Observation estimate for the heat equations with Neumann boundary condition via logarithmic convexity

Rémi Buffe, Kim Dang Phung

Observation estimate for the heat equations with Neumann boundary condition via logarithmic convexity

Rémi Buffe*, Kim Dang Phung†

Abstract .- We prove an inequality of Hölder type traducing the unique continuation property at one time for the heat equation with a potential and Neumann boundary condition. The main feature of the proof is to overcome the propagation of smallness by a global approach using a refined parabolic frequency function method. It relies with a Carleman commutator estimate to obtain the logarithmic convexity property of the frequency function.

Keywords .- heat equation with potential, logarithmic convexity, quantitative unique continuation.

1 Introduction and main result

In this paper, we establish the observation inequality at one time for the heat equation with a potential and Neumann boundary condition. The analysis is based on the parabolic frequency function method [K] adjusted for a global approach.

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with boundary $\partial\Omega$ of class C^∞. Consider in $\{(x,t) \in \Omega \times (0,T)\}$ the heat equation with a potential and Neumann boundary condition

$$\begin{cases} \partial_t u - \Delta u + au = 0 \ , & \text{in } \Omega \times (0,T) \ , \\ \partial_n u = 0 \ , & \text{on } \partial\Omega \times (0,T) \ , \\ u(\cdot,0) \in L^2(\Omega) \ . \end{cases}$$

Here, $T > 0$, $a \in L^\infty(\Omega \times (0,T))$ and n is the unit outward normal vector to $\partial\Omega$.

We propose the following result.

Theorem 1 .- *Let ω be a non-empty open subset of Ω. For any $t \in (0,T]$,*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \left(e^{K\left(1+\frac{1}{t}+t\|a\|_{L^\infty(\Omega\times(0,t))}+\|a\|^{2/3}_{L^\infty(\Omega\times(0,t))}\right)} \|u(\cdot,t)\|_{L^2(\omega)} \right)^\beta \|u(\cdot,0)\|^{1-\beta}_{L^2(\Omega)} \ .$$

Here $K > 0$ and $\beta \in (0,1)$ only depend on (Ω,ω).

Such observation estimate traduces the unique continuation property at one point in time saying that if $u = 0$ in $\omega \times \{t\}$, then u is identically null. Applications to bang-bang control and finite time stabilization are described in [PWZ] and [BuP]. Our result is an interpolation estimate which is more often used in a local way with a propagation of smallness procedure ([AEWZ], [FV]). Here the way we choose to establish our main theorem is based on a global approach.

*Institut Elie Cartan de Lorraine Université de Lorraine, Site de Nancy & Inria (Project-Team SPHINX) B.P. 70239, F-54506 Vandoeuvre-lès-Nancy Cedex, France. E-mail address: remi.buffe@univ-lorraine.fr

†Institut Denis Poisson, Université d'Orléans, Université de Tours & CNRS UMR 7013, Bâtiment de Mathématiques, Rue de Chartres, BP. 6759, 45067 Orléans, France. E-mail address: kim_dang_phung@yahoo.fr

Recall that Theorem 1 implies the observability estimate for the heat equations with a potential and Neumann boundary condition [PW]. It is well-known that the observability estimate for the heat equations can be obtained from Carleman inequalities. In the literature, at least two approaches allow to derive Carleman inequalities for parabolic equations: A local one based on the Garding inequality and interpolation estimates for the elliptic equations ([LR], [LRL], [BM]); A global one based on Morse functions and integrations by parts over $\Omega \times (0,T)$ ([FI], [FGGP]). Besides, unique continuation results can be deduced either by Carleman techniques or by logarithmic convexity of a frequency function [EFV]. Here we construct a new frequency function adapted to the global approach. Further, we explicitly give the dependence of the constants with respect to $\|a\|_{L^\infty}$ as in [FGGP], [DZZ].

2 Preliminaries

In this section we derive three propositions on which our later results will be based.

Proposition 1 .- *Let $\Omega \subset \mathbb{R}^{\mathrm{n}}$ be a bounded connected open set of class C^∞, and let ω be a non-empty open subset of Ω. Then there exist $d \in \mathbb{N}^*$, $(p_1, p_2, \cdot\cdot, p_d) \in \omega^d$ and $(\psi_1, \psi_2, \cdot\cdot, \psi_d) \in \left(C^\infty\left(\overline{\Omega}\right)\right)^d$ such that for all $i \in \{1, \cdot\cdot, d\}$*

(i) $\psi_i > 0$ in Ω, $\psi_i = 0$ on $\partial\Omega$,

(ii) the critical points of ψ_i are nondegenerate,

(iii) $\{x \in \Omega; |\nabla\psi_i(x)| = 0\} = \{p_j; j = 1, \cdot\cdot, d\}$,

(iv) p_i is the unique global maximum of ψ_i ,

(v) for any $j \in \{1, \cdot\cdot, d\}$, $\max\limits_{\overline{\Omega}} \psi_j = \max\limits_{\overline{\Omega}} \psi_i$.

Remark .- (i) implies that $\partial_n \psi_i \leq 0$; (iii) says that the criticals points of ψ_i are isolated and form a discrete set; (iii) implies that $d = \sharp\{x \in \Omega; |\nabla\psi_i(x)| = 0\}$ and $\{x \in \Omega; |\nabla\psi_i(x)| = 0\} \subset \omega$.

Proof .- The existence of Morse functions (that is C^∞ functions whose critical points are nondegenerate) which are positive in Ω and null on the boundary $\partial\Omega$ can be proved by virtue of the theorem on the density of Morse functions ([FI, page 20], [C, page 80], [TW, Chapter 14], [WW, page 433]). Next, by a small perturbation in a small neighborhood of each critical points, no two critical points share the same function value [M, Theorem 2.34]. Denote by ψ such a smooth function and let $a_1, \cdot\cdot, a_d$ be its critical points such that $\{x \in \Omega; |\nabla\psi(x)| = 0\} = \{a_j; j = 1, \cdot\cdot, d\} \subset \Omega$ and $\psi(a_1) > \psi(a_2) > \cdot\cdot > \psi(a_d)$. Now we will move the critical points following the procedure in [C, Lemma 2.68]. Introduce $p_1, \cdot\cdot, p_d$ d points in ω such that for each $i = 1, \cdot\cdot, d$, there exists $\gamma_{i,j} \in C^\infty([0,1]; \Omega)$ be such that

- $\gamma_{i,j}$ is one to one for every $j \in \{1, \cdot\cdot, d\}$,
- $\gamma_{i,j}([0,1]) \cap \gamma_{i,l}([0,1]) = \emptyset, \forall (j,l) \in \{1, \cdot\cdot, d\}$ such that $j \neq l$,
- $\gamma_{i,j}(0) = a_j$, $\forall j \in \{1, \cdot\cdot, d\}$,
- $\gamma_{i,j}(1) = \tau^{i-1}(p_j)$, $\forall j \in \{1, \cdot\cdot, d\}$.

Here τ is d-cycle, that is $\tau(p_j) = p_{j+1}$ if $j < d$ and $\tau(p_d) = p_1$, $\tau^0 = id$, $\tau^i = \tau^{i-1} \circ \tau$.

Introduce a vector field $V_i \in C^\infty(\mathbb{R}^{\mathrm{n}}; \mathbb{R}^{\mathrm{n}})$ such that $\overline{\{x \in \mathbb{R}^{\mathrm{n}}; V_i(x) \neq 0\}} \subset \Omega$ and $V_i(\gamma_{i,j}(t)) = \gamma'_{i,j}(t)$, $\forall j \in \{1, \cdot\cdot, d\}$. Let Λ_i denote the flow associated to V_i, that is $\partial_t \Lambda_i(t,x) = V_i(\Lambda_i(t,x))$ and $\Lambda_i(0,x) = x$. One has $\Lambda_i(0, a_j) = a_j$, $\Lambda_i(t, a_j) = \gamma_{i,j}(t)$ and $\Lambda_i(1, a_j) = \tau^{i-1}(p_j)$. Further, for every $t \in \mathbb{R}$, $\Lambda_i(t, \cdot)$ is a diffeomorphism on Ω and $\Lambda_i(t, \cdot)|_{\partial\Omega} = Id$. In particular, $(\Lambda_i(1, \cdot))^{-1}\left(\tau^{i-1}(p_j)\right) = a_j$.

