Discrete and gradient Elasticity Cosserat Modeling of Granular Chains

Sina Massoumi, Noël Challamel, Jean Lerbet

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First A. Sina Massoumi, Second B. Noël Challamel, Third C. Jean Lerbet

***Abstract*—** The present study theoretically investigates the free vibration problem of a discrete granular system. This micro structured system consists of uniform grains elastically connected by shear and rotation springs. Such a granular structural system is confined by discrete elastic interactions, to take into account the lateral granular contributions. This discrete repetitive system could be considered as a discrete Cosserat chain or a lattice elastic model with shear interaction. First, from the vibration equation governing the model, the natural frequencies are exactly calculated for the simply supports boundary conditions. Then it is shown that the discrete equations of this granular system, for an infinite number of grains, converge to the differential equations of the Bresse-Timoshenko beam resting on Winkler foundation. A gradient Bresse-Timoshenko model is constructed from continualization of the difference equations of the granular system. The natural frequencies of the continuous gradient Cosserat models are compared with those of the discrete Cosserat model associated with the granular chain. Scale effects and wave dispersion characteristics of the granular chain are clearly captured by the continuous gradient elasticity model.

***Keywords*—** Cosserat continuum, Gradient elasticity, Granular medium, Timoshenko beam, Vibration analysis.

I. Introduction

IN order to adapt a theory of continuous media to granular materials, it is necessary to introduce independent degrees of freedom of rotation, in addition to those conventional translation. Indeed, the relative movements between the microstructure and the average macroscopic deformations can be apprehended by additional degrees of freedom. Such enriched kinematics leads to non-classical continuous media (Cosserat-type theories [1], [2]). Voigt [3] was one of the pioneers in the development of these enriched environments. He showed the existence of stress-couples in these materials. Cosserat's continuum theories belong to the wider class of generalized continuas which introduce intrinsic length scales via higher order gradients or additional degrees of freedom ([4]-[6]). Conversely, the classical mechanics of continuous media does not incorporate a rotational interaction between the particles, and does not allow to understand the size effects in these media.

Furthermore, the Bresse-Timoshenko beam model takes into account shear stiffness and the rotational inertia of the section ([7]-[9]). The effect of shear and rotary inertia can be significant in the case of the calculation of natural frequencies for short beams or for which the shear modulus is sufficiently weak. The Bresse-Timoshenko beam model is also a generalization of the Euler-Bernoulli model, and admits kinematics with two independent fields, a field of transverse displacement and a field of rotation. Timoshenko [8], [9] calculated the exact natural frequencies for such a beam with two degrees of freedom resting on two simple supports. The calculation of natural frequencies for a Bresse-Timoshenko beam with any boundary conditions and in elastic interaction with a rigid medium is also obtained by Wang and Stephens [10], Manevich [11]-[13].

Another essential point is the equivalency of the theories of continuous beams of a one-dimensional Cosserat medium and a classical Bresse-Timoshenko medium. In fact, a Timoshenko beam is an example of a Cosserat one-dimensional continuum considering the independent double rotation-displacement kinematics (see [14], [15]). There is therefore a link maintained between the one-dimensional granular media of Cosserat and the media of Bresse-Timoshenko. The present study focuses on the vibration of the granular beam model resting on a linear Winkler foundation [16]. Note that the difference equations of this granular chain coincide with the difference equations of the granular model of Pasternak and Mühlhaus [17] in the absence of elastic foundation, but differ from the difference equations of the discrete model with shear studied by Duan et al [18] or the model formulated more recently by Bacigalupo and Gambarotta [19].

This paper is organized as follows. First, a discrete granular model is introduced from the elements interaction considering rotation and shear. Then, from the dynamic analysis of the beam, the deflection equations for the discrete granular beam and the continuum beam are obtained. The exact solution for the granular deflection equation leads to a fourth order linear difference equation. For the continuum model asymptotically by considering an infinite number of grains, the governed equation would lead to a fourth order linear differential equation. The natural frequencies are obtained for the both model and are compared together.

II. Granular Model

A granular beam of length L resting on two simple supports is modeled by a finite number of grains interacting together.

F. A. Sina Massoumi is with the University of Paris-Saclay, Univ Evry, LAMME, CNRS, 23 bvd de France, 91037 Évry Cedex, France (e-mail: sina.massoumi@etud.univ-evry.fr).

S. B. Noël Challamel, is with the Univ. Bretagne Sud UBS, IRDL, CNRS, Rue de Saint Maudé, 56321 Lorient cedex, France (e-mail: noel.challamel@univ-ubs.fr).

T. C. Jean Lerbet is with the University of Paris-Saclay, Univ Evry, LAMME, CNRS, 23 bvd de France, 91037 Évry Cedex, France (e-mail: jean.lerbet@univ-evry.fr).

