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## Dynamic Analysis of Granular Chain Using Cosserat Discrete Modeling

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**Summary** This paper deals with the free vibration of a beam which is discretized by finite rigid granular elements and compared its exact solution to continuum local elasticity. This problem, which can be considered as a simple model to rigorously study the effect of the microstructure on the dynamic behavior of a equivalent continuum structural model, can be referred to Cosserat discrete chain or a lattice elastic model with shear interaction. This micro structured system consists of uniform grains elastically connected by shear and rotation springs. First the critical frequencies of the system which change the nature of the results, are obtained. Next, the natural frequencies of such as granular model are analytically calculated whatever the considered modes for the granular beam resting on two supports, starting from the resolution of the linear difference eigenvalue problem. It is shown that the discrete equations of this granular system, for an infinite number of grains, converge to the differential equations of the Bresse-Timoshenko beam resting on Winkler foundation (such a Bresse-Timoshenko beam can be also classified as a continuous Cosserat beam model). A gradient Bresse-Timoshenko model is constructed from continualization of the difference equations of the granular system. This continuous gradient elasticity Cosserat model is obtained from a polynomial or a rational expansion of the pseudodifferential operators, stemming from the continualization process. The natural frequencies of the continuous gradient Cosserat models are compared with those of the discrete Cosserat model associated with the granular chain. Scale effects of the granular chain are clearly captured by the continuous gradient elasticity model. The results clarify the dependency of the beam dynamic responses to the beam length ratio.

**Keywords:** Cosserat continuum, Discrete Cosserat formulation, Gradient elasticity, Timoshenko beam.

### Introduction

In order to adapt a theory of continuous media to granular materials, it is necessary to introduce independent degrees of freedom of rotation, in addition to those conventional translation. Indeed, the relative movements between the microstructure and the average macroscopic deformations can be apprehended by additional degrees of freedom. Such enriched kinematics leads to non-classical continuous media or Cosserat-type theories [1], [2] which introduce intrinsic length scales via higher order gradients or additional degrees of freedom. Conversely, the classical mechanics of continuous media does not incorporate a rotational interaction between the particles, and does not allow to understand the size effects in these media.

The Bresse-Timoshenko beam model admits kinematics with two independent fields of transverse displacement and rotation [3], [4]. The exact calculation of natural frequencies for a Bresse-Timoshenko beam with any boundary conditions and in elastic interaction with a rigid medium was obtained by Wang and Stephens [5], [6]. Timoshenko beam is an example of Cosserat one-dimensional continuum considering the independent double rotation-displacement kinematics. The present study focuses on the vibration of the granular beam model resting on a linear Winkler foundation. Note that the difference equations of this granular chain coincide with the difference equations of the granular model of Pasternak and Mühlhaus [7] in the absence of elastic foundation, while differ from the ones of Duan et al [8] or Bacigalupo and Gambarotta [9].

### Granular Model

A granular beam of length  $L$  resting on two simple supports is modeled by a finite number of grains interacting together. Such a model could be presented by considering the microstructured granular chain comprising  $n+1$  rigid grain with diameter  $a$  ( $a=L/n$ ) that are connected by  $n$  shear and rotational springs, as shown in figure 1.

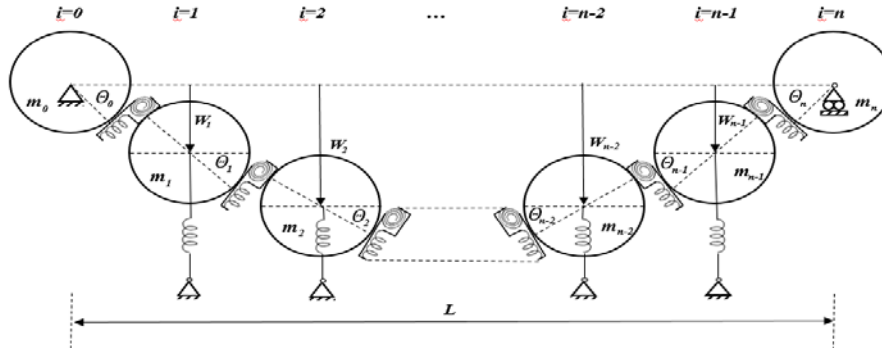


Figure 1. A discrete shear granular chain composed of  $n+1$  grain

The Lagrangian relation of the granular system may be defined as

$$L = \left[ \frac{1}{2} \sum_{i=0}^n m_i \dot{W}_i^2 + \frac{1}{2} \sum_{i=0}^n I_{m_i} \dot{\theta}_i^2 \right] - \left[ \frac{1}{2} \sum_{i=0}^{n-1} S \left( W_{i+1} - W_i - a \frac{\theta_{i+1} + \theta_i}{2} \right)^2 + \frac{1}{2} \sum_{i=0}^{n-1} C (\theta_{i+1} - \theta_i)^2 + \frac{1}{2} \sum_{i=0}^n K W_i^2 \right] \quad (1)$$

where  $S$  is the shear stiffness and defined by  $S = \frac{K_s GA}{a} = \frac{nK_s GA}{L}$ .  $G$  is the shear modulus;  $A$  is the cross-sectional area of the beam and  $K_s$  is an equivalent shear correction coefficient.  $C$  is the rotational stiffness and can be expressed as  $C = \frac{EI}{a} = \frac{nEI}{L}$ , where  $E$  is Young's modulus and  $I$  is the second moment of area.  $K=ka$  is the discrete stiffness of the elastic support.