It remains to check that $\psi_i : \overline{\Omega} \to \mathbb{R}$ given by $\psi_i(x) = \psi\left((\Lambda_i(1,\cdot))^{-1}(x)\right)$ satisfies all the required properties. Clearly, $\psi_i > 0$ in Ω, $\psi_i = 0$ on $\partial\Omega$ and ψ_i only have nondegenerate critical points given by $\{x \in \Omega; |\nabla\psi_i(x)| = 0\} = \{p_j; j = 1, \cdot\cdot, d\}$. Finally, $\max_{\overline{\Omega}}\psi_i = \max_{\overline{\Omega}}\psi$ and $\psi(a_1) = \psi\left((\Lambda_i(1,\cdot))^{-1}\left(\tau^{i-1}(p_1)\right)\right) = \psi\left((\Lambda_i(1,\cdot))^{-1}(p_i)\right) = \psi_i(p_i)$ allow to conclude that p_i is the unique global maximum of ψ_i and $\max_{\overline{\Omega}}\psi_j = \max_{\overline{\Omega}}\psi_i \ \forall i, j$. This completes the proof.

Our next result resume some identities linked to the Carleman commutator (see [P] and references therein).

Proposition 2 .- *Let*

$$\Phi(x,t) = \frac{s\varphi(x)}{\Gamma(t)},\ s > 0,\ \Gamma(t) = T - t + h,\ h > 0 \text{ and } \varphi \in C^\infty\left(\overline{\Omega}\right) .$$

Define for any $f \in H^2(\Omega)$

$$\begin{cases} \mathcal{A}_\varphi f = -\nabla\Phi\cdot\nabla f - \frac{1}{2}\Delta\Phi f\ , \\ \mathcal{S}_\varphi f = -\Delta f - \eta f \text{ where } \eta = \frac{1}{2}\partial_t\Phi + \frac{1}{4}|\nabla\Phi|^2\ , \\ \mathcal{S}'_\varphi f = -\partial_t\eta f\ . \end{cases}$$

Then we have

(i)

$$\int_\Omega \mathcal{A}_\varphi f f = -\frac{1}{2}\int_{\partial\Omega}\partial_n\Phi|f|^2$$

(ii)

$$\int_\Omega \mathcal{S}_\varphi f f = \int_\Omega |\nabla f|^2 - \int_\Omega \eta|f|^2 - \int_{\partial\Omega}\partial_n f f$$

(iii)

$$\begin{aligned} \int_\Omega \mathcal{S}'_\varphi f f + 2\int_\Omega \mathcal{S}_\varphi f \mathcal{A}_\varphi f \ &= -2\int_\Omega \nabla f\nabla^2\Phi\nabla f - \int_\Omega \nabla f\Delta\nabla\Phi f \\ &\quad -\frac{2}{\Gamma}\int_\Omega\left(\eta + \frac{1}{4}|\nabla\Phi|^2 + \frac{s}{4}\nabla\Phi\nabla^2\varphi\nabla\Phi\right)|f|^2 \\ &\quad +\text{Boundary terms} \end{aligned}$$

where

$$\begin{aligned} \text{Boundary terms} \ &= 2\int_{\partial\Omega}\partial_n f\nabla\Phi\cdot\nabla f - \int_{\partial\Omega}\partial_n\Phi|\nabla f|^2 \\ &\quad +\int_{\partial\Omega}\partial_n f\Delta\Phi f + \int_{\partial\Omega}\eta\partial_n\Phi|f|^2\ . \end{aligned}$$

Proof .- The proof of $\int_\Omega \mathcal{A}_\varphi f f$ and $\int_\Omega \mathcal{S}_\varphi f f$ is quite clear by integrations by parts. Now we compute the bracket $2\langle\mathcal{S}_\varphi f, \mathcal{A}_\varphi f\rangle$: We have from the definition of $\mathcal{S}_\varphi f$ and $\mathcal{A}_\varphi f$,

$$2\langle\mathcal{S}_\varphi f, \mathcal{A}_\varphi f\rangle = 2\int_\Omega(\Delta f + \eta f)\left(\nabla\Phi\cdot\nabla f + \frac{1}{2}\Delta\Phi f\right)$$

and four integrations by parts give

$$2\langle\mathcal{S}_\varphi f, \mathcal{A}_\varphi f\rangle = -2\int_\Omega \nabla f\nabla^2\Phi\nabla f - \int_\Omega \nabla f\Delta\nabla\Phi f - \int_\Omega \nabla\eta\cdot\nabla\Phi|f|^2 + \text{Boundary terms}\ .$$

Indeed,

$$\int_\Omega \Delta f\nabla\Phi\cdot\nabla f = \int_{\partial\Omega}\partial_n f\nabla\Phi\cdot\nabla f - \int_\Omega \nabla f\nabla^2\Phi\nabla f - \int_\Omega \nabla f\nabla^2 f\nabla\Phi\ ,$$

but

$$\int_\Omega \nabla f \nabla^2 f \nabla \Phi = \frac{1}{2}\int_{\partial\Omega} \partial_n \Phi \left|\nabla f\right|^2 - \frac{1}{2}\int_\Omega \Delta\Phi \left|\nabla f\right|^2 \ .$$

Second,

$$\int_\Omega \Delta f \Delta\Phi f = \int_{\partial\Omega} \partial_n f \Delta\Phi f - \int_\Omega \nabla f \Delta\nabla\Phi f - \int_\Omega \Delta\Phi \left|\nabla f\right|^2 \ .$$

Third,

$$2\int_\Omega \eta f \nabla\Phi\cdot\nabla f = \int_{\partial\Omega} \eta\partial_n\Phi \left|f\right|^2 - \int_\Omega \nabla\eta\cdot\nabla\Phi\left|f\right|^2 - \int_\Omega \eta\Delta\Phi\left|f\right|^2 \ .$$

This concludes to the identity

$$\begin{aligned} 2\int_\Omega \mathcal{S}_\varphi f \mathcal{A}_\varphi f - \int_\Omega \partial_t\eta\left|f\right|^2 \ &= -2\int_\Omega \nabla f\nabla^2\Phi\nabla f - \int_\Omega \nabla f\Delta\nabla\Phi f \\ &\quad + \text{Boundary terms} + \int_\Omega \left(-\partial_t\eta - \nabla\eta\cdot\nabla\Phi\right)\left|f\right|^2 \ . \end{aligned}$$

Finally, using $\partial_t\Phi = \frac{1}{\Gamma}\Phi$ and $\partial_t^2\Phi = \frac{2}{\Gamma}\partial_t\Phi$, we obtain

$$\begin{aligned} -\partial_t\eta - \nabla\eta\cdot\nabla\Phi \ &= -\tfrac{1}{2}\partial_t^2\Phi - \nabla\Phi\cdot\nabla\partial_t\Phi - \tfrac{1}{2}\nabla\Phi\nabla^2\Phi\nabla\Phi \\ &= -\tfrac{1}{\Gamma}\partial_t\Phi - \tfrac{1}{\Gamma}\left|\nabla\Phi\right|^2 - \tfrac{s}{2\Gamma}\nabla\Phi\nabla^2\varphi\nabla\Phi \\ &= -\tfrac{2}{\Gamma}\eta - \tfrac{1}{2\Gamma}\left|\nabla\Phi\right|^2 - \tfrac{s}{2\Gamma}\nabla\Phi\nabla^2\varphi\nabla\Phi \ . \end{aligned}$$

This completes the proof of (iii).

Recall the following result which is a variant of [BP, Lemma 4.3].

Proposition 3 .- *Let $h > 0$, $T > 0$ and $F_1, F_2 \geq 0$. Consider two positive functions $y, N \in C^1([0,T])$ such that*

$$\left\{ \begin{array}{l} \left|\frac{1}{2}y'(t) + N(t)\,y(t)\right| \leq F_1 y(t) \ , \\ N'(t) \leq \dfrac{1+C_0}{T-t+h}N(t) + F_2 \ , \end{array} \right. \tag{2.1}$$

where $C_0 \geq 0$. Then for any $0 \leq t_1 < t_2 < t_3 \leq T$, one has

$$y(t_2)^{1+M} \leq y(t_3)\,y(t_1)^M\, e^D$$

with

$$M = \frac{\displaystyle\int_{t_2}^{t_3} \frac{1}{(T-t+h)^{1+C_0}}dt}{\displaystyle\int_{t_1}^{t_2} \frac{1}{(T-t+h)^{1+C_0}}dt}$$

and

$$D = 2M\left(F_2(t_3-t_1)^2 + F_1(t_3-t_1)\right) \ .$$

Proof .- Set $\Gamma(t) = T - t + h$. From the second inequality of (2.1), we have

$$\left(\Gamma^{1+C_0}N\right)' \leq F_2\Gamma^{1+C_0} \ . \tag{2.2}$$

Integrating (2.2) over (t,t_2) with $t \in (t_1,t_2)$ gives

$$\left(\frac{\Gamma(t_2)}{\Gamma(t)}\right)^{1+C_0} N(t_2) \leq N(t) + F_2(t_2-t_1) \ .$$

By the first inequality of (2.1),

$$y'(t) + 2N(t)y(t) \leq 2F_1 y(t)$$

and we derive that

$$y' + \left(2 \left(\frac{\Gamma(t_2)}{\Gamma(t)} \right)^{1+C_0} N(t_2) - 2F_2(t_2 - t_1) - 2F_1 \right) y \leq 0 \text{ for } t \in (t_1, t_2) .$$

