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# A POINCARÉ TYPE INEQUALITY WITH THREE CONSTRAINTS

GISELLA CROCE AND ANTOINE HENROT

ABSTRACT. In this paper, we consider a problem in calculus of variations motivated by a quantitative isoperimetric inequality in the plane. More precisely, the aim of this article is the computation of the minimum of the variational problem

$$\inf_{u \in \mathcal{W}} \frac{\int_{-\pi}^{\pi} [(u')^2 - u^2] d\theta}{\left[ \int_{-\pi}^{\pi} |u| d\theta \right]^2}$$

where  $u \in \mathcal{W}$  is a  $H^1(-\pi, \pi)$  periodic function, with zero average on  $(-\pi, \pi)$  and orthogonal to sine and cosine.

## 1. INTRODUCTION

In this article we are interested in the following variational problem :

$$\inf_{u \in \mathcal{W}} \frac{\int_{-\pi}^{\pi} [(u')^2 - u^2] d\theta}{\left[ \int_{-\pi}^{\pi} |u| d\theta \right]^2}$$

where  $\mathcal{W}$  denotes the subspace of functions in the Sobolev space  $H^1(-\pi, \pi)$  that are  $2\pi$ -periodic, satisfying the following constraints:

$$(L1) \quad \int_{-\pi}^{\pi} u(\theta) d\theta = 0$$

$$(L2) \quad \int_{-\pi}^{\pi} u(\theta) \cos(\theta) d\theta = 0$$

$$(L3) \quad \int_{-\pi}^{\pi} u(\theta) \sin(\theta) d\theta = 0.$$

Our aim is to compute the value of the minimum and to identify the minimizer. The difficulty comes here from the nonlinear term in the denominator together with the three constraints.

This problem is motivated from a shape optimization problem and more precisely from a quantitative isoperimetric inequality. Indeed, for any open bounded set of  $\mathbb{R}^n$ , let us introduce the *isoperimetric deficit*:

$$\delta(\Omega) = \frac{P(\Omega) - P(B)}{P(B)}, \quad (1)$$

where  $|B| = |\Omega|$ . Let the *barycentric asymmetry* be defined by:

$$\lambda_0(\Omega) = \frac{|\Omega \Delta B_{x^G}|}{|\Omega|}$$

where  $B_{x^G}$  is the ball centered at the barycentre  $x^G = \frac{1}{|\Omega|} \int_{\Omega} x dx$  of  $\Omega$  and such that  $|\Omega| = |B_{x^G}|$ . Fuglede proved in [9] that there exists a positive constant (depending only on the dimension  $n$ ) such that

$$\delta(\Omega) \geq C(n) \lambda_0^2(\Omega), \quad \text{for any convex subsets } \Omega \text{ of } \mathbb{R}^n. \quad (2)$$

Now, the constant  $C(n)$  is unknown (as it is the case in most quantitative inequalities like (2)) and it would be interesting to find the best constant. This leads to consider the minimization of the ratio

$$\mathcal{G}_0(\Omega) = \frac{\delta(\Omega)}{\lambda_0^2(\Omega)}$$

among convex compact sets in the plane, in particular. In the study of this minimization problem, one is led to exclude sequences converging to the ball in the Hausdorff metric. The strategy is to prove that

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on these sequences  $\mathcal{G}_0$  is greater than 0.406 which is the value of  $\mathcal{G}_0(S)$  where  $S$  is a precise set with the shape of a stadium, as computed in [1].

If a convex planar set  $E$  has barycenter in 0, it can be parametrized in polar coordinates with respect to 0, as

$$E = \{y \in \mathbb{R}^2 : y = tx(1 + u(x)), x \in \mathbb{S}^1, t \in [0, 1]\}, \quad (3)$$

where  $u$  is a Lipschitz periodic function. Then the shape functional  $\mathcal{G}_0(E)$  can be written as a functional  $H$  of the function  $u$  describing  $E$ , as follows :

$$\mathcal{G}_0(E) = H(u) = \frac{\pi \int_{-\pi}^{\pi} [\sqrt{(1+u)^2 + u'(\theta)^2} - 1] d\theta}{2 \left[ \frac{1}{2} \int_{-\pi}^{\pi} |(1+u)^2 - 1| d\theta \right]^2}. \quad (4)$$

The constraints of area (fixed equal to  $\pi$  without loss of generality) and barycentre in 0 read in terms of a periodic  $u \in H^1(-\pi, \pi)$  as:

$$(NL1) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+u)^2 d\theta = 1;$$

$$(NL2) \quad \int_{-\pi}^{\pi} \cos(\theta)[1+u(\theta)]^3 d\theta = 0;$$

$$(NL3) \quad \int_{-\pi}^{\pi} \sin(\theta)[1+u(\theta)]^3 d\theta = 0.$$

The computation of the minimum of  $H$ , under the constraints (NL1), (NL2) and (NL3), seems very difficult. However, for sequences of sets converging to the ball in the Hausdorff metric, the limit of

$$m_\varepsilon := \inf\{H(u), \|u\|_{L^\infty} = \varepsilon, u \in H^1(-\pi, \pi) \text{ periodic, satisfying (NL1), (NL2), (NL3)}\}$$

as  $\varepsilon \rightarrow 0$ , equals the limit of the shape functional  $\mathcal{G}_0$  for these sequences. Thus, a possible strategy consists in estimating from below the minimum of  $H$  by a simpler functional, namely its linearization. Define

$$m = \inf_{u \in \mathcal{W}} \frac{\int_{-\pi}^{\pi} [(u')^2 - u^2] d\theta}{\left[ \int_{-\pi}^{\pi} |u| d\theta \right]^2} \quad (5)$$

where  $\mathcal{W}$  is the space of periodic  $H^1(0, 2\pi)$  functions satisfying the constraints:

$$(L1) \quad \int_{-\pi}^{\pi} u d\theta = 0$$

$$(L2) \quad \int_{-\pi}^{\pi} u \cos(\theta) d\theta = 0$$

$$(L3) \quad \int_{-\pi}^{\pi} u \sin(\theta) d\theta = 0.$$

In [3] we proved that

$$\liminf_{\varepsilon \rightarrow 0} m_\varepsilon \geq \frac{\pi}{4} m. \quad (6)$$

The aim of our article is to compute  $m$  and to prove that  $\liminf_{\varepsilon \rightarrow 0} m_\varepsilon \geq \frac{\pi}{4} m > 0.406$ .

We can easily get an estimate from below of  $m$  by using the Cauchy-Schwarz inequality

$$\left( \int_{-\pi}^{\pi} |u| d\theta \right)^2 \leq 2\pi \int_{-\pi}^{\pi} u^2 d\theta.$$

Then, a Wirtinger-type inequality (or Parseval formula) shows that

$$\inf_{u \in \mathcal{W}} \frac{\int_{-\pi}^{\pi} [(u')^2 - u^2] d\theta}{2\pi \int_{-\pi}^{\pi} u^2 d\theta} \geq \frac{3}{2\pi}.$$

Unfortunately this estimate on  $m$  is not sufficient to prove the above desired inequality. Moreover, at first glance, one could think that  $\cos 2\theta$  is a minimizer of the functional in (5), as for the above Wirtinger-type inequality written for functions satisfying (L1), (L2) and (L3), but this is not true. We will prove the following result:

**Theorem 1.1.** *Let  $m$  be defined by (5). Then  $m = \frac{1}{2(4-\pi)}$  and the minimizer  $u$  is the odd and  $\pi$  periodic function defined on  $[0, \pi/2]$  by  $u(\theta) = \cos \theta + \sin \theta - 1$ .*

A direct consequence of our theorem is that, by (6),  $\liminf_{\varepsilon \rightarrow 0} m_\varepsilon \geq 0.457 > 0.406$ .

The strategy to prove our result is the following. It is immediate to see that the minimization problem (5) has a solution. Thus we write the Euler equation that any minimizer  $u$  satisfies. For that purpose, we introduce the three Lagrange multipliers, related to the three constraints (L1), (L2) and (L3) : they can be written as a function of  $\text{sgn}(u)$ . Moreover we will get an explicit expression of  $u$  on any nodal domain, as a function of the endpoints of this interval. This will be done in Section 2. Technically, we will need to prove that the length of any nodal domain is less than  $\pi$ : this is used in the final argument. It turns out that it is more difficult than expected and this is done completely in Section 3. After that, the main idea consists in proving that the Lagrange multipliers are all zero and that the nodal domains have all the same length (see Section 4). Finally, in Section 5, we will prove that the number of nodal domains is precisely four and find the explicit expression of  $u$  given in the above Theorem.

We remark that the minimization problem (5) is a variant of the Wirtinger inequality :

$$\inf_{u \in H_{per}^1(-\pi, \pi): \int_{-\pi}^{\pi} u = 0} \frac{\int_{-\pi}^{\pi} |u'|^2 d\theta}{\int_{-\pi}^{\pi} |u|^2 d\theta} = 1$$

because of the constraints (L2) and (L3) and the  $L^1(-\pi, \pi)$  norm at the denominator. In the literature one can find various generalizations of the Wirtinger inequality, without our constraints (L2) and (L3). In the series of papers [2], [4], [5], [6], [7], [10], [11], [12], [14], [15], the authors consider different norms of  $u'$  and  $u$  and on the mean value of  $u$ , namely

$$\inf_{u \in W_{per}^{1,p}(-\pi, \pi): \int_{-\pi}^{\pi} |u|^{r-2} u = 0} \frac{\int_{-\pi}^{\pi} |u'|^p d\theta}{\int_{-\pi}^{\pi} |u|^q d\theta}$$

for all values of  $p, q, r$  greater than 1. We also mention [8] in which the authors study, in any dimension  $N \geq 1$ , the inequality

$$\inf_{u \in W^{1,p}(\Omega): \int_{\Omega} |u|^{p-2} u \omega = 0} \frac{\int_{\Omega} |\nabla u|^p \omega}{\int_{\Omega} |u|^p \omega}$$

with a positive log-concave weight  $\omega$ , on a convex bounded domain  $\Omega \subset \mathbb{R}^N$ .

In our article, the Rayleigh quotient that we minimise is not "too nonlinear" as in these cited papers. The difficulty comes from the orthogonality to sine and cosine.

## 2. PRELIMINARIES

**2.1. Existence of a minimizer and Euler equation.** The existence of a minimizer of the functional in (5) follows easily from the direct methods in the calculus of variations:

**Proposition 2.1.** *The minimization problem (5) has a solution  $u$ . The Euler equation satisfied by  $u$  is*

$$-u'' - u = m \cdot \text{sgn}(u) + \lambda_0 + \lambda_1 \cos \theta + \lambda_2 \sin \theta. \tag{7}$$

where the Lagrange multipliers are given by

$$\begin{aligned} \lambda_0 &= -\frac{m}{2\pi} \int_{-\pi}^{\pi} \text{sgn}(u(\theta)) d\theta \\ \lambda_1 &= -\frac{m}{\pi} \int_{-\pi}^{\pi} \text{sgn}(u(\theta)) \cos \theta d\theta \\ \lambda_2 &= -\frac{m}{\pi} \int_{-\pi}^{\pi} \text{sgn}(u(\theta)) \sin \theta d\theta. \end{aligned} \tag{8}$$

In particular, the function  $u$  is  $C^1$ .

