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# Homogeneous Bidimensional Traffic Flow Model

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**Abstract:** The object of this paper is to introduce a bidimensional model of vehicular traffic flow. The model is bidimensional in the sense that the underlying network is approximated by a continuous medium, and that traffic flow is approximated as a bidimensional fluid. The model is homogeneous in the sense that it does not presuppose local privileged directions of travel. Thus the proposed model is dynamic and recaptures several existing models. It can also be interfaced with the GSOM (Generic second order model) for traffic on networks, allowing to treat large networks with major arteries (modelled via GSOM) and dense subnetworks, modelled by the proposed bidimensional model. The bidimensional model is introduced in a discretized and phenomenological way, and is completely described in this setting.

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*Keywords:* macroscopic model, transportation, traffic, GSOM, discretization, continuum model, multimodal, anisotropic, isotropic

## 1. INTRODUCTION

The object of this paper is to present a bidimensional model of vehicular traffic. Bidimensional modelling of traffic networks relies on two approximations: the approximation of the traffic network as a continuous medium, and the approximation of traffic flow on this network as a bidimensional fluid flowing on the bidimensional medium. Such a double approximation (traffic and geometry) relies on the idea that the network is dense enough to justify its description as a bidimensional medium.

The motivation is the following. We want to be able to model traffic on very large dense networks. The difficulties to be overcome are several: complexity of the network, but also insufficiency of traffic information. Indeed on very large dense networks, traffic information, whether originating from fixed measurement stations (loops, cameras etc) or from streaming data (portables, GPS etc) does not provide a precise description of traffic flow everywhere at all times. Further, a very precise description of traffic flow, for instance on small streets, is not really necessary for most applications such as traffic management, route guidance etc. Hence the idea of bidimensional modelling which will provide approximate description of traffic on large areas with little requirements in terms of data input.

This approach has been used for solving flow problems (shortest path, max capacity) on large networks via mathematical programming as in Taguchi and Iri (1982), then to solve planification assignment problems, first static (refer to Yang et al. (1994), Wong (1998), Ho and Wong (2006)), then dynamic assignment problems (see Jiang et al. (2011), Jiang et al. (2016)). The basic idea being that the medium i.e. the network is locally isotropic, and described by local cost functions depending on the local density of vehicles.

The approach is similar to the continuum approach of pedestrian dynamics, refer to Huang et al. (2009) for instance.

The MFD (macroscopic fundamental diagram) method introduced more recently (Daganzo and Geroliminis (2008), Ramezani et al. (2015)) yields a phenomenological approach to bidimensional modeling and dynamic traffic assignment (refer to Knoop and Hoogendoorn (2015), Knoop et al. (2015) and Loder et al. (2017)). This method can be viewed as the discretization of an isotropic bidimensional model, since the basic idea of MFD is that the flow exchanged by an area is a function of the quantity of vehicles contained in this area.

All these models assume that the network is locally isotropic and traffic essentially one-directional. Saumtally (2012) introduced an anisotropic model in which the network is endowed locally with privileged directions of propagation and traffic flows along those directions while being disaggregated per destination. This model was applied to static assignment and started to be extended in a simple way to the dynamic case in Saumtally et al. (2013). This work was completed in Sossoe et al. (2015) and Sossoe (2017), and applied to dynamic traffic assignment in Sossoe and Lebacque (2017). This model will be denoted ABTM (anisotropic bidimensional traffic model) in the paper.

