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A Poincaré type inequality with three constraints

Gisella Croce · Antoine Henrot

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Abstract In this paper, we consider a problem in calculus of variations motivated by a quantitative isoperimetric inequality in the plane. More precisely, the aim of this article is the computation of the minimum of the variational problem

$$\inf_{u \in \mathcal{W}} \frac{\int_{-\pi}^{\pi} [(u')^2 - u^2] d\theta}{\left[\int_{-\pi}^{\pi} |u| d\theta \right]^2}$$

where a function $u \in \mathcal{W}$ is a $H^1(-\pi, \pi)$ periodic function, with zero average on $(-\pi, \pi)$ and orthogonal to sine and cosine.

Keywords Calculus of variations, Euler equation, Poincaré type inequality

1 Introduction

In this article we are interested in the following variational problem :

$$\inf_{u \in \mathcal{W}} \frac{\int_{-\pi}^{\pi} [(u')^2 - u^2] d\theta}{\left[\int_{-\pi}^{\pi} |u| d\theta \right]^2}$$

where \mathcal{W} denotes the subspace of functions in the Sobolev space $H^1(-\pi, \pi)$ that are 2π -periodic, satisfying the following constraints:

G. Croce
Normandie Univ, France; ULH, LMAH, F-76600 Le Havre; FR CNRS 3335, 25 rue Philippe Lebon, 76600 Le Havre, France E-mail: gisella.croce@univ-lehavre.fr

A. Henrot
Institut Élie Cartan de Lorraine UMR CNRS 7502, Université de Lorraine, BP 70239 54506 Vandoeuvre-les-Nancy Cedex, France E-mail: antoine.henrot@univ-lorraine.fr

$$(L1) \int_{-\pi}^{\pi} u(\theta) d\theta = 0$$

$$(L2) \int_{-\pi}^{\pi} u(\theta) \cos(\theta) d\theta = 0$$

$$(L3) \int_{-\pi}^{\pi} u(\theta) \sin(\theta) d\theta = 0.$$

Our aim is to compute the value of the minimum and to identify the minimizer. The difficulty comes here from the nonlinear term in the denominator of the functional together with the three constraints (L1), (L2), (L3). We will prove the following result. Let

$$m = \inf_{u \in \mathcal{W}} J(u), \quad J(u) = \frac{\int_{-\pi}^{\pi} (|u'|^2 - |u|^2)}{\left[\int_{-\pi}^{\pi} |u| \right]^2}. \quad (1)$$

Theorem 1 *Let m be defined by (1). Then $m = \frac{1}{2(4 - \pi)}$ and the minimizer u of the functional J is the odd and π periodic function defined on $[0, \pi/2]$ by $u(\theta) = \cos \theta + \sin \theta - 1$.*

We remark that the minimization problem (1) is a variant of the Wirtinger inequality :

$$\inf_{u \in H_{per}^1(-\pi, \pi): \int_{-\pi}^{\pi} u = 0} \frac{\int_{-\pi}^{\pi} |u'|^2 d\theta}{\int_{-\pi}^{\pi} |u|^2 d\theta} = 1.$$

At first glance, one could think that $\cos 2\theta$ is a minimizer of the functional in (1), as for the Wirtinger-type inequality

$$\inf_{u \in \mathcal{W}} \frac{\int_{-\pi}^{\pi} [(u')^2 - u^2] d\theta}{\int_{-\pi}^{\pi} u^2 d\theta},$$

but this is not true.

In the literature one can find various generalizations of the Wirtinger inequality, without our constraints (L2) and (L3). In the series of papers [2], [4], [5], [6], [7], [10], [11], [14], [16], [17], the authors consider different norms of u' and u and on the mean value of u , namely

$$\inf_{u \in W_{per}^{1,p}(-\pi, \pi): \int_{-\pi}^{\pi} |u|^{r-2} u = 0} \frac{\int_{-\pi}^{\pi} |u'|^p d\theta}{\int_{-\pi}^{\pi} |u|^q d\theta}$$

for all values of p, q, r greater than 1. We also mention [8] in which the authors study, in any dimension $N \geq 1$, the inequality

$$\inf_{u \in W^{1,p}(\Omega): \int_{\Omega} |u|^{p-2} u \omega = 0} \frac{\int_{\Omega} |\nabla u|^p \omega}{\int_{\Omega} |u|^p \omega}$$

with a positive log-concave weight ω , on a convex bounded domain $\Omega \subset \mathbb{R}^N$. In our article, the Rayleigh quotient that we minimise is not "too nonlinear" as in these cited papers. The difficulty comes from the orthogonality to sine and cosine.

We will explain the strategy of the proof in the next section. The proof will be developed in Sections 3, 4, 5 and 6. In the last section we will explain the geometrical motivation of this minimization problem and how we used the value of m to study a quantitative isoperimetric inequality.

2 Strategy of the proof of Theorem 1

The strategy to prove our result is the following. It is immediate to see that the minimization problem (1) has a solution. Thus we write the Euler equation that any minimizer u satisfies:

$$-u'' - u = m \cdot \operatorname{sgn}(u) + \lambda_0 + \lambda_1 \cos \theta + \lambda_2 \sin \theta.$$

For that purpose, we introduce the three Lagrange multipliers, related to the three constraints (L1), (L2) and (L3) : they can be written as a function of $\operatorname{sgn}(u)$.

$$\begin{aligned} \lambda_0 &= -\frac{m}{2\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(u(\theta)) d\theta \\ \lambda_1 &= -\frac{m}{\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(u(\theta)) \cos \theta d\theta \\ \lambda_2 &= -\frac{m}{\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(u(\theta)) \sin \theta d\theta. \end{aligned}$$

By homogeneity, we can assume that the $L^1(-\pi, \pi)$ norm of u equals 1 and, by a translation, that λ_2 is zero. Our aim is to prove that

$$\lambda_0 = \lambda_1 = 0$$

and that u has four nodal intervals, of same length. This allows us to fully determine the minimizer and compute the value of m . Indeed, by using the explicit expression of u on any of the four nodal domain, as a function of the endpoints of the interval, we can easily deduce the explicit expression of the solution u on the whole interval $[-\pi, \pi]$ and thus compute the value of m (see Section 6).

For that purpose, the most involved step is to prove that λ_1 is zero (see Proposition 4). We are now going to give an idea of the strategy.

Let $I_k = [a_{k-1}, a_k]$, $I_{k+1} = [a_k, a_{k+1}]$, $I_{k+2} = [a_{k+1}, a_{k+2}]$, $I_{k+3} = [a_{k+2}, a_{k+3}]$ four consecutive intervals of lengths $\ell_k, \ell_{k+1}, \ell_{k+2}, \ell_{k+3}$, respectively. We assume that u is alternatively positive, negative, positive and negative on these intervals. From the explicit expression of u on any nodal domain, as a function of the endpoints of the interval, it is easy to deduce an equality involving the length of an interval and its consecutive :

$$\begin{aligned}\lambda_0 \sin \frac{\ell_k + \ell_{k+1}}{2} - m \sin \frac{\ell_{k+1} - \ell_k}{2} &= \frac{\lambda_1}{2} \cos \frac{\ell_k}{2} \cos \frac{\ell_{k+1}}{2} A(I_k, I_{k+1}). \\ \lambda_0 \sin \frac{\ell_{k+1} + \ell_{k+2}}{2} + m \sin \frac{\ell_{k+2} - \ell_{k+1}}{2} &= \frac{\lambda_1}{2} \cos \frac{\ell_{k+1}}{2} \cos \frac{\ell_{k+2}}{2} A(I_{k+1}, I_{k+2}) \\ \lambda_0 \sin \frac{\ell_{k+2} + \ell_{k+3}}{2} - m \sin \frac{\ell_{k+3} - \ell_{k+2}}{2} &= \frac{\lambda_1}{2} \cos \frac{\ell_{k+2}}{2} \cos \frac{\ell_{k+3}}{2} A(I_{k+2}, I_{k+3}),\end{aligned}$$

where

$$A(I_k, I_{k+1}) = \frac{\ell_k}{\sin \ell_k} \sin a_{k-1} - \frac{\ell_{k+1}}{\sin \ell_{k+1}} \sin a_{k+1}$$

(see Section 3.2). Assuming by contradiction that $\lambda_1 \neq 0$ and that $\ell_k \neq \ell_{k+2}, \ell_{k+1} \neq \ell_{k+3}$, this 3×3 system in $(\lambda_0, \lambda_1, m)$ has necessarily a null determinant, that provides, after some manipulations the identity

$$\begin{aligned}\frac{C}{\sin \frac{\ell_{k+3}}{2} \cos \frac{\ell_k}{2}} \left[\cos \frac{\ell_{k+1}}{2} \sin \frac{\ell_{k+3}}{2} \sin \frac{\ell_k + \ell_{k+2}}{2} + \cos \frac{\ell_k}{2} \sin \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} + \ell_{k+3}}{2} \right] \\ = A(I_k, I_{k+1}) \cos \frac{\ell_{k+1}}{2} \cos \frac{\ell_{k+2}}{2} - A(I_{k+2}, I_{k+3}) \frac{\cos \frac{\ell_{k+2}}{2} \cos \frac{\ell_{k+3}}{2} \sin \frac{\ell_{k+1}}{2}}{\sin \frac{\ell_{k+3}}{2}},\end{aligned}$$

where $C = \frac{2(m + \lambda_0)}{\lambda_1}$. After proving that the length of each nodal domain is less than π , we will be able to study the sign of each term and arrive to the contradiction that C is both positive and negative.

The proof that the length of each nodal domain is strictly less than π is more difficult than we expected and is done in Section 4.

3 Preliminaries

3.1 Existence of a minimizer and Euler equation

The existence of a minimizer of the functional in (1) and the optimality conditions follow easily from the direct methods of the calculus of variations.

Notice that we will assume that

$$\int_{-\pi}^{\pi} |u| = 1 \tag{2}$$

in all the paper.

Proposition 1 *The minimization problem (1) has a solution u . If K denotes the set of points where u vanishes, then K has zero Lebesgue measure and u satisfies the following Euler equation almost everywhere in $(-\pi, \pi)$:*

$$-u'' - u = m \cdot \operatorname{sgn}(u) + \lambda_0 + \lambda_1 \cos \theta + \lambda_2 \sin \theta, \quad (3)$$

where the Lagrange multipliers are given by

$$\begin{aligned} \lambda_0 &= -\frac{m}{2\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(u(\theta)) d\theta \\ \lambda_1 &= -\frac{m}{\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(u(\theta)) \cos \theta d\theta \\ \lambda_2 &= -\frac{m}{\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(u(\theta)) \sin \theta d\theta. \end{aligned} \quad (4)$$

In particular, the function u is $C^1(-\pi, \pi)$.