Integrating over (t_1, t_2), we obtain

$$y(t_2) e^{2N(t_2) \int_{t_1}^{t_2} \left(\frac{\Gamma(t_2)}{\Gamma(t)} \right)^{1+C_0} dt} \leq y(t_1) e^{2F_2(t_2-t_1)^2 + 2F_1(t_2-t_1)} . \tag{2.3}$$

On the other hand, integrating (2.2) over (t_2, t) with $t \in (t_2, t_3)$, one has

$$N(t) \leq \left(\frac{\Gamma(t_2)}{\Gamma(t)} \right)^{1+C_0} (N(t_2) + F_2(t_3 - t_2)) .$$

By the first inequality of (2.1),

$$-y'(t) - 2N(t)y(t) \leq 2F_1 y(t)$$

and it follows that

$$0 \leq y' + \left[2 \left(\frac{\Gamma(t_2)}{\Gamma(t)} \right)^{1+C_0} (N(t_2) + F_2(t_3 - t_2)) + 2F_1 \right] y \text{ for } t \in (t_2, t_3) .$$

Integrating over (t_2, t_3) yields

$$y(t_2) \leq e^{2(N(t_2) + F_2(t_3-t_2)) \int_{t_2}^{t_3} \left(\frac{\Gamma(t_2)}{\Gamma(t)} \right)^{1+C_0} dt} y(t_3) e^{2F_1(t_3-t_2)} . \tag{2.4}$$

Combining (2.3) and (2.4), one has

$$y(t_2) \leq y(t_3) \left(\frac{y(t_1)}{y(t_2)} e^{2F_2(t_2-t_1)^2} e^{2F_1(t_2-t_1)} \right)^M e^{2F_1(t_3-t_2)} e^{2F_2(t_3-t_2) \int_{t_2}^{t_3} \left(\frac{\Gamma(t_2)}{\Gamma(t)} \right)^{1+C_0} dt}$$

which gives

$$y(t_2) \leq y(t_3) \left(\frac{y(t_1)}{y(t_2)} \right)^M e^{2F_2(t_2-t_1)^2 M} e^{2F_1(t_2-t_1)M} e^{2F_1(t_3-t_2)} e^{2F_2(t_3-t_2)(t_2-t_1)M}$$

which implies the desired estimate since $M > 1$.

3 Proof of Theorem 1

The plan of the proof of Theorem 1 is as follows. We divide it into seven steps. In Step 1, we derive some estimates on the Morse functions given in Proposition 1. In Step 2, we introduce the weight functions and establish the key properties linked to the Morse functions. In Step 3, we perform a change of function and introduce the operators described in Proposition 2. In Step 4, we construct a new frequency function adapted to our global approach. In Step 5, key estimates for the Carleman operator is provided. In Step 6, we solve a system of ordinary differential inequalities thanks to Proposition 3. In Step 7, we conclude the proof by making appear the control domain $\omega \times \{T\}$.

3.1 Step 1: The Morse functions

We have by Proposition 1, the existence of Morse functions ψ_i associated to a critical point p_i which is its unique global maximum in $\overline{\Omega}$. By Morse Lemma, there exists a neighborhood of p_i and a diffeomorphism U such that $U(p_i) = 0$ and locally

$$\psi_i\left(U^{-1}(x)\right) = \psi_i(p_i) - |x|^2$$

which implies

$$\frac{1}{4}\left|\mathrm{Jac}U^{-1}(x)\,\nabla\psi_i\left(U^{-1}(x)\right)\right|^2 = |x|^2 = \psi_i(p_i) - \psi_i\left(U^{-1}(x)\right)$$

and consequently, there are $c_1, c_2 > 0$ such that for any $i \in \{1, \cdot\cdot, d\}$, in a neighborhood of p_i

$$c_1 |\nabla\psi_i|^2 \leq \left(\max_{\overline{\Omega}}\psi_i - \psi_i\right) \leq c_2 |\nabla\psi_i|^2 \ . \tag{3.1.1}$$

Let

$$\mathcal{B}_i \text{ be a neighborhood of } \left\{x \in \Omega; |\nabla\psi_i(x)| = 0 \text{ and } \max_{\overline{\Omega}}\psi_i - \psi_i(x) = 0\right\}$$

in which (3.1.1) holds ,

$$\mathcal{C}_i \text{ be a neighborhood of } \left\{x \in \Omega; |\nabla\psi_i(x)| = 0 \text{ and } \max_{\overline{\Omega}}\psi_i - \psi_i(x) > 0\right\}$$

with $\mathcal{B}_i \cap \mathcal{C}_i = \emptyset$ in which $\psi_i - \psi_j < 0$ for some $j \neq i$. This is possible because $\psi_i(p_j) < \psi_j(p_j)$ using Proposition 1 (iv) and (v) with $\{p_j; j = 1, \cdot\cdot, d\} = \{x \in \Omega; |\nabla\psi_i(x)| = 0\}$ and $\mathcal{C}_i = \bigcup_{j\neq i}\Theta_{p_j}$ where Θ_{p_j} is a sufficiently small neighborhood of p_j. And finally let

$$\mathcal{D}_i = \Omega \setminus (\mathcal{B}_i \cup \mathcal{C}_i) \text{ be such that } \Omega = \mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{D}_i \ .$$

Proposition 4 .- *There are $c_1 > 0$ and $c_2 > 0$ such that for any $i \in \{1, \cdot\cdot, d\}$*

(i) In D_i,

$$c_1 |\nabla\psi_i|^2 \leq \left(\max_{\overline{\Omega}}\psi_i - \psi_i\right) \leq c_2 |\nabla\psi_i|^2 \ .$$

(ii) In B_i,

$$c_1 |\nabla\psi_i|^2 \leq \left(\max_{\overline{\Omega}}\psi_i - \psi_i\right) \leq c_2 |\nabla\psi_i|^2 \ .$$

(iii) In C_i,

$$c_1 |\nabla\psi_i|^2 \leq \left(\max_{\overline{\Omega}}\psi_i - \psi_i\right) \ .$$

Proof .- The inequality (ii) holds by definition of $\mathcal{B}_i$. In $\mathcal{C}_i$, we use $\max_{\overline{\Omega}}\psi_i - \psi_i \geq c > 0$ and $|\nabla\psi_i|^2 \leq \max_{\overline{\Omega}}|\nabla\psi_i|^2 \leq \frac{\max_{\overline{\Omega}}|\nabla\psi_i|^2}{c}\left(\max_{\overline{\Omega}}\psi_i - \psi_i\right)$. In $\mathcal{D}_i$, $|\nabla\psi_i| > 0$ and $\max_{\overline{\Omega}}\psi_i - \psi_i > 0$ imply the desired estimates.

3.2 Step 2: The weight functions

Introduce for any $i \in \{1, \cdot\cdot, d\}$

$$\begin{cases} \varphi_{i,1} = \psi_i - \max\limits_{\overline{\Omega}} \psi_i \ , \\ \varphi_{i,2} = -\psi_i - \max\limits_{\overline{\Omega}} \psi_i \ . \end{cases}$$

Notice that

$$\varphi_{i,1} = \varphi_{i,2} \text{ on } \partial\Omega \text{ and } \partial_n \varphi_{i,1} + \partial_n \varphi_{i,2} = 0 \text{ on } \partial\Omega \ . \tag{3.2.1}$$

Further, the link between $\varphi_{i,1}$ and ψ_i is described as follows: $|\varphi_{i,1}| = \max\limits_{\overline{\Omega}} \psi_i - \psi_i$ and $|\nabla \varphi_{i,1}|^2 = |\nabla \psi_i|^2$. Now, we are able to state the properties of $\varphi_{i,1}$ and $\varphi_{i,2}$.

