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# Isomorphic Boolean networks and dense interaction graphs

Aymeric Picard Marchetto\* and Adrien Richard\*

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## Abstract

A Boolean network (BN) with  $n$  components is a discrete dynamical system described by the successive iterations of a function  $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$ . In most applications, the main parameter is the interaction graph of  $f$ : the digraph with vertex set  $\{1, \dots, n\}$  that contains an arc from  $j$  to  $i$  if  $f_i$  depends on input  $j$ . What can be said on the set  $\mathcal{G}(f)$  of the interaction graphs of the BNs  $h$  isomorphic to  $f$ , that is, such that  $h \circ \pi = \pi \circ f$  for some permutation  $\pi$  of  $\{0, 1\}^n$ ? It seems that this simple question has never been studied. Here, we report some basic facts. First, if  $n \geq 5$  and  $f$  is neither the identity or constant, then  $\mathcal{G}(f)$  is of size at least two and contains the complete digraph on  $n$  vertices, with  $n^2$  arcs. Second, for any  $n \geq 1$ , there are  $n$ -component BNs  $f$  such that every digraph in  $\mathcal{G}(f)$  has at least  $n^2/9$  arcs.

## 1 Introduction

A *Boolean network* (network for short) with  $n$  components is a finite dynamical system defined by the successive iterations of a function

$$f : \{0, 1\}^n \rightarrow \{0, 1\}^n, \quad x = (x_1, \dots, x_n) \mapsto f(x) = (f_1(x), \dots, f_n(x)).$$

Boolean networks have many applications; in particular, they are omnipresent in the modeling of neural and gene networks (see [4] for a review).

The “network” terminology comes from the fact that the *interaction graph*  $G(f)$  of  $f$  is often considered as the main parameter of  $f$ : the vertex set is  $[n] = \{1, \dots, n\}$  and there is an arc from  $j$  to  $i$  if  $f_i$  depends on input  $j$ , that is, if there are  $x, y \in \{0, 1\}^n$  which only differ in  $x_j \neq y_j$  such that  $f_i(x) \neq f_i(y)$ . For instance, in the context of gene networks, the interaction graph is often well approximated while the actual dynamics is not. One is thus faced with the following question: *what can be said on the dynamics*

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described by  $f$  from  $G(f)$  only. There are many results in this direction (see [1] for a review). In most cases, the studied dynamical properties are invariant by isomorphism: number of fixed or periodic points, number and size of limit cycles, transient length and so on. However, the interaction graph is *not* invariant by isomorphism: even if  $f$  and  $h$  are isomorphic, their interaction graphs can be very different (by Theorem 1 below,  $G(f)$  can have  $n^2$  arcs while  $G(h)$  has a single arc). Surprisingly, this variation seems to have never been studied, and we report here some basic facts.

Given a network  $f$ , let  $\mathcal{G}(f)$  be the set of interaction graphs  $G(h)$  such that  $h$  is a network isomorphic to  $f$ , that is, such that  $h \circ \pi = \pi \circ f$  for some permutation  $\pi$  of  $\{0, 1\}^n$ . Hence, we propose to study  $\mathcal{G}(f)$ . For instance, if  $f$  is constant, we write this  $f = \text{cst}$ , then  $\mathcal{G}(f)$  contains a single digraph, the digraph on  $[n]$  without arcs, and if  $f$  is the identity, we write this  $f = \text{id}$ , then  $\mathcal{G}(f)$  also contains a single digraph, the digraph on  $[n]$  with  $n$  loops (cycles of length one) and no other arcs.

Our first result shows that, excepted few exceptions (including the two above examples),  $\mathcal{G}(f)$  always contains the complete digraph on  $[n]$ , with  $n^2$  arcs, denoted  $K_n$ .

**Theorem 1.**  $K_n \in \mathcal{G}(f)$  for all networks  $f \neq \text{cst}, \text{id}$  with  $n \geq 5$  components.

Our second result shows that, for  $n \geq 5$ , there are no networks  $f$  such that  $\mathcal{G}(f)$  only contains the complete digraph.