Proof The existence of a solution to problem (1) is straightforward using the classical methods of the calculus of variations. We are going to write the optimality condition. For this purpose, we introduce the open set $\omega = K^c = \{x \in (-\pi, \pi), u(x) \neq 0\}$. We fix now a function $\varphi \in H^1((-\pi, \pi))$ satisfying

$$\int_{-\pi}^{\pi} \varphi(x) dx = \int_{-\pi}^{\pi} \varphi(x) \cos x dx = \int_{-\pi}^{\pi} \varphi(x) \sin x dx = 0.$$

Therefore $u + t\varphi \in \mathcal{W}$, that is, it can be used as a test function for our functional $J(v) = \frac{\int_{-\pi}^{\pi} (|v'|^2 - |v|^2)}{[\int_{-\pi}^{\pi} |v|]^2}$. We observe that

$$\int_{-\pi}^{\pi} |u + t\varphi| = \int_{\omega} |u + t\varphi| + |t| \int_K |\varphi|.$$

Now, on the set ω where u is not zero, we have the expansion

$$\int_{\omega} |u + t\varphi| = \int_{\omega} |u| + t \int_{\omega} \operatorname{sign}(u)\varphi + o(t).$$

Therefore, we get

$$J(u + t\varphi) = J(u) + 2t \left[\int_{-\pi}^{\pi} (u'\varphi' - u\varphi) - \int_{\omega} m \operatorname{sign}(u)\varphi \right] - 2|t|m \int_K |\varphi| + o(t). \quad (5)$$

Let us denote by I_0 the term $\int_{-\pi}^{\pi} (u'\varphi' - u\varphi) - \int_{\omega} m \operatorname{sign}(u)\varphi$. If $I_0 > 0$, we choose $t < 0$ small enough and get a contradiction. If $I_0 < 0$, we choose $t > 0$ small enough and get the same contradiction. Therefore $I_0 = 0$ for all admissible φ providing on ω the desired Euler equation. At last, coming back

to (5) we necessarily get $\int_K |\varphi| = 0$ proving that K has zero measure and the Euler equation holds almost everywhere.

The expression of the Lagrange multipliers is obtained by integrating the Euler equation after multiplication by $1, \cos \theta, \sin \theta$.

The C^1 regularity of the function u comes from the Euler equation that shows that its second derivative is L^∞ , implying that $u \in W^{2,\infty}(-\pi, \pi) \subset C^1(-\pi, \pi)$.

Remark 1 Using (4) we see that

$$m + \lambda_0 = \frac{m}{2\pi} \int_{-\pi}^{\pi} (1 - \operatorname{sgn}(u(\theta))) d\theta > 0 \quad (6)$$

and

$$-m + \lambda_0 = -\frac{m}{2\pi} \int_{-\pi}^{\pi} (1 + \operatorname{sgn}(u(\theta))) d\theta < 0. \quad (7)$$

Up to a translation on θ , we can assume that one Lagrange multiplier is zero. Indeed, by periodicity, replacing $u(\theta)$ by $u(\theta + a)$ amounts to replace λ_2 by $\cos a \lambda_2 - \sin a \lambda_1$. Thus we can choose a such that $\lambda_2 = 0$. Therefore in the sequel, we will assume:

$$\text{the Lagrange multiplier } \lambda_2 \text{ is zero.} \quad (8)$$

We introduce the measure of the sets where u is respectively positive and negative:

$$\ell_+ = |\{x \in (-\pi, \pi) : u(x) \geq 0\}|, \quad \ell_- = |\{x \in (-\pi, \pi) : u(x) < 0\}|. \quad (9)$$

With these notations, we can rewrite $m + \lambda_0$ and $m - \lambda_0$:

$$m + \lambda_0 = \frac{m\ell_-}{\pi}, \quad m - \lambda_0 = \frac{m\ell_+}{\pi} = \frac{m}{\pi}(2\pi - \ell_-). \quad (10)$$

Note also that if $u(x)$ is a minimizer of our problem, then $-u(x)$ or $u(-x)$ or $-u(-x)$ are also minimizers. Therefore, without loss of generality, we can assume, from now on, that

$$\ell_+ \geq \ell_-.$$

3.2 Expression of the solution

As usual, we call nodal domain, each interval on which u has a constant sign. We observe that u can be zero in some points in the interior of a nodal domain.

By periodicity, there is an even number of nodal domains. A straight consequence of the Sturm-Hurwitz theorem (see [13] and [12]) applied to any minimizer (satisfying (L1), (L2), (L3)) is that

Proposition 2 *A minimizer u has at least four nodal domains.*

The main difficulty will be to prove that there are *exactly* four nodal domains with same length. It will be a consequence of the fact that the Lagrange multipliers are all zero and will be done in Section 6.

On each nodal domain, we can integrate the Euler equation and get an explicit expression of the solution. We are going to write explicitly u on two consecutive intervals $[a, b]$, $[b, c]$, where

$$u(a) = u(b) = u(c) = 0$$

and

$$u \geq 0 \text{ in } [a, b], \quad u \leq 0 \text{ in } [b, c].$$

Assume that $u \geq 0$ on (a, b) , $u(a) = u(b) = 0$. By integrating the Euler equation on $[a, b]$ and using that $u(a) = u(b) = 0$, we find

$$u(x) = A_0 \cos x + B_0 \sin x - (m + \lambda_0) - \frac{\lambda_1}{2} x \sin x, \quad x \in [a, b]$$

where

$$\begin{aligned} A_0 &= (m + \lambda_0) \frac{\cos(\frac{a+b}{2})}{\cos(\frac{b-a}{2})} - \frac{\lambda_1}{2} \frac{(b-a) \sin a \sin b}{\sin(b-a)}, \\ B_0 &= (m + \lambda_0) \frac{\sin(\frac{a+b}{2})}{\cos(\frac{b-a}{2})} + \frac{\lambda_1}{2} \frac{b \sin b \cos a - a \sin a \cos b}{\sin(b-a)}. \end{aligned} \quad (11)$$

Assume that $u \leq 0$ on an interval $[b, c]$, with $u(b) = u(c) = 0$. We find

$$u(x) = A_1 \cos x + B_1 \sin x - (-m + \lambda_0) - \frac{\lambda_1}{2} x \sin x, \quad x \in [b, c]$$

where

$$\begin{aligned} A_1 &= (-m + \lambda_0) \frac{\cos(\frac{c+b}{2})}{\cos(\frac{c-b}{2})} - \frac{\lambda_1}{2} \frac{(c-b) \sin c \sin b}{\sin(c-b)}, \\ B_1 &= (-m + \lambda_0) \frac{\sin(\frac{b+c}{2})}{\cos(\frac{c-b}{2})} + \frac{\lambda_1}{2} \frac{c \sin c \cos b - b \sin b \cos c}{\sin(c-b)}. \end{aligned} \quad (12)$$

Now we can obtain another expression of the solution on $[b, c]$ using the C^1 regularity of u in b . This gives

$$A_1 \cos b + B_1 \sin b = \lambda_0 - m + \frac{\lambda_1}{2} b \sin b$$

and

$$(A_0 - A_1) \sin b = (B_0 - B_1) \cos b.$$

We then get a different expression for A_1 and B_1 :

$$\begin{aligned} A_1 &= (\lambda_0 - m) \cos b + \frac{\lambda_1}{2} b \sin b \cos b - B_0 \sin b \cos b + A_0 \sin^2 b \\ B_1 &= (\lambda_0 - m) \sin b + \frac{\lambda_1}{2} b \sin^2 b + B_0 \cos^2 b - A_0 \sin b \cos b. \end{aligned}$$

Replacing A_0, B_0 of formulas (11) in the above expressions of A_1 and B_1 , one gets:

$$\begin{aligned} A_1 &= \lambda_0 \frac{\cos(\frac{a+b}{2})}{\cos(\frac{b-a}{2})} - m \frac{\cos(\frac{3b-a}{2})}{\cos(\frac{b-a}{2})} - \frac{\lambda_1}{2} (b-a) \frac{\sin a \sin b}{\sin(b-a)} \\ B_1 &= \lambda_0 \frac{\sin(\frac{a+b}{2})}{\cos(\frac{b-a}{2})} - m \frac{\sin(\frac{3b-a}{2})}{\cos(\frac{b-a}{2})} + \frac{\lambda_1}{2} \frac{b \sin b \cos a - a \sin a \cos b}{\sin(b-a)}. \end{aligned} \quad (13)$$

Expressions (12) and (13) of A_1 give

$$\lambda_0 \frac{\sin b \sin(\frac{c-a}{2})}{\cos(\frac{b-a}{2}) \cos(\frac{c-b}{2})} - m \frac{\sin b \sin(\frac{a+c-2b}{2})}{\cos(\frac{b-a}{2}) \cos(\frac{c-b}{2})} = \frac{\lambda_1}{2} \sin b \left[\frac{(b-a) \sin a}{\sin(b-a)} - \frac{(c-b) \sin c}{\sin(c-b)} \right].$$

Let us assume now that $b \neq 0$. If $\ell_1 = b - a$ and $\ell_2 = c - b$, this equality can be written as

$$\lambda_0 \sin \frac{\ell_1 + \ell_2}{2} - m \sin \frac{\ell_2 - \ell_1}{2} = \frac{\lambda_1}{2} \cos \frac{\ell_1}{2} \cos \frac{\ell_2}{2} \left[\frac{\ell_1}{\sin \ell_1} \sin a - \frac{\ell_2}{\sin \ell_2} \sin c \right]. \quad (14)$$

Let us assume now that $b = 0$. Expressions (12) and (13) of B_1 give

$$(-m + \lambda_0) \tan\left(\frac{c}{2}\right) + \frac{\lambda_1}{2} c = (\lambda_0 + m) \tan\left(\frac{a}{2}\right) + \frac{\lambda_1}{2} a$$

that is,

$$\lambda_0 \sin\left(\frac{c-a}{2}\right) - m \sin\left(\frac{a+c}{2}\right) + \frac{\lambda_1}{2} (c-a) \cos\left(\frac{c}{2}\right) \cos\left(\frac{a}{2}\right) = 0$$

This is exactly equation (14) written in the case $b = 0$.

Here we have assumed the lengths of the intervals not equal to π . The case of an interval of length π will be considered in Section 4.

4 The length of the nodal intervals cannot be greater than π

In this section we prove that the length of any nodal interval of the solution u is strictly less than π . We argue by contradiction, mainly by considering the integral of u on a nodal domain.

We assume that there exists a nodal interval (a, b) of length ℓ greater than π . Without loss of generality, we can assume that

- $u \geq 0$ on (a, b) ;
- $a \in [-\pi, 0]$ and $b \in (0, \pi]$ (since the function $x \mapsto u(x + \pi)$ is also a minimizer satisfying $\lambda_2 = 0$);
- $\frac{a+b}{2} \leq 0$ (since $u(-x)$ is also a minimizer); this implies $a \leq -\frac{\pi}{2}$.

In the sequel we will call *negative interval* (resp. *positive interval*) any interval where u is negative (resp. positive).

On a negative interval (a_j, b_j) of length $\ell_j \neq \pi$, we have

$$\int_{a_j}^{b_j} u(x)dx = (-m + \lambda_0) \left(2 \tan \frac{\ell_j}{2} - \ell_j \right) + \frac{\lambda_1}{2} \left(2 \sin \frac{\ell_j}{2} \cos \frac{a_j + b_j}{2} \right) \left[1 + \frac{\ell_j}{\sin \ell_j} \right], \quad (15)$$

while, on a positive interval (a_k, b_k) of length $\ell_k \neq \pi$, we have

$$\int_{a_k}^{b_k} u(x)dx = (m + \lambda_0) \left(2 \tan \frac{\ell_k}{2} - \ell_k \right) + \frac{\lambda_1}{2} \left(2 \sin \frac{\ell_k}{2} \cos \frac{a_k + b_k}{2} \right) \left[1 + \frac{\ell_k}{\sin \ell_k} \right]. \quad (16)$$

In the case of a nodal domain of length π , let $(a, a + \pi)$ be such an interval where we suppose $u \geq 0$. Now the Euler equation

$$\begin{cases} -u'' - u = m + \lambda_0 + \lambda_1 \cos x \text{ on } (a, a + \pi) \\ u(a) = 0, \quad u(a + \pi) = 0 \end{cases}$$

has not a unique solution, since 1 is an eigenvalue on the interval. Moreover, by the Fredholm alternative, the right-hand side of the equation must be orthogonal to the eigenfunction $\sin(x - a)$, providing the relation

$$m + \lambda_0 = \frac{\lambda_1}{4} \pi \sin a. \quad (17)$$

Lemma 1 *The Lagrange multiplier λ_1 is negative.*

Proof We first study the case where (a, b) has length $\ell > \pi$. Let us analyse equation (16). We recall that $m + \lambda_0 > 0$ by (6); for $\ell > \pi$, both terms $2 \tan(\ell/2) - \ell$ and $1 + \ell/\sin \ell$ are negative. If $\lambda_1 \geq 0$, the integral is negative: this is a contradiction with the sign of u on (a, b) .