Proposition 5 .- *There are $c_1, \cdot\cdot, c_6 > 0$ all positive constants such that for any $i \in \{1, \cdot\cdot, d\}$*

(i) *In D_i,*

$$c_1 |\nabla \varphi_{i,1}|^2 \leq |\varphi_{i,1}| \leq c_2 |\nabla \varphi_{i,1}|^2 \ .$$

(ii) *In B_i,*

$$c_1 |\nabla \varphi_{i,1}|^2 \leq |\varphi_{i,1}| \leq c_2 |\nabla \varphi_{i,1}|^2 \ .$$

(iii) *In C_i,*

$$c_1 |\nabla \varphi_{i,1}|^2 \leq |\varphi_{i,1}| \ .$$

(iv) *There is $j \in \{1, \cdot\cdot, d\}$ with $j \neq i$ such that*

$$\varphi_{i,1} - \varphi_{j,1} \leq -c_3 \text{ in } \mathcal{C}_i \ .$$

(v)

$$c_4 |\nabla \varphi_{i,2}|^2 \leq |\varphi_{i,2}| \text{ in } \Omega \text{ and } |\varphi_{i,2}| \leq c_5 |\nabla \varphi_{i,2}|^2 \text{ in a neighborhood of } \partial\Omega \ .$$

(vi)

$$\varphi_{i,2} - \varphi_{i,1} \leq -c_6 \text{ outside a neighborhood of } \partial\Omega \ .$$

Proof .- By the properties of the Morse functions described in Proposition 4, we deduce $(i) - (ii)$ and (iii). The inequality (iv) holds from the definition of $\mathcal{C}_i$ and Proposition 1 (v). Next, we start to prove (v) by seeing that $|\nabla \varphi_{i,2}|^2 \leq c \leq \frac{c}{\max\limits_{\overline{\Omega}} \psi_i} |\varphi_{i,2}|$. Since $|\nabla \varphi_{i,2}| = |\nabla \psi_i| > 0$ in a neighborhood of $\partial\Omega$, we have $|\varphi_{i,2}| \leq c \leq c_5 |\nabla \varphi_{i,2}|^2$. This completes the proof of (v). Finally, since $\psi_i > 0$ outside a neighborhood of $\partial\Omega$, we get $0 < c \leq \psi_i$ and $\varphi_{i,2} - \varphi_{i,1} = -2\psi_i \leq -2c = -c_6$, that is (vi).

3.3 Step 3: Change of functions

Introduce for any $(x,t) \in \Omega \times [0,T]$ and any $i \in \{1, \cdot\cdot, d\}$

$$\begin{cases} \Phi_i(x,t) = \dfrac{s}{\Gamma(t)} \varphi_{i,1}(x) \ , \\ \Phi_{d+i}(x,t) = \dfrac{s}{\Gamma(t)} \varphi_{i,2}(x) \ . \end{cases}$$

with $s \in (0,1]$ and $\Gamma(t) = T - t + h$, $h \in (0,1]$.

Let $\boldsymbol{f} = (f_i)_{1 \leq i \leq 2d}$ where $f_i = u e^{\Phi_i/2}$. We look for the equation solved by f_i by computing $e^{\Phi_i/2} (\partial_t - \Delta) \left(e^{-\Phi_i/2} f_i \right)$. Introduce

$$\begin{cases} \mathcal{A}_{\varphi_i} f_i = -\nabla \Phi_i \cdot \nabla f_i - \frac{1}{2} \Delta \Phi_i f_i \ , \\ \mathcal{S}_{\varphi_i} f_i = -\Delta f_i - \eta_i f_i \text{ where } \eta_i = \frac{1}{2} \partial_t \Phi_i + \frac{1}{4} |\nabla \Phi_i|^2 \ . \end{cases}$$

Let $\mathcal{S}\boldsymbol{f} = (\mathcal{S}_{\varphi_i} f_i)_{1\leq i\leq 2d}$, $\mathcal{A}\boldsymbol{f} = (\mathcal{A}_{\varphi_i} f_i)_{1\leq i\leq 2d}$, and $F = (-af_i)_{1\leq i\leq 2d}$. We find that

$$\left\{ \begin{array}{l} \partial_t \boldsymbol{f} + \mathcal{S}\boldsymbol{f} = \mathcal{A}\boldsymbol{f} + F \ , \\ \partial_n f_i - \frac{1}{2}\partial_n \Phi_i f_i = 0 \text{ on } \partial\Omega \times (0,T) \ . \end{array} \right. \tag{3.3.1}$$

Let $\langle \cdot, \cdot \rangle$ denote the usual scalar product in $\left(L^2(\Omega)\right)^{2d}$ and let $\|\cdot\|$ be its corresponding norm. Now, we claim that

$$\left\{ \begin{array}{l} \langle \mathcal{A}\boldsymbol{f}, \boldsymbol{f} \rangle = 0 \ , \\ \langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle = \displaystyle\sum_{i=1,..,2d} \int_\Omega |\nabla f_i|^2 - \int_\Omega \eta_i |f_i|^2 \ , \\ \dfrac{d}{dt} \langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle = - \displaystyle\sum_{i=1,..,2d} \int_\Omega \partial_t \eta_i |f_i|^2 + 2 \langle \mathcal{S}\boldsymbol{f}, \partial_t \boldsymbol{f} \rangle := \langle \mathcal{S}'\boldsymbol{f}, \boldsymbol{f} \rangle + 2 \langle \mathcal{S}\boldsymbol{f}, \partial_t \boldsymbol{f} \rangle \ . \end{array} \right. \tag{3.3.2}$$

Indeed, applying Proposition 2 $(i)-(ii)$ and using the Robin boundary condition for f_i, all the boundary terms appearing in the integrations by parts can be dropped since for any $i \in \{1, \cdot\cdot, d\}$

$$\Phi_i = \Phi_{d+i} \text{ and } \partial_n \Phi_i + \partial_n \Phi_{d+i} = 0 \text{ on } \partial\Omega \times (0,T) \ , \tag{3.3.3}$$

by (3.2.1). To establish the last identity in (3.3.2), we compute $\frac{d}{dt} \langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle$ as follows:

$$\begin{aligned} \frac{d}{dt} \langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle &= \frac{d}{dt} \left(\sum_{i=1,..,2d} \int_\Omega |\nabla f_i|^2 - \int_\Omega \eta_i |f_i|^2 \right) \\ &= 2 \langle \mathcal{S}\boldsymbol{f}, \partial_t \boldsymbol{f} \rangle - \sum_{i=1,..,2d} \int_\Omega \partial_t \eta_i |f_i|^2 + 2 \sum_{i=1,..,2d} \int_{\partial\Omega} \partial_n f_i \partial_t f_i \end{aligned}$$

by an integration by parts. But, by using the Robin boundary condition for $f_i = u e^{\Phi_i/2}$ in (3.3.1), we have

$$\sum_{i=1,..,2d} \int_{\partial\Omega} \partial_n f_i \partial_t f_i = \sum_{i=1,..,2d} \int_{\partial\Omega} \frac{1}{2} \partial_n \Phi_i \left(u \partial_t u + |u|^2 \frac{1}{2} \partial_t \Phi_i \right) e^{\Phi_i} = 0$$

since for any $i \in \{1, \cdot\cdot, d\}$, $\Phi_{d+i} = \Phi_i$ and $\partial_n \Phi_i + \partial_n \Phi_{d+i} = 0$ on $\partial\Omega \times (0,T)$.

3.4 Step 4: Energy estimates

By a standard energy method, we have

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{f}\|^2 + \langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle = \langle F, \boldsymbol{f} \rangle \ ,$$

and by introducing the frequency function

$$\mathbf{N}(t) = \frac{\langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle}{\|\boldsymbol{f}\|^2}$$

it holds

$$\mathbf{N}'(t) \|\boldsymbol{f}\|^2 \leq \langle \mathcal{S}'\boldsymbol{f}, \boldsymbol{f} \rangle + 2 \langle \mathcal{S}\boldsymbol{f}, \mathcal{A}\boldsymbol{f} \rangle + \|F\|^2 \ .$$

Indeed, for the energy identity we use the first equality of (3.3.1) and $\langle \mathcal{A}\boldsymbol{f}, \boldsymbol{f} \rangle = 0$. For the inequality of the derivative of the frequency function, we use $\frac{d}{dt} \langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle = \langle \mathcal{S}'\boldsymbol{f}, \boldsymbol{f} \rangle + 2 \langle \mathcal{S}\boldsymbol{f}, \partial_t \boldsymbol{f} \rangle$ (see (3.3.2)) and replace $\partial_t \boldsymbol{f}$ by $\mathcal{A}\boldsymbol{f} - \mathcal{S}\boldsymbol{f} + F$ in order to get

$$\begin{aligned} \mathbf{N}'(t) \|\boldsymbol{f}\|^4 &= \left(\langle \mathcal{S}'\boldsymbol{f}, \boldsymbol{f} \rangle + 2 \langle \mathcal{S}\boldsymbol{f}, \partial_t \boldsymbol{f} \rangle \right) \|\boldsymbol{f}\|^2 - \langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle \left(-2 \langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle + 2 \langle F, \boldsymbol{f} \rangle \right) \\ &= \left(\langle \mathcal{S}'\boldsymbol{f}, \boldsymbol{f} \rangle + 2 \langle \mathcal{S}\boldsymbol{f}, \mathcal{A}\boldsymbol{f} \rangle \right) \|\boldsymbol{f}\|^2 - 2 \|\mathcal{S}\boldsymbol{f}\|^2 \|\boldsymbol{f}\|^2 + 2 \langle \mathcal{S}\boldsymbol{f}, F \rangle \|\boldsymbol{f}\|^2 \\ &\quad + 2 \langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle^2 - 2 \langle \mathcal{S}\boldsymbol{f}, \boldsymbol{f} \rangle \langle F, \boldsymbol{f} \rangle \\ &= \left(\langle \mathcal{S}'\boldsymbol{f}, \boldsymbol{f} \rangle + 2 \langle \mathcal{S}\boldsymbol{f}, \mathcal{A}\boldsymbol{f} \rangle \right) \|\boldsymbol{f}\|^2 - 2 \left\| \mathcal{S}\boldsymbol{f} - \tfrac{1}{2} F \right\|^2 \|\boldsymbol{f}\|^2 + \tfrac{1}{2} \|F\|^2 \|\boldsymbol{f}\|^2 \\ &\quad + 2 \left\langle \mathcal{S}\boldsymbol{f} - \tfrac{1}{2} F, \boldsymbol{f} \right\rangle^2 - \tfrac{1}{2} \langle F, \boldsymbol{f} \rangle^2 \ . \end{aligned}$$

By Cauchy-Schwarz, we obtain the desired estimate for $\mathbf{N}'(t)$.