Such a model could be presented by considering the microstructured granular chain comprising *n+1* rigid grain with diameter *a* (*a=L/n*) that are connected by n shear and rotational springs, as shown in figure 1. It is assumed that the elastic support springs are located at the center of each rigid grain. Each grain has two degrees of freedom which are denoted by W_i for the deflection and Θ_i for the rotation. This model is slightly different with the one of Challamel et al. (2014) where the nodal kinematics and the Winkler elastic foundation are located at the grain interface. The aim of this paper consists in finding the vibration equation of this granular chain and then trying to obtain the natural frequencies.

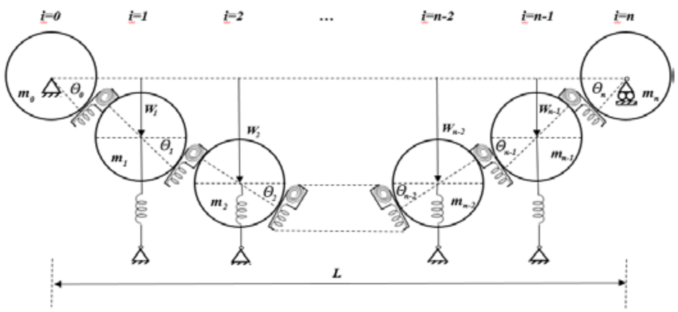

Fig. 1 A discrete shear granular chain composed of n+1 grains of diameter a and mass m; L=n.a

The total energy function of the system is given by

$$U = \mathrm{U_s} + \boldsymbol{U_b} + \boldsymbol{U_{Winkler}} \quad \textbf{(1)}$$

where $\mathrm{U_s}$, $\mathrm{U_b}$ and $\mathrm{U_{Winkler}}$ are respectively the strain energy due to deformed shear, rotational springs and Winkler elastic foundations and are defined as follows

$$\mathrm{U_s} = \frac{1}{2}\sum_{\mathrm{i=0}}^{\mathrm{n-1}} \mathrm{S}\left(\mathrm{W_{i+1}} - \mathrm{W_i} - \mathrm{a}\frac{\Theta_{\mathrm{i+1}} + \Theta_{\mathrm{i}}}{2}\right)^2 \quad \textbf{(2)}$$

$$\boldsymbol{U_b} = \frac{1}{2}\sum_{i=0}^{n-1} \boldsymbol{C}(\boldsymbol{\Theta_{i+1}} - \boldsymbol{\Theta_i})^2 \quad \textbf{(3)}$$

$$\boldsymbol{U_{Winkler}} = \frac{1}{2}\sum_{i=0}^{n} \boldsymbol{K W_i^2} \quad \textbf{(4)}$$

where *S* is the shear stiffness and defined by $S = \frac{K_sGA}{a} = \frac{nK_sGA}{L}$. *G* is the shear modulus; *A* is the cross-sectional area of the beam and K_s is an equivalent shear correction coefficient. *C* is the rotational stiffness and can be expressed as $C = \frac{EI}{a} = \frac{nEI}{L}$, where *E* is Young's modulus and *I* is the second moment of area. *K=ka* is the discrete stiffness of the elastic support and is attached to the center of each grain.

The kinematic variables are measured at nodes *i* located at the center of each grain, which is consistent with the approach followed for instance by Pasternak and Mühlhaus [17].

The total kinetic energy of the granular model may be expressed as follows:

$$T = \frac{1}{2}\sum_{i=0}^{n} m_i \dot{W}_i^{\,2} + \frac{1}{2}\sum_{i=0}^{n} I_{m_i}\dot{\Theta}_i^2 \quad \textbf{(5)}$$

where $I_{m_0} = I_{m_n} = \frac{1}{2}I_{m_i}$ and $I_{m_i} = \frac{\rho IL}{n} = \rho I a$ for i in [1,n-1] are the second moment of inertia of the beam segment and. m_i is the mass term for each grain that is defined for the inter grains by $m_i = \rho a$ and the half value for the boundaries.

The Lagrangian relation of the granular system may be defined as $L = T - (U_s + U_b + U_{Winkler})$. The Euler-Lagrange of such a granular system are given by

$$\frac{\partial L}{\partial W_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{W}_i}\right) \;;\; \frac{\partial L}{\partial \Theta_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Theta}_i}\right) \text{ with } i = 0, 1, 2, \dots, n \quad \textbf{(6)}$$

Using (6) based on the energy function leads to the following difference equation system

$$\begin{gathered} K_sGA(\delta_2 W) - K_sGA(\delta_1\theta) - kW_i - \rho A\ddot{W}_i = 0 \\ EI(\delta_2\theta) + K_sGA(\delta_1 W) - K_sGA(\delta_0\theta) - \rho I\ddot{\theta}_i = 0 \end{gathered} \quad \textbf{(7)}$$