In the case of a nodal domain of length π , one has $\lambda_1 < 0$ by equation (17), since $m + \lambda_0 > 0$ and $\sin a < 0$.

Lemma 2 *The Lagrange multiplier λ_1 satisfies*

$$\left| \frac{\lambda_1}{2} \right| \leq 2 \sin \left(\frac{\ell_-}{2} \right) \frac{m}{\pi} = \frac{2}{\ell_-} \sin \frac{\ell_-}{2} (m + \lambda_0) < m + \lambda_0, \quad (18)$$

where ℓ_- is the measure of $\{x : u(x) < 0\}$.

Proof Let us introduce the two numbers:

$$\ell_-^b = |\{t > b, u(t) < 0\}|, \quad \ell_-^a = |\{t < a, u(t) < 0\}|.$$

Obviously $\ell_-^a + \ell_-^b = \ell_-$. By (6), $m + \lambda_0 = \frac{m \ell_-}{\pi}$. Now, since $\lambda_1 < 0$ by the previous lemma,

$$\left| \frac{\lambda_1}{2} \right| = \frac{m}{2\pi} \int_{-\pi}^{\pi} \text{sign}(u) \cos t dt$$

and

$$\int_{-\pi}^{\pi} \text{sign}(u) \cos t dt = \int_{-\pi}^a \text{sign}(u) \cos t dt + \int_a^b \cos t dt + \int_b^{\pi} \text{sign}(u) \cos t dt.$$

By the bathtub principle (see [15]), the value of $\int_{-\pi}^a \text{sign}(u) \cos t dt$ is maximum when we choose $\text{sign}(u) = -1$ on the left, namely on $(-\pi, -\pi + \ell^b]$ (because \cos is increasing on $[-\pi, a]$) and similarly for the last integral. Therefore, we get

$$\left| \frac{\pi \lambda_1}{m} \right| \leq - \int_{-\pi}^{-\pi + \ell^b} \cos t dt + \int_{-\pi + \ell^b}^{\pi + \ell^a} \cos t dt - \int_{\pi - \ell^a}^{\pi} \cos t dt = 2(\sin \ell_-^a + \sin \ell_-^b). \quad (19)$$

Since

$$\sin \ell_-^a + \sin \ell_-^b = 2 \sin \left(\frac{\ell_-}{2} \right) \cos \left(\frac{\ell_-^a - \ell_-^b}{2} \right) \leq 2 \sin \left(\frac{\ell_-}{2} \right) \leq \ell_-,$$

we finally get estimate (18), using (6) and (19).

Let us introduce the following positive quantity :

$$A = \frac{|\lambda_1/2|}{m + \lambda_0}. \quad (20)$$

Our strategy to get a contradiction is based on the following

Proposition 3 *If $A < \frac{2}{\pi}$ or $A \cos(\frac{a+b}{2}) < \frac{2}{\pi}$, then we cannot have a nodal interval of length $\ell \geq \pi$.*

Proof Let us start with the case $b - a = \ell = \pi$. In that case we have $A = \frac{2}{\pi |\sin a|}$ by (17). Therefore the assumption $A < \frac{2}{\pi}$ provides immediately a contradiction. In the same way, if $A \cos(\frac{a+b}{2}) < \frac{2}{\pi}$ we deduce $|\sin a| > \cos(\frac{a+b}{2}) = \cos(\frac{a+a+\pi}{2}) = -\sin a$ that is also a contradiction.

Now, let us assume that $b - a = \ell > \pi$. We use the following claim: the function $g : \ell \mapsto \ell - 2 \tan(\ell/2) + \frac{4}{\pi} \sin(\ell/2)[1 + \ell/\sin \ell]$ is positive on $(\pi, 2\pi)$. Indeed g is positive if and only if $k(t) = t \cos t - \sin t + \frac{1}{\pi}[2t + \sin(2t)]$ is negative on $(\frac{\pi}{2}, \pi)$. Observe that $k(\frac{\pi}{2}) = 0$ and $k(\pi) < 0$. Now, the derivative of k , $k'(t) = -t \sin(t) + \frac{2}{\pi}[1 + \cos(2t)]$, is the difference between two functions which intersect in only one point t_0 . Since k' is negative near $\frac{\pi}{2}$ and positive near π , k is minimal at t_0 and therefore $k < 0$ on $(\frac{\pi}{2}, \pi)$.

We are able to get a contradiction by using the expression (16) of the integral of u on the interval (a, b) , that can be written as

$$\int_a^b u(x) dx = (m + \lambda_0) \left(2 \tan \frac{\ell}{2} - \ell - 2A \cos \frac{a+b}{2} \sin \frac{\ell}{2} \left[1 + \frac{\ell}{\sin \ell} \right] \right).$$

Therefore, if $A < \frac{2}{\pi}$ or $A \cos(\frac{a+b}{2}) < \frac{2}{\pi}$, we obtain $\int_a^b u(x) dx \leq -g(\ell) \leq 0$ (note that $1 + \ell/\sin \ell < 0$ for $\ell > \pi$). Thus we have the desired contradiction.

We are now going to find some estimates on A , in order to apply Proposition 3. This is quite technical and for that reason, we postpone all these computations to the Appendix. After proving an estimate on ℓ_- (see Proposition 6), we distinguish the cases where u has at least 6 nodal domains (see Propositions 7 and 8) and u has exactly 4 nodal domains (see Proposition 9).

5 The Lagrange multipliers are zero and the nodal domains have same length

Now we enter into the heart of the paper. We are going to prove that the Lagrange multipliers λ_0 and λ_1 are zero (we already know that $\lambda_2 = 0$) and that all the nodal domains have the same length. For that purpose, we will use the relation (14) on different intervals.

Theorem 2 *The Lagrange multipliers λ_0, λ_1 are equal to zero and all the nodal intervals have the same length.*

The proof will be done in two main steps. First, we prove that $\lambda_1 = 0$ in Proposition 4. Then, we prove that $\lambda_0 = 0$ and the nodal domains have same length in Proposition 5.

Let us first introduce some notations and give a preliminary lemma. Let $I_k = [a_{k-1}, a_k]$ and $I_{k+1} = [a_k, a_{k+1}]$ be two consecutive intervals of length respectively ℓ_k, ℓ_{k+1} . We introduce:

$$A(I_k, I_{k+1}) = \frac{\ell_k}{\sin \ell_k} \sin a_{k-1} - \frac{\ell_{k+1}}{\sin \ell_{k+1}} \sin a_{k+1}.$$

Note that, using $a_{k-1} = a_k - \ell_k$ and $a_{k+1} = a_k + \ell_{k+1}$ we can also write

$$A(I_k, I_{k+1}) = \left(\frac{\ell_k}{\tan \ell_k} - \frac{\ell_{k+1}}{\tan \ell_{k+1}} \right) \sin a_k - (\ell_k + \ell_{k+1}) \cos a_k. \quad (21)$$

Lemma 3 *There exist three consecutive intervals, say I_j, I_{j+1}, I_{j+2} , such that $A(I_j, I_{j+1}) \geq 0$, $A(I_{j+1}, I_{j+2}) \geq 0$ and there exist three consecutive intervals, say I_i, I_{i+1}, I_{i+2} , such that $A(I_i, I_{i+1}) < 0$, $A(I_{i+1}, I_{i+2}) < 0$.*

Proof Let us consider $I_i = (a_{i-1}, a_i), I_{i+1} = (a_i, a_{i+1}), I_{i+2} = (a_{i+1}, a_{i+2})$, with $a_i < 0 < a_{i+1}$. Without loss of generality we can assume that $I_i \cup I_{i+1} \cup I_{i+2} \subset [-\pi, \pi]$ (up to consider $u(-x)$ instead of $u(x)$). Since $\sin a_{i-1} < 0$ and $\sin a_{i+1} > 0$

$$A(I_i, I_{i+1}) = \frac{\ell_i}{\sin \ell_i} \sin a_{i-1} - \frac{\ell_{i+1}}{\sin \ell_{i+1}} \sin a_{i+1} < 0.$$

Since $\sin a_i < 0$ and $\sin a_{i+2} > 0$

$$A(I_{i+1}, I_{i+2}) = \frac{\ell_{i+1}}{\sin \ell_{i+1}} \sin a_i - \frac{\ell_{i+2}}{\sin \ell_{i+2}} \sin a_{i+2} < 0.$$

Let us consider $I_j = (a_{j-1}, a_j)$, $I_{j+1} = (a_j, a_{j+1})$, $I_{j+2} = (a_{j+1}, a_{j+2})$, with $a_j < -\pi < a_{j+1}$. Assume that $I_j \cup I_{j+1} \cup I_{j+2} \subset [-2\pi, 0]$. Since $\sin a_{j-1} > 0$ and $\sin a_{j+1} < 0$

$$A(I_j, I_{j+1}) = \frac{\ell_j}{\sin \ell_j} \sin a_{j-1} - \frac{\ell_{j+1}}{\sin \ell_{j+1}} \sin a_{j+1} > 0.$$

Since $\sin a_j > 0$ and $\sin a_{j+2} < 0$

$$A(I_{j+1}, I_{j+2}) = \frac{\ell_{j+1}}{\sin \ell_{j+1}} \sin a_j - \frac{\ell_{j+2}}{\sin \ell_{j+2}} \sin a_{j+2} > 0.$$

Proposition 4 *The Lagrange multiplier λ_1 is zero.*

Proof Let $I_k = [a_{k-1}, a_k]$, $I_{k+1} = [a_k, a_{k+1}]$, $I_{k+2} = [a_{k+1}, a_{k+2}]$, $I_{k+3} = [a_{k+2}, a_{k+3}]$ four consecutive intervals of lengths $\ell_k, \ell_{k+1}, \ell_{k+2}, \ell_{k+3}$, respectively. We assume that u is alternatively positive, negative, positive and negative. In Section 3 we have seen that (see (14))

$$\lambda_0 \sin \frac{\ell_k + \ell_{k+1}}{2} - m \sin \frac{\ell_{k+1} - \ell_k}{2} = \frac{\lambda_1}{2} \cos \frac{\ell_k}{2} \cos \frac{\ell_{k+1}}{2} A(I_k, I_{k+1}). \quad (22)$$

We can reproduce this identity for the other intervals:

$$\lambda_0 \sin \frac{\ell_{k+1} + \ell_{k+2}}{2} + m \sin \frac{\ell_{k+2} - \ell_{k+1}}{2} = \frac{\lambda_1}{2} \cos \frac{\ell_{k+1}}{2} \cos \frac{\ell_{k+2}}{2} A(I_{k+1}, I_{k+2}) \quad (23)$$

$$\lambda_0 \sin \frac{\ell_{k+2} + \ell_{k+3}}{2} - m \sin \frac{\ell_{k+3} - \ell_{k+2}}{2} = \frac{\lambda_1}{2} \cos \frac{\ell_{k+2}}{2} \cos \frac{\ell_{k+3}}{2} A(I_{k+2}, I_{k+3}). \quad (24)$$

Assume by contradiction that $\lambda_1 \neq 0$. We divide the proof into three cases, according to the lengths of the nodal intervals.