Urban areas, especially in Europe, alternate sectors in which the surface network is very structured, with obvious privileged directions of travel, and unstructured sectors in which traffic travels in any direction. Refer to Figure 1 (adapted from Saumtally (2012)), which shows anisotropic areas (in which there are two privileged travel directions depicted by blue arrows), and an isotropic area in which

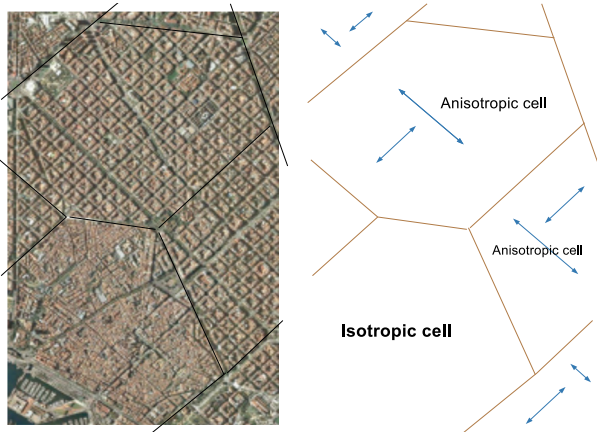


Fig. 1. Part of Barcelona network, with anisotropic and isotropic areas.

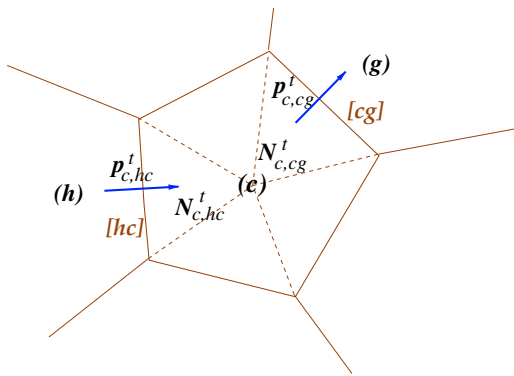


Fig. 2. Cell: boundary flows and stock variables.

there are no privileged travel directions. Thus it is necessary to extend the ABTM model developed in Saumtally et al. (2013), Saumtally (2012), Sossoe and Lebacque (2017) and Sossoe et al. (2015) in order to accommodate also locally isotropic networks.

The development of this extension constitutes the main result introduced in this paper. The new model actually encompasses the anisotropic bidimensional model and also the MFD. It will be denoted GBTM (generic bidimensional traffic model) in the paper. In the second section of the paper we will describe the model. The third section addresses some specific points and applications.

## 2. PHENOMENOLOGICAL MODEL.

As mentioned in the introduction, the global framework of the GBTM model described in this paper is the same as in the ABTM model developed in Sossoe and Lebacque (2017). The network is divided into polygonal cells; and the model calculates dynamically the flows through the cell boundaries and the numbers of vehicles inside each cell. The main innovation in the GBTM model described in this section is that there are no privileged traffic flow directions in cells. The division of a network into cells is exemplified in Figures 1 and 7.

### 2.1 Main notations

The setting is illustrated by the following Figure 2. Let us introduce the notations.

- Cells  $(c)$ ,  $c \in \mathcal{C}$ . The typical order of magnitude of the size of a cell is 1 - 5 km.
- $(t) \stackrel{def}{=} [t\Delta t, (t+1)\Delta t]$  with  $t \in \mathcal{T}$ : the current time-step. The typical time-step duration is 1 - 5 mn.
- $d \in \mathcal{D}$ : the indices of final destinations of traffic  $(d)$ , according to which the traffic is disaggregated.
- $c \in \mathcal{C}$ : the cell indices. A cell with index  $c$  is denoted  $(c)$ . A cell  $(h)$  is a predecessor of cell  $(c)$  if traffic can flow from  $(h)$  to  $(c)$ , i.e.  $(h) \in \text{Pred}(c)$ . A cell  $(g)$  is a successor of cell  $(c)$  if traffic can flow from  $(c)$  to  $(g)$ , i.e.  $(g) \in \text{Succ}(c)$ . A special cell  $(\omega)$  accounts for areas outside of the cells of the network.
- Stock variables of traffic in cells:
  - $N_{c,cg}^{d,t}$  = vehicles present at the begin of time-step  $(t)$  in cell  $(c)$ , heading for cell  $(g) \in \text{Succ}(c)$ , with final destination  $(d)$ ;
  - $N_{c,hc}^{d,t}$  = vehicles present at the begin of time-step  $(t)$  in cell  $(c)$ , which entered cell  $(c)$  from cell  $(h) \in \text{Pred}(c)$ , with final destination  $(d)$ .