The proof of this proposition is straightforward. In particular, the expression of the Lagrange multipliers is obtained by integrating the Euler equation after multiplication by  $1, \cos \theta, \sin \theta$ . Moreover, the  $C^1$  regularity of the function  $u$  comes from the Euler equation that shows that its second derivative is  $L^\infty$ , implying that  $u \in W^{2,\infty} \subset C^1$ .

Notice that one can assume that

$$\int_{-\pi}^{\pi} |u| = 1 \quad (9)$$

by homogeneity. It will be done in all the paper.

**Remark 2.2.** Using (8) we see that

$$m + \lambda_0 = \frac{m}{2\pi} \int_{-\pi}^{\pi} (1 - \operatorname{sgn}(u(\theta))) d\theta > 0 \quad (10)$$

and

$$-m + \lambda_0 = -\frac{m}{2\pi} \int_{-\pi}^{\pi} (1 + \operatorname{sgn}(u(\theta))) d\theta < 0. \quad (11)$$

Up to a translation on  $\theta$ , we can assume that one Lagrange multiplier is zero. Indeed, by periodicity, replacing  $u(\theta)$  by  $u(\theta + a)$  amounts to replace  $\lambda_2$  by  $\cos a \lambda_2 - \sin a \lambda_1$ . Thus we can choose  $a$  such that  $\lambda_2 = 0$ . Therefore in the sequel, we will assume:

$$\text{the Lagrange multiplier } \lambda_2 = 0. \quad (12)$$

We introduce the measure of the sets where  $u$  is positive and negative:

$$\ell_+ = |\{x \in (-\pi, \pi) : u(x) \geq 0\}|, \quad \ell_- = |\{x \in (-\pi, \pi) : u(x) < 0\}| \quad (13)$$

With these notations, we can rewrite  $m + \lambda_0$  and  $m - \lambda_0$ :

$$m + \lambda_0 = \frac{m\ell_-}{\pi}, \quad m - \lambda_0 = \frac{m\ell_+}{\pi} = \frac{m}{\pi} (2\pi - \ell_-). \quad (14)$$

Note also that if  $u(x)$  is a minimizer of our problem, then  $-u(x)$  or  $u(-x)$  or  $-u(-x)$  are also minimizers. Therefore, without loss of generality, we can assume, from now on, that  $\ell_+ \geq \ell_-$ .

**2.2. Expression of the solution.** As usual, we call nodal domain, each interval on which  $u$  has a constant sign. By periodicity, there is an even number of nodal domains. Moreover, we will prove below (see Proposition 2.3) that there is *at least four nodal domains*. The main difficulty will be to prove that there are *exactly* four nodal domains with same length. It will be a consequence that the Lagrange multipliers are all zero and will be done in the last section.

On each nodal domain, we can integrate the Euler equation and get an explicit expression of the solution. We are going to write explicitly  $u$  on two consecutive intervals  $[a, b], [b, c]$ , where

$$u(a) = u(b) = u(c) = 0$$

and

$$u \geq 0 \text{ in } [a, b], \quad u \leq 0 \text{ in } [b, c].$$

Here we assume the lengths of the intervals not equal to  $\pi$ . The case of an interval of length  $\pi$  will be considered in section 3.

By integrating the Euler equation on  $[a, b]$  and using that  $u(a) = u(b) = 0$ , we find

$$u(x) = A_0 \cos x + B_0 \sin x - (m + \lambda_0) - \frac{\lambda_1}{2} x \sin x, \quad x \in [a, b] \quad (15)$$

where

$$\begin{aligned} A_0 &= (m + \lambda_0) \frac{\cos(\frac{a+b}{2})}{\cos(\frac{b-a}{2})} - \frac{\lambda_1}{2} \frac{(b-a) \sin a \sin b}{\sin(b-a)}, \\ B_0 &= (m + \lambda_0) \frac{\sin(\frac{a+b}{2})}{\cos(\frac{b-a}{2})} + \frac{\lambda_1}{2} \frac{b \sin b \cos a - a \sin a \cos b}{\sin(b-a)}. \end{aligned} \quad (16)$$

By integrating the Euler equation on  $[b, c]$ , and using that  $u(b) = u(c) = 0$ , we find

$$u(x) = A_1 \cos x + B_1 \sin x - (-m + \lambda_0) - \frac{\lambda_1}{2} x \sin x, \quad x \in [b, c]$$

where

$$\begin{aligned} A_1 &= (-m + \lambda_0) \frac{\cos(\frac{c+b}{2})}{\cos(\frac{c-b}{2})} - \frac{\lambda_1}{2} \frac{(c-b) \sin c \sin b}{\sin(c-b)}, \\ B_1 &= (-m + \lambda_0) \frac{\sin(\frac{b+c}{2})}{\cos(\frac{c-b}{2})} + \frac{\lambda_1}{2} \frac{c \sin c \cos b - b \sin b \cos c}{\sin(c-b)}. \end{aligned} \quad (17)$$

Now we can obtain another expression of the solution on  $[b, c]$  using the  $C^1$  regularity of  $u$  in  $b$ . This gives

$$A_1 \cos b + B_1 \sin b = \lambda_0 - m + \frac{\lambda_1}{2} b \sin b$$

and

$$(A_0 - A_1) \sin b = (B_0 - B_1) \cos b.$$

We then get a different expression for  $A_1$  and  $B_1$ :

$$A_1 = (\lambda_0 - m) \cos b + \frac{\lambda_1}{2} b \sin b \cos b - B_0 \sin b \cos b + A_0 \sin^2 b$$

$$B_1 = (\lambda_0 - m) \sin b + \frac{\lambda_1}{2} b \sin^2 b + B_0 \cos^2 b - A_0 \sin b \cos b.$$

Replacing  $A_0, B_0$  of formulas (16) in the above expressions of  $A_1$  and  $B_1$ , one gets:

$$\begin{aligned} A_1 &= \lambda_0 \frac{\cos(\frac{a+b}{2})}{\cos(\frac{b-a}{2})} - m \frac{\cos(\frac{3b-a}{2})}{\cos(\frac{b-a}{2})} - \frac{\lambda_1}{2} (b-a) \frac{\sin a \sin b}{\sin(b-a)} \\ B_1 &= \lambda_0 \frac{\sin(\frac{a+b}{2})}{\cos(\frac{b-a}{2})} - m \frac{\sin(\frac{3b-a}{2})}{\cos(\frac{b-a}{2})} + \frac{\lambda_1}{2} \frac{b \sin b \cos a - a \sin a \cos b}{\sin(b-a)}. \end{aligned} \quad (18)$$

Expressions (17) and (18) of  $A_1$  give

$$\lambda_0 \frac{\sin b \sin(\frac{c-a}{2})}{\cos(\frac{b-a}{2}) \cos(\frac{c-b}{2})} - m \frac{\sin b \sin(\frac{a+c-2b}{2})}{\cos(\frac{b-a}{2}) \cos(\frac{c-b}{2})} = \frac{\lambda_1}{2} \sin b \left[ \frac{(b-a) \sin a}{\sin(b-a)} - \frac{(c-b) \sin c}{\sin(c-b)} \right].$$

Let us assume now that  $b \neq 0$ . If  $\ell_1 = b - a$  and  $\ell_2 = c - b$ , this equality can be written as

$$\lambda_0 \sin \frac{\ell_1 + \ell_2}{2} - m \sin \frac{\ell_2 - \ell_1}{2} = \frac{\lambda_1}{2} \cos \frac{\ell_1}{2} \cos \frac{\ell_2}{2} \left[ \frac{\ell_1}{\sin \ell_1} \sin a - \frac{\ell_2}{\sin \ell_2} \sin c \right]. \quad (19)$$

Let us assume now that  $b = 0$ . Expressions (17) and (18) of  $B_1$  give

$$(-m + \lambda_0) \tan\left(\frac{c}{2}\right) + \frac{\lambda_1}{2} c = (\lambda_0 + m) \tan\left(\frac{a}{2}\right) + \frac{\lambda_1}{2} a$$

that is,

$$\lambda_0 \sin\left(\frac{c-a}{2}\right) - m \sin\left(\frac{a+c}{2}\right) + \frac{\lambda_1}{2} (c-a) \cos\left(\frac{c}{2}\right) \cos\left(\frac{a}{2}\right) = 0$$

This is exactly equation (19) written in the case  $b = 0$ .

As a first consequence of the previous formulae, let us prove the following

**Proposition 2.3.** *A minimizer  $u$  has at least four nodal domains.*

*Proof.* Assume, by contradiction, that the minimizer has only two nodal domains, say  $(a, b) \subset [-\pi, \pi]$  where  $u$  is positive and its complement where  $u$  is negative. First of all, from  $\lambda_2 = 0$  and Proposition 2.1, we infer

$$0 = - \int_{-\pi}^a \sin x dx + \int_a^b \sin x dx - \int_b^\pi \sin x dx = 2(\cos a - \cos b).$$

Therefore  $a = -b$ . Using this equality, we deduce from (10) and (11) in the same way that

$$m + \lambda_0 = \frac{2m}{\pi} (\pi - b) \text{ and } -m + \lambda_0 = -\frac{2mb}{\pi}$$

and from (8),

$$\frac{\lambda_1}{2} = -\frac{2m}{\pi} \sin b.$$

We observe that  $b \neq \frac{\pi}{2}$ , otherwise the function  $u(x) \cos x$  would be positive on  $(-\pi, \pi)$  contradicting the constraint (L2). Formulae (16) and (18) give

$$A_0 = \frac{2m}{\pi \cos b} (\pi - b - b \sin^2 b), \quad B_0 = 0$$

and

$$A_1 = \frac{2m}{\pi \cos b} [(\pi - b) \sin^2 b - b], \quad B_1 = -2m \sin b.$$

Since  $B_0 = 0$ , we deduce from the expression of  $u$  on this interval (see (15)) that  $u$  is even on  $(-b, b)$ . In particular, since  $u$  is  $C^1$ , one has  $u'(-b) = -u'(b)$ . Therefore, on the intervals  $[b - 2\pi, -b]$  and  $[b, -b + 2\pi]$ , where  $u$  is negative,  $u$  solves the same ordinary differential equation, with  $u(b) = u(-b) = 0$  and opposite values for the derivative in  $b, -b$ . This implies that  $u$  is even in the whole interval  $(-\pi, \pi)$ .

By (L2),  $0 = \int_0^\pi u(x) \cos x dx$ , that is,

$$0 = A_0 \frac{b + \sin b \cos b}{2} + A_1 \frac{\pi}{2} - A_1 \frac{b + \sin b \cos b}{2} - \frac{B_1}{2} \sin^2 b + \frac{\lambda_1}{8} \pi - 2m \sin b.$$

Replacing the above expressions of  $A_0, A_1, B_1$  yields

$$0 = m \sin b \left[ -\frac{3}{2} + (\pi - 2b) \tan(b) \right].$$

Setting  $x = \frac{\pi}{2} - b$  we are led to the equation  $x/\tan x = 3/4$ . Since  $x \mapsto x/\tan x$  is even and decreasing from 1 to 0, this equation has two solutions in  $(-\frac{\pi}{2}, \frac{\pi}{2})$  yielding the only two possible values  $b \simeq 0.726$  and  $b \simeq 2.4155$ .