1. Let us assume that $\ell_k \neq \ell_{k+2}$ and $\ell_{k+1} \neq \ell_{k+3}$. Equations (22), (23) can be seen as a system in λ_0 and m from which we get

$$\lambda_0 = \frac{\lambda_1}{2} \cos \frac{\ell_{k+1}}{2} \frac{\cos \frac{\ell_k}{2} \sin \frac{\ell_{k+2} - \ell_{k+1}}{2} A(I_k, I_{k+1}) + \cos \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} - \ell_k}{2} A(I_{k+1}, I_{k+2})}{\sin \ell_{k+1} \sin \frac{\ell_{k+2} - \ell_k}{2}}$$

$$m = \frac{\lambda_1}{2} \cos \frac{\ell_{k+1}}{2} \frac{-\cos \frac{\ell_k}{2} \sin \frac{\ell_{k+2} + \ell_{k+1}}{2} A(I_k, I_{k+1}) + \cos \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} + \ell_k}{2} A(I_{k+1}, I_{k+2})}{\sin \ell_{k+1} \sin \frac{\ell_{k+2} - \ell_k}{2}}.$$

We observe that

$$\lambda_0 + m = \frac{\lambda_1}{2} \frac{\cos \frac{\ell_k}{2} \cos \frac{\ell_{k+2}}{2}}{\sin \frac{\ell_{k+2} - \ell_k}{2}} [A(I_{k+1}, I_{k+2}) - A(I_k, I_{k+1})]. \quad (25)$$

Similarly, if one chooses equations (23), (24) to solve with respect to λ_0 , m , he gets

$$\lambda_0 + m = \frac{\lambda_1}{2} \frac{\cos \frac{\ell_{k+2}}{2}}{\sin \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+3} - \ell_{k+1}}{2}} \times$$

$$\times \left[\cos \frac{\ell_{k+1}}{2} \sin \frac{\ell_{k+3}}{2} A(I_{k+1}, I_{k+2}) - \cos \frac{\ell_{k+3}}{2} \sin \frac{\ell_2}{2} A(I_{k+2}, I_{k+3}) \right]. \quad (26)$$

We now set $C = \frac{2(m + \lambda_0)}{\lambda_1}$. We use (25) to get $A(I_{k+1}, I_{k+2})$ in terms of $A(I_k, I_{k+1})$:

$$A(I_{k+1}, I_{k+2}) = A(I_k, I_{k+1}) + \frac{C \sin \frac{\ell_{k+2} - \ell_k}{2}}{\cos \frac{\ell_k}{2} \cos \frac{\ell_{k+2}}{2}}. \quad (27)$$

We also use (26) to get $A(I_{k+1}, I_{k+2})$ in terms of $A(I_{k+2}, I_{k+3})$:

$$A(I_{k+1}, I_{k+2}) = \frac{\tan \frac{\ell_{k+1}}{2}}{\tan \frac{\ell_{k+3}}{2}} A(I_{k+2}, I_{k+3}) + \frac{C \tan \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+3} - \ell_{k+1}}{2}}{\sin \frac{\ell_{k+3}}{2} \cos \frac{\ell_{k+1}}{2}}. \quad (28)$$

Since m is non-zero, the 3×3 determinant of the system in $(m, \lambda_0, \lambda_1)$ given by equations (22), (23), (24) has to be equal to zero. Now, the computation of this determinant with respect to its third column gives the following equality after some simplification:

$$\begin{aligned} & A(I_k, I_{k+1}) \cos \frac{\ell_k}{2} \cos \frac{\ell_{k+1}}{2} \sin \ell_{k+2} \sin \frac{\ell_{k+1} - \ell_{k+3}}{2} \\ & + A(I_{k+2}, I_{k+3}) \cos \frac{\ell_{k+2}}{2} \cos \frac{\ell_{k+3}}{2} \sin \ell_{k+1} \sin \frac{\ell_{k+2} - \ell_k}{2} \\ & - A(I_{k+1}, I_{k+2}) \cos \frac{\ell_{k+1}}{2} \cos \frac{\ell_{k+2}}{2} \times \\ & \times \left(\sin \frac{\ell_{k+1} - \ell_{k+3}}{2} \sin \frac{\ell_k + \ell_{k+2}}{2} + \sin \frac{\ell_{k+2} - \ell_k}{2} \sin \frac{\ell_{k+1} + \ell_{k+3}}{2} \right) = 0. \end{aligned} \quad (29)$$

Now we replace $A(I_{k+1}, I_{k+2})$ in (29) by using both (27) (for the first term), (28) (for the second) and we get, after use of trigonometric formulae

$$\begin{aligned} 0 &= A(I_k, I_{k+1}) \cos \frac{\ell_{k+1}}{2} \cos \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} - \ell_{k+3}}{2} \sin \frac{\ell_{k+2} - \ell_k}{2} + \\ & + A(I_{k+2}, I_{k+3}) \cos \frac{\ell_{k+2}}{2} \cos \frac{\ell_{k+3}}{2} \sin \frac{\ell_{k+3} - \ell_{k+1}}{2} \sin \frac{\ell_{k+2} - \ell_k}{2} \frac{\sin \frac{\ell_{k+1}}{2}}{\sin \frac{\ell_{k+3}}{2}} \\ & - C \frac{\sin \frac{\ell_{k+1} - \ell_{k+3}}{2} \sin \frac{\ell_{k+2} - \ell_k}{2}}{\sin \frac{\ell_{k+3}}{2} \cos \frac{\ell_k}{2}} \left[\cos \frac{\ell_{k+1}}{2} \sin \frac{\ell_{k+3}}{2} \sin \frac{\ell_k + \ell_{k+2}}{2} + \cos \frac{\ell_k}{2} \sin \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} + \ell_{k+3}}{2} \right]. \end{aligned} \quad (30)$$

Simplifying by $\sin \frac{\ell_{k+1} - \ell_{k+3}}{2} \sin \frac{\ell_{k+2} - \ell_k}{2}$ (this is possible since we are assuming $\ell_{k+2} \neq \ell_k$ and $\ell_{k+3} \neq \ell_{k+1}$) we finally get

$$\begin{aligned} & \frac{C}{\sin \frac{\ell_{k+3}}{2} \cos \frac{\ell_k}{2}} \quad (31) \\ & \times \left[\cos \frac{\ell_{k+1}}{2} \sin \frac{\ell_{k+3}}{2} \sin \frac{\ell_k + \ell_{k+2}}{2} + \cos \frac{\ell_k}{2} \sin \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} + \ell_{k+3}}{2} \right] \\ & = A(I_k, I_{k+1}) \cos \frac{\ell_{k+1}}{2} \cos \frac{\ell_{k+2}}{2} - A(I_{k+2}, I_{k+3}) \frac{\cos \frac{\ell_{k+2}}{2} \cos \frac{\ell_{k+3}}{2} \sin \frac{\ell_{k+1}}{2}}{\sin \frac{\ell_{k+3}}{2}}. \end{aligned}$$

Note that C has the same sign as λ_1 , since by definition of λ_0 (see section 3), $\lambda_0 + m > 0$. Moreover, in equation (31), the coefficients of C , $A(I_k, I_{k+1})$ and $-A(I_{k+2}, I_{k+3})$ are all positive, since the length of each nodal domain is less than π , as we have seen in Section 4.

Now we claim that we can choose four consecutive intervals such that

– $A(I_k, I_{k+1})$ is positive and $A(I_{k+2}, I_{k+3})$ is negative (with u positive on I_k).

and we can choose four intervals such that

– $A(I_j, I_{j+1})$ is negative and $A(I_{j+2}, I_{j+3})$ is positive (with u positive on I_j).

If this claim is true, we get a contradiction since (31) would show that C (and then λ_1) is both positive and negative. To prove our claim, we set

$$\mathcal{I}_- = \{(I_k, I_{k+1}) : A(I_k, I_{k+1}) < 0\}, \quad \mathcal{I}_+ = \{(I_k, I_{k+1}) : A(I_k, I_{k+1}) \geq 0\}.$$

We have seen in Lemma 3 that both \mathcal{I}_- and \mathcal{I}_+ contain pairs of consecutive intervals (or triplet of intervals). Let us now consider the last triplet of intervals for which $A < 0$. the second one. Let I_{k-1}, I_k, I_{k+1} be these three intervals. Therefore $A(I_{k+1}, I_{k+2}) \geq 0$. If $u > 0$ on I_{k-1} we are done, because we can consider $(I_{k-1}, I_k), (I_{k+1}, I_{k+2})$. If u is negative on I_{k-1} we have u negative on $I_{k-1}, I_{k+1}, I_{k+3} \dots$ and positive on $I_k, I_{k+2}, I_{k+4} \dots$. We can consider (I_k, I_{k+1}) for which $A < 0$. If $A(I_{k+2}, I_{k+3}) \geq 0$ we are done. If $A(I_{k+2}, I_{k+3}) \leq 0$, then $A(I_{k+3}, I_{k+4}) \geq 0$ (otherwise the last triplet in \mathcal{I}_- would be $I_{k+2}, I_{k+3}, I_{k+4}$). If $A(I_{k+4}, I_{k+5}) \geq 0$ we are done. If $A(I_{k+4}, I_{k+5}) \leq 0$, then $A(I_{k+5}, I_{k+6}) \geq 0 \dots$ after some steps we will get necessarily four consecutive intervals $I_{m-2}, I_{m-1}, I_m, I_{m+1}$ (with u positive on I_{m-2}) such that $A(I_{m-2}, I_{m-1}) < 0, A(I_m, I_{m+1}) \geq 0$ (because we have to stop before the first triplet of \mathcal{I}_+). Therefore, we have proved the first part of our claim. The second part is proved exactly in the same way, starting from the last triplet in \mathcal{I}_+ .

In conclusion, we get a contradiction.

2. Assume $\ell_k = \ell_{k+2}$. Since the left-hand sides of equations (22) and (23) coincide, the right-hand sides are equal that implies necessarily (since we are assuming $\lambda_1 \neq 0$)

$$\frac{\ell_k}{\sin \ell_k} (\sin a_{k-1} + \sin a_{k+2}) = \frac{\ell_{k+1}}{\sin \ell_{k+1}} (\sin a_k + \sin a_{k+1}).$$

Now, we observe that since $\ell_k = \ell_{k+2}$, one has $(a_{k-1} + a_{k+2})/2 = (a_k + a_{k+1})/2$. Replacing $\sin a_{k-1} + \sin a_{k+2}$ by $2 \sin(a_{k-1} + a_{k+2})/2 \cos(a_{k+2} - a_{k-1})/2$ and $\sin a_k + \sin a_{k+1}$ by $2 \sin(a_k + a_{k+1})/2 \cos(a_{k+1} - a_k)/2$ the above equality gives

$$\frac{\ell_k}{\sin \ell_k} \cos\left(\ell_k + \frac{\ell_{k+1}}{2}\right) = \frac{\ell_{k+1}}{\sin \ell_{k+1}} \cos \frac{\ell_{k+1}}{2} = \frac{\ell_{k+1}}{2 \sin \frac{\ell_{k+1}}{2}}. \quad (32)$$

Assuming ℓ_k fixed, we can study the function

$$g : x \mapsto \ell_k \sin x \cos(\ell_k + x) - x \sin \ell_k.$$

Since $g'(x)$ is negative and $g(0) = 0$, it is not possible to find $\ell_{k+1} > 0$ such that (32) holds. Therefore we have a contradiction.

3. Assuming $\ell_{k+1} = \ell_{k+3}$ we get a contradiction in the same way as in the previous case ($\ell_k = \ell_{k+2}$).

Therefore, we conclude that necessarily $\lambda_1 = 0$.

To finish the proof of Theorem 2, we need the following proposition.