They constitute the basic dynamic variables. We define also

$$N_c^{d,t} \stackrel{def}{=} \sum_{(g) \in \text{Succ}(c)} N_{c,cg}^{d,t} + \sum_{(h) \in \text{Pred}(c)} N_{c,hc}^{d,t} \quad (1)$$

- $\ell_{c,cg}$  = total length of lanes in cell  $(c)$  which are available for traffic  $(c) \rightarrow (g)$ , if  $(g) \in \text{Succ}(c)$ .
- $\ell_{c,hc}$  = total length of lanes in cell  $(c)$  which are available for traffic  $(h) \rightarrow (c)$ , if  $(h) \in \text{Pred}(c)$ .
- Densities in cell  $(c)$  are now defined as:

$$\left\{ \begin{array}{l} \rho_{c,cg}^{d,t} \stackrel{def}{=} N_{c,cg}^{d,t} / \ell_{c,cg} \quad \text{and} \quad \rho_{c,cg}^t \stackrel{def}{=} \sum_{d \in \mathcal{D}} \rho_{c,cg}^{d,t} \\ \rho_{c,hc}^{d,t} \stackrel{def}{=} N_{c,hc}^{d,t} / \ell_{c,hc} \quad \text{and} \quad \rho_{c,hc}^t \stackrel{def}{=} \sum_{d \in \mathcal{D}} \rho_{c,hc}^{d,t} \end{array} \right. \quad (2)$$

Note that the couples  $cg$ ,  $hc$  are connected to the cell boundaries  $(c) - (g)$  and  $(h) - (c)$  respectively.

- $\nu_{cg}^c$  = number of lanes available inside cell  $(c)$  for traffic flowing  $(c) \rightarrow (g)$ . Note that in general it is possible that  $\nu_{cg}^c \neq \nu_{cg}^g$ .
- $\nu_{hc}^c$  = number of lanes available inside cell  $(c)$  for traffic flowing  $(h) \rightarrow (c)$ . In general it is possible that  $\nu_{hc}^c \neq \nu_{hc}^h$ .

### 2.2 Through-flows between cells: boundary flows.

In order to derive conservation equations for the stock variables  $N_{c,cg}^{d,t}$  and  $N_{c,hc}^{d,t}$ , for all  $c \in \mathcal{C}$ , all  $d \in \mathcal{D}$  and all  $(g) \in \text{Succ}(c)$ ,  $(h) \in \text{Pred}(c)$ , it is necessary to calculate the flows  $(c) \rightarrow (g)$  and  $(h) \rightarrow (c)$ .

We denote  $\Delta_e^c(\cdot)$  and  $\Sigma_e^c(\cdot)$  the lane specific equilibrium supply and demand functions in cell  $(c)$  (refer to Lebacque (1995) and Lebacque and Khoshyaran (2013) for these concepts). Figure 3 illustrates these functions. For each cell  $(c)$  we define the boundary supplies and demands (Lebacque (1995), Sossoe (2017)) during time-step  $(t)$ :

$$\left\{ \begin{array}{l} \delta_{c,cg}^t \stackrel{def}{=} \nu_{cg}^c \Delta_e^c(\rho_{c,cg}^t) \\ \sigma_{c,hc}^t \stackrel{def}{=} \nu_{hc}^c \Sigma_e^c(\rho_{c,hc}^t) \end{array} \right. \quad (3)$$

We obtain the through-flows  $(c) \rightarrow (g)$  during time-step  $(t)$  following the min principle (Lebacque (1995), Sossoe (2017)):

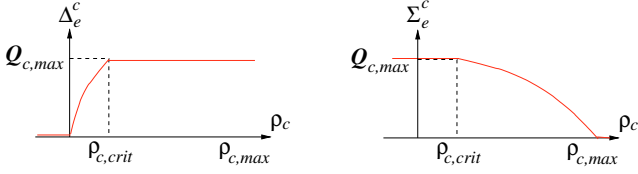


Fig. 3. Lane equilibrium functions.