Since

$$\|F\|^2 \leq \|a\|_\infty^2 \|\boldsymbol{f}\|^2$$

where $\|a\|_\infty = \|a\|_{L^\infty(\Omega\times(0,T))}$, we obtain the following system of ordinary differential inequalities

$$\left\{\begin{array}{l} \left|\frac{1}{2}\frac{d}{dt}\|\boldsymbol{f}\|^2 + \mathbf{N}(t)\|\boldsymbol{f}\|^2\right| \leq \|a\|_\infty \|\boldsymbol{f}\|^2 \ , \\ \mathbf{N}'(t) \leq \dfrac{\langle \mathcal{S}'\boldsymbol{f},\boldsymbol{f}\rangle + 2\langle \mathcal{S}\boldsymbol{f},\mathcal{A}\boldsymbol{f}\rangle}{\|\boldsymbol{f}\|^2} + \|a\|_\infty^2 \ . \end{array}\right. \tag{3.4.1}$$

3.5 Step 5: Carleman commutator estimates

We claim that for some $s \in (0,1]$ sufficiently small, $\eta_i \leq 0$ and $\langle \mathcal{S}\boldsymbol{f},\boldsymbol{f}\rangle \geq 0$ and

$$\langle \mathcal{S}'\boldsymbol{f},\boldsymbol{f}\rangle + 2\langle \mathcal{S}\boldsymbol{f},\mathcal{A}\boldsymbol{f}\rangle \leq \frac{1+C_0}{\Gamma}\langle \mathcal{S}\boldsymbol{f},\boldsymbol{f}\rangle + \frac{C}{h^2}\|\boldsymbol{f}\|^2 \ ,$$

where $C_0 \in (0,1)$ and $C>0$ do not depend on $h \in (0,1]$.

Indeed, observe that

$$\eta_i = \frac{1}{2}\partial_t\Phi_i + \frac{1}{4}|\nabla\Phi_i|^2 = \left|\begin{array}{ll} \frac{s}{4\Gamma^2}\left(-2|\varphi_{i,1}| + s|\nabla\varphi_{i,1}|^2\right) & \text{if } i\in\{1,\cdot\cdot,d\} \\ \frac{s}{4\Gamma^2}\left(-2|\varphi_{i-d,2}| + s|\nabla\varphi_{i-d,2}|^2\right) & \text{if } i\in\{d+1,\cdot\cdot,2d\} \end{array}\right. \leq 0$$

for $s\in(0,1]$ sufficiently small since $|\nabla\varphi_{i,j}|^2 \leq c|\varphi_{i,j}|$ for any $i\in\{1,\cdot\cdot,d\}$, any $j\in\{1,2\}$ by Proposition 5 $(i)-(iii)$ and (v). This concludes the proof that $\langle \mathcal{S}\boldsymbol{f},\boldsymbol{f}\rangle \geq 0$ for s small.

By Proposition 2 (iii),

$$\begin{array}{rl} \langle \mathcal{S}'\boldsymbol{f},\boldsymbol{f}\rangle + 2\langle \mathcal{S}\boldsymbol{f},\mathcal{A}\boldsymbol{f}\rangle & = -2\displaystyle\sum_{i=1,..,2d}\int_\Omega \nabla f_i\nabla^2\Phi_i\nabla f_i - \sum_{i=1,..,2d}\int_\Omega \nabla f_i\Delta\nabla\Phi_i f_i \\ & \quad -\dfrac{2}{\Gamma}\displaystyle\sum_{i=1,..,2d}\int_\Omega\left(\eta_i + \frac{1}{4}|\nabla\Phi_i|^2 + \frac{s}{4}\nabla\Phi_i\nabla^2\varphi_i\nabla\Phi_i\right)|f_i|^2 \\ & \quad +\text{Boundary terms} \end{array} \tag{3.5.1}$$

where $\varphi_i = \varphi_{i,1}$ for $i\in\{1,\cdot\cdot,d\}$, $\varphi_i = \varphi_{i-d,2}$ for $i\in\{d+1,\cdot\cdot,2d\}$, and

$$\begin{array}{rl} \text{Boundary terms} & = 2\displaystyle\sum_{i=1,..,2d}\int_{\partial\Omega}\partial_n f_i\nabla\Phi_i\cdot\nabla f_i - \sum_{i=1,..,2d}\int_{\partial\Omega}\partial_n\Phi_i|\nabla f_i|^2 \\ & \quad + \displaystyle\sum_{i=1,..,2d}\int_{\partial\Omega}\partial_n f_i\Delta\Phi_i f_i + \sum_{i=1,..,2d}\int_{\partial\Omega}\eta_i\partial_n\Phi_i|f_i|^2 \ . \end{array} \tag{3.5.2}$$

First we estimate the contribution of the gradient terms:

$$\begin{array}{rl} \displaystyle\sum_{i=1,..,2d}\left(-2\int_\Omega\nabla f_i\nabla^2\Phi_i\nabla f_i - \int_\Omega\nabla f_i\Delta\nabla\Phi_i f_i\right) & \leq \dfrac{cs}{\Gamma}\displaystyle\sum_{i=1,..,2d}\int_\Omega|\nabla f_i|^2 + \frac{cs}{\Gamma}\|\boldsymbol{f}\|^2 \\ & \leq \dfrac{cs}{\Gamma}\displaystyle\sum_{i=1,..,2d}\int_\Omega|\nabla f_i|^2 + \frac{c}{h}\|\boldsymbol{f}\|^2 \end{array} \tag{3.5.3}$$

for $s\in(0,1]$, using Cauchy-Schwarz, $\left|2\nabla^2\Phi_i\right| \leq \frac{cs}{\Gamma}$, and $|\Delta\nabla\Phi_i| \leq \frac{cs}{\Gamma} \leq \frac{1}{h}$.

Next we check the contribution of the boundary terms. We claim that

$$\sum_{i=1,..,2d} \int_{\partial\Omega} \eta_i \partial_n \Phi_i \left|f_i\right|^2 = 0 \ .$$

Indeed, $\eta_i = \frac{1}{2}\partial_t \Phi_i + \frac{1}{4}\left|\nabla \Phi_i\right|^2$ implies

$$\begin{aligned}\sum_{i=1,..,2d} \int_{\partial\Omega} \eta_i \partial_n \Phi_i \left|f_i\right|^2 &= \sum_{i=1,..,d} \int_{\partial\Omega} \left(\frac{1}{2}\partial_t \Phi_i + \frac{1}{4}\left|\nabla \Phi_i\right|^2\right) \partial_n \Phi_i \left|u\right|^2 e^{\Phi_i} \\ &\quad + \sum_{i=1,..,d} \int_{\partial\Omega} \left(\frac{1}{2}\partial_t \Phi_i + \frac{1}{4}\left|\nabla \Phi_i\right|^2\right) \partial_n \Phi_{d+i} \left|u\right|^2 e^{\Phi_i}\end{aligned}$$

where we used $\Phi_{d+i} = \Phi_i$ on $\partial\Omega \times (0,T)$ and $|\nabla \Phi_{d+i}| = |\nabla \Phi_i|$ on $\partial\Omega \times (0,T)$. Since $\partial_n \Phi_i + \partial_n \Phi_{d+i} = 0$ on $\partial\Omega \times (0,T)$, this completes the claim. We also have

$$2 \sum_{i=1,..,2d} \int_{\partial\Omega} \partial_n f_i \nabla \Phi_i \cdot \nabla f_i - \sum_{i=1,..,2d} \int_{\partial\Omega} \partial_n \Phi_i \left|\nabla f_i\right|^2 = 0 \ .$$