Proposition 5 *The Lagrange multiplier λ_0 is zero and the nodal domains have same length.*

Proof Since $\lambda_1 = 0$ by the previous proposition, from section 3 we deduce that

$$\lambda_0 \sin\left(\frac{\ell_k + \ell_{k+1}}{2}\right) - m \sin\left(\frac{\ell_{k+1} - \ell_k}{2}\right) = 0, \quad (33)$$

$$\lambda_0 \sin\left(\frac{\ell_{k+1} + \ell_{k+2}}{2}\right) + m \sin\left(\frac{\ell_{k+2} - \ell_{k+1}}{2}\right) = 0.$$

The determinant of this homogeneous system is zero, as $m \neq 0$. This means

$$\sin\left(\frac{\ell_{k+1} - \ell_k}{2}\right) \sin\left(\frac{\ell_{k+1} + \ell_{k+2}}{2}\right) = \sin\left(\frac{\ell_{k+1} - \ell_{k+2}}{2}\right) \sin\left(\frac{\ell_k + \ell_{k+1}}{2}\right)$$

that is,

$$\cos\left(\frac{2\ell_{k+1} - \ell_k + \ell_{k+2}}{2}\right) = \cos\left(\frac{2\ell_{k+1} + \ell_k - \ell_{k+2}}{2}\right)$$

which implies $\ell_k - \ell_{k+2} = -\ell_k + \ell_{k+2}$, that is, $\ell_k = \ell_{k+2}$. With the same argument on $\ell_{k+1}, \ell_{k+2}, \ell_{k+3}$ we find $\ell_{k+1} = \ell_{k+3}$.

Therefore all the intervals where u is positive have the same length, say ℓ_1 ; all the intervals where u is negative have the same length, say ℓ_2 . The sum of these lengths give $n(\ell_1 + \ell_2) = 2\pi$.

On the other hand, $\lambda_0 = -\frac{m}{2\pi} \int_0^{2\pi} \text{sign}(u) = -\frac{m}{2\pi} n(\ell_1 - \ell_2)$ which gives us

$$\ell_2 - \ell_1 = \frac{2\pi\lambda_0}{mn}. \quad (34)$$

If we replace this equality in (33), we have

$$\lambda_0 \sin\left(\frac{\pi}{n}\right) = m \sin\left(\frac{\pi\lambda_0}{mn}\right).$$

We now study the function $f(x) = m \sin\left(\frac{\pi x}{mn}\right) - x \sin\left(\frac{\pi}{n}\right)$, for $x \in [0, m)$ (recall that $\lambda_0 < m$). Since $f(0) = 0 = f(m)$, $f'(0) > 0$, $f'(m) < 0$ and $f''(x) < 0$, we deduce that the only zero on f is zero. This means that $f(\lambda_0) = 0$ if and only if $\lambda_0 = 0$. This argument proves that $\lambda_0 = 0$. We deduce from (34) that $\ell_1 = \ell_2$.

6 Conclusion

We are now in position to prove Theorem 1. We have seen in the previous section that all the nodal intervals have same length, say $\ell < \pi$, and the Lagrange multipliers $\lambda_0, \lambda_1, \lambda_2$ are all zero. We recall that if (a, b) is an interval where $u \geq 0$, one has

$$u(x) = A_0 \cos x + B_0 \sin x - m, \quad A_0 = m \frac{\cos(\frac{a+b}{2})}{\cos \frac{\ell}{2}}, \quad B_0 = m \frac{\sin(\frac{a+b}{2})}{\cos \frac{\ell}{2}};$$

if (b, c) is an interval where $u \leq 0$, one has

$$u(x) = A_1 \cos x + B_1 \sin x + m, \quad A_1 = -m \frac{\cos(\frac{b+c}{2})}{\cos \frac{\ell}{2}}, \quad B_1 = -m \frac{\sin(\frac{b+c}{2})}{\cos \frac{\ell}{2}}$$

(see subsection 3.2). The solution of the system

$$\begin{cases} u(x) = A_0 \cos x + B_0 \sin x - m = 0 \\ u(x) = A_1 \cos x + B_1 \sin x + m = 0 \end{cases}$$

is

$$\begin{cases} \cos(x - \varphi_0) = \cos \frac{\ell}{2}, \quad \tan(\varphi_0) = \tan\left(\frac{a+b}{2}\right) \\ \cos(x - \varphi_1) = \cos \frac{\ell}{2}, \quad \tan(\varphi_1) = \tan\left(\frac{b+c}{2}\right) \end{cases}$$

with $x - \varphi_0 = \pm(\pi + x - \varphi_1) + 2k\pi$. Since $x = a + \ell$, the only possible solution is $\ell = \frac{\pi}{2}$. Therefore $u(x)$ is symmetric with respect to a , and

$$u(x) = \begin{cases} m \cos x + m \sin x - m, & x \in [a, a + \frac{\pi}{2}] \\ -m \cos x - m \sin x + m, & x \in [a + \frac{\pi}{2}, a + \pi] \end{cases}.$$

We now compute m defined in (1). Recalling that $\int_0^1 |u| = 1$ (see (2)), we have

$$\left[\int_0^{2\pi} |u| \right]^2 = 16 \left[\int_a^{a+\frac{\pi}{2}} |u| \right]^2 = 16m^2 \left(2 - \frac{\pi}{2}\right)^2 = 1.$$

Therefore $m = \frac{1}{2(4 - \pi)}$.

7 Motivations and final remarks

The minimization of the functional in (1) is motivated by a shape optimization problem and more precisely from a quantitative isoperimetric inequality. Indeed, for any open bounded set of \mathbb{R}^n , let us introduce the *isoperimetric deficit*:

$$\delta(\Omega) = \frac{P(\Omega) - P(B)}{P(B)}, \quad (35)$$

where $|B| = |\Omega|$. Let the *barycentric asymmetry* be defined by:

$$\lambda_0(\Omega) = \frac{|\Omega \Delta B_{x^G}|}{|\Omega|}$$

where B_{x^G} is the ball centered at the barycentre $x^G = \frac{1}{|\Omega|} \int_{\Omega} x \, dx$ of Ω and such that $|\Omega| = |B_{x^G}|$. Fuglede proved in [9] that there exists a positive constant (depending only on the dimension n) such that

$$\delta(\Omega) \geq C(n) \lambda_0^2(\Omega), \quad \text{for any convex subsets } \Omega \text{ of } \mathbb{R}^n. \quad (36)$$

Now, the constant $C(n)$ is unknown (as it is the case in most quantitative inequalities like (36)) and it would be interesting to find the best constant. This leads to consider the minimization of the ratio

$$\mathcal{G}_0(\Omega) = \frac{\delta(\Omega)}{\lambda_0^2(\Omega)}$$

among convex compact sets in the plane, in particular. In the study of this minimization problem, one is led to exclude sequences converging to the ball in the Hausdorff metric. The strategy is to prove that on these sequences \mathcal{G}_0 is greater than 0.406 which is the value of $\mathcal{G}_0(S)$ where S is a precise set with the shape of a stadium, as computed in [1].

If a convex planar set E has barycenter in 0, it can be parametrized in polar coordinates with respect to 0, as

$$E = \{y \in \mathbb{R}^2 : y = tx(1 + u(x)), x \in \mathbb{S}^1, t \in [0, 1]\}, \quad (37)$$

where u is a Lipschitz periodic function. Then the shape functional $\mathcal{G}_0(E)$ can be written as a functional H of the function u describing E , as follows :

$$\mathcal{G}_0(E) = H(u) = \frac{\pi \int_{-\pi}^{\pi} \left[\sqrt{(1+u)^2 + u'(\theta)^2} - 1 \right] d\theta}{2 \left[\frac{1}{2} \int_{-\pi}^{\pi} |(1+u)^2 - 1| d\theta \right]^2}. \quad (38)$$

The constraints of area (fixed equal to π without loss of generality) and barycentre in 0 read in terms of a periodic $u \in H^1(-\pi, \pi)$ as:

$$(NL1) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+u)^2 d\theta = 1;$$

$$(NL2) \quad \int_{-\pi}^{\pi} \cos(\theta) [1+u(\theta)]^3 d\theta = 0;$$

$$(NL3) \quad \int_{-\pi}^{\pi} \sin(\theta) [1+u(\theta)]^3 d\theta = 0.$$

The computation of the minimum of H , under the constraints (NL1), (NL2) and (NL3), seems very difficult. However, for sequences of sets converging to the ball in the Hausdorff metric, the limit of

$$m_\varepsilon := \inf\{H(u), \|u\|_{L^\infty} = \varepsilon, u \in H^1(-\pi, \pi) \text{ periodic, satisfying (NL1), (NL2), (NL3)}\}$$

as $\varepsilon \rightarrow 0$, equals the limit of the shape functional \mathcal{G}_0 for these sequences. Thus, a possible strategy consists in estimating from below the minimum of H by a simpler functional, namely its linearization. Define

$$m = \inf_{u \in \mathcal{W}} \frac{\int_{-\pi}^{\pi} [(u')^2 - u^2] d\theta}{\left[\int_{-\pi}^{\pi} |u| d\theta \right]^2}$$

where \mathcal{W} is the space of periodic $H^1(0, 2\pi)$ functions satisfying the constraints:

$$(L1) \quad \int_{-\pi}^{\pi} u d\theta = 0$$

$$(L2) \quad \int_{-\pi}^{\pi} u \cos(\theta) d\theta = 0$$

$$(L3) \quad \int_{-\pi}^{\pi} u \sin(\theta) d\theta = 0.$$

In [3] we proved that

$$\liminf_{\varepsilon \rightarrow 0} m_\varepsilon \geq \frac{\pi}{4} m. \quad (39)$$

The value of m found in Theorem 1 allows us to conclude that

$$\liminf_{\varepsilon \rightarrow 0} m_\varepsilon \geq \frac{\pi}{4} m > 0.406.$$

Remark 2 We observe that one can easily get an estimate from below of m by using the Cauchy-Schwarz inequality

$$\left(\int_{-\pi}^{\pi} |u| d\theta \right)^2 \leq 2\pi \int_{-\pi}^{\pi} u^2 d\theta.$$

Then, a Wirtinger-type inequality (or Parseval formula) shows that

$$m \geq \inf_{u \in \mathcal{W}} \frac{\int_{-\pi}^{\pi} [(u')^2 - u^2] d\theta}{2\pi \int_{-\pi}^{\pi} u^2 d\theta} \geq \frac{3}{2\pi}.$$

Unfortunately this estimate on m is not sufficient to prove the desired inequality $\liminf_{\varepsilon \rightarrow 0} m_\varepsilon > 0.406$.

Remark 3 One could be tempted by looking for an approximation of the value of m , considering the subset of \mathcal{W} composed by piecewise affine functions, which are 0 on the same set of zeros as a minimizer u . Unfortunately this strategy would give an estimate from above of m . Instead, we need an estimate from below for our quantitative isoperimetric inequality.

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Conflict of interest

The authors declare that they have no conflict of interest.

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8 Appendix

In this section we prove the most technical results of Section 4, under the assumptions given at the beginning of that section. We recall that our aim is to prove that $A < \frac{2}{\pi}$ and thus to get a contradiction, as explained in Proposition 3.

Let us denote by m_j the midpoint of a nodal interval (a_j, b_j) .