Considering two fictitious grains connected by two shear and rotational springs to the first and last grains and also by introducing the following pseudo-difference operators

$$\delta_0 W_i = \frac{W_{i+1}+2W_i+W_{i-1}}{4},\ \delta_1 W_i = \frac{W_{i+1}-W_{i-1}}{2a},\ \delta_2 W_i = \frac{W_{i+1}-2W_i+W_{i-1}}{a^2} \quad \textbf{(8)}$$

Assuming a harmonic motion $W_i = w_i e^{j\omega t}$ and $\Theta_i = \theta_i e^{j\omega t}$ with $j^2=-1$, **(7)** may be written in a matrix form

$$\begin{pmatrix} \boldsymbol{EI\delta_2 - K_sGA\delta_0 + \rho I\omega^2} & \boldsymbol{K_sGA\delta_1} \\ \boldsymbol{K_sGA\delta_1} & \boldsymbol{k - K_sGA\delta_2 - \rho A\omega^2} \end{pmatrix}\begin{pmatrix}\boldsymbol{\theta}\\ \boldsymbol{w}\end{pmatrix}_i = \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix} \quad \textbf{(9)}$$

These difference equations system **(9)** have been obtained by Pasternak and Mühlhaus [17] when neglecting the elastic Winkler foundation *(k=0)*. With consideration of an infinite number of grains ($n \to \infty$) referring to the continuum beam, it converges to the coupled system differential equations of (10) which has been obtained by Bresse (1859) and Timoshenko in absence of Winkler foundation *(k=0)* and assuming the shear correction factor be unity *(K_s=1)*. This equation valid for a Bresse-Timoshenko beam on elastic foundation have been also obtained by Wang and Stephens [10] and Manevich [11].

$$\begin{pmatrix} \boldsymbol{EI\partial_x^2 - K_sGA + \rho I\omega^2} & \boldsymbol{K_sGA\partial_x} \\ \boldsymbol{K_sGA\partial_x} & \boldsymbol{k - K_sGA\partial_x^2 - \rho A\omega^2} \end{pmatrix}\begin{pmatrix}\boldsymbol{\theta}\\ \boldsymbol{w}\end{pmatrix} = \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix} \quad (10)$$

It is possible to introduce the following pseudo-differential operators

$$\begin{aligned} \boldsymbol{\delta_0} &= \frac{\boldsymbol{e^{a\partial_x} + 2 + e^{-a\partial_x}}}{\boldsymbol{4}} = \mathbf{cosh^2}\left(\frac{\boldsymbol{a\partial_x}}{\boldsymbol{2}}\right) \\ \boldsymbol{\delta_1} &= \frac{\boldsymbol{e^{a\partial_x} - e^{-a\partial_x}}}{\boldsymbol{2a}} = \frac{\mathbf{sinh}\,(\boldsymbol{a\partial_x})}{\boldsymbol{a}} \\ \boldsymbol{\delta_2} &= \frac{\boldsymbol{e^{a\partial_x} - 2 + e^{-a\partial_x}}}{\boldsymbol{a^2}} = \frac{\boldsymbol{4}}{\boldsymbol{a^2}}\mathbf{sinh^2}\left(\frac{\boldsymbol{a\partial_x}}{\boldsymbol{2}}\right) \end{aligned} \quad (11)$$

The relation could be obtained between these operators as

$$\delta_2\delta_0 = \delta_0\delta_2 = {\delta_1}^2 \tag{12}$$

With consideration determinant of the matrix in (10) equal to zero and using the property of (12) gives the fourth-order difference equation for the deflection as follows:

$$[EI{\delta_2}^2 + \left(\rho I\omega^2 - \frac{kEI}{K_sGA} + \frac{EI\rho\omega^2}{K_sG}\right)\delta_2 + (k - \rho A\omega^2)\delta_0 - \frac{k\rho I\omega^2}{K_sGA} + \frac{\rho^2 I\omega^4}{K_sG}]w_i = 0 \tag{13}$$

Equation (13) is slightly different from the fourth-order difference equation obtained by Duan et al. [18] for (*k=0*).

Considering infinite number of grains ($n \to \infty$) for the continuum beam, the fourth-order differential equation valid for a Bresse-Timoshenko beam on Winkler elastic foundation is given by (14) which also could be compared well by [10], [11] and [20].

$$\frac{d^4w}{dx^4} + \left(\frac{\rho\omega^2}{E}\left(1 + \frac{E}{k_sG}\right) - \frac{k}{k_sGA}\right)\frac{d^2w}{dx^2} - (\frac{\rho\omega^2}{E}\left(\frac{A}{I} + \frac{k}{k_sGA} - \frac{\rho\omega^2}{k_sG}\right) - \frac{k}{EI})]w = 0 \tag{14}$$

III. Exact Solution

A. Resolution of The Difference Equation

The exact solution for the fourth-order linear difference eigenvalue problem of (13) is investigated (see [21], [22] for the general solution of linear difference equations).

