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# MULTIPLE SCATTERING BY A RANDOM DISTRIBUTION OF SPHERICAL FLUID CAVITIES IN A POROUS MEDIUM, 2020

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## ABSTRACT

A longitudinal wave (fast or slow) or a transverse wave incident on a spherical inhomogeneity localized in a fluid-saturated poro-elastic medium causes the scattering by this inhomogeneity of three waves (fast, slow and transverse). After having dealt with the problem of scattering by an inhomogeneity, we examine that of the multiple scattering by a random distribution of spheres of equal radii, in the low frequency regime known as the Rayleigh regime. In this context, the influence of certain effective quantities (wavenumbers, densities and moduli) of the poro-elastic matrix/fluid cavities continuity conditions is highlighted. Some analytical and numerical results will be discussed.

## 1. INTRODUCTION

In nature or in the manufacturing industry, heterogeneous porous materials constitute a vast field of study. We can distinguish in particular the materials formed from a poro-elastic matrix in which are randomly distributed cavities (scatterers) of size much larger than that of the pores and, generally, of very varied, non-canonical shapes. This last point makes an analytical study of the propagation of poroelastic waves delicate, especially since sometimes, it is necessary to take into account a probable interpenetration of the scatterers. In the present study, we assume that the scatterers are identical spheres of radius  $a$ . The dilute medium assumption is considered - low concentration, scatterers sufficiently distant from each other - so that the theory developed by Linton and Martin [1] can be applied. The acoustic scattering by a spherical inhomogeneity in a poroelastic matrix obeying Biot's theory [2] is the subject of Ref. [3]. Here, the inhomogeneities are spherical cavities and we examine only the question of the multiple scattering of the fast incident wave by multiple spheres.

## 2. SCATTERING BY A SPHERICAL INHOMOGENEITY

Let us consider a spherical fluid cavity of radius  $a$  centered at the origin of the axes and localized in a poroelastic matrix (the saturating fluid is the same as in the cavity). The physical properties of the poroelastic matrix and of the fluid are given in Table 1. A full description of the theory and of the wave propagation can be found in Biot [2]

and Berryman [4] ( $e^{-i\omega\tau}$  dependence with  $\omega$  the angular frequency and  $\tau$  the time is assumed everywhere). Such medium sustains two longitudinal waves (fast and slow) of respective wavenumber  $k_1$  and  $k_2$ , and a transverse wave of wavenumber  $k_t$ . The wavenumber in the fluid will be referred to as  $k_f$ . When a plane harmonic wave of type  $\alpha$  ( $= 1, 2$ ) is incident on the sphere, it gives rise to scattered spherical waves of type  $\beta$  ( $= 1, 2$  or  $t$ ).

Let  $U_r$  be the radial displacement for the solid,  $P$  the pore pressure and  $\sigma_{rr}$ ,  $\sigma_{r\theta}$  the components of the stress tensor in the poroelastic medium. In the fluid cavity, the pressure is  $P_f$  and the radial displacement  $u_r$ . The application of the boundary conditions at  $r = a$

$$(U_r)_\alpha + (U_r)_{sc} = u_r \quad (1)$$

$$(P)_\alpha + (P)_{sc} = (P)_f \quad (2)$$

$$(\sigma_{rr})_\alpha + (\sigma_{rr})_{sc} = -P_f \quad (3)$$

$$(\sigma_{r\theta})_\alpha + (\sigma_{r\theta})_{sc} = 0 \quad (4)$$

yields a linear system of 4 equations with 4 unknowns, of the form

$$A_n \vec{X}_n^\alpha = \vec{S}_n^\alpha, \quad (5)$$

where the letter  $\alpha$  refers to the type of the incident longitudinal wave ( $\alpha = 1, 2$ ) and  $sc$  to the scattered parts. The matrix  $A_n$  and the vectors  $\vec{X}_n^\alpha$  and  $\vec{S}_n^\alpha$  are detailed in the Appendix. For  $\alpha = 1$  et 2, the far-field scattered amplitudes are given by

$$\frac{1}{a} f^{\alpha\beta}(\theta) = \frac{1}{ix_\beta} \sum_{n=0}^{\infty} (2n+1) t_n^{\alpha\beta} P_n(\cos\theta) \quad (6)$$

where  $\beta = 1, 2, t$  and where  $P_n(\cos\theta)$  is the Legendre polynomial of degree  $n$  and  $\theta$  the angle of the scattering.