**Theorem 2.**  $\mathcal{G}(f) \neq \{K_n\}$  for all networks  $f$  with  $n \geq 5$  components.

From these theorems we deduce the following property, which might seem innocent but that doesn't seem to have a one-line proof: *If  $f$  is a network with  $n \geq 5$  components, then  $|\mathcal{G}(f)| = 1$  if and only if  $f = \text{id}$  or  $f = \text{cst}$ .*

Even if, for  $n \geq 5$ ,  $\mathcal{G}(f)$  cannot only contain the complete digraph, using a well-known isoperimetric inequality in hypercubes, we show that, at least,  $\mathcal{G}(f)$  can only contain digraphs with many arcs.

**Theorem 3.** For every  $n \geq 1$ , there is a network  $f$  with  $n$  components such that every digraph in  $\mathcal{G}(f)$  has at least  $n^2/9$  arcs.

Concerning short term perspectives, we checked by computer that the first theorem holds for  $n = 3$  (for  $n = 2$  it fails; see the appendix) and that the second holds for  $n = 2, 3$ . For  $n = 4$  this is much time-consuming and we didn't do it, hoping to find instead dedicated arguments, since those given for  $n \geq 5$  do not work. A more involved perspective, related to the third theorem, is to prove that, for any  $\epsilon > 0$  and  $n$  large enough, any digraph in  $\mathcal{G}(f)$  has at least  $(1 - \epsilon)n^2$  arcs for some  $f$ . We also want to study networks  $f$  such that  $\mathcal{G}(f)$  is very large and ask: do we have, for any  $\epsilon > 0$  and  $n$  large enough,  $|\mathcal{G}(f)|/2^{n^2} > 1 - \epsilon$  for some  $f$ ?

The three theorems above are proved in the following three sections. Before going on, we give some basic definitions. An element  $x$  of  $\{0, 1\}^n$  is a

configuration, and elements in  $[n]$  are *components*. We set  $\mathbf{1}(x) = \{i \in [n] \mid x_i = 1\}$  and  $\mathbf{0}(x) = [n] \setminus \mathbf{1}(x)$ . The *weight* of  $x$  is  $w(x) = |\mathbf{1}(x)|$ . We denote by  $e_i$  the configuration such that  $\mathbf{1}(e_i) = \{i\}$ . For  $x, y \in \{0, 1\}^n$ , the sum  $x + y$  is applied component-wise modulo two. Hence  $x$  and  $x + e_i$  only differ in component  $i$ . We denote by  $\mathbf{1}$  (resp.  $\mathbf{0}$ ) the configuration of weight  $n$  (resp.  $0$ ). Thus  $x$  and  $x + \mathbf{1}$  differ in every component. Given  $A \subseteq \{0, 1\}^n$ , we set  $A + x = \{a + x \mid a \in A\}$ . Let  $f$  be a  $n$ -component network. A *fixed point* is a configuration  $x$  such that  $f(x) = x$ . Let  $\Gamma(f)$  be the digraph with vertex set  $\{0, 1\}^n$  and an arc from  $x$  to  $f(x)$  for every  $x \in \{0, 1\}^n$ . A *limit cycle* of  $f$  is a cycle of  $\Gamma(f)$ . Hence fixed points correspond to limit cycles of length one. An *independent set* of  $f$  is an independent set of  $\Gamma(f)$ , equivalently, it is a set  $A \subseteq \{0, 1\}^n$  such that  $f(A) \cap A = \emptyset$ .

## 2 Proof of Theorem 1

We proceed by showing that, for  $n \geq 5$  and  $f \neq \text{cst}, \text{id}$ , we have  $K_n \in \mathcal{G}(f)$  if at least one of the following three conditions holds:  $f$  has at least  $2n$  fixed points; or  $f$  has at least  $n$  limit cycles of length  $\geq 3$ ; or  $f$  has an independent set of size  $\geq 2n$ . We then prove that, because  $n \geq 5$ , at least one of the three conditions holds, and Theorem 1 follows.