The boundary conditions of the system could be supposed as follows [25]:

$$\begin{aligned} &At\ \ i = 0: \quad w_0 = 0\,;\ \delta_2 w_0 = 0 \\ &At\ \ i = n: \quad w_n = 0\,;\ \delta_2 w_n = 0 \end{aligned} \tag{15}$$

The non-dimensional quantities may be introduced

$$\Omega^2 = \frac{\omega^2\rho AL^4}{EI}\ ,\ \mu_s = \frac{E}{K_s\,G}\ , r = \sqrt{\frac{I}{A}}\ ,\ r^* = \frac{r}{L}\ ,\ k^* = \frac{kL^4}{EI} \tag{16}$$

Ω is a dimensionless frequency; μ_s is inversely proportional to the shear stiffness; and r^* is proportional to the rotatory inertia. The solution of the linear difference equation is thought in the $w_i = B\lambda^i$ form, where B is a constant. Therefore, the characteristic equation could be obtained as

$$(\lambda + \frac{1}{\lambda})^2 + \left(\lambda + \frac{1}{\lambda}\right)\epsilon + \tau = 0 \tag{17}$$

where the parameters ϵ and τ can be defined as

$$\epsilon = \left[\frac{r^{*2}\Omega^2}{n^2}\left(1 + \mu_s - \frac{1}{4r^{*2}n^2}\right) - \frac{r^{*2}k^*\mu_s}{n^2} + \frac{k^*}{4n^4} - 4\right], \tag{18}$$

$$\tau = \left[\frac{r^{*2}\Omega^2}{n^4}(-\mu_s r^{*2}k^* + \mu_s r^{*2}\Omega^2) + 2\left(\frac{k^*}{4n^4} - \frac{\Omega^2}{4n^4}\right) - 2(\frac{r^{*2}\Omega^2}{n^2}(1 + \mu_s) - \frac{r^{*2}k^*\mu_s}{n^2}) + 4\right]$$

Solving (17) leads to the following equation

$$\lambda + \frac{1}{\lambda} = \frac{-\epsilon \pm \sqrt{\epsilon^2 - 4\,\tau}}{2} \tag{19}$$

Equation (19) admits four solutions written

$$\lambda_{1,2} = \frac{-\epsilon + \sqrt{\epsilon^2 - 4\,\tau}}{4} \pm \sqrt{(\frac{\epsilon - \sqrt{\epsilon^2 - 4\,\tau}}{4})^2 - 1} \tag{20}$$

$$\lambda_{3,4} = \frac{-\epsilon - \sqrt{\epsilon^2 - 4\,\tau}}{4} \pm j\sqrt{1 - (\frac{\epsilon + \sqrt{\epsilon^2 - 4\,\tau}}{4})^2} \tag{21}$$

where $j^2 = -1$. On the other hand, it is important to notice that according to (19) the results of $\lambda + \frac{1}{\lambda}$ are in the ranges of $(-\infty,\ -2]$ or $[2,\ +\infty)$.

The limited cases when $\lambda + \frac{1}{\lambda} = \pm 2$ would be happened for $\lambda = \pm 1$ which refers to the critical frequencies. The critical frequencies of the system are inconsistent condition with the general supposed form of the deflection and would be obtained by assuming:

$$\tau = \pm 2\epsilon - 4 \tag{22}$$

Replacing τ and ϵ by using (18) ones could be obtained as:

$$(\frac{r^{*2}\Omega_{cr}^2}{n^2} - 4)(\frac{\mu_s r^{*2}}{n^2}(\Omega_{cr}^2 - k^*) - 4) = 0 \tag{23}$$

$$\frac{1}{n^4}((\Omega_{cr}^2 - k^*)(\mu_s\, r^{*4}\Omega_{cr}^2 - 1)) = 0 \tag{24}$$

Therefore, two branches of critical frequencies would be obtained as follows

$$\Omega_{cr_{1,1}} = \frac{2n}{r^*}\quad ,\quad \Omega_{cr_{1,2}} = \sqrt{\frac{4n^2}{\mu_s r^{*2}} + k^*} \tag{25}$$

$$\Omega_{cr_{2,1}} = \sqrt{k^*}\quad ,\quad \Omega_{cr_{2,2}} = \sqrt{\frac{1}{\mu_s r^{*4}}} \tag{26}$$

The critical frequencies of the first branch depend on the grain number (microstructure parameter), mechanical properties and beam geometry (macrostructure parameters) while the second branch critical frequencies are only defined as a function of the beam mechanical properties and geometry. On the other hand, comparing these critical values with the those of the Timoshenko continuum beam resting on the Winkler foundations [10], leads to the equivalency of the second branch critical values (26) to the Timoshenko continuum beam's. For infinite number of grains, since the first branch critical frequencies ($\Omega_{cr_{1,1}}$ and $\Omega_{cr_{1,2}}$) leads to infinite values and consequently disappear, so only the second branch would remain. These critical values could be shown as follows

$$\omega_{cr_{2,1}} = \omega_{cr_{Timoshenko\,1}} = \sqrt{\frac{k}{\rho A}} \quad , \tag{27}$$

$$\omega_{cr_{2,2}} = \omega_{cr_{Timoshenko\,2}} = \sqrt{\frac{K_s\,GA}{\rho I}}$$

It can be obtained, the behavior of the beam deflection solution would be separated by the critical frequencies into different regimes and depending on the frequencies values, the results would be in distinct manner.