## 3. MULTIPLE SCATTERING EQUATIONS

According to the theory of Lloyd and Berry [5] applied first to the classical physics by Linton and Martin [1], and extended later by Luppé, Conoir and Norris [6], a wave incident on an assembly of randomly distributed sphere, with the wavenumber  $k_\alpha$  gives rise to an effective wavenumber  $\zeta_\alpha$  of the form

$$\zeta_\alpha^2 = k_\alpha^2 + n_0 \delta_1^\alpha + n_0^2 (\delta_2^{\alpha,0} + \delta_2^{\alpha,c}) + O(n_0^3). \quad (7)$$

This expansion with respect to  $n_0$  - the number of spheres per unit volume - assumes that the concentration of scatterer is low (dilute case), so that terms of order  $n_0^3$  and the followings can be neglected. At very low frequency, the first terms appearing in this expansion, which are infinite series, can be truncated. We get

$$\delta_1^\alpha = \frac{4\pi}{ik_\alpha} [t_0^{\alpha\alpha} + 3t_1^{\alpha\alpha} + \dots], \quad (8)$$

$$\delta_2^{\alpha,0} = -\frac{1}{2} \left(\frac{4\pi}{k_\alpha}\right)^4 [K_{00} t_0^{\alpha\alpha} t_0^{\alpha\alpha} + 2K_{01} t_0^{\alpha\alpha} t_1^{\alpha\alpha} + K_{11} t_1^{\alpha\alpha} t_1^{\alpha\alpha} + \dots], \quad (9)$$

$$\delta_2^{\alpha,c} = -\frac{1}{2} \left(\frac{4\pi}{k_\alpha}\right)^4 \sum_{\beta \neq \alpha} \frac{2k_\alpha^3}{k_\beta(k_\alpha^2 - k_\beta^2)} \times [K_{00}^{\alpha\beta} t_0^{\beta\alpha} t_0^{\alpha\beta} + K_{01}^{\alpha\beta} t_0^{\beta\alpha} t_1^{\alpha\beta} + K_{10}^{\alpha\beta} t_1^{\beta\alpha} t_0^{\alpha\beta} + K_{11}^{\alpha\beta} t_1^{\beta\alpha} t_1^{\alpha\beta} + \dots]. \quad (10)$$

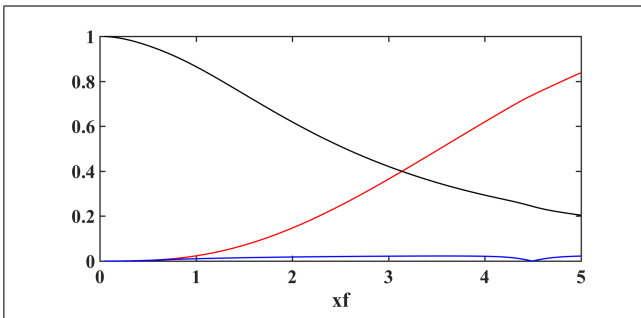
The dots in the brackets mean that, the scattering coefficients  $t_{n>1}^{\alpha\beta}$ , which are very small with respect to  $t_{n \leq 1}^{\alpha\beta}$ , have been neglected ( $t_0^{\alpha\beta}$  and  $t_1^{\alpha\beta}$  are  $O(x_\alpha^3)$  whereas  $t_{n>1}^{\alpha\beta}$  is  $O(x_\alpha^5)$ ) (see figure 1, 2 and 3). The coefficients  $K_{nm}$  and  $K_{nm}^{\alpha\beta}$  [6] are such that  $K_{00} = 0$ ,  $K_{01} = K_{10} = 3/(16\pi^2)$ ,  $K_{11} = 3/4\pi^2$  and