**Lemma 1.** *Let  $f$  be a network with  $n$  components, which is not the identity. If  $f$  has at least  $2n$  fixed points then  $K_n \in \mathcal{G}(f)$ .*

*Proof.* Since  $f \neq \text{id}$  we have  $f(c) \neq c$  for some  $c$ , and since  $f$  has at least  $2n$  fixed points, it has  $2n - 1$  fixed points distinct from  $f(c)$ , say  $a^0, a^1, \dots, a^n, b^3, \dots, b^n$ . Let  $\pi$  be any permutation of  $\{0, 1\}^n$  such that  $\pi(a^0) = \mathbf{0}$ ,  $\pi(a^i) = e_i$  for  $1 \leq i \leq n$ ,  $\pi(b^i) = e_1 + e_2 + e_i$  for  $3 \leq i \leq n$ ,  $\pi(c) = e_1 + e_2$  and  $\pi(f(c)) = e_1 + e_2 + \mathbf{1}$ . Let  $h = \pi \circ f \circ \pi^{-1}$ . We will prove that  $G(h) = K_n$ . For  $i \in [n]$ , we have  $h(\mathbf{0}) = \pi(f(a^0)) = \pi(a^0) = \mathbf{0}$  and  $h(e_i) = \pi(f(a^i)) = \pi(a^i) = e_i$ , hence  $h(\mathbf{0})$  and  $h(e_i)$  differ in component  $i$ , and we deduce that  $G(h)$  has an arc from  $i$  to itself. It remains to prove that  $G(h)$  has an arc from  $i$  to  $j$  for distinct  $i, j \in [n]$ . We have  $h(e_1 + e_2) = \pi(f(c)) = e_1 + e_2 + \mathbf{1}$ . Hence  $h(e_2) = e_2$  and  $h(e_1 + e_2)$  differ in every component  $j \neq 1$ , and thus  $G(h)$  has an arc from  $1$  to every  $j \neq 1$ . We prove similarly that  $G(h)$  has an arc from  $2$  to every  $j \neq 2$ . For  $3 \leq i \leq n$ , we have  $h(e_1 + e_2 + e_i) = \pi(f(b^i)) = \pi(b^i) = e_1 + e_2 + e_i$ , so  $h(e_1 + e_2 + e_i)$  and  $h(e_1 + e_2)$  differ in every component  $j \neq i$ , and we deduce that  $G(h)$  has an arc from  $i$  to every  $j \neq i$ .  $\square$

**Lemma 2.** *Let  $f$  be a network with  $n$  components. If  $f$  has at least  $n$  limit cycles of length  $\geq 3$ , then  $K_n \in \mathcal{G}(f)$ .*

*Proof.* Suppose that  $f$  has  $n$  limit cycles of length  $\geq 3$ ; this implies  $n \geq 4$ . Let  $a^1, \dots, a^n$  be configurations inside distinct limit cycles of  $f$  of length

$\geq 3$ . For  $i \in [n]$  let  $b^i = f(a^i)$  and  $c^i = f(b^i)$ . Then  $a^1, \dots, a^n, b^1, \dots, b^n, c^1, \dots, c^n$  are all distinct. For  $i \in [n]$ , let  $x^i = e_{i-1} + e_i$ ,  $y^i = e_{i-1}$  and  $z^i = e_{i-1} + \mathbf{1}$ , where  $e_0$  means  $e_n$ . Since  $n \geq 4$ , the configurations  $x^1, \dots, x^n, y^1, \dots, y^n, z^1, \dots, z^n$  are all distinct. Hence there is a permutation  $\pi$  of  $\{0, 1\}^n$  such that, for  $i \in [n]$ ,  $\pi(a^i) = x^i$ ,  $\pi(b^i) = y^i$  and  $\pi(c^i) = z^i$ . Let  $h = \pi \circ f \circ \pi^{-1}$ . For  $i \in [n]$  we have  $h(e_{i-1} + e_i) = h(x^i) = \pi(f(a^i)) = \pi(b^i) = y^i = e_{i-1}$  and  $h(e_{i-1}) = h(y^i) = \pi(f(b^i)) = \pi(c^i) = z^i = e_{i-1} + \mathbf{1}$ . Hence  $h(e_{i-1} + e_i)$  and  $h(e_{i-1})$  differ in every component, and since  $e_{i-1} + e_i$  and  $e_{i-1}$  only differ in component  $i$ , we deduce that  $G(h)$  has an arc from  $i$  to every  $j \in [n]$ . Thus  $G(h) = K_n$ .  $\square$

**Lemma 3.** *Let  $f$  be a non-constant network with  $n \geq 5$  components. If  $f$  has an independent set of size  $\geq 2n$ , then  $K_n \in \mathcal{G}(f)$ .*

*Proof.* Let  $A$  be an independent set of  $f$ . We first prove:

(1) *If  $|A| \geq n + k$  and  $|f(A)| = 2k$  for some  $1 \leq k \leq n$ , then  $K_n \in \mathcal{G}(f)$ .*