Indeed, since $\nabla \Phi_i = \partial_n \Phi_i \overrightarrow{n}$ on $\partial\Omega \times (0,T)$ and $\partial_n f_i = \frac{1}{2}\partial_n \Phi_i f_i$,

$$\begin{aligned}2 \sum_{i=1,..,2d} \int_{\partial\Omega} \partial_n f_i \nabla \Phi_i \cdot \nabla f_i &= 2 \sum_{i=1,..,2d} \int_{\partial\Omega} \partial_n \Phi_i \left|\frac{1}{2}\partial_n \Phi_i f_i\right|^2 \\ &= 2 \sum_{i=1,..,d} \int_{\partial\Omega} \left(\partial_n \Phi_i + \partial_n \Phi_{d+i}\right) \left|\frac{1}{2}\partial_n \Phi_i f_i\right|^2 \\ &= 0\end{aligned}$$

where we used (3.3.3). For the second contribution, it holds

$$\left|\nabla f_i\right|^2 = \left|\nabla u + u\frac{1}{2}\nabla \Phi_i\right|^2 e^{\Phi_i} = \left|\partial_\tau u \overrightarrow{\tau} + u\frac{1}{2}\partial_n \Phi_i \overrightarrow{n}\right|^2 e^{\Phi_i} = \left(\left|\partial_\tau u\right|^2 + \left|\frac{1}{2}u\partial_n \Phi_i\right|^2\right) e^{\Phi_i}$$

on $\partial\Omega \times (0,T)$. We then conclude that $-\sum_{i=1,..,2d} \int_{\partial\Omega} \partial_n \Phi_i \left|\nabla f_i\right|^2 = 0$ using (3.3.3). The last boundary term is treated as follows. Using $\partial_n f_i = \frac{1}{2}\partial_n \Phi_i f_i$, $|\Delta \Phi_i| \leq \frac{cs}{\Gamma}$ and (3.3.3), we have

$$\begin{aligned}\sum_{i=1,..,2d} \int_{\partial\Omega} \partial_n f_i \Delta \Phi_i f_i &= \sum_{i=1,..,2d} \int_{\partial\Omega} \frac{1}{2}\partial_n \Phi_i \Delta \Phi_i \left|f_i\right|^2 \\ &\leq \frac{cs}{\Gamma} \sum_{i=1,..,d} \int_{\partial\Omega} \left|\partial_n \Phi_i\right| \left|f_i\right|^2 = \frac{cs}{\Gamma} \sum_{i=1,..,d} \int_{\partial\Omega} \left(-\partial_n \Phi_i\right) \left|f_i\right|^2\end{aligned}$$

since $\partial_n \psi_i \leq 0$ and, by an integration by parts

$$\begin{aligned}\int_{\partial\Omega} \left(-\partial_n \Phi_i\right) \left|f_i\right|^2 &= -2 \int_\Omega \nabla f_i \cdot \nabla \Phi_i f_i - \int_\Omega \Delta \Phi_i \left|f_i\right|^2 \\ &\leq \int_\Omega \left|\nabla f_i\right|^2 + \int_\Omega \left|\nabla \Phi_i\right|^2 \left|f_i\right|^2 + \frac{cs}{h} \left\|\boldsymbol{f}\right\|^2\end{aligned}$$

using Cauchy-Schwarz and $|\Delta \Phi_i| \leq \frac{cs}{\Gamma} \leq \frac{cs}{h}$, which implies that

$$\sum_{i=1,..,2d} \int_{\partial\Omega} \partial_n f_i \Delta \Phi_i f_i \leq \frac{cs}{\Gamma} \sum_{i=1,..,d} \int_\Omega \left|\nabla f_i\right|^2 + \frac{c^2 s^2}{h^2} \left\|\boldsymbol{f}\right\|^2 + \frac{cs}{\Gamma} \sum_{i=1,..,d} \int_\Omega \left|\nabla \Phi_i\right|^2 \left|f_i\right|^2 \ .$$

One can conclude for the contribution of the boundary terms that for any $s \in (0,1]$

$$\text{Boundary terms} \leq \frac{cs}{\Gamma} \sum_{i=1,..,d} \int_\Omega \left|\nabla f_i\right|^2 + \frac{c^2}{h^2} \left\|\boldsymbol{f}\right\|^2 + \frac{cs}{\Gamma} \sum_{i=1,..,d} \int_\Omega \left|\nabla \Phi_i\right|^2 \left|f_i\right|^2 \ . \tag{3.5.4}$$

Consequently, from (3.5.1)-(3.5.2)-(3.5.3)-(3.5.4), we obtain that for any $h \in (0,1]$ and any $s \in (0,1]$

$$\begin{aligned}\langle \mathcal{S}'\boldsymbol{f},\boldsymbol{f}\rangle + 2\langle \mathcal{S}\boldsymbol{f},\mathcal{A}\boldsymbol{f}\rangle \quad &\leq \frac{cs}{\Gamma}\sum_{i=1,..,2d}\int_\Omega |\nabla f_i|^2 + \frac{c^2}{h^2}\|\boldsymbol{f}\|^2 \\ &-\frac{2}{\Gamma}\sum_{i=1,..,2d}\int_\Omega \left(\eta_i + \frac{1}{4}|\nabla\Phi_i|^2 + \frac{s}{4}\nabla\Phi_i\nabla^2\varphi_i\nabla\Phi_i\right)|f_i|^2 \\ &+\frac{cs}{\Gamma}\sum_{i=1,..,d}\int_\Omega |\nabla\Phi_i|^2|f_i|^2\end{aligned}$$

which gives that for any $s \in (0,1]$ sufficiently small,

$$\begin{aligned}\langle \mathcal{S}'\boldsymbol{f},\boldsymbol{f}\rangle + 2\langle \mathcal{S}\boldsymbol{f},\mathcal{A}\boldsymbol{f}\rangle \quad &\leq \frac{Cs}{\Gamma}\sum_{i=1,..,2d}\int_\Omega |\nabla f_i|^2 + \frac{C}{h^2}\|\boldsymbol{f}\|^2 \\ &-\frac{2}{\Gamma}\sum_{i=1,..,2d}\int_\Omega \left(\eta_i + \frac{1}{8}|\nabla\Phi_i|^2\right)|f_i|^2 \ .\end{aligned} \tag{3.5.5}$$

Indeed, $-\frac{s}{4}\nabla\Phi_i\nabla^2\varphi_i\nabla\Phi_i \leq \frac{s}{4}\left|\nabla^2\varphi_i\right||\nabla\Phi_i|^2 \leq cs|\nabla\Phi_i|^2$.

It remains to prove that

$$-\frac{2}{\Gamma}\sum_{i=1,..,2d}\int_\Omega \left(\eta_i + \frac{1}{8}|\nabla\Phi_i|^2\right)|f_i|^2 \leq C\|\boldsymbol{f}\|^2 + \frac{2-s/c}{\Gamma}\sum_{i=1,..,2d}\int_\Omega (-\eta_i)|f_i|^2 \ . \tag{3.5.6}$$

By Proposition 5 (i) and (ii), $|\varphi_{i,1}| \leq \frac{c}{2}|\nabla\varphi_{i,1}|^2$ in $\mathcal{B}_i \cup \mathcal{D}_i$. This implies that for any $i \in \{1,\cdot\cdot,d\}$

$$-|\nabla\Phi_i|^2 = -\frac{s^2}{\Gamma^2}|\nabla\varphi_{i,1}|^2 \leq -\frac{2s^2}{c\Gamma^2}|\varphi_{i,1}| = \frac{4s}{c}\left(-\frac{s}{2\Gamma^2}|\varphi_{i,1}|\right) \leq \frac{4s}{c}\eta_i \ .$$

Therefore, we get that for any $i \in \{1,\cdot\cdot,d\}$

$$-\frac{1}{4}\int_{\mathcal{B}_i\cup\mathcal{D}_i}|\nabla\Phi_i|^2|f_i|^2 \leq \frac{s}{c}\int_{\mathcal{B}_i\cup\mathcal{D}_i}\eta_i|f_i|^2$$

which yields

$$-\frac{2}{\Gamma}\sum_{i=1,..,d}\int_{\mathcal{B}_i\cup\mathcal{D}_i}\left(\eta_i + \frac{1}{8}|\nabla\Phi_i|^2\right)|f_i|^2 \leq \frac{2-s/c}{\Gamma}\sum_{i=1,..,d}\int_{\mathcal{B}_i\cup\mathcal{D}_i}(-\eta_i)|f_i|^2 \ .$$

By Proposition 5 (iv), there is $c_3 > 0$ such that $\varphi_{i,1} - \varphi_{j,1} \leq -c_3$ in $\mathcal{C}_i$ for some $j \neq i$. Therefore, using $\left|\eta_i + \frac{1}{8}|\nabla\Phi_i|^2\right| \leq \frac{c}{\Gamma^2}$ and $|f_i|^2 = e^{s(\varphi_{i,1}-\varphi_{j,1})\frac{1}{\Gamma}}|f_j|^2$, it holds

$$-\frac{2}{\Gamma}\sum_{i=1,..,d}\int_{\mathcal{C}_i}\left(\eta_i + \frac{1}{8}|\nabla\Phi_i|^2\right)|f_i|^2 \leq \frac{2c}{\Gamma^3}e^{-c_3\frac{s}{\Gamma}}\left(\sum_{i=1,..,d}\int_{\bigcup_{j\neq i}\Theta_{p_j}}|f_j|^2\right) \leq C_s\|\boldsymbol{f}\|^2 \ .$$