Lemma 4 *Let (a_j, b_j) be a negative interval. If $m_j \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then*

$$A \leq \frac{l_j^2(2\pi - \ell_-)}{12\ell_- |\cos(m_j)|}. \quad (40)$$

For a positive interval (a_k, b_k) , whose midpoint $m_k \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, one has

$$A \leq \frac{l_k^2}{12 \cos(m_k)}. \quad (41)$$

Proof For a negative interval (a_j, b_j) , as soon as $\frac{a_j + b_j}{2} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the fact that $\int_{a_j}^{b_j} u < 0$ implies

$$\frac{|\lambda_1/2|}{m - \lambda_0} \leq \frac{h(\ell_j)}{|\cos(\frac{a_j+b_j}{2})|},$$

where $h(x) = \frac{2 \tan(\frac{x}{2}) - x}{2 \sin(\frac{x}{2}) (1 + \frac{x}{\sin x})}$. We claim that for $x \in [0, \pi]$, one has $h(x) \leq \frac{x^2}{12}$. By using (6), (7), (10) and this bound on h we have the following estimate on A :

$$A \leq \frac{l_j^2(2\pi - \ell_-)}{12\ell_- |\cos(\frac{a_j+b_j}{2})|}.$$

We now prove our claim, that is, the bound on h . The statement is equivalent to the positivity of $f(x) = \frac{x^2}{12}(x + \sin x) - 2 \sin(x/2) + x \cos(x/2)$ in $[0, \pi]$. Observe that $f(0) = 0$. The result will follow if we prove that the derivative of f is positive, that is, $k(x) = 3x + 2 \sin(x) + x \cos(x) - 6 \sin(x/2)$ is positive. We remark that $k(0) = 0, k(\frac{\pi}{2}) > 0, k(\pi) > 0$. We will split the analysis into two cases.

1. If $x \in [0, \frac{\pi}{2}]$ it is easy to see that $k'' < 0$ and therefore $k(x) \geq 0$ in $[0, \frac{\pi}{2}]$.
2. In the case where $x \in [\frac{\pi}{2}, \pi]$, k is decreasing. Indeed it is easy to see that k' is negative, since k' is convex, $k'(\pi) = 0, k'(\frac{\pi}{2}) < 0$.

In the same way, using (16), we can prove that, on a positive interval (a_k, b_k) (m_k being its midpoint):

$$A \leq \frac{l_k^2}{12 \cos(m_k)},$$

when $m_k \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Estimate of A involves the midpoint of nodal domains, as we have just seen. The next lemma gives us important information about nodal domains whose midpoint is between $-\pi/2$ and $\pi/2$ or outside this interval.

Lemma 5 *There are no negative intervals on the "left" of $I_0 = (a, b)$ whose midpoint is between $-\pi/2$ and $\pi/2$. There is at most one negative interval on the "right" of I_0 whose midpoint lies between $-\pi/2$ and $\pi/2$.*

Proof Since $a \leq -\frac{\pi}{2}$, there are no negative intervals on the "left" of I_0 whose midpoint is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

We prove the second statement by contradiction. If there were two negative intervals, say I_1 and I_3 with a positive interval $I_2 = (a_2, b_2)$ between them, its midpoint would satisfy $m_2 \leq \frac{\pi - \ell_2}{2}$. Moreover, since $b_2 \leq \frac{\pi}{2}$, and $0 \leq b < a_2 = b_2 - \ell_2 \leq \frac{\pi}{2} - \ell_2$, we infer $\ell_2 \leq \frac{\pi}{2}$. Now using (41), we see that

$$A \leq \frac{\ell_2^2}{12 \cos(m_2)} \leq \frac{\ell_2^2}{12 \sin(\ell_2/2)}.$$

Now, it is immediate to check that $x \mapsto x^2/\sin(x/2)$ is increasing. Therefore the previous inequality would imply for $\ell_2 \leq \frac{\pi}{2}$: $A \leq \pi^2/(24\sqrt{2}) < \frac{2}{\pi}$ implying $\int_{a_2}^{b_2} u dt < 0$, that is, a contradiction.

We are now going to find a lower bound for ℓ_- , the measure of $\{u < 0\}$. We know that

$$\int_{\{u>0\}} u(x) dx + \int_{\{u<0\}} u(x) dx = 0,$$

while

$$\int_{-\pi}^{\pi} |u(x)| dx = \int_{\{u>0\}} u(x) dx - \int_{\{u<0\}} u(x) dx = 1.$$

Therefore

$$\int_{\{u>0\}} u(x) dx = - \int_{\{u<0\}} u(x) dx = \frac{1}{2}. \quad (42)$$

The following proposition gives a lower bound for ℓ_- .

Proposition 6 $\ell_- \geq 1.55$.

Proof We start with a simple estimate on m that will be useful in the proof. Using the explicit function given in Theorem 1 as a test function in the functional defined by (1), we get

$$m \leq \frac{1}{2(4 - \pi)}. \quad (43)$$

We now prove the estimate of the statement. We can assume that the length of all negative intervals is less than 1.6 (otherwise, if there is a negative interval of length $\ell_j \geq 1.6$, the estimate $\ell_- \geq 1.55$ is straightforward). By Lemma 5, we can split the analysis into two cases:

1. no negative interval has its midpoint in $[-\frac{\pi}{2}, \frac{\pi}{2}]$;
2. the midpoint of only one negative interval, say I_1 , belongs to $[-\frac{\pi}{2}, \frac{\pi}{2}]$;

that we are now going to develop.

1. Assume that no negative interval has its midpoint in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We are going to estimate the sum of $\int_{I_j} (-u(x))dx$ on all negative intervals I_j . We observe that on all (but possibly one) negative intervals I_j whose midpoint $m_j \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have

$$\int_{I_j} -u(x)dx \leq (m - \lambda_0)[2 \tan(\ell_j/2) - \ell_j] \quad (44)$$

by (15), since the second term of the right hand side is negative. Now, it is easy to prove that

$$2 \tan(x/2) - x \leq \frac{0.45x^3}{4}, \quad 0 \leq x \leq 0.8. \quad (45)$$

Therefore inequalities (44) and (42) imply

$$\frac{1}{2} = \sum_j \int_{I_j} (-u(x))dx \leq \frac{0.45(m - \lambda_0)}{4} \sum_j \ell_j^3.$$

The maximization of the convex function $t \in \mathbb{R}^+ \rightarrow t^3$ and the fact that the sum of the lengths ℓ_j of all negative intervals is ℓ_- give

$$\sum_j \ell_j^3 \leq \ell_-^3.$$

By (10) and (43), we end up with

$$\frac{1}{2} \leq \frac{0.45(2\pi - \ell_-)}{8\pi(4 - \pi)} \ell_-^3.$$

This polynomial inequality provides finally the inequality $\ell_- \geq 1.74$.

2. Assume that the midpoint of one interval, say I_1 , belongs to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Notice that by Lemma 5, such a negative interval is unique. We are going to estimate the sum of $\int_{I_j} (-u(x))dx$ on all negative intervals I_j . For $j \neq 1$, the midpoint of I_j does not belong to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore the integral $\int_{I_j} (-u(x))dx, j \neq 1$ can be estimated as in the previous case, that is, using (15) and (45). Instead, $\int_{I_1} (-u(x))dx$ can be estimated by

$$(m - \lambda_0)[2 \tan(\ell_1/2) - \ell_1] + P, \quad P = -\frac{\lambda_1}{2} 2 \sin\left(\frac{\ell_1}{2}\right) \cos m_1 \left(1 + \frac{\ell_1}{\sin \ell_1}\right) \quad (46)$$

by formula (15). We thus obtain, for the sum of $\int_{I_j} (-u(x))dx$ on all negative intervals I_j

$$\frac{1}{2} = \sum_j \int_{I_j} (-u(x))dx \leq \frac{0.45(m - \lambda_0)}{4} \sum_j \ell_j^3 + P,$$

using (42) for the first equality. By the same argument as in the previous case, we get

$$\frac{1}{2} \leq \frac{0.45(2\pi - \ell_-)}{8\pi(4 - \pi)} \ell_-^3 + P. \quad (47)$$

We are going to distinguish two cases, according to the values of ℓ_1 :

(a) Assume $\ell_1 \geq 0.228$. In this case, we look at the first negative interval

I_{-1} on the left of I_0 . Since $b \leq \frac{\pi - \ell_1}{2} \leq \frac{\pi}{2} - 0.114$ and $\ell \geq \pi$, we have

$$m_{-1} = a - \frac{\ell_{-1}}{2} = b - \ell - \frac{\ell_{-1}}{2} \leq -0.114 - \frac{\pi + \ell_{-1}}{2}.$$

If $\ell_{-1} \geq \frac{\pi}{2}$, then $\ell_- \geq \frac{\pi}{2} + 0.228 > 1.75$. If, on the contrary, $\ell_{-1} < \frac{\pi}{2}$,

then $m_{-1} \geq -\frac{5\pi}{4}$ necessarily (otherwise $a = m_{-1} + \frac{\ell_{-1}}{2} < -\pi$ which is impossible). Therefore,

$$|\cos(m_{-1})| \geq \min \left\{ \frac{\sqrt{2}}{2}, \left| \cos \left(0.114 + \frac{\pi + \ell_{-1}}{2} \right) \right| \right\} := C.$$

Using this estimate and $\ell_- \geq 0.228 + \ell_{-1}$ in (40) we get

$$A \leq \frac{\ell_{-1}^2(2\pi - 0.228 - \ell_{-1})}{12(0.228 + \ell_{-1})C}.$$

As a function of ℓ_{-1} , the right-hand side is increasing and for $\ell_{-1} \leq 1.322$ we get the inequality $A \leq 0.636 < \frac{2}{\pi}$, that is, a contradiction.

Therefore, in this case, we deduce $\ell_{-1} \geq 1.322$ and $\ell_- \geq \ell_1 + \ell_{-1} \geq 1.55$.

(b) Assume $\ell_1 \leq 0.228$. Lemma 2 and inequalities (10) and (43) give

$$-\frac{\lambda_1}{2} \leq m + \lambda_0 = m \frac{\ell_-}{\pi} \leq \frac{\ell_-}{2\pi(4 - \pi)}.$$

We can use these inequalities to estimate the positive term P defined by (46) :

$$P \leq \frac{\ell_-}{2\pi(4 - \pi)} 2 \sin(0.114) \left(1 + \frac{0.228}{\sin(0.228)} \right).$$

Therefore we get from (47)

$$\frac{1}{2} \leq \frac{0.45(2\pi - \ell_-)}{8\pi(4 - \pi)} \ell_-^3 + \frac{\ell_-}{2\pi(4 - \pi)} 2 \sin(0.114) \left(1 + \frac{0.228}{\sin(0.228)} \right).$$

This polynomial inequality implies $\ell_- \geq 1.55$.

In the next two propositions we will analyse the case where there would be at least 6 nodal domains.

Proposition 7 *Assume that u has at least 6 nodal domains and $b \geq \frac{\pi}{2}$. Then $A < 2/\pi$ and the length of no nodal interval is greater than π .*

Proof We will consider the two negative intervals next to $I_0 = (a, b)$, namely I_{-1} on the left of I_0 and I_1 on the right, and prove that both have lengths ℓ_{-1} and ℓ_1 respectively greater than 1.17.

- Assume by contradiction that $\ell_{-1} \leq 1.17$. On one hand, the midpoint m_{-1} of I_{-1} satisfies

$$m_{-1} = a - \frac{\ell_{-1}}{2} \leq -\frac{\pi}{2} - \frac{\ell_{-1}}{2}.$$

On the other hand, $m_{-1} \geq -\frac{5\pi}{4}$ (otherwise $a = m_{-1} + \frac{\ell_{-1}}{2} < -\pi$ which is impossible). Therefore we have

$$|\cos(m_{-1})| \geq \min \left\{ \frac{1}{\sqrt{2}}, \left| \cos \left(\frac{\pi + \ell_{-1}}{2} \right) \right| \right\} = \min \left\{ \frac{1}{\sqrt{2}}, \sin \left(\frac{\ell_{-1}}{2} \right) \right\}.$$

Plugging this estimate in (40) and using Proposition 6 yields

$$A \leq \frac{\ell_{-1}^2(2\pi - 1.55)}{12 \cdot 1.55 \cdot \min \left\{ \frac{1}{\sqrt{2}}, \sin(\ell_{-1}/2) \right\}}. \quad (48)$$

Now the right-hand side is increasing in ℓ_{-1} and its value for $\ell_{-1} = 1.17$ is less than $0.631 < \frac{2}{\pi}$ that is absurd, proving the claim.