Suppose these condition holds. One easily check that there is an independent set  $A$  with  $|A| = n + k$  and  $|f(A)| = 2k$  for some  $1 \leq k \leq n$ . Let us write  $f(A) = \{a^1, \dots, a^{2k}\}$ , and let  $A_p = f^{-1}(a^p) \cap A$  for  $p \in [2k]$ . Let  $X_1, \dots, X_{2k}$  be disjoint subsets of  $\{0, 1\}^n$  of size  $|A_1|, \dots, |A_{2k}|$  such that, for all  $i \in [n]$ , there is  $p \in [k]$  and  $x \in X_{2p-1}$  with  $x + e_i \in X_{2p}$ ; that these sets exist is the “technical” part of the proof, given by Lemma 8 in appendix.

Let  $X = X_1 \cup \dots \cup X_{2k}$ , let  $Y$  be the set of  $y \in \{0, 1\}^n$  with  $y_1 = 0$ , and let  $Y'$  be the set of  $y \in Y$  such that  $y, y + \mathbf{1} \notin X$ . Since  $n \geq 5$  and  $n \geq k$ :

$$|Y'| \geq |Y| - |X| = 2^{n-1} - (n + k) \geq 2^{n-1} - 2n \geq n \geq k.$$

Thus there are  $k$  distinct configurations in  $y^1, \dots, y^k \in Y'$  and by construction,  $Y'' = \{y^1, \dots, y^k\}$  and  $Y'' + \mathbf{1}$  are disjoint and disjoint from  $X$ .

Hence there is a permutation  $\pi$  of  $\{0, 1\}^n$  such that, for all  $p \in [k]$ ,  $\pi(a^{2p-1}) = y^p$ ,  $\pi(a^{2p}) = y^p + \mathbf{1}$ ,  $\pi(A_{2p-1}) = X_{2p-1}$  and  $\pi(A_{2p}) = X_{2p}$ . Let  $h = \pi \circ f \circ \pi^{-1}$ . By construction, for every  $i \in [n]$  there is  $p \in [k]$  and  $x \in X_{2p-1}$  with  $x + e_i \in X_{2p}$ . Since  $\pi^{-1}(x) \in A_{2p-1}$  and  $\pi^{-1}(x + e_i) \in A_{2p}$ , we have  $h(x) \in \pi(f(A_{2p-1})) = \{\pi(a^{2p-1})\} = \{y^p\}$  and  $h(x + e_i) \in \pi(f(A_{2p})) = \{\pi(a^{2p})\} = \{y^p + \mathbf{1}\}$ . Thus  $h(x)$  and  $h(x + e_i)$  differ in every component, and we deduce that  $G(h) = K_n$ . This proves (1).

We next prove another condition on  $A$  to obtain the complete digraph.

(2) *If  $|A| > n$  and  $|f(A)| = 1$ , then  $K_n \in \mathcal{G}(f)$ .*

Let  $a \in \{0, 1\}^n$  such that  $f(A) = \{a\}$ . Since  $f \neq \text{cst}$ , there is  $b \in \{0, 1\}^n$  with  $f(b) \neq a$ , and thus  $b \notin A$ . We consider three cases.

Suppose first that  $f(a) \neq a$ . Since  $|A| > n$ , there are  $n$  configurations  $a^1, \dots, a^n$  in  $A$  distinct from  $f(a)$ . Then  $a^1, \dots, a^n, a, f(a)$  are all distinct, so

there is a permutation  $\pi$  with  $\pi(a) = \mathbf{0}$ ,  $\pi(f(a)) = \mathbf{1}$  and  $\pi(a^i) = e_i$  for  $1 \leq i \leq n$ . Let  $h = \pi \circ f \circ \pi^{-1}$ . For  $i \in [n]$ , we have  $h(e_i) = \pi(f(a^i)) = \pi(a) = \mathbf{0}$  and  $h(\mathbf{0}) = \pi(f(a)) = \mathbf{1}$ , so  $h(e_i)$  and  $h(\mathbf{0})$  differ in every component, and thus  $G(h) = K_n$ .