For finite number of grains, four regimes would be occurred categorized as follows: when $0 < \Omega < \Omega_{cr_{2,2}}$ there are two exponential terms and two travelling waves since $\lambda_{1,2}$ are real and $\lambda_{3,4}$ are imaginary. In this case the deflection equation form would be obtained from (38).

When $\Omega_{cr_{2,2}} < \Omega < \Omega_{cr_{1,2}}$, $\lambda_{1,2,3,4}$ are all imaginary and therefore all terms represent travelling waves and for this case the deflection equation form would be obtained from (39) .

For $\Omega_{cr_{1,2}} < \Omega < \Omega_{cr_{1,1}}$ again there are two exponential terms and two travelling waves since $\lambda_{1,2}$ are imaginary and $\lambda_{3,4}$ are real and the deflection equation form would be obtained from (40).

Finally, for $\Omega_{cr_{1,2}} < \Omega$, since all parameters of $\lambda_{1,2,3,4}$ are real, thus whole terms represent exponential terms which leads to the deflection equation form of (41).

For specific value of grain number (n_s) $\Omega_{cr_{1,1}}$ and $\Omega_{cr_{2,2}}$ would be equal together. This leads to the reduction of the four regimes to three.

$$n_s = \frac{1}{2r^*\sqrt{\mu_s}} = \frac{L}{2}\sqrt{\frac{K_s\,GA}{EI}} \tag{28}$$

The results are shown for a case study of 50 grains and the dimensionless parameters of $\mu_s = 4.28$, $r^* = 0.029$, $k^* = 15$ in figure 2. In this example the values of the critical frequencies are respectively $\Omega_{cr_{1,1}} = 3464.1$, $\Omega_{cr_{1,2}} = 1673.7$, $\Omega_{cr_{2,2}} = 3.87$ and $\Omega_{cr_{2,2}} = 579.8$.

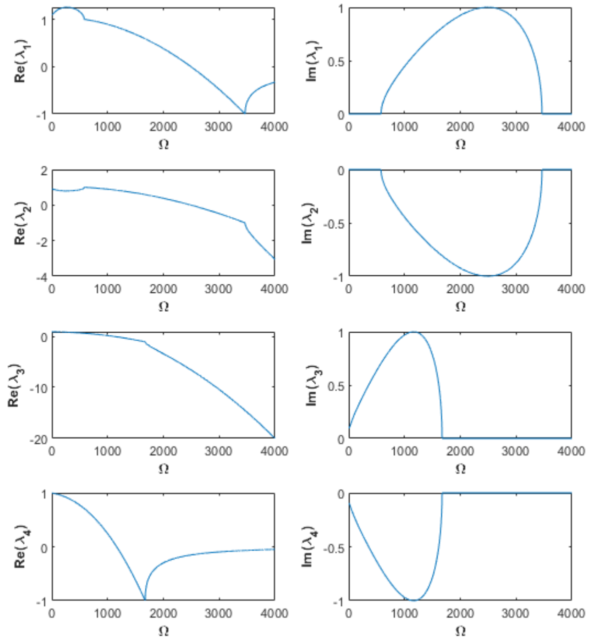

Fig. 2 Schematic behavior of the wave vector regarding to the eigen frequencies for finite grain number (n=50). The left and right panels correspond to the real and imaginary part of the wave vector.

For infinite number of grains, since the first two critical values converge to the infinite, so the previous different regimes reduce to two regimes: when $0 < \Omega < \Omega_{cr_{2,2}}$ there are two exponential terms and two travelling waves as $\lambda_{1,2}$ are real and $\lambda_{3,4}$ are imaginary and thus the deflection equation form would be obtained from (38).

When $\Omega_{cr_{2,2}} < \Omega$, all $\lambda_{1,2,3,4}$ are all imaginary and all terms of represent travelling waves. This case leads to the deflection equation form of (39). These two regimes correspond to the ones obtained for the continuum beam of Timoshenko resting on Winkler foundations.