Since we have chosen the normalization  $\int_{-\pi}^\pi |u(x)| dx = 1$ , we get

$$\frac{1}{2} = \int_0^\pi |u(x)| dx = \int_0^b u(x) dx - \int_b^\pi u(x) dx = \frac{4m}{\pi} [(\pi - 2b) \tan b + \sin^2 b - b(\pi - b)].$$

Plugging the value of  $b$  obtained above, we would get  $m = 2.1003$ . Now, the minimum of the functional defined in (5) must be less or equal to the functional evaluated in the function  $u$  described in Theorem 1.1, that is,  $m \leq \frac{1}{2(4-\pi)}$ . This gives a contradiction with the value of  $m$  found above.  $\square$

### 3. THE LENGTH OF THE NODAL INTERVALS CANNOT BE GREATER THAN $\pi$

In this section we prove that the length of any nodal interval of the solution  $u$  is strictly less than  $\pi$ . We argue by contradiction: we will use different analytic tools to get the desired contradiction, mainly by considering the integral of  $u$  on a nodal domain.

We assume that there exists a nodal interval  $(a, b)$  of length  $\ell$  greater than  $\pi$ . Without loss of generality, we can assume that

- $u > 0$  on  $(a, b)$ ;
- $a \in [-\pi, 0]$  and  $b \in (0, \pi]$  (the function  $x \mapsto u(x + \pi)$  is also a minimizer satisfying  $\lambda_2 = 0$ );
- $\frac{a+b}{2} \leq 0$  (since  $u(-x)$  is also a minimizer); this implies  $a \leq -\frac{\pi}{2}$ .

In the sequel we will call *negative interval* (resp. *positive interval*) any interval where  $u$  is negative (resp. positive).

On a negative interval  $(a_j, b_j)$  of length  $\ell_j \neq \pi$ , we have

$$\int_{a_j}^{b_j} u(x) dx = (-m + \lambda_0) \left( 2 \tan \frac{\ell_j}{2} - \ell_j \right) + \frac{\lambda_1}{2} \left( 2 \sin \frac{\ell_j}{2} \cos \frac{a_j + b_j}{2} \right) \left[ 1 + \frac{\ell_j}{\sin \ell_j} \right], \quad (20)$$

while, on a positive interval  $(a_k, b_k)$  of length  $\ell_k \neq \pi$ , we have

$$\int_{a_k}^{b_k} u(x) dx = (m + \lambda_0) \left( 2 \tan \frac{\ell_k}{2} - \ell_k \right) + \frac{\lambda_1}{2} \left( 2 \sin \frac{\ell_k}{2} \cos \frac{a_k + b_k}{2} \right) \left[ 1 + \frac{\ell_k}{\sin \ell_k} \right]. \quad (21)$$

In the case of a nodal domain of length  $\pi$ , let  $(a, a + \pi)$  be such an interval where we suppose  $u > 0$ . Now the Euler equation

$$\begin{cases} -u'' - u = m + \lambda_0 + \lambda_1 \cos x & \text{on } (a, a + \pi) \\ u(a) = 0, \quad u(a + \pi) = 0 \end{cases}$$

has not a unique solution since 1 is an eigenvalue on the interval. Moreover, by the Fredholm alternative, the right-hand side of the equation must be orthogonal to the eigenfunction  $\sin(x - a)$  providing the relation

$$m + \lambda_0 = \frac{\lambda_1}{4} \pi \sin a. \quad (22)$$

**Lemma 3.1.** *The Lagrange multiplier  $\lambda_1$  is negative.*

*Proof.* We first study the case where  $(a, b)$  has length  $\ell > \pi$ . Let us analyse equation (21). We recall that  $m + \lambda_0 > 0$  by (10); for  $\ell > \pi$ , both terms  $2 \tan(\ell/2) - \ell$  and  $1 + \ell/\sin \ell$  are negative. If  $\lambda_1 \geq 0$ , the integral is negative: this is a contradiction with the sign of  $u$  on  $(a, b)$ .

In the case of a nodal domain of length  $\pi$ , one has  $\lambda_1 < 0$  by equation (22), since  $m + \lambda_0 > 0$  and  $\sin a < 0$ .  $\square$

**Lemma 3.2.** *The Lagrange multiplier  $\lambda_1$  satisfies*

$$\left| \frac{\lambda_1}{2} \right| \leq 2 \sin \left( \frac{\ell_-}{2} \right) \frac{m}{\pi} = \frac{2}{\ell_-} \sin \frac{\ell_-}{2} (m + \lambda_0) < m + \lambda_0, \quad (23)$$

where  $\ell_-$  is the measure of  $\{x : u(x) < 0\}$ .

*Proof.* Let us introduce the two numbers:

$$\ell_-^b = |\{t > b, u(t) < 0\}|, \quad \ell_-^a = |\{t < a, u(t) < 0\}|.$$

Obviously  $\ell_-^a + \ell_-^b = \ell_-$ . By (10),  $m + \lambda_0 = \frac{m\ell_-}{\pi}$ . Now, since  $\lambda_1 < 0$  by the previous lemma,

$$\left| \frac{\lambda_1}{2} \right| = \frac{m}{2\pi} \int_{-\pi}^{\pi} \text{sign}(u) \cos t dt$$

and

$$\int_{-\pi}^{\pi} \text{sign}(u) \cos t dt = \int_{-\pi}^a \text{sign}(u) \cos t dt + \int_a^b \cos t dt + \int_b^{\pi} \text{sign}(u) \cos t dt.$$

By the bathtub principle (see [13]), the value of  $\int_{-\pi}^a \text{sign}(u) \cos t dt$  is maximum when we choose  $\text{sign}(u) = -1$  on the left, namely on  $(-\pi, -\pi + \ell_-^b]$  (because the  $\cos$  is increasing on  $[-\pi, a]$ ) and similarly for the last integral. Therefore, we get that

$$\left| \frac{\pi\lambda_1}{m} \right| \leq - \int_{-\pi}^{-\pi + \ell_-^b} \cos t dt + \int_{-\pi + \ell_-^b}^{\pi + \ell_-^a} \cos t dt - \int_{\pi - \ell_-^a}^{\pi} \cos t dt = 2(\sin \ell_-^a + \sin \ell_-^b). \quad (24)$$

Since

$$\sin \ell_-^a + \sin \ell_-^b = 2 \sin \left( \frac{\ell_-}{2} \right) \cos \left( \frac{\ell_-^a - \ell_-^b}{2} \right) \leq 2 \sin \left( \frac{\ell_-}{2} \right) \leq \ell_-$$

we finally get estimate (23), using (10) and (24).  $\square$

Let us introduce the following quantity :

$$A = \frac{|\lambda_1/2|}{m + \lambda_0}. \quad (25)$$

Our strategy to prove that there cannot be an interval of length greater than  $\pi$  consists in proving that  $A < \frac{2}{\pi}$  or  $A \cos(\frac{a+b}{2}) < \frac{2}{\pi}$ . Indeed, in the case where  $b - a = \ell > \pi$ , these estimates imply that  $\int_a^b u dt < 0$ , that is, a contradiction. This is due to the following Lemma 3.3. In the case where  $b - a = \ell = \pi$ , one has  $A = \frac{2}{\pi |\sin a|}$  by (22). Therefore proving that  $A < \frac{2}{\pi}$  provides a contradiction in this case too.

**Lemma 3.3.** *The function  $g : \ell \mapsto \ell - 2 \tan(\ell/2) + \frac{4}{\pi} \sin(\ell/2)[1 + \ell/\sin \ell]$  is positive on  $(\pi, 2\pi)$ .*

*Proof.* Indeed  $g$  is positive if and only if  $k(t) = t \cos t - \sin t + \frac{1}{\pi}[2t + \sin(2t)]$  is negative on  $(\frac{\pi}{2}, \pi)$ . Observe that  $k(\frac{\pi}{2}) = 0$  and  $k(\pi) < 0$ . Now, the derivative of  $k$ ,  $k'(t) = -t \sin(t) + \frac{2}{\pi}[1 + \cos(2t)]$ , is the difference between two functions which intersect in only one point  $t_0$ . Since  $k'$  is negative near  $\frac{\pi}{2}$  and positive near  $\pi$ ,  $k$  is minimal at  $t_0$  and therefore  $k < 0$  on  $(\frac{\pi}{2}, \pi)$ .  $\square$

We are now going to find some estimates on  $A$ , taking into account the midpoint of a nodal interval  $(a_j, b_j)$ , that we will denote by  $m_j$ .

**Lemma 3.4.** *Let  $(a_j, b_j)$  be a negative interval. If  $m_j \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , then*

$$A \leq \frac{l_j^2(2\pi - \ell_-)}{12\ell_- |\cos(m_j)|}. \quad (26)$$

For a positive interval  $(a_k, b_k)$ , whose midpoint  $m_k \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  one has

$$A \leq \frac{l_k^2}{12 \cos(m_k)}. \quad (27)$$



*Proof.* For a negative interval  $(a_j, b_j)$ , as soon as  $\frac{a_j + b_j}{2} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  the fact that  $\int_{a_j}^{b_j} u < 0$  implies

$$\frac{|\lambda_1/2|}{m - \lambda_0} \leq \frac{h(\ell_j)}{|\cos(\frac{a_j + b_j}{2})|}$$

where  $h(x) = \frac{2 \tan(\frac{x}{2}) - x}{2 \sin(\frac{x}{2}) (1 + \frac{x}{\sin x})}$ . We claim that for  $x \in [0, \pi]$ , one has  $h(x) \leq \frac{x^2}{12}$ . By using (10), (11), (14) and this bound on  $h$  we have the following estimate on  $A$ :

$$A \leq \frac{l_j^2 (2\pi - \ell_-)}{12\ell_- |\cos(\frac{a_j + b_j}{2})|}.$$

We now prove our claim, that is, the bound on  $h$ . The statement is equivalent to the positivity of  $f(x) = \frac{x^2}{12}(x + \sin x) - 2 \sin(x/2) + x \cos(x/2)$  in  $[0, \pi]$ . Observe that  $f(0) = 0$ . The result will follow if we prove that the derivative of  $f$  is positive, that is,  $k(x) = 3x + 2 \sin(x) + x \cos(x) - 6 \sin(x/2)$  is positive. We remark that  $k(0) = 0, k(\frac{\pi}{2}) > 0, k(\pi) > 0$ . We will split the analysis into two cases.

- (1) If  $x \in [0, \frac{\pi}{2}]$  it is easy to see that  $k'' < 0$  and therefore  $k(x) \geq 0$  in  $[0, \frac{\pi}{2}]$ .
- (2) In the case where  $x \in [\frac{\pi}{2}, \pi]$ ,  $k$  is decreasing. Indeed it is easy to see that  $k'$  is negative, since  $k'$  is convex,  $k'(\pi) = 0, k'(\frac{\pi}{2}) < 0$ .