Suppose now that  $f(a) = a$  and  $f(b) = b$ . Let  $a^1, \dots, a^n \in A$ , all distinct. Then  $a^1, \dots, a^n, a, b$  are all distinct since  $b = f(b) \neq a$ . So there is a permutation  $\pi$  with  $\pi(a) = \mathbf{1}$ ,  $\pi(b) = \mathbf{0}$  and  $\pi(a^i) = e_i$  for  $i \in [n]$ . Let  $h = \pi \circ f \circ \pi^{-1}$ . For all  $i \in [n]$ , we have  $h(e_i) = \pi(f(a^i)) = \pi(a) = \mathbf{1}$  and  $h(\mathbf{0}) = \pi(f(b)) = \pi(b) = \mathbf{0}$ , so  $h(e_i)$  and  $h(\mathbf{0})$  differ in every component, and thus  $G(h) = K_n$ .

Suppose finally that  $f(a) = a$  and  $f(b) \neq b$ . Since  $|A| > n$ , there is  $A' \subseteq A \setminus \{f(b)\}$  of size  $n$ . Then  $A' \cup \{b\}$  is an independent set of size  $n + 1$  and  $|f(A' \cup \{b\})| = |\{a, f(b)\}| = 2$  so  $K_n \in \mathcal{G}(f)$  by (1). This proves (2).

We can now conclude the proof. Suppose that  $|A| \geq 2n$ . Then we can choose  $A$  so that  $|A| = 2n$ . Suppose, for a contradiction, that  $K_n \notin \mathcal{G}(f)$ . If  $|f(A)|$  is even then  $K_n \in \mathcal{G}(f)$  by (1) and if  $|f(A)| = 1$  then  $K_n \in \mathcal{G}(f)$  by (2). Thus  $|f(A)| = 2k + 1$  for some  $1 \leq k < n$ . Let us write  $f(A) = \{a^1, \dots, a^{2k+1}\}$ , and let  $A_p = f^{-1}(a^p) \cap A$  for  $1 \leq p \leq 2k + 1$ . Suppose, without loss, that  $|A_1| \leq |A_2| \leq \dots \leq |A_{2k+1}|$ . Then  $A' = A \setminus A_1$  is an independent set with  $|f(A')| = 2k$ . Setting  $m = n + k - 1$ , if  $|A'| > m$ , then  $K_n \in \mathcal{G}(f)$  by (2). So  $2k|A_2| \leq |A'| \leq m$ , thus  $|A_1| \leq |A_2| \leq m/2k$ . We deduce that  $2n = |A| = |A_1| + |A'| \leq m/2k + m$ . However, one easily checks that  $2n > m/2k + m$  when  $1 \leq k < n$ , a contradiction. Thus  $K_n \in \mathcal{G}(f)$   $\square$

We are ready to prove Theorem 1. Let  $f \neq \text{cst, id}$  with  $n \geq 5$  components. Suppose, for a contradiction, that  $K_n \notin \mathcal{G}(f)$ . Let  $F$  be the set of fixed points of  $f$ , and let  $L$  be a smallest subset of  $\{0, 1\}^n$  intersecting every limit cycle of  $f$  of length  $\geq 3$ . Let  $\Gamma'$  be obtained from  $\Gamma(f)$  by deleting the vertices in  $F \cup L$ ; then  $\Gamma'$  has only cycles of length two, thus it is bipartite. Since  $K_n \notin \mathcal{G}(f)$ , by Lemmas 1 and 2, we have  $|F| < 2n$  and  $|L| < n$ , thus  $\Gamma'$  has at least  $N = 2^n - 3n + 2$  vertices. Since  $\Gamma'$  is bipartite, it has an independent set  $A$  of size  $\geq N/2$ . Then  $A$  is an independent set of  $f$  and since  $K_n \notin \mathcal{G}(f)$ , we deduce from Lemma 3 that  $|A| < 2n$ . Thus  $2n > N/2$ , that is,  $7n > 2^n + 3$ , which is false since  $n \geq 5$ . Thus  $K_n \in \mathcal{G}(f)$ .

### 3 Proof of Theorem 2

We first give a necessary and sufficient condition for the presence of a digraph in  $\mathcal{G}(f)$  that misses some arc with distinct ends. It has been obtained with Kévin Perrot, whom we thank, and its proof is in appendix. Given an  $n$ -component network  $f$ , and  $1 \leq k \leq 2^{n-1}$ , a  $k$ -nice set of  $f$  is a set  $A \subseteq \{0, 1\}^n$  of size  $2k$  with  $|f^{-1}(A)|$  and  $|f^{-1}(A) \cap A|$  even.