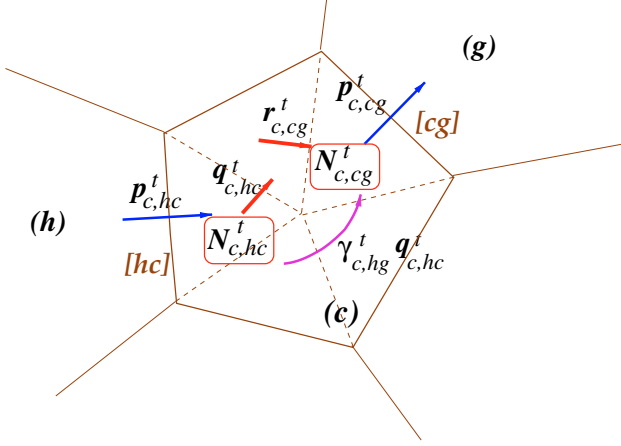


Fig. 4. Internal flows in a cell.

$$\begin{cases} p_{c,cg}^t = \min[\delta_{c,cg}^t, \sigma_{g,cg}^t] & (4.1) \\ p_{c,cg}^{d,t} = \chi_{c,cg}^{d,t} p_{c,cg}^t & (4.2) \\ \text{with } \chi_{c,cg}^{d,t} \stackrel{def}{=} \rho_{c,cg}^{d,t} / \rho_{c,cg}^t & (4.3) \end{cases} \quad (4)$$

The total through-flow  $p_{c,cg}^t$  ( $c \rightarrow g$ ) is disaggregated per destination ( $d$ ), and the partial flows are denoted  $p_{c,cg}^{d,t}$ . The composition coefficient  $\chi_{c,cg}^{d,t}$  denotes the fraction of the traffic ( $c \rightarrow g$ ) with destination ( $d$ ).

Actually (4.2) means that destination is considered as a passive attribute of traffic and that the underlying traffic flow model belongs to the GSOM (Generic traffic flow model) family, refer to Lebacque et al. (2007), (Lebacque and Khoshyaran, 2013).

Finally, note that by conservation of traffic:

$$p_{c,cg}^t = p_{g,cg}^t \quad \text{and} \quad p_{c,cg}^{d,t} = p_{g,cg}^{d,t} \quad (5)$$

Note that

$$\begin{aligned} \chi_{c,cg}^{d,t} p_{g,cg}^t &\leq \nu_{cg}^c \rho_{c,cg}^{d,t} \Delta_e^c(\rho_{c,cg}^t) / \rho_{c,cg}^t \\ &\leq \nu_{cg}^c \rho_{c,cg}^{d,t} V_{c,max} < +\infty \end{aligned}$$

(since  $\partial_\rho \Delta_e^c(\rho)|_{\rho=0} = V_{c,max}$ ). Thus the composition coefficients in (4.3) induce no division by 0 problems in (4.2) and the flows  $p_{c,cg}^{d,t}$  are always well defined.

Remark: if  $c = \omega$ , the demand  $\delta_{c,cg}^t$  and the supply  $\sigma_{c,gc}^t$  are not calculated according to (3) but are given as external boundary data.

### 2.3 Internal cell flows

Refer to Figure 4 for this subsection. If we consider a cell ( $c$ ), the vehicles having entered the cell coming from a predecessor cell ( $h$ ) must be routed towards the successor

cells ( $g$ ). This internal assignment is the result of three competing processes:

- the demand of vehicles having entered the cell coming from the predecessor cell ( $h$ ),
- the supply available for vehicles bound to exit ( $c$ ) towards the successor cells ( $g$ );
- the route choice of drivers in order to reach their destination ( $d$ ). This route choice is expressed by assignment coefficients.

