leq i \leq nd}.$$

\mathcal{R}_n define naturally the classes of the vertices. The central limit theorem guarantees that the class i_n^* where we have exactly n coordinates equal to k for any k in $\mathbb{Z}/d\mathbb{Z}$. is a good candidate. By choosing fixed and different proportions of every element of $\mathbb{Z}/d\mathbb{Z}$ for i_n^* , one can obtain different universality results.

- Let $(H_n)_{n \geq 1}$ be a sequence of non-commutative and finite groups and $(A_n)_{n \geq 1}$ such that A_n is a conjugation invariant subset of H_n .⁹
 - G_n be the Cayley graph generated by H_n .
 - I_n is the set of conjugacy classes
 - $V_n^i = i$

In this case, G -invariant random variables are conjugation invariant variables. The choice of i_n^* is specific to the choice of G_n .

- Dihedral group \mathbb{D}_{2n} For our study, one can admit that

$$\mathbb{D}_{2n} = \{s_0, \dots, s_{n-1}, r_0, \dots, r_{n-1}\}$$

and

$$r_i r_j = r_{i+j}, \quad r_i s_j = s_{i+j}, \quad s_i r_j = s_{i-j}, \quad s_i s_j = r_{i-j}.$$

Here, (i, j) are in $\mathbb{Z}/n\mathbb{Z}$. One can choose either

- G_n : the Cayley graph generated by $\{s_i, 0 \leq i \leq n\}$,
- $I_n = \{r, s\}$,
- $V_n^s = \{s_i, 0 \leq i \leq n\}$ is the set of transpositions and $V_n^r = \{r_i, 0 \leq i \leq n\}$ is the set of rotations,
- $i_n^* = r$

or keep the same graph and choose conjugacy classes as classes (as in the previous examples) In the second case, we require that $f(\sigma_n)$ convergences for any sequence of rotations and a transposition does not change a lot the statistic.

- Colored permutations: A less trivial example is the set of signed permutations and more generally the set of colored permutations. Given two positive integers n and m ,

⁹ i.e. if $\sigma \in A_n$ then $\bar{\sigma} \subset H_n$.

a colored permutation is a map $\pi = (\sigma, \phi)$ such that $\sigma \in \mathfrak{S}_n$ and $\phi \in \{1, \dots, n\}^{\{1, \dots, m\}}$. A subsequence $\pi(x_1), \dots, \pi(x_k)$ of π is called increasing of length $m(k - 1) + p$ if $\sigma(x_1) < \sigma(x_2) < \dots < \sigma(x_k)$ and $\phi(x_1) = \phi(x_2) = \dots = \phi(x_k) = p$. We denote by $\text{LIS}(\pi)$ the length of a longest increasing subsequence.

Theorem 31. *Let $(\pi_n = (\sigma_n, \phi_n))_{n \geq 1}$ be a sequence of random colored permutations and assume that:*

- σ_n is independent of ϕ_n ,
 - ϕ_n is distributed according to the uniform distribution,
 - σ_n is conjugation invariant,
 - $\frac{\#\sigma_n}{n^6} \xrightarrow{\mathbb{P}} 0$.
- then,

$$\mathbb{P}\left(\frac{\text{LIS}(\pi_n) - 2\sqrt{nm}}{m^{\frac{2}{3}}\sqrt[6]{nm}} < s\right) \rightarrow F_2^m(s). \tag{24}$$

Proof. The uniform case is proved by Borodin [7]. To apply our theorem, choose the graph where two colored permutations are related by an edge if only the first components differ by a transposition, i.e.

- $E_n := \{((\sigma, \phi), (\sigma \circ (i, j), \phi)); i \neq j\}$,
- $I_n = \mathbb{Y}_n$,
- $V_n^i = \{(\sigma, \phi); \bar{\sigma} = i\}$,
- $i_n^* = (n, \mathbf{0})$. \square