In the same way, using (21), we can prove that, on a positive interval  $(a_k, b_k)$  ( $m_k$  being its midpoint):

$$A \leq \frac{l_k^2}{12 \cos(m_k)},$$

when  $m_k \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . □

**Lemma 3.5.** *There are no negative intervals on the "left" of  $I_0 = (a, b)$  whose midpoint is between  $-\pi/2$  and  $\pi/2$ . There is at most one negative interval on the "right" of  $I_0$  whose midpoint lies between  $-\pi/2$  and  $\pi/2$ .*

*Proof.* Since  $a \leq -\frac{\pi}{2}$ , there are no negative intervals on the "left" of  $I_0$  whose midpoint is between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$ .

We prove the second statement by contradiction. If there were two negative intervals, say  $I_1$  and  $I_3$  with a positive interval  $I_2 = (a_2, b_2)$  between them, its midpoint would satisfy  $m_2 \leq \frac{\pi - \ell_2}{2}$ . Moreover, since  $b_2 \leq \frac{\pi}{2}$ , and  $0 \leq b < a_2 = b_2 - \ell_2 \leq \frac{\pi}{2} - \ell_2$ , we infer  $\ell_2 \leq \frac{\pi}{2}$ . Now using (27), we see that

$$A \leq \frac{\ell_2^2}{12 \cos(m_2)} \leq \frac{\ell_2^2}{12 \sin(\ell_2/2)}.$$

Now, it is immediate to check that  $x \mapsto x^2 / \sin(x/2)$  is increasing. Therefore the previous inequality would imply for  $\ell_2 \leq \frac{\pi}{2}$ :  $A \leq \pi^2 / (24\sqrt{2}) < \frac{2}{\pi}$  implying  $\int_{a_2}^{b_2} u dt < 0$ , that is, a contradiction. □

We are now going to find a lower bound for  $\ell_-$ , the measure of  $\{u < 0\}$ . We know that  $\int_{\{u > 0\}} u(x) dx + \int_{\{u < 0\}} u(x) dx = 0$ , while  $\int_{-\pi}^{\pi} |u(x)| dx = \int_{\{u > 0\}} u(x) dx - \int_{\{u < 0\}} u(x) dx = 1$ . Therefore

$$\int_{\{u > 0\}} u(x) dx = - \int_{\{u < 0\}} u(x) dx = \frac{1}{2}. \quad (28)$$

On all (but possibly one) negative intervals  $I_j$  whose midpoint  $m_j \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , we have

$$\int_{I_j} -u(x) dx \leq (m - \lambda_0)[2 \tan(\ell_j/2) - \ell_j]. \quad (29)$$

The following proposition gives a lower bound for  $\ell_-$ .

**Proposition 3.6.**  $\ell_- \geq 1.55$ .

*Proof.* There is only one possibility between

- there is a negative interval of length  $\ell_j \geq 1.6$ ;
- the length of all negative intervals is less than 1.6.

The first possibility implies that  $\ell_- \geq 1.6 > 1.55$ . We need to study the second possibility. We will split the analysis into two cases: in the first case we will assume that no negative interval has its midpoint in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ . In the second case we will assume that the midpoint of one interval, say  $I_1$ , is before  $\frac{\pi}{2}$ . This is possible due to Lemma 3.5.

- (1) Assume that no negative interval has its midpoint in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ . We observe that

$$2 \tan(x/2) - x \leq \frac{0.45x^3}{4}, \quad 0 \leq x \leq 0.8.$$

Using this inequality in (29) and summing up on all negative intervals  $I_j$ , we get

$$\frac{1}{2} = \sum_j \int_{I_j} (-u(x)) dx \leq \frac{0.45(m - \lambda_0)}{4} \sum_j \ell_j^3.$$

Now, the maximization of the convex function  $t \in \mathbb{R}^+ \rightarrow t^3$  and the fact that the sum of the lengths  $\ell_j$  of all negative intervals is  $\ell_-$  give

$$\sum_j \ell_j^3 \leq \ell_-^3.$$

Moreover, consider the functional defined by (5). Using as a test function the explicit function given in Theorem (1.1) we get an estimate of  $m$ :

$$m \leq \frac{1}{2(4 - \pi)}. \quad (30)$$

Recalling that  $m - \lambda_0 = \frac{m}{\pi}(2\pi - \ell_-)$  by (14), we end up with

$$\frac{1}{2} \leq \frac{0.45(2\pi - \ell_-)}{8\pi(4 - \pi)} \ell_-^3.$$

This polynomial inequality provides finally the inequality  $\ell_- \geq 1.74$ .

- (2) Assume that the midpoint of one interval, say  $I_1$ , is before  $\frac{\pi}{2}$ . In the above computation of  $\sum_j \int_{I_j} (-u(x)) dx$ , we need to add the following positive term

$$P = -\frac{\lambda_1}{2} 2 \sin\left(\frac{\ell_1}{2}\right) \cos m_1 \left(1 + \frac{\ell_1}{\sin \ell_1}\right) \quad (31)$$

coming from the integral on  $I_1$  (see (20), second term of the right hand side). In other words

$$\frac{1}{2} \leq \frac{0.45(2\pi - \ell_-)}{8\pi(4 - \pi)} \ell_-^3 + P. \quad (32)$$

We are going to distinguish two cases, according to the values of  $\ell_1$ :

- (a) Assume  $\ell_1 \geq 0.228$ . In this case, we look at the first negative interval  $I_{-1}$  on the left of  $I_0$ .

Since  $b \leq \frac{\pi - \ell_1}{2} \leq \frac{\pi}{2} - 0.114$  and  $\ell \geq \pi$ , we have

$$m_{-1} = a - \frac{\ell_{-1}}{2} = b - \ell - \frac{\ell_{-1}}{2} \leq -0.114 - \frac{\pi + \ell_{-1}}{2}.$$

If  $\ell_{-1} \geq \frac{\pi}{2}$ , then  $\ell_- \geq \frac{\pi}{2} + 0.228 > 1.75$ . If, on the contrary,  $\ell_{-1} < \frac{\pi}{2}$ , then  $m_{-1} \geq -\frac{5\pi}{4}$  necessarily (otherwise  $a = m_{-1} + \frac{\ell_{-1}}{2} < -\pi$  which is impossible). Therefore,

$$|\cos(m_{-1})| \geq \min \left\{ \frac{\sqrt{2}}{2}, \left| \cos \left( 0.114 + \frac{\pi + \ell_{-1}}{2} \right) \right| \right\} := C.$$

Using this estimate and  $\ell_- \geq 0.228 + \ell_{-1}$  in (26) we get

$$A \leq \frac{\ell_{-1}^2(2\pi - 0.228 - \ell_{-1})}{12(0.228 + \ell_{-1})C}.$$

As a function of  $\ell_{-1}$ , the right-hand side is increasing and for  $\ell_{-1} \leq 1.322$  we get the inequality  $A \leq 0.636 < \frac{2}{\pi}$ , that is, a contradiction. Therefore, in this case, we deduce  $\ell_{-1} \geq 1.322$  and  $\ell_- \geq \ell_1 + \ell_{-1} \geq 1.55$ .

(b) Assume that  $\ell_1 \leq 0.228$ . Lemma 3.2 and inequalities (14) and (30) give

$$-\frac{\lambda_1}{2} \leq m + \lambda_0 = m \frac{\ell_-}{\pi} \leq \frac{\ell_-}{2\pi(4-\pi)}.$$

We can use these inequalities to estimate the positive term  $P$  defined by (31) :

$$P \leq \frac{\ell_-}{2\pi(4-\pi)} 2 \sin(0.114) \left( 1 + \frac{0.228}{\sin(0.228)} \right).$$

Therefore we get from (32)

$$\frac{1}{2} \leq \frac{0.45(2\pi - \ell_-)}{8\pi(4-\pi)} \ell_-^3 + \frac{\ell_-}{2\pi(4-\pi)} 2 \sin(0.114) \left( 1 + \frac{0.228}{\sin(0.228)} \right).$$

This polynomial inequality implies  $\ell_- \geq 1.55$ . □

We are now in position to prove the main result of this section.

**Theorem 3.7.** *There is no nodal interval of length greater than  $\pi$ .*

*Proof.* We will divide the proof in two cases: the solution  $u$  has at least 6 nodal intervals and the solution  $u$  has only 4 nodal intervals.

(1) We are going to prove that there cannot be 6 nodal intervals or more. We recall that  $a \leq -\frac{\pi}{2}$ .

We divide the proof into two cases according to the values of  $b$ .

(a) Assume that  $b \geq \frac{\pi}{2}$ . We will consider the two negative intervals next to  $I_0 = (a, b)$ , namely  $I_{-1}$  on the left to  $I_0$  and  $I_1$  on the right, and prove that both have lengths greater than 1.17.

(i) Assume by contradiction that  $\ell_{-1} \leq 1.17$ . On one hand

$$m_{-1} = a - \frac{\ell_{-1}}{2} \leq -\frac{\pi}{2} - \frac{\ell_{-1}}{2}.$$

On the other hand,  $m_{-1} \geq -\frac{5\pi}{4}$  (otherwise  $a = m_{-1} + \frac{\ell_{-1}}{2} < -\pi$  which is impossible).

Therefore we have

$$|\cos(m_{-1})| \geq \min \left\{ \frac{1}{\sqrt{2}}, \left| \cos \left( \frac{\pi + \ell_{-1}}{2} \right) \right| \right\} = \min \left\{ \frac{1}{\sqrt{2}}, \sin \left( \frac{\ell_{-1}}{2} \right) \right\}.$$

Plugging this estimate in (26) and using Proposition 3.6 yields

$$A \leq \frac{\ell_{-1}^2(2\pi - 1.55)}{12 \cdot 1.55 \cdot \min \left\{ \frac{1}{\sqrt{2}}, \sin(\ell_{-1}/2) \right\}}. \quad (33)$$

Now the right-hand side is increasing in  $\ell_{-1}$  and its value for  $\ell_{-1} = 1.17$  is less than  $0.631 < \frac{2}{\pi}$  that is absurd, proving the claim.

(ii) Assume by contradiction that  $\ell_1 \leq 1.17$ . In this case we are going to use that  $b \geq \frac{\pi}{2}$  to estimate  $m_1$ :

$$\frac{\pi}{2} + \frac{\ell_1}{2} \leq b + \frac{\ell_1}{2} = m_1 = b + \frac{\ell_1}{2} < \pi + \frac{1.17}{2} < \frac{5\pi}{4}.$$

One can now repeat the above argument to get a contradiction.

We observe that  $\ell_- \geq \ell_{-1} + \ell_1 \geq 2.34$ .

With the same arguments of the two steps above, we can prove that  $\ell_{-1} \geq \frac{\pi}{2}$  and  $\ell_1 \geq \frac{\pi}{2}$ .

Indeed it is sufficient to replace 1.17 by  $\frac{\pi}{2}$  and 1.55 by 2.34 in the estimate of  $A$ . This gives

$A \leq 0.49 < \frac{2}{\pi}$ , that is a contradiction. We deduce that  $\ell_{-1} \geq \frac{\pi}{2}$  and  $\ell_1 \geq \frac{\pi}{2}$  that implies  $\ell_- \geq \pi$ : this is a contradiction, since  $\ell_+ > \pi$ .