Therefore, $\lambda_{1,2}$ can be rewritten for $\Omega < \Omega_{cr_{2,2}}$ and $\Omega_{cr_{1,2}} < \Omega$ ([23] for a similar presentation applied to the finite difference formulation of Euler-Bernoulli beams) as

$$\lambda_{1,2} = \cosh\vartheta \pm \sinh\vartheta \tag{29}$$

where

$$\cos\varphi = \frac{-\epsilon}{4} - \frac{1}{2}\sqrt{\left(\frac{-\epsilon}{2}\right)^2 - \tau} \tag{30}$$

$$\cosh\vartheta = \frac{-\epsilon}{4} + \frac{1}{2}\sqrt{\left(\frac{-\epsilon}{2}\right)^2 - \tau} = \frac{-\epsilon}{2} - \cos\varphi \tag{31}$$

while $\lambda_{1,2}$ would be obtained for $\Omega_{cr_{2,2}} < \Omega < \Omega_{cr_{1,2}}$

$$\lambda_{1,2} = \cos\vartheta \pm j\sin\vartheta \tag{32}$$

where

$$\boldsymbol{\cos\vartheta = \frac{-\epsilon}{4} + \frac{1}{2}\sqrt{\left(\frac{-\epsilon}{2}\right)^2 - \tau}} \tag{33}$$

On the other hand, $\lambda_{3,4}$ would be defined for $\Omega < \Omega_{cr_{1,1}}$ by

$$\boldsymbol{\lambda_{3,4} = cos\varphi \pm jsin\varphi} \tag{34}$$

where

$$\boldsymbol{\varphi = arccos\,(\frac{-\epsilon}{4} - \frac{1}{2}\sqrt{\left(\frac{-\epsilon}{2}\right)^2 - \tau})} \tag{35}$$

And for $\Omega_{cr_{1,1}} < \Omega$

$$\boldsymbol{\lambda_{3,4} = cosh\varphi \pm sinh\varphi} \tag{36}$$

$$\boldsymbol{\varphi = arccosh\,(\frac{-\epsilon}{4} - \frac{1}{2}\sqrt{\left(\frac{-\epsilon}{2}\right)^2 - \tau})} \tag{37}$$

There are three possible general solutions for w_i depending on the critical values of the frequencies which may be represented as

$$w_i = A_1 \cos i\varphi + A_2 \sin i\varphi + A_3 \cosh i\vartheta + A_4 \sinh i\vartheta \quad (\Omega < \Omega_{cr_{2,2}}) \tag{38}$$

$$w_i = B_1 \cos i\varphi + B_2 \sin i\varphi + B_3 \cos i\vartheta + B_4 \sin i\vartheta \quad (\Omega_{cr_{2,2}} < \Omega < \Omega_{cr_{1,1}}) \tag{39}$$

$$w_i = C_1 \cosh i\varphi + C_2 \sinh i\varphi + C_3 \cos i\vartheta + C_4 \sin i\vartheta \quad (\Omega_{cr_{1,1}} < \Omega < \Omega_{cr_{1,2}}) \tag{40}$$

$$w_i = D_1 \cosh i\varphi + D_2 \sinh i\varphi + D_3 \cosh i\vartheta + D_4 \sinh i\vartheta \quad (\Omega_{cr_{1,2}} < \Omega) \tag{41}$$

Considering simply supported boundary conditions, the deflection of each grain could be obtained by following equation while $\Omega < \Omega_{cr_{1,2}}$

$$\boldsymbol{w_i = \mathrm{B}\sin\left(\frac{ip\pi}{n}\right)} \tag{42}$$

where p is the mode number or natural number ($1 \le p < n$ for w_i) and i is the grain number ($0 \le i \le n$). Using non-dimensional eigenfrequency parameters and (35)

$$\boldsymbol{\begin{aligned}\left[\frac{\mu_s r^{*4}}{n^4}\right]\Omega^4 + \Bigg[\frac{2r^{*2}}{n^2}\left(1 + \mu_s - \frac{1}{4r^{*2}n^2}\right) cos\left(\frac{p\pi}{n}\right) - \frac{\mu_s r^{*4}k^*}{n^4} \\ - \frac{1}{2n^4} - \frac{2r^{*2}}{n^2}(1+\mu_s)\Bigg]\Omega^2 \\ + \Bigg[2(-\frac{r^{*2}k^*\mu_s}{n^2} + \frac{k^*}{4n^4} \\ - 4)\,cos\left(\frac{p\pi}{n}\right) + \frac{k^*}{2n^4} + \frac{2r^{*2}k^*\mu_s}{n^2} + 4 \\ + 4(cos\left(\frac{p\pi}{n}\right))^2\Bigg] = 0\end{aligned}} \tag{43}$$

The natural frequencies of the granular chain represented in figure 1 could be presented in a single form

$$\boldsymbol{\omega = \frac{\gamma}{L^2}\sqrt{\frac{EI}{\rho A}}} \tag{44}$$

where the dimensionless parameter γ would be substituted by the following equation:

$$\gamma = \sqrt{-\frac{n^2}{\mu_s r^{*2}}\left(1 + \mu_s - \frac{1}{4r^{*2}n^2}\right) cos\left(\frac{p\pi}{n}\right) + \frac{k^*}{2} + \frac{1}{4\mu_s r^{*4}} + \frac{n^2}{\mu_s r^{*2}}(1+\mu_s) \pm \sqrt{(\frac{n^2}{\mu_s r^{*2}})^2 + \cdots}}. \tag{45}$$