- Assume by contradiction that $\ell_1 \leq 1.17$. In this case we are going to use that $b \geq \frac{\pi}{2}$ to estimate m_1 :

$$\frac{\pi}{2} + \frac{\ell_1}{2} \leq b + \frac{\ell_1}{2} = m_1 = b + \frac{\ell_1}{2} < \pi + \frac{1.17}{2} < \frac{5\pi}{4}.$$

One can now repeat the above argument to get a contradiction.

By the previous estimates on ℓ_{-1}, ℓ_1 , we get $\ell_- \geq \ell_{-1} + \ell_1 \geq 2.34$.

With the same arguments of the two steps above, we can prove that $\ell_{-1} \geq \frac{\pi}{2}$ and $\ell_1 \geq \frac{\pi}{2}$. Indeed it is sufficient to replace 1.17 by $\frac{\pi}{2}$ and 1.55 by 2.34 in the estimate of A . This gives $A \leq 0.49 < \frac{2}{\pi}$, that is a contradiction. We deduce that $\ell_{-1} \geq \frac{\pi}{2}$ and $\ell_1 \geq \frac{\pi}{2}$ that implies $\ell_- \geq \pi$: this is a contradiction, since $\ell_+ > \pi$.

Proposition 8 *Assume that u has at least 6 nodal domains and $b < \frac{\pi}{2}$. Then $A < 2/\pi$ and the length of no nodal interval is greater than π .*

The proof of this proposition follows analogous arguments to the previous one, but uses three intervals on the right of (a, b) , since we do not know the position of m_1 .

Proof One can prove that $\ell_{-1} \geq 1.17$, following exactly the same argument as in the proof of Proposition 7. We will work with the intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $I_3 = [a_3, b_3]$, with $b_1 = a_2, b_2 = a_3$, on the right of (a, b) , with $b = a_1$.

1. Assume that $b_2 \leq \frac{\pi}{2}$. Observe that $0 < m_2 \leq \frac{\pi}{2} - \frac{\ell_2}{2}$ where we have used that $b > 0$. Using (41) we get

$$A \leq \frac{\ell_2^2}{12 \sin(\ell_2/2)}.$$

Since $b_2 \leq \frac{\pi}{2}$ and $0 \leq b < a_2 = b_2 - \ell_2 \leq \frac{\pi}{2} - \ell_2$, one has $\ell_2 \leq \frac{\pi}{2}$. Therefore the right hand side of the above estimate of A is less than $\frac{\pi}{2}$, that is a contradiction.

2. Assume that $b_2 = a_3 \geq \frac{\pi}{2}$. We start by proving that $\ell_- \geq 1.93$. Assume by contradiction that $\ell_- \leq 1.93$. On one hand we have $m_3 = a_3 + \frac{\ell_3}{2} \geq \frac{\pi}{2} + \frac{\ell_3}{2}$. On the other hand we claim that $m_3 < \frac{5\pi}{4}$. We have $m_3 = b + \ell_1 + \ell_2 + \frac{\ell_3}{2}$.

Observe that

- $b < \frac{\pi}{2}$ by hypothesis,
- $\ell_2 \leq \pi - 1.55$ since $\ell_2 + \ell \leq \ell_+ = 2\pi - \ell_- \leq 2\pi - 1.55$,
- $\ell_1 + \frac{\ell_3}{2} \leq 0.76$ since $\ell_{-1} + \ell_1 + \frac{\ell_3}{2} \leq \ell_- \leq 1.93$ and $\ell_{-1} \geq 1.17$.

Thus $m_3 < \frac{3\pi}{2} - 1.55 + 0.76 < \frac{5\pi}{4}$. We deduce from the previous bounds on m_3 that:

$$|\cos m_3| \geq \min \left\{ \frac{1}{\sqrt{2}}, \left| \cos \left(\frac{\pi + \ell_3}{2} \right) \right| \right\}$$

which provides, as in case (a)(i), $\ell_3 \geq 1.17$.

We deduce that $\ell_- \geq \ell_{-1} + \ell_3 > 1.93$.

We are going to prove that $\ell_{-1} \geq 1.532$. Assume that this is not the case. Using the same argument as in step (a)(i), one has a contradiction from the following estimate :

$$A \leq \frac{1.532^2(2\pi - 1.93)}{12 \cdot 1.93 \cdot \min \left\{ \frac{1}{\sqrt{2}}, \sin(1.532/2) \right\}} < \frac{2}{\pi}. \quad (49)$$

It is easy to prove that $\ell_- > 2.67$. Indeed, if this is not the case, replacing 1.93 by 2.67 in (49), we get a contradiction since A is still less than $\frac{2}{\pi}$.

We are going to prove that $\ell_{-1} > \frac{\pi}{2}$. Assume that this is not the case. By the same argument as in step (a)(i), one has a contradiction from estimate

$$A \leq \frac{(\pi/2)^2(2\pi - 2.67)}{12 \cdot 2.67 \cdot \sqrt{2}/2} < \frac{2}{\pi}.$$

We are going to prove that $\ell_3 \geq \frac{\pi}{2}$. Assume that this is not true. Recall

that $\ell_- < \pi$. As above, on one hand we have $m_3 = a_3 + \frac{\ell_3}{2} \geq \frac{\pi}{2} + \frac{\ell_3}{2}$. On the other hand $m_3 < \frac{5\pi}{4}$. Indeed $m_3 = b + \ell_1 + \ell_2 + \frac{\ell_3}{2}$. Observe that

- $b < \frac{\pi}{2}$ by hypothesis,
- $\ell_2 \leq \pi - 2.67$
- $\ell_1 + \frac{\ell_3}{2} \leq \pi - \frac{\pi}{2}$

Thus $m_3 < 2\pi - 2.67 < \frac{5\pi}{4}$. We deduce from the previous bounds on m_3 that: $|\cos m_3| \geq \frac{1}{\sqrt{2}}$. This gives a contradiction since

$$A \leq \frac{(\pi/2)^2(2\pi - 2.67)}{12 \cdot 2.67 \cdot \sqrt{2}/2} < \frac{2}{\pi}.$$

Therefore $\ell_3 > \frac{\pi}{2}$.

This last estimate implies $\pi \geq \ell_- > \ell_{-1} + \ell_3 > \pi$, that is, a contradiction.

We consider now the case of four nodal domains.

Proposition 9 *Assume that u has 4 nodal domains. Then $A < 2/\pi$ and the length of no nodal interval is greater than π .*

Proof Assume now that u has four nodal domains: $I_{-1} = I_3, I_1$ (two negative ones) and I_0, I_2 (two positive ones), that we write as $I_0 = (a_0, a_1), I_1 = (a_1, a_2), I_2 = (a_2, a_3), I_{-1} = I_3 = (a_3 - 2\pi, a_0)$. We assume that m_0, m_1, m_2, m_3 are the midpoints of each interval and $\ell_0, \ell_1, \ell_2, \ell_3$ are the lengths.

We are going to work with a more explicit expression of λ_1, λ_2 . Without loss of generality, we can assume that the lengths satisfy $0 < \ell_1 \leq \ell_3 < \pi$ (up to replacing $u(x)$ by $u(-x)$).

As in the proof of Proposition 7, we can get the lower bound $\ell_3 \geq 1.17$. We can assume $-\pi \leq m_3 \leq -\frac{\pi}{2}$ by Lemma 5. From $\lambda_2 = 0$ we have

$$0 = \int_{-\pi}^{\pi} \text{sign}(u(x)) \sin x dx = 2(\cos a_0 - \cos a_1 + \cos a_2 - \cos a_3). \quad (50)$$

Gathering $-\cos a_1 + \cos a_2$ on the one hand and $\cos a_0 - \cos a_3$ on the other hand, the right hand side of (50) can be rewritten

$$\sin \frac{\ell_3}{2} \sin m_3 + \sin \frac{\ell_1}{2} \sin m_1 = 0. \quad (51)$$

In the same way, coming back to the definition of λ_1 (see (4)), we have

$$\frac{\lambda_1}{2} = \frac{2m}{\pi} \left(\sin \frac{\ell_3}{2} \cos m_3 + \sin \frac{\ell_1}{2} \cos m_1 \right). \quad (52)$$

Observing that $\cos m_3 < 0$, by (51) $\cos m_3$ can be rewritten as

$$\cos m_3 = -\sqrt{1 - \frac{\sin^2 \frac{\ell_1}{2} \sin^2 m_1}{\sin^2 \frac{\ell_3}{2}}}. \quad (53)$$

By using (52), (53) and (6), the quantity A defined in (20) can be rewritten as a function of ℓ_1, ℓ_3, m_1 as

$$A = A(\ell_1, \ell_3, m_1) = 2 \frac{\sqrt{\sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2} \sin^2 m_1} - \sin \frac{\ell_1}{2} \cos m_1}{\ell_1 + \ell_3}. \quad (54)$$

We use now the integral of u on the interval I_3 . We recall that

$$\int_{I_3} (-u(x)) dx = \frac{m\ell_+}{\pi} \left(2 \tan \frac{\ell_3}{2} - \ell_3 \right) - \frac{\lambda_1}{2} \left[2 \sin \frac{\ell_3}{2} \cos m_3 \left(1 + \frac{\ell_3}{\sin \ell_3} \right) \right].$$

Moreover $0 < \int_{I_3} (-u(x)) dx \leq \frac{1}{2}$ by (42). We now use the expression of $\lambda_1/2$ given in (52) and replace $\sin \frac{\ell_1}{2} \cos m_1$ by $\pm \sqrt{\sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3}$ (with a + if $m_1 \leq \frac{\pi}{2}$ and a - if $\frac{\pi}{2} \leq m_1 \leq 1.9$), thanks to (51). This provides an expression of the integral as a function of the three variables ℓ_1, ℓ_3, m_3 :

$$0 < \int_{I_3} (-u(x)) dx = \frac{m}{\pi} I_{\pm}(\ell_1, \ell_3, m_3).$$

More precisely, we have

$$\begin{aligned} I_{\pm}(\ell_1, \ell_3, m_3) &= (2\pi - \ell_1 - \ell_3) \left(2 \tan \frac{\ell_3}{2} - \ell_3 \right) - \dots \\ &4 \left(\sin \frac{\ell_3}{2} \cos m_3 \pm \sqrt{\sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3} \right) \sin \frac{\ell_3}{2} \cos m_3 \left(1 + \frac{\ell_3}{\sin \ell_3} \right) \end{aligned}$$

(with a + if $m_1 \leq \frac{\pi}{2}$ and a - if $\frac{\pi}{2} \leq m_1 \leq 1.9$). By Lemma 6 below, the function I_{\pm} is negative, and thus we get a contradiction.