$$K_{00}^{\alpha\beta} = \frac{1}{16\pi^2}, \quad (11)$$

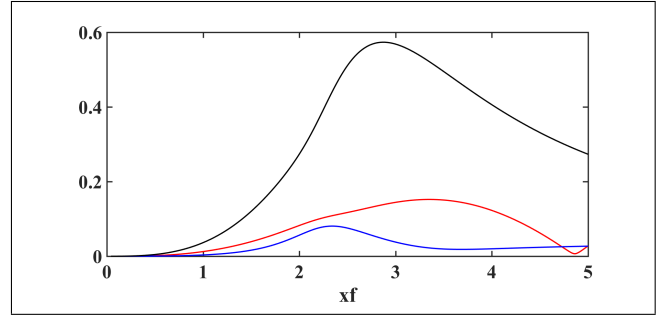
$$K_{01}^{\alpha\beta} = \frac{3}{16\pi^2} \frac{k_\alpha}{k_\beta} (= K_{10}^{\alpha\beta}), \quad (12)$$

$$K_{11}^{\alpha\beta} = \frac{3}{16\pi^2} [1 + 2\left(\frac{k_\alpha}{k_\beta}\right)^2]. \quad (13)$$

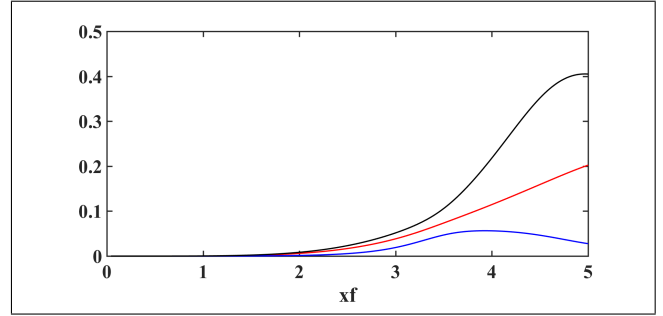
The terms  $\delta_1^\alpha$  (figure 4) and  $\delta_2^{\alpha,0}$  (figure 5) involve only scattering coefficients  $t_0^{\alpha\alpha}$  and  $t_1^{\alpha\alpha}$  accounting for the scattering of an  $\alpha$ -wave into an  $\alpha$ -wave. For its part, the term  $\delta_2^{\alpha,c}$  (fig 6) accounts for mode conversions of an  $\alpha$ -wave into an  $\beta$ -wave, with  $\beta$  different from  $\alpha$ . As shown by the plot in fig 4 and 5, the contribution of the  $\delta_2^{\alpha,0}$  is very small compared to the  $\delta_1^\alpha$  term. The same remark holds for the  $\delta_2^{\alpha,c}$ . The comparison between the modulus of the effective wavenumber at order 1 ( $k_\alpha + n_0\delta_1^\alpha$ ), at order 2 ( $k_1^2 + n_0\delta_1^1 + n_0^2(\delta_2^{\alpha,0} + \delta_2^{\alpha,c})$ ) and the wavenumber  $k_1$  (free of spheres) is shown in fig(7).



**Figure 1.** Scattering amplitude versus  $xf$ ;  $t_0^{11}$ (red),  $t_0^{12}$ (blue) and  $t_0^{1t}$ (black)



**Figure 2.** Scattering amplitude versus  $xf$ ;  $t_1^{11}$ (red),  $t_1^{12}$ (blue) and  $t_1^{1t}$ (black)



**Figure 3.** Scattering amplitude versus  $xf$ ;  $t_2^{11}$ (red),  $t_2^{12}$ (blue) and  $t_2^{1t}$ (black)

#### 4. APPENDIX

The elements of the matrix  $A_n$  are the  $a_{n,ij}$ s, ( $i, j = 1, 2, \dots, 4$ ) given by