From the Euler-Lagrange of the granular system by assuming a harmonic motion  $W_i = w_i e^{i\omega t}$  and  $\theta_i = \theta_i e^{i\omega t}$  with  $j^2 = -1$ , the following deflection equation could be obtained

$$[EI\delta_2^2 + \left(\rho I \omega^2 - \frac{kEI}{K_s GA} + \frac{EI\rho\omega^2}{K_s G}\right)\delta_2 + (k - \rho A \omega^2)\delta_0 - \frac{k\rho I \omega^2}{K_s GA} + \frac{\rho^2 I \omega^4}{K_s G}]w_i = 0 \quad (2)$$

The pseudo-difference parameters are defined as:

$$\delta_0 W_i = \frac{W_{i+1} + 2W_i + W_{i-1}}{4}, \quad \delta_1 W_i = \frac{W_{i+1} - W_{i-1}}{2a}, \quad \delta_2 W_i = \frac{W_{i+1} - 2W_i + W_{i-1}}{a^2} \quad (3)$$

The fourth-order difference equation (2) is equivalent to the one of Challamel et al. [10] in the static range ( $\omega = 0$ ). Considering infinite number of grains ( $n \rightarrow \infty$ ) for the continuum beam, the fourth-order differential equation valid for a Bresse-Timoshenko beam on Winkler elastic foundation is given by (4) which also could be compared well by [5], [6].

$$\frac{d^4 w}{dx^4} + \left(\frac{\rho\omega^2}{E} \left(1 + \frac{E}{k_s G}\right) - \frac{k}{k_s GA}\right) \frac{d^2 w}{dx^2} - \left(\frac{\rho\omega^2}{E} \left(\frac{A}{I} + \frac{k}{k_s GA} - \frac{\rho\omega^2}{k_s G}\right) - \frac{k}{EI}\right) w = 0 \quad (4)$$

### Exact solution

For simply supported boundary conditions by assuming  $w_0 = 0$ ;  $\delta_1 \theta_0 = 0$  and  $w_n = 0$ ;  $\delta_1 \theta_n = 0$  the frequencies of the discrete system could be obtained by (5). The results are shown in figure 2.

$$\omega = \frac{1}{L^2} \sqrt{\frac{EI}{\rho A} \left[ \frac{n^2}{\mu_s r^2} \left(1 + \mu_s - \frac{1}{4r^2 n^2}\right) \cos\left(\frac{p\pi}{n}\right) + \frac{k}{2} + \frac{1}{4\mu_s r^2} + \frac{n^2}{\mu_s r^2} (1 + \mu_s) \pm \sqrt{\left(\frac{n^2}{\mu_s r^2} \left(1 + \mu_s - \frac{1}{4r^2 n^2}\right) \cos\left(\frac{p\pi}{n}\right) - \frac{k}{2} - \frac{1}{4\mu_s r^2} - \frac{n^2}{\mu_s r^2} (1 + \mu_s)\right)^2 - \frac{n^4}{\mu_s r^4} \left(-\frac{r^2 k \mu_s}{n^2} + \frac{k}{4n^2} - 4\right) 2 \cos\left(\frac{p\pi}{n}\right) + \frac{k}{2n^2} + \frac{2r^2 k \mu_s}{n^2} + 4 + 4 \left(\cos\left(\frac{p\pi}{n}\right)\right)^2} \right]} \quad (5)$$

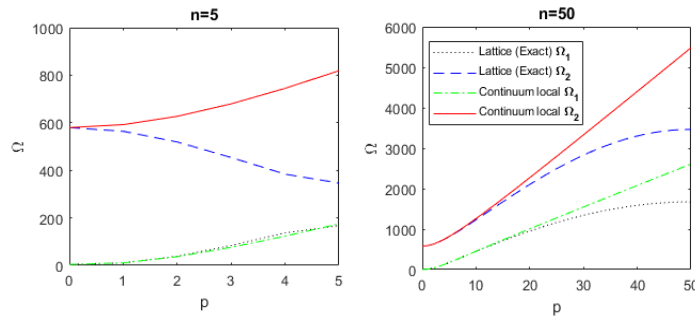


Figure 2. Comparison of the natural frequencies for the discrete exact and continuum solutions with respect to the mode number ( $p$ ) and grain number ( $n \in \{5, 50\}$ ) for  $\mu_s = 4.28$ ,  $r^* = 0.029$  and  $k^* = 15$

### CONCLUSIONS

This paper investigates the macroscopic free vibration behavior of a discrete granular system resting on a Winkler elastic foundation. It is shown that the discrete deflection equation of this granular system (Cosserat chain) is mathematically equivalent to the finite difference formulation of a shear deformable Bresse-Timoshenko beam resting on Winkler foundation. Next, the natural frequencies of such a granular model with simply supported ends are first analytically investigated, whatever considered modes through the resolution of a linear difference equation. The scale effects of the granular chain are clearly captured by the continuous gradient elasticity model. This scale effect is related to the grain size with respect to the total length of the Cosserat chain.

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