We have thus proved that in the case  $b \geq \frac{\pi}{2}$  one cannot have 6 or more nodal domains.

(b) Assume that  $b < \frac{\pi}{2}$ . One can prove that  $\ell_{-1} \geq 1.17$ , following exactly the same argument as in step (i) above. Now since we do not know the position of  $m_1$ , we work with the intervals  $I_1 = [a_1, b_1]$ ,  $I_2 = [a_2, b_2]$ ,  $I_3 = [a_3, b_3]$ , with  $b_1 = a_2, b_2 = a_3$ .

- (i) Assume that  $b_2 \leq \frac{\pi}{2}$ . Observe that  $0 < m_2 \leq \frac{\pi}{2} - \frac{\ell_2}{2}$  where we have used that  $b > 0$ . Using (27) we get

$$A \leq \frac{\ell_2^2}{12 \sin(\ell_2/2)}.$$

Since  $b_2 \leq \frac{\pi}{2}$  and  $0 \leq b < a_2 = b_2 - \ell_2 \leq \frac{\pi}{2} - \ell_2$ , one has  $\ell_2 \leq \frac{\pi}{2}$ . Therefore the right hand side of the above estimate of  $A$  is less than  $\frac{2}{\pi}$ , that is a contradiction.

- (ii) Assume that  $b_2 = a_3 \geq \frac{\pi}{2}$ . We start by proving that  $\ell_- \geq 1.93$ . Assume by contradiction that  $\ell_- \leq 1.93$ . On one hand we have  $m_3 = a_3 + \frac{\ell_3}{2} \geq \frac{\pi}{2} + \frac{\ell_3}{2}$ . On the other hand we claim that  $m_3 < \frac{5\pi}{4}$ . We have  $m_3 = b + \ell_1 + \ell_2 + \frac{\ell_3}{2}$ . Observe that

- $b < \frac{\pi}{2}$  by hypothesis,
- $\ell_2 \leq \pi - 1.55$  since  $\ell_2 + \ell \leq \ell_+ = 2\pi - \ell_- \leq 2\pi - 1.55$ ,
- $\ell_1 + \frac{\ell_3}{2} \leq 0.76$  since  $\ell_{-1} + \ell_1 + \frac{\ell_3}{2} \leq \ell_- \leq 1.93$  and  $\ell_{-1} \geq 1.17$ .

Thus  $m_3 < \frac{3\pi}{2} - 1.55 + 0.76 < \frac{5\pi}{4}$ . We deduce from the previous bounds on  $m_3$  that:

$$|\cos m_3| \geq \min \left\{ \frac{1}{\sqrt{2}}, \left| \cos \left( \frac{\pi + \ell_3}{2} \right) \right| \right\}$$

which provides, as in case (a)(i),  $\ell_3 \geq 1.17$ .

We deduce that  $\ell_- \geq \ell_{-1} + \ell_3 > 1.93$ .

We are going to prove that  $\ell_{-1} \geq 1.532$ . Assume that this is not the case. Using the same argument as in step (a)(i), one has a contradiction from the following estimate :

$$A \leq \frac{1.532^2(2\pi - 1.93)}{12 \cdot 1.93 \cdot \min \left\{ \frac{1}{\sqrt{2}}, \sin(1.532/2) \right\}} < \frac{2}{\pi}. \quad (34)$$

It is easy to prove that  $\ell_- > 2.67$ . Indeed, if this is not the case, replacing 1.93 by 2.67 in (34), we get a contradiction since  $A$  is still less than  $\frac{2}{\pi}$ .

We are going to prove that  $\ell_{-1} > \frac{\pi}{2}$ . Assume that this is not the case. By the same argument as in step (a)(i), one has a contradiction from estimate

$$A \leq \frac{(\pi/2)^2(2\pi - 2.67)}{12 \cdot 2.67 \cdot \sqrt{2}/2} < \frac{2}{\pi}.$$

We are going to prove that  $\ell_3 \geq \frac{\pi}{2}$ . Assume that this is not true. Recall that  $\ell_- < \pi$ .

As above, on one hand we have  $m_3 = a_3 + \frac{\ell_3}{2} \geq \frac{\pi}{2} + \frac{\ell_3}{2}$ . On the other hand  $m_3 < \frac{5\pi}{4}$ .

Indeed  $m_3 = b + \ell_1 + \ell_2 + \frac{\ell_3}{2}$ . Observe that

- $b < \frac{\pi}{2}$  by hypothesis,
- $\ell_2 \leq \pi - 2.67$
- $\ell_1 + \frac{\ell_3}{2} \leq \pi - \frac{\pi}{2}$

Thus  $m_3 < 2\pi - 2.67 < \frac{5\pi}{4}$ . We deduce from the previous bounds on  $m_3$  that:

$$|\cos m_3| \geq \frac{1}{\sqrt{2}}. \text{ This gives a contradiction since}$$

$$A \leq \frac{(\pi/2)^2(2\pi - 2.67)}{12 \cdot 2.67 \cdot \sqrt{2}/2} < \frac{2}{\pi}.$$

Therefore  $\ell_3 > \frac{\pi}{2}$ .

This last estimate implies  $\pi \geq \ell_- > \ell_{-1} + \ell_3 > \pi$ . Therefore there cannot be 6 or more nodal intervals.

- (2) It remains to consider the case of 4 nodal intervals:  $I_{-1} = I_3, I_1$  (two negative ones) and  $I_0, I_2$  (two positive ones). The strategy for the proof is similar, but now we are going to work with a more explicit expression of  $\lambda_1, \lambda_2$ . Without loss of generality, we can assume that the lengths

satisfy  $0 < \ell_1 \leq \ell_3 < \pi$  (up to replacing  $u(x)$  by  $u(-x)$ ). Moreover, as before, we can get the lower bound  $\ell_3 \geq 1.17$  (see (b), (ii)). Let us rewrite the four intervals as  $I_0 = (a_0, a_1)$ ,  $I_1 = (a_1, a_2)$ ,  $I_2 = (a_2, a_3)$ ,  $I_{-1} = I_3 = (a_3 - 2\pi, a_0)$  and  $m_0, m_1, m_2, m_3$  are the midpoints of each interval. We can assume  $-\pi \leq m_3 \leq -\frac{\pi}{2}$  by Lemma 3.5. From  $\lambda_2 = 0$  we have

$$0 = \int_{-\pi}^{\pi} \text{sign}(u(x)) \sin x dx = 2(\cos a_0 - \cos a_1 + \cos a_2 - \cos a_3). \quad (35)$$

Gathering  $-\cos a_1 + \cos a_2$  on the one hand and  $\cos a_0 - \cos a_3$  on the other hand, the right hand side of (35) can be rewritten

$$\sin \frac{\ell_3}{2} \sin m_3 + \sin \frac{\ell_1}{2} \sin m_1 = 0. \quad (36)$$

In the same way, coming back to the definition of  $\lambda_1$ , see (8), we have

$$\frac{\lambda_1}{2} = \frac{2m}{\pi} \left( \sin \frac{\ell_3}{2} \cos m_3 + \sin \frac{\ell_1}{2} \cos m_1 \right). \quad (37)$$

Observing that  $\cos m_3 < 0$ , by (36)  $\cos m_3$  can be rewritten as

$$\cos m_3 = -\sqrt{1 - \frac{\sin^2 \frac{\ell_1}{2} \sin^2 m_1}{\sin^2 \frac{\ell_3}{2}}}. \quad (38)$$

By using (37), (38) and (10), the quantity  $A$  defined in (25) can be rewritten as a function of  $\ell_1, \ell_3, m_1$  as

$$A = A(\ell_1, \ell_3, m_1) = 2 \frac{\sqrt{\sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2} \sin^2 m_1} - \sin \frac{\ell_1}{2} \cos m_1}{\ell_1 + \ell_3}. \quad (39)$$

We need to distinguish the cases  $m_1 \leq \frac{\pi}{2}$  and  $m_1 > \frac{\pi}{2}$ . In the first case, we will prove that we can assume  $\ell_1 \leq \frac{\pi}{6}$  while, in the second case, we can assume  $\ell_1 \leq 0.62$ .

**(a):** Let us start with the first case  $m_1 \leq \frac{\pi}{2}$  and let us first look at the dependence of  $A$  with respect to  $m_1$ . The derivative of  $A(\ell_1, \ell_3, m_1)$  with respect to  $m_1$  has the same sign as

$$\sin m_1 \left( 1 - \frac{\cos m_1 \sin \frac{\ell_1}{2}}{\sqrt{\sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2} \sin^2 m_1}} \right),$$

that is, the same sign as

$$\sin m_1 \left( \sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2} \right).$$

The above quantity is positive, since  $0 \leq m_1 \leq \frac{\pi}{2}$  and  $\ell_3 \geq \ell_1$ . Therefore, if  $m_1 \leq \frac{\pi}{2}$

$$A(\ell_1, \ell_3, m_1) \leq A(\ell_1, \ell_3, \frac{\pi}{2}) = 2 \frac{\sqrt{\sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2}}}{\ell_1 + \ell_3}. \quad (40)$$

Assume by contradiction that  $\ell_1 > \frac{\pi}{6}$ . We are going to prove that  $A < \frac{2}{\pi}$ , thus reaching a contradiction. We set

$$x = \frac{\ell_3 - \ell_1}{2} \quad \text{and} \quad y = \frac{\ell_3 + \ell_1}{2}$$

and observe that  $0 \leq x < \frac{\pi}{2}$ ,  $\frac{1.17}{2} \leq y < \frac{\pi}{2}$ . The quantity  $A(\ell_1, \ell_3, \frac{\pi}{2})$  can be rewritten as a function of  $x, y$ :

$$G(x, y) = \frac{\sqrt{\sin x \sin y}}{y}, \quad (x, y) \in \left[0, \frac{\pi}{2}\right] \times \left[\frac{1.17}{2}, \frac{\pi}{2}\right].$$

The map  $x \mapsto G(x, y)$  is increasing while  $y \mapsto G(x, y)$  is decreasing on this set. We remark that the assumption  $\ell_1 > \frac{\pi}{6}$  is equivalent to  $y > x + \frac{\pi}{6}$ . In the rectangle  $\left[0, \frac{\pi}{2}\right] \times \left[\frac{1.17}{2}, \frac{\pi}{2}\right]$

this implies that  $x \in \left[0, \frac{\pi}{3}\right]$ . Now, it can be checked that

$$\forall t \in \left[0, \frac{\pi}{3}\right], \quad G(t, t + \frac{\pi}{6}) \leq \frac{2}{\pi}.$$

Indeed the function  $t \mapsto G(t, t + \frac{\pi}{6})$  is first increasing then decreasing and satisfies the above inequality for its maximum that is approximatively at 0.6627206. From the properties of  $G$ , we infer that  $G(x, y) < \frac{2}{\pi}$  for all  $(x, y)$  such that  $y > x + \frac{\pi}{6}, y \in [\frac{1.17}{2}, \frac{\pi}{2}]$ ,  $x \in [0, \frac{\pi}{3}]$ , that is,  $\ell_1 > \frac{\pi}{6}$ . We have thus proved that  $A < \frac{2}{\pi}$  as soon as  $\ell_1 > \frac{\pi}{6}$ , that is a contradiction. Therefore we can assume from now on that  $\ell_1 \leq \frac{\pi}{6}$  when  $m_1 \leq \frac{\pi}{2}$ .