$$a_{11} = (1 + \gamma_1)x_1 h'_n(x_1) \quad (14)$$

$$a_{12} = (1 + \gamma_2)x_2 h'_n(x_2) \quad (15)$$

$$a_{13} = (1 + \gamma_t)\bar{n}h_n(x_t) \quad (16)$$

$$a_{14} = -x_f j'_n(x_f) \quad (17)$$

$$a_{21} = \frac{\rho_f}{\rho_f} h_n(x_1) \quad (18)$$

$$a_{22} = \frac{\rho_f}{\rho_f} h_n(x_2) \quad (19)$$

$$a_{23} = 0 \quad (20)$$

$$a_{24} = -j_n(x_f) \quad (21)$$

$$a_{31} = [2\bar{n} - \frac{\rho_1}{\rho_t} x_t^2] h_n(x_1) - 4x_1 h'_n(x_1) \quad (22)$$

$$a_{32} = [2\bar{n} - \frac{\rho_2}{\rho_t} x_t^2] h_n(x_2) - 4x_2 h'_n(x_2) \quad (23)$$

$$a_{33} = 2\bar{n}[x_t h'_n(x_t) - h_n(x_t)] \quad (24)$$

$$a_{34} = \frac{\rho_f}{\rho_t} x_t^2 j_n(x_f) \quad (25)$$

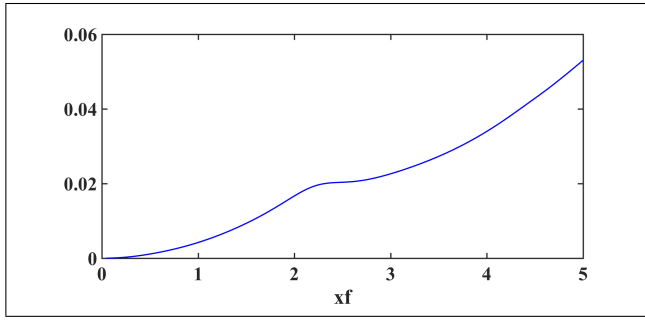
$$a_{41} = h_n(x_1) - x_1 h'_n(x_1) \quad (26)$$

$$a_{42} = h_n(x_2) - x_2 h'_n(x_2) \quad (27)$$

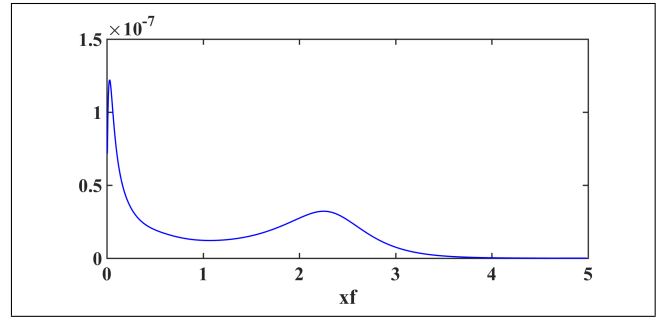
$$a_{43} = x_t h'_n(x_t) + [1 - \bar{n} + \frac{x_t^2}{2}] h_n(x_t) \quad (28)$$

$$a_{44} = 0 \quad (29)$$

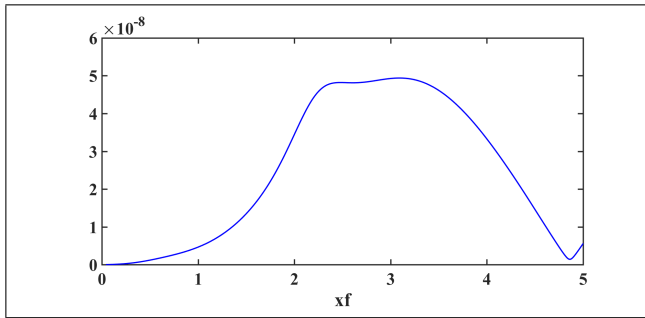
For convenience, we introduced the notations  $x_j = k_j a$  ( $j = f, 1, 2, t$ ) and  $\bar{n} = n(n+1)$ . The components of the