We will now describe these processes. Let  $\mu_{cg}^c$  and  $\mu_{gc}^c$  the number of lanes inside cell ( $c$ ) dedicated to traffic ( $c \rightarrow g$ ) respectively ( $g \rightarrow c$ ). Thus  $\ell_{k\ell}^c / \mu_{k\ell}^c$  can be considered as the approximate mean length used in cell ( $c$ ) by traffic ( $k \rightarrow \ell$ ) (with one of the cells ( $k$ ), ( $\ell$ ) being ( $c$ ), the other being a neighboring cell).

The internal cell supply  $\Sigma_{c,cg}^t$  and demands  $\Delta_{c,hc}^t$  follow:

$$\begin{cases} \Sigma_{c,cg}^t \stackrel{def}{=} \mu_{cg}^c \Sigma_e^c(\rho_{c,cg}^t) \\ \Delta_{c,hc}^t \stackrel{def}{=} \mu_{hc}^c \Delta_e^c(\rho_{c,hc}^t) \end{cases} \quad (6)$$

The basic assignment coefficients are:

- $\Gamma_{c,gh}^{d,t}$  = the fraction of traffic ( $h \rightarrow c$ ) (with  $h \in \text{Pred}(c)$ ), with final destination ( $d$ ),  $d \in \mathcal{D}$ , which chose the cell ( $g$ ) with  $g \in \text{Succ}(c)$  to exit ( $c$ ).

These assignment coefficients can be derived from reactive or predictive DTA (dynamic traffic assignment) schemes. The assignment coefficients result from the expression of individual travel choices. They satisfy the following:

$$\begin{cases} \sum_{g \in \text{Succ}(c)} \Gamma_{c,hg}^{d,t} = 1 \\ \Gamma_{c,hg}^{d,t} \geq 0 \end{cases} \quad \forall c \in \mathcal{C}, d \in \mathcal{D}, h \in \text{Pred}(c)$$

The global assignment coefficients in cell ( $c$ ) during time step ( $t$ ) result:

$$\gamma_{c,hg}^t \stackrel{def}{=} \sum_{d \in \mathcal{D}} \chi_{c,hc}^{d,t} \Gamma_{c,gh}^{d,t} \quad \forall g \in \text{Succ}(c), h \in \text{Pred}(c) \quad (7)$$

Let us define the internal cell flows:

- $q_{c,hc}^t$  = flow leaving  $N_{c,hc}^t$  during time step ( $t$ ). This flow is limited by the demand  $\Delta_{c,hc}^t$ .
- $r_{c,cg}^t$  = flow joining  $N_{c,cg}^t$  during time step ( $t$ ). This flow is limited by the supply  $\Sigma_{c,cg}^t$ .

These flows are also connected through the assignment coefficients, expressing route choice, and conservation of traffic. Thus the internal flows satisfy the following constraints (which express the three competing processes mentioned at the beginning of the subsection 2.3):

$$\begin{cases} 0 \leq q_{c,hc}^t \leq \Delta_{c,hc}^t & \forall h \in \text{Pred}(c) \\ 0 \leq r_{c,cg}^t \leq \Sigma_{c,cg}^t & \forall g \in \text{Succ}(c) \\ r_{c,cg}^t = \sum_{h \in \text{Pred}(c)} \gamma_{c,hg}^t q_{c,hc}^t & \forall g \in \text{Succ}(c) \end{cases} \quad (8)$$