**(b):** Let us now assume that  $m_1 > \frac{\pi}{2}$ . Let us suppose first that  $m_1 \geq 1.9$ , therefore  $|\cos m_1| \geq |\cos 1.9|$  and, following (26) we infer

$$A(\ell_1, \ell_3, m_1) \leq \frac{\ell_1^2(2\pi - \ell_1 - \ell_3)}{12(\ell_1 + \ell_3)|\cos 1.9|}.$$

Now this expression is decreasing in  $\ell_3$  and increasing in  $\ell_1$ , thus it is always less than its value for  $\ell_1 = \ell_3 < \frac{\pi}{2}$ :

$$A(\ell_1, \ell_3, m_1) \leq A(\ell_1, \ell_1, 1.9) = \frac{\ell_1(\pi - \ell_1)}{12|\cos 1.9|} \leq \frac{\pi^2}{48|\cos 1.9|} < \frac{2}{\pi}$$

and the contradiction is obtained in this case. Therefore, we assume now  $\frac{\pi}{2} \leq m_1 \leq 1.9$ .

Expressing  $m_3$  in terms of  $m_1$ , we have  $m_3 = m_1 - \frac{\ell_-}{2} - \ell_0 \leq 1.9 - \frac{\ell_-}{2} - \pi$ , thus  $|\sin m_3| \leq \sin(1.9 - \frac{\ell_-}{2})$  (recall that  $-\pi \leq m_3 \leq -\frac{\pi}{2}$ ). By identity (36), we have

$$|\sin 1.9| \sin \frac{\ell_1}{2} \leq \sin m_1 \sin \frac{\ell_1}{2} = |\sin m_3| \sin \frac{\ell_3}{2} \leq \sin \frac{\ell_3}{2} \sin \left(1.9 - \frac{\ell_-}{2}\right).$$

This implies, by (38)

$$|\cos m_3| = \sqrt{1 - \frac{\sin^2 \frac{\ell_1}{2} \sin^2 m_1}{\sin^2 \frac{\ell_3}{2}}} \geq \sqrt{1 - \frac{\sin^2(1.9 - \frac{\ell_-}{2})}{\sin^2 1.9}}.$$

Therefore, using (26) we can estimate  $A$  from above by

$$A \leq \frac{\ell_3^2(2\pi - \ell_-)}{12\ell_- \sqrt{1 - \frac{\sin^2(1.9 - \frac{\ell_-}{2})}{\sin^2 1.9}}}.$$

As a function of  $\ell_-$  the right-hand side is clearly decreasing. Now, if  $\ell_1 \geq 0.62$  (and then  $\ell_- \geq 1.17 + 0.62$ ), we can see that  $A < \frac{2}{\pi}$  for any  $\ell_3 \in [1.17, \pi - \ell_1]$  giving the desired contradiction. Therefore, when  $m_1 > \frac{\pi}{2}$ , we can assume  $\ell_1 \leq 0.62$ .

We use now the integral of  $u$  on the interval  $I_3$ . We recall that

$$\int_{I_3} (-u(x)) dx = \frac{m\ell_+}{\pi} \left(2 \tan \frac{\ell_3}{2} - \ell_3\right) - \frac{\lambda_1}{2} \left[2 \sin \frac{\ell_3}{2} \cos m_3 \left(1 + \frac{\ell_3}{\sin \ell_3}\right)\right].$$

Moreover  $0 < \int_{I_3} (-u(x)) dx \leq \frac{1}{2}$  by (28). We use the same idea as before, using the expression of  $\lambda_1/2$  given in (37) and replacing  $\sin \frac{\ell_1}{2} \cos m_1$  by  $\pm \sqrt{\sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3}$  (with a + if  $m_1 \leq \frac{\pi}{2}$  and a - if  $\frac{\pi}{2} \leq m_1 \leq 1.9$ ), thanks to (36). This provides an expression of the integral as a function of the three variables  $\ell_1, \ell_3, m_3$ :  $\int_{I_3} (-u(x)) dx = \frac{m}{\pi} I(\ell_1, \ell_3, m_3)$ . More precisely, we have

$$\begin{aligned} I(\ell_1, \ell_3, m_3) &= (2\pi - \ell_1 - \ell_3) \left(2 \tan \frac{\ell_3}{2} - \ell_3\right) - \dots \\ &4 \left( \sin \frac{\ell_3}{2} \cos m_3 \pm \sqrt{\sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3} \right) \sin \frac{\ell_3}{2} \cos m_3 \left(1 + \frac{\ell_3}{\sin \ell_3}\right). \end{aligned}$$

We look first at the dependence with respect to  $m_3$ . The derivative of  $I(\ell_1, \ell_3, m_3)$  with respect to  $m_3$  has the sign of

$$\pm \sin m_3 \left( \left( \sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3 \right)^{1/4} \pm \frac{\sin \frac{\ell_3}{2} \cos m_3}{\left( \sin^2 \frac{\ell_1}{2} - \sin^2 \frac{\ell_3}{2} \sin^2 m_3 \right)^{1/4}} \right)^2.$$

Since  $-\pi \leq m_3 \leq -\frac{\pi}{2}$ ,  $\sin m_3$  is negative. Therefore  $m_3 \mapsto I(\ell_1, \ell_3, m_3)$  is decreasing if  $m_1 \leq \frac{\pi}{2}$  and increasing if  $m_1 \geq \frac{\pi}{2}$ . Now  $m_3$  varies between  $-\pi$  and its maximal value  $m_3^*$  that is such  $|\sin m_3| \sin \frac{\ell_3}{2} = \sin \frac{\ell_1}{2}$  by (36). Therefore

$$I(\ell_1, \ell_3, m_3) \leq \max \{I(\ell_1, \ell_3, -\pi), I(\ell_1, \ell_3, m_3^*)\}.$$

Let us study the last two functions. First of all, if  $m_1 \leq \frac{\pi}{2}$  we have

$$I(\ell_1, \ell_3, -\pi) = (2\pi - \ell_1 - \ell_3) \left( 2 \tan \frac{\ell_3}{2} - \ell_3 \right) + 4 \left( \sin \frac{\ell_1}{2} - \sin \frac{\ell_3}{2} \right) \sin \frac{\ell_3}{2} \left( 1 + \frac{\ell_3}{\sin \ell_3} \right).$$

The derivative with respect to  $\ell_1$  is  $\ell_3 - 2 \tan \frac{\ell_3}{2} + 2 \cos \frac{\ell_1}{2} \sin \frac{\ell_3}{2} \left( 1 + \frac{\ell_3}{\sin \ell_3} \right)$  that is decreasing in  $\ell_1$ , thus greater than its value for  $\ell_1 = \frac{\pi}{6}$  (recall that we can assume  $\ell_1 \leq \frac{\pi}{6}$ , as proved above):

$$\frac{\partial I(\ell_1, \ell_3, -\pi)}{\partial \ell_1} \geq \ell_3 - 2 \tan \frac{\ell_3}{2} + 2 \cos \frac{\pi}{12} \sin \frac{\ell_3}{2} \left( 1 + \frac{\ell_3}{\sin \ell_3} \right) > 0.$$

This shows that  $I(\ell_1, \ell_3, -\pi)$  is increasing in  $\ell_1$ . This implies that

- for  $1.17 \leq \ell_3 \leq \frac{5\pi}{6}$ ,  $I(\ell_1, \ell_3, -\pi) \leq I\left(\frac{\pi}{6}, \ell_3, -\pi\right)$ ;
- for  $\ell_3 \in \left[\frac{5\pi}{6}, \pi\right)$ ,  $I(\ell_1, \ell_3, -\pi) \leq I(\pi - \ell_3, \ell_3, -\pi)$ .

The functions on the right hand side of the above inequalities are negative. Thus we reach a contradiction with  $I(\ell_1, \ell_3, m_3) > 0$ .

In the case  $m_1 > \frac{\pi}{2}$  we consider

$$I(\ell_1, \ell_3, m_3^*) = (2\pi - \ell_1 - \ell_3) \left( 2 \tan \frac{\ell_3}{2} - \ell_3 \right) - 4 \left( \sin^2 \frac{\ell_3}{2} - \sin^2 \frac{\ell_1}{2} \right) \left( 1 + \frac{\ell_3}{\sin \ell_3} \right).$$

This is a convex function with respect to  $\ell_1$ . Therefore

$$I(\ell_1, \ell_3, m_3^*) \leq \max \{I(0, \ell_3, m_3^*), I(0.62, \ell_3, m_3^*)\}.$$

Now the functions  $\ell_3 \mapsto I(0, \ell_3, m_3^*)$  and  $\ell_3 \mapsto I(0.62, \ell_3, m_3^*)$  are decreasing and negative. We therefore arrive to a contradiction. This finishes the proof of that case and therefore, the whole proof of the theorem.  $\square$

#### 4. THE LAGRANGE MULTIPLIERS ARE ZERO AND THE NODAL DOMAINS HAVE SAME LENGTH

Now we enter into the heart of the paper. We are going to prove that the Lagrange multipliers  $\lambda_0$  and  $\lambda_1$  are zero (we already know that  $\lambda_2 = 0$ ) and that all the nodal domains have the same length. For that purpose, we will use the relation (19) on different intervals.

**Theorem 4.1.** *The Lagrange multipliers  $\lambda_0, \lambda_1$  are equal to zero and all the nodal intervals have the same length.*

The proof will be done in two main steps. First, we prove that  $\lambda_1 = 0$  in Proposition 4.3. Then, we prove that  $\lambda_0 = 0$  and the nodal domains have same length in Proposition 4.4.