By Proposition 5 (v), $|\varphi_{i,2}| \leq c_5|\nabla\varphi_{i,2}|^2$ in a neighborhood ϑ of $\partial\Omega$ and similarly one can deduce that,

$$-\frac{2}{\Gamma}\sum_{i=d+1,..,2d}\int_\vartheta\left(\eta_i + \frac{1}{8}|\nabla\Phi_i|^2\right)|f_i|^2 \leq \frac{2-s/c}{\Gamma}\sum_{i=d+1,..,2d}\int_\vartheta(-\eta_i)|f_i|^2 \ .$$

By Proposition 5 (vi), there is $c_6 > 0$ such that $\varphi_{i,2} - \varphi_{i,1} \leq -c_6$ outside the neighborhood ϑ of $\partial\Omega$ which implies

$$-\frac{2}{\Gamma}\sum_{i=d+1,..,2d}\int_{\Omega\setminus\vartheta}\left(\eta_i + \frac{1}{8}|\nabla\Phi_i|^2\right)|f_i|^2 \leq \frac{2c}{\Gamma^3}e^{-c_6\frac{s}{\Gamma}}\sum_{i=1,..,d}\int_{\Omega\setminus\vartheta}|f_i|^2 \leq C_s\|\boldsymbol{f}\|^2 \ .$$

This completes the proof of (3.5.6).

Consequently, by (3.5.5) and (3.5.6) one can conclude that for any $h \in (0,1]$ and any $s \in (0,1]$ sufficiently small,

$$\langle \mathcal{S}' \boldsymbol{f}, \boldsymbol{f} \rangle + 2 \langle \mathcal{S} \boldsymbol{f}, \mathcal{A} \boldsymbol{f} \rangle \leq \frac{Cs}{\Gamma} \sum_{i=1,..,2d} \int_\Omega |\nabla f_i|^2 + \frac{C_s}{h^2} \|\boldsymbol{f}\|^2 + \frac{2 - s/c}{\Gamma} \sum_{i=1,..,2d} \int_\Omega (-\eta_i) |f_i|^2$$

which implies

$$\langle \mathcal{S}' \boldsymbol{f}, \boldsymbol{f} \rangle + 2 \langle \mathcal{S} \boldsymbol{f}, \mathcal{A} \boldsymbol{f} \rangle \leq \frac{1 + C_0}{\Gamma} \langle \mathcal{S} \boldsymbol{f}, \boldsymbol{f} \rangle + \frac{C}{h^2} \|\boldsymbol{f}\|^2$$

with $C_0 \in (0,1)$ and $C > 0$. Finally, the system (3.4.1) of ordinary differential inequalities becomes

$$\begin{cases} \left| \frac{1}{2} \frac{d}{dt} \|\boldsymbol{f}\|^2 + \mathbf{N}(t) \|\boldsymbol{f}\|^2 \right| \leq \|a\|_\infty \|\boldsymbol{f}\|^2 \ , \\ \mathbf{N}'(t) \leq \frac{1 + C_0}{\Gamma} \mathbf{N}(t) + \|a\|_\infty^2 + \frac{C}{h^2} \ . \end{cases}$$

3.6 Step 6: Solve ODE

Let $h \in (0,1]$ and $\ell > 1$ such that $\ell h < \min(1/2, T/4)$. Applying Proposition 3 with $t_3 = T$, $t_2 = T - \ell h$, and $t_1 = T - 2\ell h$, we obtain that

$$y(T - \ell h)^{1 + M_\ell} \leq y(T)\, y(T - 2\ell h)^{M_\ell}\, e^{D_\ell}$$

where $D_\ell = 2M_\ell \left(F_2 (2\ell h)^2 + F_1 (2\ell h) \right)$, $M_\ell = \frac{(\ell+1)^{C_0} - 1}{1 - \left(\frac{\ell+1}{2\ell+1}\right)^{C_0}} \leq \frac{(\ell+1)^{C_0}}{1 - \left(\frac{2}{3}\right)^{C_0}}$ if $C_0 > 0$.

From now, $y(t) = \|\boldsymbol{f}(\cdot, t)\|^2$, N is the frequency function $\mathbf{N}$, $F_1 = \|a\|_\infty$ and $F_2 = \|a\|_\infty^2 + \frac{C}{h^2}$: We have by the above Proposition 3 and Step 5,

$$\left(\|\boldsymbol{f}(\cdot, T - \ell h)\|^2 \right)^{1 + M_\ell} \leq \|\boldsymbol{f}(\cdot, T)\|^2 \left(\|\boldsymbol{f}(\cdot, T - 2\ell h)\|^2 \right)^{M_\ell} K_\ell \tag{3.6.1}$$

where $K_\ell = e^{D_\ell}$ with $D_\ell = 2M_\ell \left(\left(\|a\|_\infty^2 + \frac{C}{h^2} \right) (2\ell h)^2 + \|a\|_\infty (2\ell h) \right)$. Notice that when $\|a\|_\infty^{2/3} h < 1$, then the following upper bound for K_ℓ holds

$$K_\ell \leq e^{C_\ell \left(1 + \|a\|_\infty^{2/3}\right)} \ . \tag{3.6.2}$$

Indeed, $D_\ell \leq 2M_\ell \left(1 + 4C\ell^2 + 2\|a\|_\infty^2 (2\ell h)^2 \right)$ and $h^2 \|a\|_\infty^2 = \|a\|_\infty^{2/3} \left(\|a\|_\infty^{2/3} h \right)^2 \leq \|a\|_\infty^{2/3}$.

3.7 Step 7: Make appear ω

It is well-known that for any $0 \leq t_1 \leq t_2 \leq T$,

$$\|u(\cdot, t_2)\|_{L^2(\Omega)} \leq e^{(t_2 - t_1)\|a\|_\infty} \|u(\cdot, t_1)\|_{L^2(\Omega)} \tag{3.7.1}$$

where $\|a\|_\infty = \|a\|_{L^\infty(\Omega \times (0,T))}$.

Observe that

$$\|f_1\|_{L^2(\Omega)}^2 \leq \|\boldsymbol{f}\|^2 \leq 2 \sum_{i=1,..,d} \|f_i\|_{L^2(\Omega)}^2$$

since $\varphi_{i,2} \leq \varphi_{i,1}$ on Ω. Therefore, (3.6.1) becomes

$$\begin{aligned}\left(\left\|f_1\left(\cdot, T-\ell h\right)\right\|_{L^2(\Omega)}^2\right)^{1+M_\ell} &\leq 2 \sum_{i=1,..,d}\left\|f_i\left(\cdot, T\right)\right\|_{L^2(\Omega)}^2 \\ &\quad \times\left(2 \sum_{i=1,..,d}\left\|f_i\left(\cdot, T-2\ell h\right)\right\|_{L^2(\Omega)}^2\right)^{M_\ell} K_\ell \ .\end{aligned} \tag{3.7.2}$$

First, notice that from (3.7.1), using $\Phi_i \leq 0$,

$$\left\|f_i\left(\cdot, T-2\ell h\right)\right\|_{L^2(\Omega)}^2 \leq e^{2T\|a\|_\infty} \int_\Omega \left|u\left(\cdot, 0\right)\right|^2 \ . \tag{3.7.3}$$

Second, we make appear $\omega_{i,r} = \{x; |x-p_i| < r\} \subset \omega$ from $\left\|f_i\left(\cdot, T\right)\right\|_{L^2(\Omega)}^2$ as follows:

$$\begin{aligned}\left\|f_i\left(\cdot, T\right)\right\|_{L^2(\Omega)}^2 &= \int_{\omega_{i,r}} \left|u\left(\cdot, T\right)\right|^2 e^{\frac{s}{h}\varphi_{i,1}} + \int_{\Omega\setminus\omega_{i,r}} \left|u\left(\cdot, T\right)\right|^2 e^{\frac{s}{h}\varphi_{i,1}} \\ &\leq \int_\omega \left|u\left(\cdot, T\right)\right|^2 + e^{-\frac{s\mu}{h}} e^{2T\|a\|_\infty} \int_\Omega \left|u\left(\cdot, 0\right)\right|^2\end{aligned} \tag{3.7.4}$$

because on $\Omega\setminus\omega_{i,r}$, $\varphi_{i,1} \leq -\mu$ for some $\mu > 0$ and we used (3.7.1). Third, from (3.7.1) with $\ell h <$min$(1/2, T/4)$ and $-\varphi_{1,1} \leq c$ it holds

$$\begin{aligned}\int_\Omega \left|u\left(\cdot, T\right)\right|^2 &\leq e^{2\ell h\|a\|_\infty} \int_\Omega \left|u\left(\cdot, T-\ell h\right)\right|^2 e^{\frac{s}{(\ell+1)h}\varphi_{1,1}} e^{-\frac{s}{(\ell+1)h}\varphi_{1,1}} \\ &\leq e^{T\|a\|_\infty} e^{\frac{sc}{(\ell+1)h}} \left\|f_1\left(\cdot, T-\ell h\right)\right\|_{L^2(\Omega)}^2 \ .\end{aligned} \tag{3.7.5}$$