Following (Lebacque and Khoshyaran, 2005), we use an optimality in order to derive the internal flows. Thus the internal flows  $q_{c,hc}^t$ ,  $r_{c,cg}^t$  are obtained as the solutions of the following optimization problem:

$$\begin{aligned} \text{Max} \left( \begin{array}{l} (q_{hc})_{h \in \text{Pred}(c)} \\ (r_{cg})_{g \in \text{Succ}(c)} \end{array} \right) & \left[ \sum_{h \in \text{Pred}(c)} \Phi_{hc}^c(q_{hc}) + \dots \right. \\ & \left. \dots + \sum_{g \in \text{Succ}(c)} \Psi_{cg}^c(r_{cg}) \right] \quad (9) \\ \begin{array}{l} 0 \leq q_{hc} \leq \Delta_{c,hc}^t \quad \forall h \in \text{Pred}(c) \\ 0 \leq r_{cg} \leq \Sigma_{c,cg}^t \quad \forall g \in \text{Succ}(c) \\ r_{cg} = \sum_{h \in \text{Pred}(c)} \gamma_{c,hg}^t q_{hc} \quad \forall g \in \text{Succ}(c) \end{array} \end{aligned}$$

Thus if  $q_{hc}^*$ ,  $r_{cg}^*$  denote the solutions of (9),  $q_{c,hc}^t = q_{hc}^*$ ,  $r_{c,cg}^t = r_{cg}^*$ . The functions  $\Phi_{hc}^c(\cdot)$  and  $\Psi_{cg}^c(\cdot)$  should be positive, concave and increasing. They express the global behavior of the intersections in cell (c). A simple model for these functions, already used in several applications (see for instance Sossoe et al. (2015)), would be:

$$\begin{aligned} \Phi_{hc}^c(q_{hc}) &= (q_{hc}/\mu_{hc}^c Q_{c,max}) - (q_{hc})^2/2 \\ \Psi_{cg}^c(r_{cg}) &= (r_{cg}/\mu_{cg}^c Q_{c,max}) - (r_{cg})^2/2 \end{aligned}$$

#### 2.4 Global traffic conservation updates for a cell

The conservation of vehicles implies, for all  $h \in \text{Pred}(c)$  and all  $g \in \text{Succ}(c)$ :

$$\begin{cases} N_{c,hc}^{t+1} = N_{c,hc}^t + \Delta t (p_{c,hc}^t - q_{c,hc}^t) \\ N_{c,cg}^{t+1} = N_{c,cg}^t + \Delta t (r_{c,cg}^t - p_{c,cg}^t) \end{cases} \quad (10)$$

If we disaggregate the flows by destination, the following conservation equation is satisfied for all  $d \in \mathcal{D}$ :

$$\begin{cases} N_{c,hc}^{d,t+1} = N_{c,hc}^{d,t} + \Delta t (p_{c,hc}^{d,t} - \chi_{c,hc}^{d,t} q_{c,hc}^t) \\ N_{c,cg}^{d,t+1} = N_{c,cg}^{d,t} + \Delta t \left( \sum_{h \in \text{Pred}(c)} \chi_{c,hc}^{d,t} \Gamma_{c,hg}^{d,t} q_{c,hc}^t - p_{c,cg}^{d,t} \right) \end{cases} \quad (11)$$

The model, with its main equations (3), (4), (6), (5), (9), (10), (11), is thus complete. As mentioned before, the cell ( $\omega$ ), which represents the domain outside of the modelled network, provides the external demand as well as the supply for the network outflow.

### 3. SPECIFIC QUESTIONS AND ASPECTS.

#### 3.1 Consistency checks.

Let us first check that (10) can be derived from (11).

$$\begin{aligned} \sum_{d \in \mathcal{D}} \sum_{h \in \text{Pred}(c)} \chi_{c,hc}^{d,t} \Gamma_{c,hg}^{d,t} q_{c,hc}^t \\ = \sum_{h \in \text{Pred}(c)} \gamma_{c,hc}^t q_{c,hc}^t = r_{c,cg}^t \end{aligned}$$

Thus summing (11) over  $d \in \mathcal{D}$  we obtain (10).