Let us first introduce some notations and give a preliminary lemma. Let  $I_k = [a_{k-1}, a_k]$  and  $I_{k+1} = [a_k, a_{k+1}]$  be two consecutive intervals of length respectively  $\ell_k, \ell_{k+1}$ . We introduce:

$$A(I_k, I_{k+1}) = \frac{\ell_k}{\sin \ell_k} \sin a_{k-1} - \frac{\ell_{k+1}}{\sin \ell_{k+1}} \sin a_{k+1}.$$

Note that, using  $a_{k-1} = a_k - \ell_k$  and  $a_{k+1} = a_k + \ell_{k+1}$  we can also write

$$A(I_k, I_{k+1}) = \left( \frac{\ell_k}{\tan \ell_k} - \frac{\ell_{k+1}}{\tan \ell_{k+1}} \right) \sin a_k - (\ell_k + \ell_{k+1}) \cos a_k. \quad (41)$$

**Lemma 4.2.** *There exist three consecutive intervals  $I_j, I_{j+1}, I_{j+2}$  such that  $A(I_j, I_{j+1}) \geq 0$ ,  $A(I_{j+1}, I_{j+2}) \geq 0$  and there exist three consecutive intervals  $I_i, I_{i+1}, I_{i+2}$  such that  $A(I_i, I_{i+1}) < 0$ ,  $A(I_{i+1}, I_{i+2}) < 0$ .*

*Proof.* Let us consider  $I_i = (a_{i-1}, a_i), I_{i+1} = (a_i, a_{i+1}), I_{i+2} = (a_{i+1}, a_{i+2})$ , with  $a_i < 0 < a_{i+1}$ . Without loss of generality we can assume that  $I_i \cup I_{i+1} \cup I_{i+2} \subset [-\pi, \pi]$  (up to consider  $u(-x)$  instead of  $u(x)$ ). Since  $\sin a_{i-1} < 0$  and  $\sin a_{i+1} > 0$

$$A(I_i, I_{i+1}) = \frac{\ell_i}{\sin \ell_i} \sin a_{i-1} - \frac{\ell_{i+1}}{\sin \ell_{i+1}} \sin a_{i+1} < 0.$$

Since  $\sin a_i < 0$  and  $\sin a_{i+2} > 0$

$$A(I_{i+1}, I_{i+2}) = \frac{\ell_{i+1}}{\sin \ell_{i+1}} \sin a_i - \frac{\ell_{i+2}}{\sin \ell_{i+2}} \sin a_{i+2} < 0.$$

Let us consider  $I_j = (a_{j-1}, a_j), I_{j+1} = (a_j, a_{j+1}), I_{j+2} = (a_{j+1}, a_{j+2})$ , with  $a_j < -\pi < a_{j+1}$ . Assume that  $I_j \cup I_{j+1} \cup I_{j+2} \subset [-2\pi, 0]$ . Since  $\sin a_{j-1} > 0$  and  $\sin a_{j+1} < 0$

$$A(I_j, I_{j+1}) = \frac{\ell_j}{\sin \ell_j} \sin a_{j-1} - \frac{\ell_{j+1}}{\sin \ell_{j+1}} \sin a_{j+1} > 0.$$

Since  $\sin a_j > 0$  and  $\sin a_{j+2} < 0$

$$A(I_{j+1}, I_{j+2}) = \frac{\ell_{j+1}}{\sin \ell_{j+1}} \sin a_j - \frac{\ell_{j+2}}{\sin \ell_{j+2}} \sin a_{j+2} > 0.$$

□

**Proposition 4.3.** *The Lagrange multiplier  $\lambda_1$  is zero.*

*Proof.* Let  $I_k = [a_{k-1}, a_k], I_{k+1} = [a_k, a_{k+1}], I_{k+2} = [a_{k+1}, a_{k+2}], I_{k+3} = [a_{k+2}, a_{k+3}]$  four consecutive intervals of lengths  $\ell_k, \ell_{k+1}, \ell_{k+2}, \ell_{k+3}$ , respectively. We assume that  $u$  is alternatively positive, negative, positive and negative. In Section 2 we have seen that (see (19))

$$\lambda_0 \sin \frac{\ell_k + \ell_{k+1}}{2} - m \sin \frac{\ell_{k+1} - \ell_k}{2} = \frac{\lambda_1}{2} \cos \frac{\ell_k}{2} \cos \frac{\ell_{k+1}}{2} A(I_k, I_{k+1}). \quad (42)$$

We can reproduce this identity for the other intervals:

$$\lambda_0 \sin \frac{\ell_{k+1} + \ell_{k+2}}{2} + m \sin \frac{\ell_{k+2} - \ell_{k+1}}{2} = \frac{\lambda_1}{2} \cos \frac{\ell_{k+1}}{2} \cos \frac{\ell_{k+2}}{2} A(I_{k+1}, I_{k+2}) \quad (43)$$

$$\lambda_0 \sin \frac{\ell_{k+2} + \ell_{k+3}}{2} - m \sin \frac{\ell_{k+3} - \ell_{k+2}}{2} = \frac{\lambda_1}{2} \cos \frac{\ell_{k+2}}{2} \cos \frac{\ell_{k+3}}{2} A(I_{k+2}, I_{k+3}). \quad (44)$$

Assume by contradiction that  $\lambda_1 \neq 0$ . We divide the proof into three cases, according to the lengths of the nodal intervals.

- (1) Let us assume that  $\ell_k \neq \ell_{k+2}$  and  $\ell_{k+1} \neq \ell_{k+3}$ . Equations (42), (43) can be seen as a system in  $\lambda_0$  and  $m$  from which we get

$$\lambda_0 = \frac{\lambda_1}{2} \cos \frac{\ell_{k+1}}{2} \frac{\cos \frac{\ell_k}{2} \sin \frac{\ell_{k+2} - \ell_{k+1}}{2} A(I_k, I_{k+1}) + \cos \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} - \ell_k}{2} A(I_{k+1}, I_{k+2})}{\sin \ell_{k+1} \sin \frac{\ell_{k+2} - \ell_k}{2}}$$

$$m = \frac{\lambda_1}{2} \cos \frac{\ell_{k+1}}{2} \frac{\cos \frac{\ell_k}{2} \sin \frac{\ell_{k+2} + \ell_{k+1}}{2} A(I_k, I_{k+1}) + \cos \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} + \ell_k}{2} A(I_{k+1}, I_{k+2})}{\sin \ell_{k+1} \sin \frac{\ell_{k+2} - \ell_k}{2}}.$$

We observe that

$$\lambda_0 + m = \frac{\lambda_1}{2} \frac{\cos \frac{\ell_k}{2} \cos \frac{\ell_{k+2}}{2}}{\sin \frac{\ell_{k+2} - \ell_k}{2}} [A(I_{k+1}, I_{k+2}) - A(I_k, I_{k+1})]. \quad (45)$$

Similarly, if one chooses equations (43), (44) to solve with respect to  $\lambda_0, m$ , he gets

$$\lambda_0 + m = \frac{\lambda_1}{2} \frac{\cos \frac{\ell_{k+2}}{2}}{\sin \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+3} - \ell_{k+1}}{2}} \left[ \cos \frac{\ell_{k+1}}{2} \sin \frac{\ell_{k+3}}{2} A(I_{k+1}, I_{k+2}) - \cos \frac{\ell_{k+3}}{2} \sin \frac{\ell_2}{2} A(I_{k+2}, I_{k+3}) \right]. \quad (46)$$

We now set  $C = \frac{2(m + \lambda_0)}{\lambda_1}$ . We use (45) to get  $A(I_{k+1}, I_{k+2})$  in terms of  $A(I_k, I_{k+1})$ :

$$A(I_{k+1}, I_{k+2}) = A(I_k, I_{k+1}) + \frac{C \sin \frac{\ell_{k+2} - \ell_k}{2}}{\cos \frac{\ell_k}{2} \cos \frac{\ell_{k+2}}{2}}. \quad (47)$$



We also use (46) to get  $A(I_{k+1}, I_{k+2})$  in terms of  $A(I_{k+2}, I_{k+3})$ :

$$A(I_{k+1}, I_{k+2}) = \frac{\tan \frac{\ell_{k+1}}{2}}{\tan \frac{\ell_{k+3}}{2}} A(I_{k+2}, I_{k+3}) + \frac{C \tan \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+3} - \ell_{k+1}}{2}}{\sin \frac{\ell_{k+3}}{2} \cos \frac{\ell_{k+1}}{2}}. \quad (48)$$

Since  $m$  is non-zero, the  $3 \times 3$  determinant of the system in  $(m, \lambda_0, \lambda_1)$  given by equations (42), (43), (44) has to be equal to zero. Now, the computation of this determinant with respect to its third column gives the following equality after some simplification:

$$A(I_k, I_{k+1}) \cos \frac{\ell_k}{2} \cos \frac{\ell_{k+1}}{2} \sin \ell_{k+2} \sin \frac{\ell_{k+1} - \ell_{k+3}}{2} + A(I_{k+2}, I_{k+3}) \cos \frac{\ell_{k+2}}{2} \cos \frac{\ell_{k+3}}{2} \sin \ell_{k+1} \sin \frac{\ell_{k+2} - \ell_k}{2} - A(I_{k+1}, I_{k+2}) \cos \frac{\ell_{k+1}}{2} \cos \frac{\ell_{k+2}}{2} \left( \sin \frac{\ell_{k+1} - \ell_{k+3}}{2} \sin \frac{\ell_k + \ell_{k+2}}{2} + \sin \frac{\ell_{k+2} - \ell_k}{2} \sin \frac{\ell_{k+1} + \ell_{k+3}}{2} \right) = 0. \quad (49)$$

Now we replace  $A(I_{k+1}, I_{k+2})$  in (49) by using both (47) (for the first term), (48) (for the second) and we get, after use of trigonometric formulae

$$0 = A(I_k, I_{k+1}) \cos \frac{\ell_{k+1}}{2} \cos \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} - \ell_{k+3}}{2} \sin \frac{\ell_{k+2} - \ell_k}{2} + A(I_{k+2}, I_{k+3}) \cos \frac{\ell_{k+2}}{2} \cos \frac{\ell_{k+3}}{2} \sin \frac{\ell_{k+3} - \ell_{k+1}}{2} \sin \frac{\ell_{k+2} - \ell_k}{2} \frac{\sin \frac{\ell_{k+1}}{2}}{\sin \frac{\ell_{k+3}}{2}} - C \frac{\sin \frac{\ell_{k+1} - \ell_{k+3}}{2} \sin \frac{\ell_{k+2} - \ell_k}{2}}{\sin \frac{\ell_{k+3}}{2} \cos \frac{\ell_k}{2}} \left[ \cos \frac{\ell_{k+1}}{2} \sin \frac{\ell_{k+3}}{2} \sin \frac{\ell_k + \ell_{k+2}}{2} + \cos \frac{\ell_k}{2} \sin \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} + \ell_{k+3}}{2} \right]. \quad (50)$$

Simplifying by  $\sin \frac{\ell_{k+1} - \ell_{k+3}}{2} \sin \frac{\ell_{k+2} - \ell_k}{2}$  (this is possible since we are assuming  $\ell_{k+2} \neq \ell_k$  and  $\ell_{k+3} \neq \ell_{k+1}$ ) we finally get

$$\frac{C}{\sin \frac{\ell_{k+3}}{2} \cos \frac{\ell_k}{2}} \left[ \cos \frac{\ell_{k+1}}{2} \sin \frac{\ell_{k+3}}{2} \sin \frac{\ell_k + \ell_{k+2}}{2} + \cos \frac{\ell_k}{2} \sin \frac{\ell_{k+2}}{2} \sin \frac{\ell_{k+1} + \ell_{k+3}}{2} \right] = A(I_k, I_{k+1}) \cos \frac{\ell_{k+1}}{2} \cos \frac{\ell_{k+2}}{2} - A(I_{k+2}, I_{k+3}) \frac{\cos \frac{\ell_{k+2}}{2} \cos \frac{\ell_{k+3}}{2} \sin \frac{\ell_{k+1}}{2}}{\sin \frac{\ell_{k+3}}{2}}. \quad (51)$$

Note that  $C$  has the same sign as  $\lambda_1$ , since by definition of  $\lambda_0$  (see section 2),  $\lambda_0 + m > 0$ . Moreover, in Equation (51), the coefficients of  $C$ ,  $A(I_k, I_{k+1})$  and  $-A(I_{k+2}, I_{k+3})$  are all positive. Now we claim that we can choose four consecutive intervals such that

- $A(I_k, I_{k+1})$  is positive and  $A(I_{k+2}, I_{k+3})$  is negative (with  $u$  positive on  $I_k$ ),
- and we can choose four intervals such that
- $A(I_j, I_{j+1})$  is negative and  $A(I_{j+2}, I_{j+3})$  is positive (with  $u$  positive on  $I_j$ ).