Combining the above four facts (3.7.2), (3.7.3), (3.7.4) and (3.7.5), we can deduce that

$$\begin{aligned}&\left(\int_\Omega \left|u\left(\cdot, T\right)\right|^2\right)^{1+M_\ell} \\ &\leq e^{\frac{sc\left(1+M_\ell\right)}{(\ell+1)h}} e^{T\|a\|_\infty(1+M_\ell)} \left(\left\|f_1\left(\cdot, T-\ell h\right)\right\|_{L^2(\Omega)}^2\right)^{1+M_\ell} \\ &\leq e^{\frac{sc\left(1+M_\ell\right)}{(\ell+1)h}} e^{T\|a\|_\infty(1+M_\ell)} \left(2 \sum_{i=1,..,d}\left\|f_i\left(\cdot, T-2\ell h\right)\right\|_{L^2(\Omega)}^2\right)^{M_\ell} K_\ell \\ &\quad \times\left(2 \sum_{i=1,..,d}\left\|f_i\left(\cdot, T\right)\right\|_{L^2(\Omega)}^2\right) \\ &\leq e^{\frac{sc\left(1+M_\ell\right)}{(\ell+1)h}} e^{T\|a\|_\infty(1+M_\ell)} \left(2de^{2T\|a\|_\infty} \int_\Omega \left|u\left(\cdot, 0\right)\right|^2\right)^{M_\ell} K_\ell \\ &\quad \times 2d\left(\int_\omega \left|u\left(\cdot, T\right)\right|^2 + e^{-\frac{s\mu}{h}} e^{2T\|a\|_\infty} \int_\Omega \left|u\left(\cdot, 0\right)\right|^2\right) \ .\end{aligned}$$

We will choose $\ell > 1$ large enough in order that $\frac{sc(1+M_\ell)}{(\ell+1)h} - \frac{s\mu}{h} \leq -\frac{s\mu}{2h}$ that is $\frac{c(1+M_\ell)}{(\ell+1)} \leq \frac{\mu}{2}$. This is possible because $M_\ell \leq \frac{(\ell+1)^{C_0}}{1-\left(\frac{2}{3}\right)^{C_0}}$ with $C_0 \in (0,1)$. Therefore, combining with the upper bound for K_ℓ (see (3.6.2)), there are $M > 0$ and $c > 0$, such that for any $h > 0$ satisfying $\ell h <$min$(1/2, T/4)$ and $\|a\|_\infty^{2/3} h < 1$, we have

$$\begin{aligned}\left(\int_\Omega \left|u\left(\cdot, T\right)\right|^2\right)^{1+M} &\leq e^{c\left(1+T\|a\|_\infty+\|a\|_\infty^{2/3}\right)} \left(\int_\Omega \left|u\left(\cdot, 0\right)\right|^2\right)^M \\ &\quad \times\left(e^{\frac{s\mu}{2h}} \int_\omega \left|u\left(\cdot, T\right)\right|^2 + e^{-\frac{s\mu}{2h}} \int_\Omega \left|u\left(\cdot, 0\right)\right|^2\right) \ .\end{aligned}$$

On the other hand, using (3.7.1), for any $h \geq \min(1/(2\ell), T/(4\ell))$,

$$\int_\Omega |u(\cdot,T)|^2 \leq e^{2T\|a\|_\infty} \int_\Omega |u(\cdot,0)|^2 e^{-\frac{s\mu}{2h}} e^{\frac{s\mu}{2}\left(2\ell+\frac{4\ell}{T}\right)} ,$$

and for any h such that $1 \leq \|a\|_\infty^{2/3} h$,

$$\int_\Omega |u(\cdot,T)|^2 \leq e^{2T\|a\|_\infty} \int_\Omega |u(\cdot,0)|^2 e^{-\frac{s\mu}{2h}} e^{\frac{s\mu}{2}\|a\|_\infty^{2/3}} .$$

Consequently, one can conclude that for any $h > 0$, it holds

$$\begin{aligned}\left(\int_\Omega |u(\cdot,T)|^2\right)^{1+M} &\leq e^{c\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3}\right)} \left(\int_\Omega |u(\cdot,0)|^2\right)^M \\ &\quad \times \left(e^{\frac{c}{h}} \int_\omega |u(\cdot,T)|^2 + e^{-\frac{1}{h}} \int_\Omega |u(\cdot,0)|^2\right) .\end{aligned}$$

Now, choose $h > 0$ such that

$$e^{-\frac{1}{h}} e^{c\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3}\right)} \left(e^{2T\|a\|_\infty} \int_\Omega |u(\cdot,0)|^2\right)^{1+M} = \frac{1}{2} \left(\int_\Omega |u(\cdot,T)|^2\right)^{1+M} ,$$

we obtain the desired estimate for some $M_1 > 1$ and $c_1 > 0$

$$\left(\int_\Omega |u(\cdot,T)|^2\right)^{1+M_1} \leq e^{c_1\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3}\right)} \int_\omega |u(\cdot,T)|^2 \left(\int_\Omega |u(\cdot,0)|^2\right)^{M_1} .$$

This completes the proof.

References

[AEWZ] J. Apraiz, L. Escauriaza, G. Wang and C. Zhang, Observability inequalities and measurable sets, J. Eur. Math. Soc. (JEMS) 16 (11) (2014), 2433–2475.

[BP] C. Bardos and K.D. Phung, Observation estimate for kinetic transport equations by diffusion approximation, C. R. Math. Acad. Sci. Paris 355 (2017), no.6, 640–664.

[BuP] R. Buffe and K.D. Phung, A spectral inequality for degenerate operators and applications, C. R. Math. Acad. Sci. Paris 356 (11-12) (2018) 1131–1155.

[BM] N. Burq and I. Moyano, Propagation of smallness and control for heat equations, ArXiv:1912.07402.

[C] J.-M. Coron, Control and Nonlinearity, vol. 136 of Mathematical Surveys and Monographs, AMS, Providence, RI, 2007.

[DZZ] T. Duyckaerts, X. Zhang and E. Zuazua, On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008) 1–41.

[EFV] L. Escauriaza, F.J. Fernández and S. Vessella, Doubling properties of caloric functions, Appl. Anal. 85 (2006) 205–223.

[FGGP] E. Fernández-Cara, M. González-Burgos, S. Guerrero and J.-P. Puel, Null controllability of the heat equation with boundary Fourier conditions: the linear case, ESAIM Control Optim. Calc. Var. 12(3) (2006) 442–465.

[FV] E. Francini and S. Vessella, Carleman estimates for the parabolic transmission problem and Hölder propagation of smallness across an interface, J. Differential Equations 265 (2018) 2375–2430.

[FI] A.V. Fursikov and O.Y. Imanuvilov, Controllability of evolution equations. Lecture Notes Series, 34. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[K] K. Kurata, On a backward estimate for solutions of parabolic differential equations and its application to unique continuation. Spectral and scattering theory and applications, 247–257, Adv. Stud. Pure Math., 23, Math. Soc. Japan, Tokyo, 1994.

[LRL] J. Le Rousseau and G. Lebeau, On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations, ESAIM Control Optim. Calc. Var., 18 (2012), 712–747.

[LR] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur, Communications in Partial Differential Equation, 20 (1995) 335–356.

[M] Y. Matsumoto, An introduction to Morse Theory, American Mathematical Society, 2002

[P] K.D. Phung, Carleman commutator approach in logarithmic convexity for parabolic equations, Mathematical Control and Related Fields 8 (3-4) (2018) 899–933.

[PW] K. D. Phung and G. Wang, An observability estimate for parabolic equations from a measurable set in time and its applications, J. Eur. Math. Soc. (JEMS) 15 (2) (2013) 681–703.

[PWZ] K. D. Phung, L. Wang and C. Zhang, Bang-bang property for time optimal control of semilinear heat equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (3) (2014) 477–499.

[TW] M. Tucsnak and G. Weiss, Observation and control for operator semigroups, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 2009.

[WW] G. Wang and L. Wang, The Carleman inequality and its application to periodic optimal control governed by semilinear parabolic differential equations. J. Optim. Theory Appl. 118 (2003), no. 2, 429–461.