By considering low mode number ($p<<n$) and for the continuum case when $n \to \infty$, the assumption of $cos\left(\frac{p\pi}{n}\right) \sim 1 - \frac{1}{2}(\frac{p\pi}{n})^2$ could be applied to (43). This leads to

$$\boldsymbol{\begin{aligned}\Omega^4 - \left[\frac{p^2\pi^2}{\mu_s r^{*2}}(1+\mu_s) + k^* + \frac{1}{\mu_s r^{*4}}\right]\Omega^2 \\ + \left[(\frac{k^* p^2\pi^2}{r^{*2}} + \frac{p^4\pi^4}{\mu_s r^{*4}}) + \frac{k^*}{\mu_s r^{*4}}\right] = 0\end{aligned}} \tag{46}$$

Solving the quartic equation of (46) leads to the eigenfrequency values of the continuous beam that would be obtained again by (44) and with γ expressed as follows:

$$\gamma = \sqrt{\frac{p^2\pi^2}{2\mu_s r^{*2}}(1+\mu_s) + \frac{k^*}{2} + \frac{1}{2\mu_s r^{*4}} \pm \sqrt{(\frac{p^2\pi^2}{2\mu_s r^{*2}}(1+\mu_s) + \frac{k^*}{2} + \frac{1}{2\mu_s r^{*4}})^2 + \cdots)}} \tag{47}$$

The sensitivity analysis is performed for the granular chain by assuming the following set of dimensionless parameters

$$\mu_s = 4.28,\ r^* = 0.029,\ k^* = 15 \tag{48}$$

In figure 4, the frequency results obtained by the exact solution of the discrete lattice model have been compared with those of Duan et al. [18] for four grain number values *(n=5; n=20; n=35; n=50)*. For some values of mode number *(p)* and grain number *(n)*, the two branch frequencies of Duan et al. [18] contain real and imaginary part that physically doesn't have any sense. Therefore, this model can't cover all eigenfrequencies existed for the granular beam for certain amounts of mode number *(p)* and grain number *(n)*.

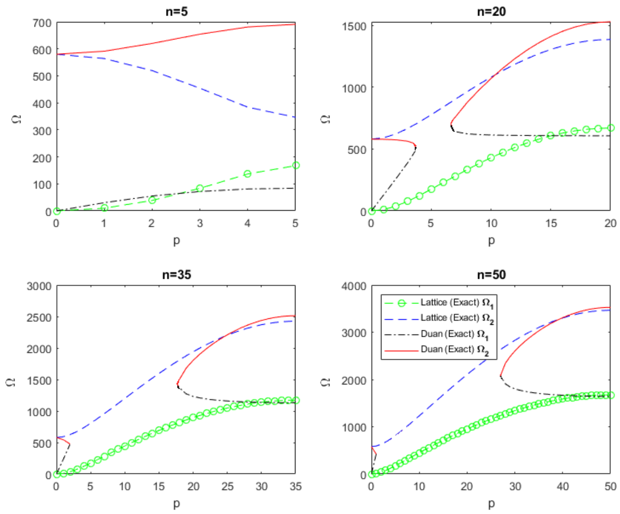

Fig. 4 Comparison of the natural frequencies for the proposed lattice model and the model of Duan et al. (2013) by using lattice exact solution with respect to the mode number (p) and grain number ($n \in \{5, 20, 35, 50\}$) for $\mu_s = 4.28$, $r^* = 0.029$ and $k^* = 0$

B. Continuum Solution

In the limit case for the continuum beam, the fourth-order differential equation including the Winkler elastic foundation could be considered in dimensionless form

$$\frac{d^4\bar{w}}{d\bar{x}^4} + [r^{*2}\Omega^2(1+\mu_s) - r^{*2}k^*\mu_s]\frac{d^2\bar{w}}{d\bar{x}^2} - \left[r^{*2}\Omega^2\left(\mu_s r^{*2}k^* + \frac{1}{r^{*2}} - \mu_s r^{*2}\Omega^2\right) - k^*\right]\bar{w} = 0 \qquad (49)$$

Equation (49) is obtained by Wang and Stephens [10] and the non-dimensional parameters can be introduced

$$\bar{x} = \frac{x}{L}\ ,\ \bar{w} = \frac{w}{L}\ , \frac{d^2\bar{w}}{d\bar{x}^2} = L\frac{d^2w}{dx^2}\ ,\ \frac{d^4\bar{w}}{d\bar{x}^4} = L^3\frac{d^4w}{dx^4} \qquad (50)$$