If this claim is true, we get a contradiction since (51) would show that  $C$  (and then  $\lambda_1$ ) is both positive and negative. To prove our claim, we set

$$\mathcal{I}_- = \{(I_k, I_{k+1}) : A(I_k, I_{k+1}) < 0\}, \quad \mathcal{I}_+ = \{(I_k, I_{k+1}) : A(I_k, I_{k+1}) \geq 0\}.$$

We have seen in Lemma 4.2 that both  $\mathcal{I}_-$  and  $\mathcal{I}_+$  contain pairs of consecutive intervals (or triplet of intervals). Let us now consider the last triplet of intervals for which  $A < 0$ . Let  $I_{k-1}, I_k, I_{k+1}$  be these three intervals. Therefore  $A(I_{k+1}, I_{k+2}) \geq 0$ . If  $u > 0$  on  $I_{k-1}$  we are done, because we can consider  $(I_{k-1}, I_k), (I_{k+1}, I_{k+2})$ . If  $u$  is negative on  $I_{k-1}$  we have  $u$  negative on  $I_{k-1}, I_{k+1}, I_{k+3} \dots$  and positive on  $I_k, I_{k+2}, I_{k+4} \dots$ . We can consider  $(I_k, I_{k+1})$  for which  $A < 0$ . If  $A(I_{k+2}, I_{k+3}) \geq 0$  we are done. If  $A(I_{k+2}, I_{k+3}) \leq 0$ , then  $A(I_{k+3}, I_{k+4}) \geq 0$  (otherwise the last triplet in  $\mathcal{I}_-$  would be  $I_{k+2}, I_{k+3}, I_{k+4}$ ). If  $A(I_{k+4}, I_{k+5}) \geq 0$  we are done. If  $A(I_{k+4}, I_{k+5}) \leq 0$ , then  $A(I_{k+5}, I_{k+6}) \geq 0 \dots$  after some steps we will get necessarily four consecutive intervals  $I_{m-2}, I_{m-1}, I_m, I_{m+1}$  (with  $u$  positive on  $I_{m-2}$ ) such that  $A(I_{m-2}, I_{m-1}) < 0$ ,  $A(I_m, I_{m+1}) \geq 0$  (because we have to stop before the first triplet of  $\mathcal{I}_+$ ). Therefore, we have proved the first part of our claim. The second part is proved exactly in the same way, starting from the last triplet in  $\mathcal{I}_+$ .

In conclusion, we get a contradiction.

- (2) Assume  $\ell_k = \ell_{k+2}$ . Since the left-hand sides of equations (42) and (43) coincide, the right-hand sides are equal that implies necessarily (since we are assuming  $\lambda_1 \neq 0$ )

$$\frac{\ell_k}{\sin \ell_k} (\sin a_{k-1} + \sin a_{k+2}) = \frac{\ell_{k+1}}{\sin \ell_{k+1}} (\sin a_k + \sin a_{k+1}).$$

Now, we observe that since  $\ell_k = \ell_{k+2}$ , one has  $(a_{k-1} + a_{k+2})/2 = (a_k + a_{k+1})/2$ . Replacing  $\sin a_{k-1} + \sin a_{k+2}$  by  $2 \sin(a_{k-1} + a_{k+2})/2 \cos(a_{k+2} - a_{k-1})/2$  and  $\sin a_k + \sin a_{k+1}$  by  $2 \sin(a_k + a_{k+1})/2 \cos(a_{k+1} - a_k)/2$  the above equality gives

$$\frac{\ell_k}{\sin \ell_k} \cos(\ell_k + \frac{\ell_{k+1}}{2}) = \frac{\ell_{k+1}}{\sin \ell_{k+1}} \cos \frac{\ell_{k+1}}{2} = \frac{\ell_{k+1}}{2 \sin \frac{\ell_{k+1}}{2}}. \quad (52)$$

Assuming  $\ell_k$  fixed, we can study the function

$$g : x \mapsto \ell_k \sin x \cos(\ell_k + x) - x \sin \ell_k.$$

Since  $g'(x)$  is negative and  $g(0) = 0$ , it is not possible to find  $\ell_{k+1} > 0$  such that (52) holds. Therefore we have a contradiction.

(3) Assuming  $\ell_{k+1} = \ell_{k+3}$  we get a contradiction in the same way as in the previous case ( $\ell_k = \ell_{k+2}$ ). Therefore, we conclude that necessarily  $\lambda_1 = 0$ .  $\square$

To finish the proof of Theorem 4.1, we need the following proposition.

**Proposition 4.4.** *The Lagrange multiplier  $\lambda_0$  is zero and the nodal domains have same length.*

*Proof.* Since  $\lambda_1 = 0$  by the previous proposition, from section 2 we deduce that

$$\begin{aligned} \lambda_0 \sin\left(\frac{\ell_k + \ell_{k+1}}{2}\right) - m \sin\left(\frac{\ell_{k+1} - \ell_k}{2}\right) &= 0, \\ \lambda_0 \sin\left(\frac{\ell_{k+1} + \ell_{k+2}}{2}\right) + m \sin\left(\frac{\ell_{k+2} - \ell_{k+1}}{2}\right) &= 0. \end{aligned} \quad (53)$$

The determinant of this homogeneous system is zero, as  $m \neq 0$ . This means

$$\sin\left(\frac{\ell_{k+1} - \ell_k}{2}\right) \sin\left(\frac{\ell_{k+1} + \ell_{k+2}}{2}\right) = \sin\left(\frac{\ell_{k+1} - \ell_{k+2}}{2}\right) \sin\left(\frac{\ell_k + \ell_{k+1}}{2}\right)$$

that is,

$$\cos\left(\frac{2\ell_{k+1} - \ell_k + \ell_{k+2}}{2}\right) = \cos\left(\frac{2\ell_{k+1} + \ell_k - \ell_{k+2}}{2}\right)$$

which implies  $\ell_k - \ell_{k+2} = -\ell_k + \ell_{k+2}$ , that is,  $\ell_k = \ell_{k+2}$ . With the same argument on  $\ell_{k+1}, \ell_{k+2}, \ell_{k+3}$  we find  $\ell_{k+1} = \ell_{k+3}$ .

Therefore all the intervals where  $u$  is positive have the same length, say  $\ell_1$ ; all the intervals where  $u$  is negative have the same length, say  $\ell_2$ . The sum of these lengths give  $n(\ell_1 + \ell_2) = 2\pi$ .

On the other hand,  $\lambda_0 = -\frac{m}{2\pi} \int_0^{2\pi} \text{sign}(u) = -\frac{m}{2\pi} n(\ell_1 - \ell_2)$  which gives us

$$\ell_2 - \ell_1 = \frac{2\pi\lambda_0}{mn}. \quad (54)$$

If we replace this equality in (53), we have

$$\lambda_0 \sin\left(\frac{\pi}{n}\right) = m \sin\left(\frac{\pi\lambda_0}{mn}\right).$$

We now study the function  $f(x) = m \sin\left(\frac{\pi x}{mn}\right) - x \sin\left(\frac{\pi}{n}\right)$ , for  $x \in [0, m)$  (recall that  $\lambda_0 < m$ ). Since  $f(0) = 0 = f(m)$ ,  $f'(0) > 0$ ,  $f'(m) < 0$  and  $f''(x) < 0$ , we deduce that the only zero on  $f$  is zero. This means that  $f(\lambda_0) = 0$  if and only if  $\lambda_0 = 0$ . This argument proves that  $\lambda_0 = 0$ . We deduce from (54) that  $\ell_1 = \ell_2$ .  $\square$

## 5. CONCLUSION

We are now in position to prove Theorem 1.1.

We have seen in the previous section that all the nodal intervals have same length, say  $\ell$ , and the Lagrange multipliers  $\lambda_0, \lambda_1, \lambda_2$  are all zero. We recall that if  $(a, b)$  is an interval where  $u \geq 0$ , one has

$$u(x) = A_0 \cos x + B_0 \sin x - m, \quad A_0 = m \frac{\cos(\frac{a+b}{2})}{\cos \frac{\ell}{2}}, \quad B_0 = m \frac{\sin(\frac{a+b}{2})}{\cos \frac{\ell}{2}};$$

if  $(b, c)$  is an interval where  $u \leq 0$ , one has

$$u(x) = A_1 \cos x + B_1 \sin x + m, \quad A_1 = -m \frac{\cos(\frac{b+c}{2})}{\cos \frac{\ell}{2}}, \quad B_1 = -m \frac{\sin(\frac{b+c}{2})}{\cos \frac{\ell}{2}}$$

(see subsection 2.2). The solution of the system

$$\begin{cases} u(x) = A_0 \cos x + B_0 \sin x - m = 0 \\ u(x) = A_1 \cos x + B_1 \sin x + m = 0 \end{cases}$$

is

$$\begin{cases} \cos(x - \varphi_0) = \cos \frac{\ell}{2}, \quad \tan(\varphi_0) = \tan\left(\frac{a+b}{2}\right) \\ \cos(x - \varphi_1) = \cos \frac{\ell}{2}, \quad \tan(\varphi_1) = \tan\left(\frac{b+c}{2}\right) \end{cases}$$

with  $x - \varphi_0 = \pm(\pi + x - \varphi_1) + 2k\pi$ . Since  $x = a + \ell$ , the only possible solution is  $\ell = \frac{\pi}{2}$ . Therefore  $u(x)$  is symmetric with respect to  $a$ , and

$$u(x) = \begin{cases} m \cos x + m \sin x - m, & x \in [a, a + \frac{\pi}{2}] \\ -m \cos x - m \sin x + m, & x \in [a + \frac{\pi}{2}, a + \pi] \end{cases} .$$

We now compute  $m$  defined in (5). Recalling that  $\int_0^1 |u| = 1$  (see (9)), we have

$$\left[ \int_0^{2\pi} |u| \right]^2 = 16 \left[ \int_a^{a+\frac{\pi}{2}} |u| \right]^2 = 16m^2 \left( 2 - \frac{\pi}{2} \right)^2 = 1 .$$

Therefore  $m = \frac{1}{2(4 - \pi)} \approx 0.58$  and  $\frac{\pi}{4}m \approx 0.457$ .

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*G. Croce:* NORMANDIE UNIV, FRANCE; ULH, LMAH, F-76600 LE HAVRE; FR CNRS 3335, 25 RUE PHILIPPE LEBON, 76600 LE HAVRE, FRANCE.

*E-mail address:* gisella.croce@univ-lehavre.fr

*A. Henrot:* INSTITUT ÉLIE CARTAN DE LORRAINE UMR CNRS 7502, UNIVERSITÉ DE LORRAINE, BP 70239 54506 VANDOEUVRE-LES-NANCY CEDEX, FRANCE.

*E-mail address:* antoine.henrot@univ-lorraine.fr