For our examples, a trivial choice of G -invariant elements is the uniform measure, or the uniform measure on a given class. Since d is constant in classes, a natural way to generalize Ewens measures is the following. Given $q \in \mathbb{R}_+$, the probability measure satisfying

$$\mathbb{P}(\rho_{G,q,n} = \sigma) = \frac{q^{d(\sigma)}}{\sum_{\sigma' \in V_n} q^{d(\sigma')}}$$

is G -invariant and for any statistic f such that $f(\rho_{G,0,n})$ converges, one can obtain a non-empty universality result around $\rho_{G,0,n}$ since

$$err_n := q \mapsto \mathbb{E}(d_F(f(T^{d(\rho_{G,q,n})}(\rho_{G,q,n}), f(\rho_{G,q,n}))))$$

is continuous and $err_n(0) = 0$. In fact, in the case of permutations, Ewens and Mallows measures with Kendall tau distance are particular cases of $\rho_{G,q,n}$.

4.3. Infinite case

We take now $G_n = G$ an infinite graph. Example of “nice graphs”:

- The infinite d -regular tree \mathfrak{T}_d .
- The set of words of a finite alphabet of length d .
- The free group \mathcal{F}_d with its natural Cayley graph.
- The Cayley graph of \mathcal{B}_d , the Artin Braid group.
- The Cayley graph of an infinite and finitely generated group $H = \langle x_1, x_2, \dots, x_n \rangle$.

The classes here are indexed by \mathbb{N} according to the distance to the root (or the identity). Let G be such that

$$0 < \liminf_{n \rightarrow \infty} \frac{\log(\text{card}(\{x; d(x) = n\}))}{n} = \limsup_{n \rightarrow \infty} \frac{\log(\text{card}(\{x; d(x) = n\}))}{n} = \log(\lambda) < \infty.$$

In this case, Mallows laws are well-defined and G -invariant. It is the case for the first three examples.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Ewens measures

Definition 32. Let θ be a non-negative real number. We say that a random permutation $\sigma_{Ew,\theta,n}$ follows the Ewens distribution with parameter θ if for all $\sigma \in \mathfrak{S}_n$,

$$\mathbb{P}(\sigma_{Ew,\theta,n} = \sigma) = \frac{\theta^{\#(\sigma)-1}}{\prod_{i=1}^{n-1} (\theta + i)}. \tag{25}$$

Note that when $\theta = 1$, the Ewens distribution is just the uniform distribution on \mathfrak{S}_n , whereas when $\theta = 0$ we have the uniform distribution on permutations having a unique cycle. For general θ , the Ewens distribution is clearly conjugation invariant since it only involves the cycle structure of θ .

We want to recall the interpretation of the Ewens distribution via a nice stochastic process known as “the Chinese restaurant process”. Suppose that there are an infinite number of circular tables with infinite capacity.

- At $t = 0$, all tables are empty.
- At $t = 1$, the person “1” comes and sits in the first table.



- At $t = 2$, the person “2” comes and sits in the table near person 1 with probability $\frac{1}{1+\theta}$

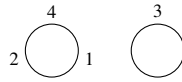


and sits alone in a new table with probability $\frac{\theta}{1+\theta}$.

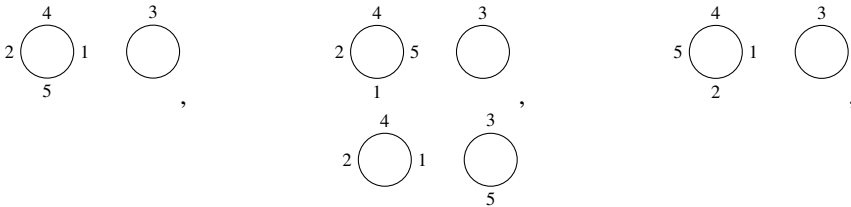


- At $t = n$, the person “ n ” comes, she/he chooses to sit alone in a new table with probability $\frac{\theta}{\theta+n-1}$ and in an occupied table i with probability $\frac{|B_i|}{\theta+n-1}$, where B_i is the number of persons at the table i . In this case, she/he chooses her/his position uniformly in gaps between two persons.

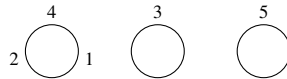
For example, if we have the following configuration



at $t = 5$, the probability to switch to each of the following configurations

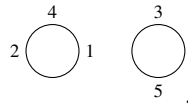


is $\frac{1}{\theta+4}$ and the probability to switch to



is equal to $\frac{\theta}{4+\theta}$.