For simply supported beam, the solution can be proposed by the form $\bar{w}(\bar{x}) = \sin(p\pi\bar{x})$. Substituting in (49) leads to the following quartic frequency equation.

$$[\mu_s r^{*4}]\Omega^4 - \left[r^{*2}\left(\mu_s r^{*2}k^* + \frac{1}{r^*}\right) + r^{*2}p^2\pi^2(1+\mu_s)\right]\Omega^2 + [r^{*2}k^*\mu_s p^2\pi^2 + k^* + p^4\pi^4] = 0 \qquad (51)$$

The results for the two branches of eigenfrequencies have been shown in figure 5, both for the equivalent continuum beam and the exact discrete one with respect to the mode number *(p)* and for four grain number values (*n=5; n=20; n=35; n=50*). It can be concluded that the increase rate in the frequencies of the second branch is more pronounced in comparison with the first branch. Furthermore, the exact solution of the discrete model always predicts lower frequencies than the continuum one. As it is expected by increasing the ratio of *n/p* the results of the discrete model converge to the continuous ones. The coincidence of the results could be considered happen for the second branch when the ratio of *n/p* is typically higher than approximate value of *5* while this approximate limit value is typically *3* for the first branch. In figure 6, the effect of length ratio (beam thickness/beam length) regarding to the grain number has been studied for two typical mode number (*p=1* and *p=10)*. The minimum values of required grain number (*n**) have been also determined and reported when the difference of the discrete and continuum results start to be smaller than *1%*. It can be concluded generally that in order to achieve the continuum results from discrete solution, whether the length ratio decrease or the mode number increase, the grain number value needs to increase.

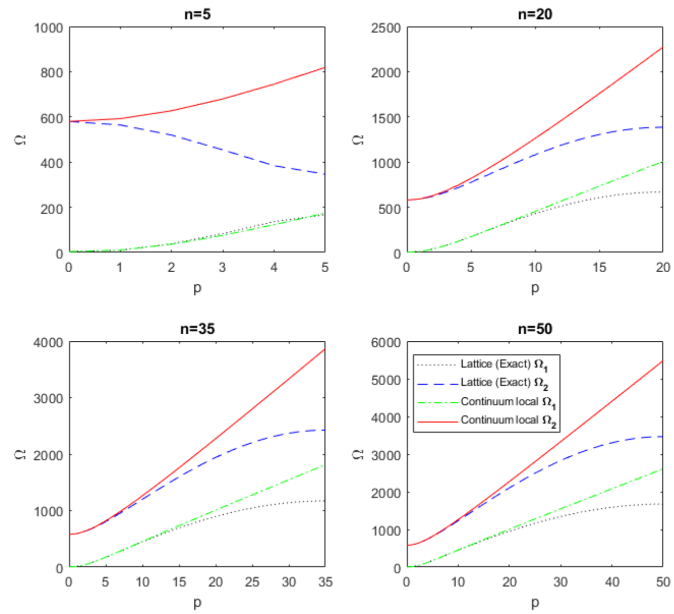

Fig. 5 Comparison of the natural frequencies for the discrete exact and continuum solutions with respect to the mode number (p) and grain number ($n \in \{5, 20, 35, 50\}$) for $\mu_s = 4.28$, $r^* = 0.029$ and $k^* = 15$

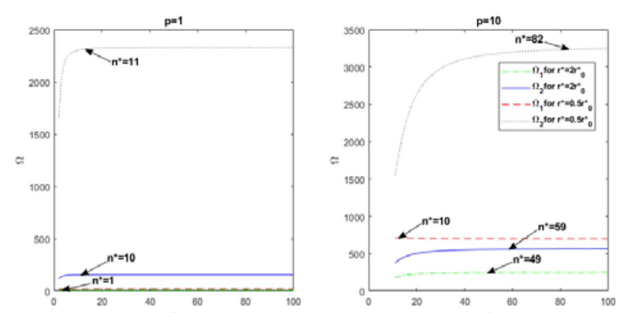

Fig. 6 Analysis of the grain number effect on the frequencies with respect to the length ratio ($r^*_0 = 0.029$) for $\mu_s = 4.28$ and $k^* = 15$

IV. Conclusion

This paper investigates the macroscopic free vibration behavior of a discrete granular system resting on a Winkler elastic foundation. This microstructured system consists of uniform grains elastically connected by shear and rotation springs. It is shown that the discrete deflection equation of this granular system (Cosserat chain) is mathematically equivalent to the finite difference formulation of a shear deformable Bresse-Timoshenko beam resting on Winkler foundation. Next, the natural frequencies of such a granular model with simply supported ends are first analytically investigated, whatever considered modes through the resolution of a linear difference equation. The scale effects of the granular chain are clearly captured by the continuous gradient elasticity model. This scale effect is related to the grain size with respect to the total length of the Cosserat chain.

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