We recall that we assume that u has four nodal domains: $I_{-1} = I_3, I_1$ (two negative ones) and I_0, I_2 (two positive ones). We assume that m_0, m_1, m_2, m_3 are the midpoints of each interval and $\ell_0, \ell_1, \ell_2, \ell_3$ are the lengths. We recall that

$$A = A(\ell_1, \ell_3, m_1) = 2 \frac{\sqrt{\sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2} \sin^2 m_1} - \sin \frac{\ell_1}{2} \cos m_1}{\ell_1 + \ell_3}.$$

(see 54) and

$$\cos m_3 = -\sqrt{1 - \frac{\sin^2 \frac{\ell_1}{2} \sin^2 m_1}{\sin^2 \frac{\ell_3}{2}}}. \quad (55)$$

Lemma 6 *Assume $\ell_3 \geq 1.17$ and $0 < \ell_1 \leq \ell_3 < \pi$. Let m_3^* be the solution to $|\sin m_3| \sin \frac{\ell_3}{2} = \sin \frac{\ell_1}{2}$. Assume that $-\pi \leq m_3 \leq m_3^*$.*

1. *Let*

$$I_+(\ell_1, \ell_3, m_3) = (2\pi - \ell_1 - \ell_3) \left(2 \tan \frac{\ell_3}{2} - \ell_3 \right) - \dots \\ 4 \left(\sin \frac{\ell_3}{2} \cos m_3 + \sqrt{\sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3} \right) \sin \frac{\ell_3}{2} \cos m_3 \left(1 + \frac{\ell_3}{\sin \ell_3} \right).$$

Assume $\ell_1 \leq \frac{\pi}{6}$. Then $I_+(\ell_1, \ell_3, m_3) < 0$.

2. *Let*

$$I_-(\ell_1, \ell_3, m_3) = (2\pi - \ell_1 - \ell_3) \left(2 \tan \frac{\ell_3}{2} - \ell_3 \right) - \dots \\ 4 \left(\sin \frac{\ell_3}{2} \cos m_3 - \sqrt{\sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3} \right) \sin \frac{\ell_3}{2} \cos m_3 \left(1 + \frac{\ell_3}{\sin \ell_3} \right).$$

Assume $\ell_1 \leq 0.62$. Then $I_-(\ell_1, \ell_3, m_3) < 0$.

The proof of Lemma 6 is based on the following estimates on ℓ_1 .

Lemma 7 *Assume that u has 4 nodal domains. Assume that $m_1 \leq \frac{\pi}{2}$. Then $\ell_1 \leq \frac{\pi}{6}$.*

Proof Let us first look at the dependence of A with respect to m_1 . The derivative of $A(\ell_1, \ell_3, m_1)$ with respect to m_1 has the same sign as

$$\sin m_1 \left(1 - \frac{\cos m_1 \sin \frac{\ell_1}{2}}{\sqrt{\sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2} \sin^2 m_1}} \right),$$

that is, the same sign as

$$\sin m_1 \left(\sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2} \right).$$

The above quantity is positive, since $0 \leq m_1 \leq \frac{\pi}{2}$ and $\ell_3 \geq \ell_1$. Therefore, if $m_1 \leq \frac{\pi}{2}$

$$A(\ell_1, \ell_3, m_1) \leq A(\ell_1, \ell_3, \frac{\pi}{2}) = 2 \frac{\sqrt{\sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2}}}{\ell_1 + \ell_3}. \quad (56)$$

Assume by contradiction that $\ell_1 > \frac{\pi}{6}$. We are going to prove that $A < \frac{2}{\pi}$, thus reaching a contradiction. We set

$$x = \frac{\ell_3 - \ell_1}{2} \quad \text{and} \quad y = \frac{\ell_3 + \ell_1}{2}$$

and observe that $0 \leq x < \frac{\pi}{2}$, $\frac{1.17}{2} \leq y < \frac{\pi}{2}$. The quantity $A(\ell_1, \ell_3, \frac{\pi}{2})$ can be rewritten as a function of x, y :

$$G(x, y) = \frac{\sqrt{\sin x \sin y}}{y}, \quad (x, y) \in \left[0, \frac{\pi}{2}\right] \times \left[\frac{1.17}{2}, \frac{\pi}{2}\right].$$

The map $x \mapsto G(x, y)$ is increasing while $y \mapsto G(x, y)$ is decreasing on this set. We remark that the assumption $\ell_1 > \frac{\pi}{6}$ is equivalent to $y > x + \frac{\pi}{6}$. In the rectangle $\left[0, \frac{\pi}{2}\right] \times \left[\frac{1.17}{2}, \frac{\pi}{2}\right]$ this implies that $x \in \left[0, \frac{\pi}{3}\right]$. Now, it can be checked that

$$\forall t \in \left[0, \frac{\pi}{3}\right], \quad G(t, t + \frac{\pi}{6}) \leq \frac{2}{\pi}.$$

Indeed the function $t \mapsto G(t, t + \frac{\pi}{6})$ is first increasing then decreasing and satisfies the above inequality for its maximum that is approximately at 0.6627206. From the properties of G , we infer that $G(x, y) < \frac{2}{\pi}$ for all (x, y) such that $y > x + \frac{\pi}{6}$, $y \in [\frac{1.17}{2}, \frac{\pi}{2}]$, $x \in [0, \frac{\pi}{3}]$, that is, $\ell_1 > \frac{\pi}{6}$. We have thus proved that $A < \frac{2}{\pi}$ as soon as $\ell_1 > \frac{\pi}{6}$, that is a contradiction. Therefore ℓ_1 is less or equal $\frac{\pi}{6}$.

Lemma 8 *Assume that u has 4 nodal domains. Assume that $m_1 > \frac{\pi}{2}$. Then $\ell_1 \leq 0.62$.*

Proof Let us suppose first that $m_1 \geq 1.9$, therefore $|\cos m_1| \geq |\cos 1.9|$ and, following (40) we infer

$$A(\ell_1, \ell_3, m_1) \leq \frac{l_1^2(2\pi - \ell_1 - \ell_3)}{12(\ell_1 + \ell_3)|\cos 1.9|}.$$

Now, this expression is decreasing in ℓ_3 and increasing in ℓ_1 , thus it is always less than its value for $\ell_1 = \ell_3 < \frac{\pi}{2}$:

$$A(\ell_1, \ell_3, m_1) \leq A(\ell_1, \ell_1, 1.9) = \frac{l_1(\pi - \ell_1)}{12|\cos 1.9|} \leq \frac{\pi^2}{48|\cos 1.9|} < \frac{2}{\pi}$$

and the contradiction is obtained in this case.

We can therefore assume that $\frac{\pi}{2} \leq m_1 \leq 1.9$. Expressing m_3 in terms of m_1 , we have $m_3 = m_1 - \frac{\ell_-}{2} - \ell_0 \leq 1.9 - \frac{\ell_-}{2} - \pi$, thus $|\sin m_3| \leq \sin(1.9 - \frac{\ell_-}{2})$ (recall that $-\pi \leq m_3 \leq -\frac{\pi}{2}$). By identity (51), we have

$$|\sin 1.9| \sin \frac{\ell_1}{2} \leq \sin m_1 \sin \frac{\ell_1}{2} = |\sin m_3| \sin \frac{\ell_3}{2} \leq \sin \frac{\ell_3}{2} \sin \left(1.9 - \frac{\ell_-}{2}\right).$$

This implies, by (53)

$$|\cos m_3| = \sqrt{1 - \frac{\sin^2 \frac{\ell_1}{2} \sin^2 m_1}{\sin^2 \frac{\ell_3}{2}}} \geq \sqrt{1 - \frac{\sin^2(1.9 - \frac{\ell_-}{2})}{\sin^2 1.9}}.$$

Therefore, using (40) we can estimate A from above by

$$A \leq \frac{\ell_3^2(2\pi - \ell_-)}{12\ell_- \sqrt{1 - \frac{\sin^2(1.9 - \frac{\ell_-}{2})}{\sin^2 1.9}}}.$$

As a function of ℓ_- the right-hand side is clearly decreasing. Now, if $\ell_1 \geq 0.62$ (and then $\ell_- \geq 1.17 + 0.62$), we can see that $A < \frac{2}{\pi}$ for any $\ell_3 \in [1.17, \pi - \ell_1]$ giving the desired contradiction.

We are now able to prove Lemma 6.

Proof (of Lemma 6)

1. We look first at the dependence with respect to m_3 . The derivative of $I_+(\ell_1, \ell_3, m_3)$ with respect to m_3 has the sign of

$$\sin m_3 \left(\left(\sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3 \right)^{1/4} + \frac{\sin \frac{\ell_3}{2} \cos m_3}{\left(\sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3 \right)^{1/4}} \right)^2.$$

Since $-\pi \leq m_3 \leq -\frac{\pi}{2}$, $\sin m_3$ is negative. Therefore $m_3 \mapsto I_+(\ell_1, \ell_3, m_3)$ is decreasing, which implies

$$I_+(\ell_1, \ell_3, m_3) \leq I_+(\ell_1, \ell_3, -\pi).$$

Now,

$$I_+(\ell_1, \ell_3, -\pi) = (2\pi - \ell_1 - \ell_3) \left(2 \tan \frac{\ell_3}{2} - \ell_3 \right) + 4 \left(\sin \frac{\ell_1}{2} - \sin \frac{\ell_3}{2} \right) \sin \frac{\ell_3}{2} \left(1 + \frac{\ell_3}{\sin \ell_3} \right).$$

The derivative with respect to ℓ_1 is $\ell_3 - 2 \tan \frac{\ell_3}{2} + 2 \cos \frac{\ell_1}{2} \sin \frac{\ell_3}{2} \left(1 + \frac{\ell_3}{\sin \ell_3} \right)$

that is decreasing in ℓ_1 , thus greater than its value for $\ell_1 = \frac{\pi}{6}$:

$$\frac{\partial I_+(\ell_1, \ell_3, -\pi)}{\partial \ell_1} \geq \ell_3 - 2 \tan \frac{\ell_3}{2} + 2 \cos \frac{\pi}{12} \sin \frac{\ell_3}{2} \left(1 + \frac{\ell_3}{\sin \ell_3} \right) > 0.$$

This shows that $I_+(\ell_1, \ell_3, -\pi)$ is increasing in ℓ_1 . We deduce that

- for $1.17 \leq \ell_3 \leq \frac{5\pi}{6}$, $I_+(\ell_1, \ell_3, -\pi) \leq I_+\left(\frac{\pi}{6}, \ell_3, -\pi\right)$;
- for $\ell_3 \in \left[\frac{5\pi}{6}, \pi\right)$, $I_+(\ell_1, \ell_3, -\pi) \leq I_+(\pi - \ell_3, \ell_3, -\pi)$.

The functions on the right hand side of the above inequalities are negative.

2. We look first at the dependence with respect to m_3 . The derivative of $I_-(\ell_1, \ell_3, m_3)$ with respect to m_3 has the sign of

$$-\sin m_3 \left(\left(\sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3 \right)^{1/4} - \frac{\sin \frac{\ell_3}{2} \cos m_3}{\left(\sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3 \right)^{1/4}} \right)^2.$$

Since $-\pi \leq m_3 \leq -\frac{\pi}{2}$, $\sin m_3$ is negative. Therefore $m_3 \mapsto I_-(\ell_1, \ell_3, m_3)$ is increasing which implies

$$I_-(\ell_1, \ell_3, m_3) \leq I_-(\ell_1, \ell_3, m_3^*).$$

Now

$$I_-(\ell_1, \ell_3, m_3^*) = (2\pi - \ell_1 - \ell_3) \left(2 \tan \frac{\ell_3}{2} - \ell_3 \right) - 4 \left(\sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2} \right) \left(1 + \frac{\ell_3}{\sin \ell_3} \right).$$

This is a convex function with respect to ℓ_1 . Therefore

$$I_-(\ell_1, \ell_3, m_3^*) \leq \max \{ I_-(0, \ell_3, m_3^*), I(0.62, \ell_3, m_3^*) \}.$$

Now the functions $\ell_3 \mapsto I_-(0, \ell_3, m_3^*)$ and $\ell_3 \mapsto I(0.62, \ell_3, m_3^*)$ are decreasing and negative.