To obtain the associated permutation to a configuration one reads the elements on each non-empty circle counterclockwise to get a cycle. For example, to the configuration



we associate the permutation $(1, 4, 2)(3, 5)$. For further reading, we refer to Aldous [2], McCullagh [24] and Chafaï et al. [11].

Using the Chinese restaurant process description of the Ewens distribution, it is obvious to see that the number of cycles $\#(\sigma_{Ew,\theta,n})$ is the sum of n independent Bernoulli random variables with parameters $\{\frac{\theta}{\theta+i}\}_{0 \leq i \leq n-1}$. The following classic result is immediate.

Proposition 33.

$$\mathbb{E}(\#_1(\sigma_{Ew,\theta,n})) = \frac{n\theta}{n-1+\theta} \quad \text{and} \quad \mathbb{E}(\#(\sigma_{Ew,\theta,n})) = 1 + \sum_{i=2}^n \frac{\theta}{i-1+\theta} \leq 2 + \theta \log(n).$$

In particular, for the uniform distribution, we have

$$\frac{\#(\sigma_{unif,n})}{\log(n)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 1. \tag{26}$$

Proof. Since the number of cycles of the uniform law is the sum of n independent random Bernoulli variables of parameters $1, \frac{1}{2}, \dots, \frac{1}{n}$ and using Chebyshev’s inequality, we obtain

$$\mathbb{P}\left(\left|\frac{\#(\sigma_{unif,n})}{\log(n)} - 1\right| > \alpha\right) \leq \frac{\frac{\sum_{i=1}^n \frac{i-1}{i^2}}{\log(n)^2}}{\left(\alpha + 1 - \frac{\sum_{i=1}^n \frac{1}{i}}{\log n}\right)^2} = O\left(\frac{1}{\log(n)}\right). \quad \square$$

Remark 34. This convergence holds almost surely. The proof uses martingale techniques. One can find a proof of this result in [11].

In our work, we need the following two results.

Corollary 35. Let $(\theta_n)_{n \geq 1}$ be a sequence of non-negative real numbers such that:

$$\lim_{n \rightarrow \infty} \frac{\theta_n \log(n)}{n^{\frac{1}{\alpha}}} = 0. \tag{27}$$

Then $(\sigma_{Ew,\theta_n,n})_{n \geq 1}$ satisfies $(\mathcal{H}_{inv,\alpha}^{\mathbb{P}})$.

Proposition 36. For any $\theta \geq 0, \alpha > 0$ and $p \geq [1, \infty[$, $(\sigma_{Ew,\theta,n})_{n \geq 1}$ satisfies $(\mathcal{H}_{inv,\alpha}^{\mathbb{L}^p})$.

Proof. Using Bernstein inequality, if $\theta \geq 1$,

$$\begin{aligned} \mathbb{P}(\#(\sigma_{Ew,\theta,n}) > (3p + 1)\theta \log(n) + 2) &\leq \mathbb{P}(\#(\sigma_{Ew,\theta,n}) > \mathbb{E}(\#(\sigma_{Ew,\theta,n})) + 3p\theta \log(n)) \\ &\leq \exp\left(\frac{-\frac{9}{2}\theta^2 p^2 \log(n)^2}{\text{var}(\#(\sigma_{Ew,\theta,n})) + \frac{3p}{3}\theta \log(n)}\right) \\ &\leq \exp\left(\frac{-\frac{9}{2}\theta^2 p^2 \log(n)^2}{(p + 1)\theta \log(n) + 2}\right) \\ &= O\left(n^{-\frac{9}{4}\theta \frac{p^2}{p+1}}\right) = O\left(n^{-\frac{9p}{8}}\right). \end{aligned}$$

Consequently,

$$\mathbb{E}(\#(\sigma_{Ew,\theta,n})^p) \leq ((3p + 1)\theta \log(n) + 2)^p + n^p O(n^{-\frac{9p}{8}}) = O(\log^p(n)).$$

When $\theta < 1$, one can conclude since $\mathbb{E}(\#(\sigma_{Ew,\theta,n})^p) < \mathbb{E}(\#(\sigma_{Ew,1,n})^p)$. \square

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