Let us now sum (10) over  $h \in \text{Pred}(c)$  and over  $g \in \text{Succ}(c)$ . Using the definition (1) it follows:

$$N_c^{t+1} = N_c^t + \Delta t \left[ \sum_{h \in \text{Pred}(c)} p_{c,hc}^t - \sum_{g \in \text{Succ}(c)} p_{c,cg}^t \right] \quad (12)$$

because the internal cell flows cancel each other out. (12) expresses of course the global traffic flow conservation for

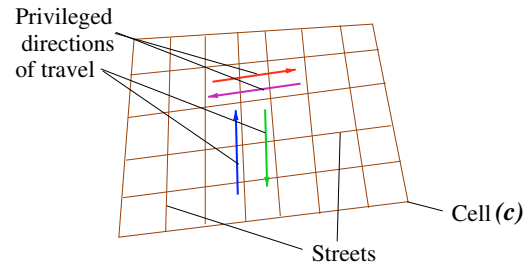


Fig. 5. An anisotropic cell.

the cell. But it can also be interpreted in terms of MFD if we aggregate all boundary demands as a single cell demand. In this sense the proposed model encompasses the MFD models because it recaptures it if the demand functions are simplified (expressed in terms of the total number of vehicles  $N_c^t$  only).

#### 3.2 The CFL condition.

In the one-dimensional case (for instance when discretizing the GSOM model) this condition expresses that within a time-step traffic waves originating from different cell boundaries do not interact. In the present case cells are bidimensional. The time step duration  $\Delta t$  must be less than the minimum mean crossing time of any cell (c). Generalizing the CFL in the one-dimensional case we posit that the CFL condition in the bidimensional case can be expressed as follows. The time step duration  $\Delta t$  must be less than the minimum mean crossing time of any cell (c). The mean crossing time must be understood as the travel time from a ( $h$ ) – ( $c$ ) boundary to a ( $c$ ) – ( $g$ ) boundary (with  $h \in \text{Pred}(c)$  and over  $g \in \text{Succ}(c)$ ).

It must be noted that traffic waves originating from distinct interfaces ( $h_1$ ) – ( $c$ ), ( $h_2$ ) – ( $c$ ), sharing a corner may interact at this corner inside the cell (c). Similarly waves crossing the cell (c) from interface ( $h_1$ ) – ( $c$ ) to ( $c$ ) – ( $g_1$ ) can interact with waves crossing the cell (c) from interface ( $h_2$ ) – ( $c$ ) to ( $c$ ) – ( $g_2$ ). Nevertheless these interactions will have little impact on the cell inflows or outflows.

#### 3.3 Connection with the anisotropic bidimensional model ABTM

In the anisotropic bidimensional model ABTM, cells have typically four sides and traffic flows in the cell according to four privileged directions, as depicted in Figure 5. Now let us consider the setting of Figure 6. There is a privileged travel direction say ( $i$ ) corresponding to ( $h$ ) → ( $g$ ) traffic inside (c), and a privileged travel direction ( $j$ ) corresponding to ( $g$ ) → ( $h$ ) traffic inside (c). Let us carry out the aggregations

$$\begin{cases} N_{c,i}^t \stackrel{def}{=} N_{c,hc}^t + N_{c,cg}^t \\ N_{c,i}^{d,t} \stackrel{def}{=} N_{c,hc}^{d,t} + N_{c,cg}^{d,t} \end{cases} \quad (13)$$

$N_{c,i}^t$  denotes the number of vehicles in cell (c) at the beginning of time-step ( $t$ ) which have the direction ( $i$ ). This number can be disaggregated per destination  $d$ , yielding the  $N_{c,i}^{d,t}$ . If we apply equations (9), (10), (11) of the GBTM to the directional traffic variables  $N_{c,i}^t$ ,  $N_{c,i}^{d,t}$ ,





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