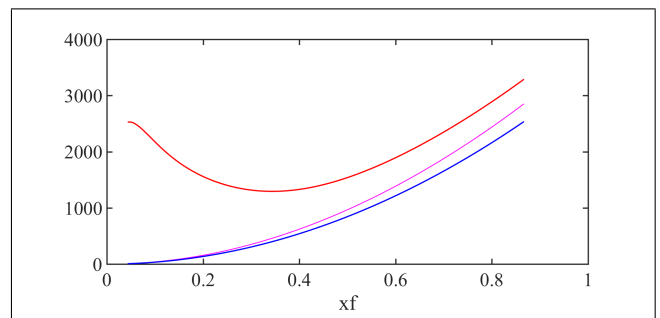
**Figure 4.** Modulus of the coefficient  $\delta_1^1$  versus  $x_f$



**Figure 6.** Modulus of the coefficient  $\delta_2^{1,c}$  versus  $x_f$



**Figure 5.** Modulus of the coefficient  $\delta_2^{1,0}$  versus  $x_f$



**Figure 7.** Comparison between the modulus of  $\zeta_1^2$  at order 1(magenta); at order 2(red) and  $k_1^2$  (blue) versus  $x_f$

vector  $\vec{S}_n^\alpha$  are given by

$$\vec{S}_n^\alpha = \begin{pmatrix} -(1 + \gamma_\alpha)x_\alpha j_n'(x_\alpha) \\ -\frac{\rho_{f\alpha}}{\rho_f} j_n(x_\alpha) \\ (-2\bar{n} + \frac{\rho_\alpha}{\rho_t} x_t^2) j_n(x_\alpha) + 4x_\alpha j_n'(x_\alpha) \\ -j_n(x_\alpha) + x_\alpha j_n'(x_\alpha) \end{pmatrix} \quad (30)$$

and those of vector  $\vec{X}_n^\alpha$  which contains the unknowns by

$$\vec{X}_n^\alpha = \begin{pmatrix} t_n^{\alpha 1} \\ t_n^{\alpha 2} \\ t_n^{\alpha t} \\ B_n^{\alpha 0} \end{pmatrix}. \quad (31)$$

$j_n$  is the spherical Bessel function and  $h_n$  is the spherical Hankel function of the first kind ( $h_n \equiv h_n^1$ ). The dimensionless parameters  $\gamma_\alpha$  and the mass densities  $\rho_{f1}$ ,  $\rho_{f2}$  and  $\rho_j$  with  $j = 1, 2, t$  are related to the properties of the poroelastic medium presented in Table 1 -  $K_f$ ,  $\rho_f$  and  $\eta$  are the bulk modulus, the mass density and the kinematic viscosity of the saturating fluid, resp.,  $\phi$  is the porosity,  $\tau$  the tortuosity,  $\kappa$  the permeability,  $a_p$  the mean radius of the pores,  $a$  is the radius of the sphericals scatters,  $K_s$  and  $\rho_s$  are the bulk modulus and the mass density of the solid, resp.,  $K_b$  the bulk modulus of the dried porous medium,  $\mu$  the shear modulus [7].

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Parameter	Unit	Value
$K_s$	Pa	$3.66 \times 10^{10}$
$K_b$	Pa	$9.47 \times 10^9$
$K_f$	Pa	$2.22 \times 10^9$
$\mu$	Pa	$7.63 \times 10^9$
$\rho_s$	$\text{kg m}^{-3}$	2760
$\rho_f$	$\text{kg m}^{-3}$	1000
$\phi$	-	0.402
$k$	$\text{m}^2$	$1.68 \times 10^{-11}$
$a_p$	m	$3.26 \times 10^{-5}$
$a$	-	$1 \times 10^{-2}$
$\eta$	$\text{kg m}^{-1} \text{s}^{-1}$	$1.14 \times 10^{-3}$
$\tau$	-	1.89

**Table 1.** Parameters of the poroelastic medium.