**Lemma 4.** *Let  $f$  be an  $n$ -component network and distinct  $i, j \in [n]$ . Some digraph in  $\mathcal{G}(f)$  has no arc from  $j$  to  $i$  if and only if  $f$  has a  $(2^{n-2})$ -nice set.*

Hence, to prove Theorem 2, it is sufficient to prove that, for  $n \geq 5$ ,  $f$  has always a  $(2^{n-2})$ -nice set. To use induction, it is more convenient to prove something stronger: if  $n \geq 4$  and  $8 \leq k \leq 2^{n-1}$  then  $f$  has always a  $k$ -nice set. In particular, if  $n \geq 5$  then  $2^{n-2} \geq 8$  thus  $f$  has a  $(2^{n-2})$ -nice set and we are done. So it remains to prove:

**Lemma 5.** *Let  $f$  be a network with  $n \geq 4$  components and  $8 \leq k \leq 2^{n-1}$ . Then  $f$  has a  $k$ -nice set.*

*Proof.* We proceed by induction on  $k$ , decreasing from  $2^{n-1}$  to 8. For the base case, observe that  $\{0, 1\}^n$  is a  $(2^{n-1})$ -nice set of  $f$ . Suppose that  $8 < k \leq 2^{n-1}$ , and suppose that  $f$  has a  $k$ -nice set  $A$ . We will prove that  $f$  has a  $(k-1)$ -nice set included in  $A$ , thus completing the induction step. For that, we will define an equivalence relation on  $A$ , with at most 8 equivalence classes, and show that, since  $A \geq 18$ , there always are two equivalent elements  $x, y \in A$  such that  $A \setminus \{x, y\}$  is a  $(k-1)$ -nice set of  $f$ . This relation is defined through 3 binary properties on the elements of  $A$ , defined below.

Let  $\alpha, \beta, \gamma : A \rightarrow \{0, 1\}$  be defined by: for all  $x \in A$ ,

1.  $\alpha(x) = 1$  if and only if  $f(x) \in A$ ,
2.  $\beta(x) = 1$  if and only if  $|f^{-1}(x)|$  is even,
3.  $\gamma(x) = 1$  if and only if  $|(f^{-1}(x) \cap A) \setminus \{x\}|$  is even.

We say that  $x, y \in A$  are *equivalent* if  $\alpha(x) = \alpha(y)$  and  $\beta(x) = \beta(y)$  and  $\gamma(x) = \gamma(y)$ . We say that  $x, y \in A$  are *independent* if  $f(x) \neq y$  and  $f(y) \neq x$ . It is straightforward (and annoying) to check that if  $x, y \in A$  are equivalent and independent, then  $A \setminus \{x, y\}$  is a  $(k-1)$ -nice of  $f$ .

Suppose now that  $f$  has no  $(k-1)$ -nice set. By the above property, there is no equivalence class containing two independent elements. Let  $A_0 = \alpha^{-1}(0)$  and  $A_1 = \alpha^{-1}(1)$ . Each of  $A_0, A_1$  contains at most 4 equivalence classes. If  $x, y \in A_0$ , then  $f(x), f(y) \notin A$ , and thus  $x$  and  $y$  are independent. We deduce that each of the classes contained in  $A_0$  is of size  $< 2$ , and thus  $|A_0| \leq 4$ . Also, one easily checks that any class of size  $\geq 4$  contains two independent elements. Hence each of the classes contained in  $A_1$  is of size  $< 4$ , and thus  $|A_1| \leq 12$ . But then  $|A| = |A_1| + |A_2| \leq 16$ , a contradiction. Thus  $f$  has a  $(k-1)$ -nice set.  $\square$

## 4 Proof of Theorem 3

If  $1 \leq n \leq 9$  and  $f$  is the identity on  $\{0, 1\}^n$ , then  $\mathcal{G}(f)$  contains a unique digraph, which has  $n = n^2/n \geq n^2/9$  vertices. This proves the theorem for  $n \leq 9$ . We treat the other cases with an explicit construction.

Given  $A \subseteq \{0, 1\}^n$  with  $\mathbf{0} \in A$ , we denote by  $f^A$  the  $n$ -component network such that  $f^A(A) = \{\mathbf{0}\}$  and  $f^A(b) = b$  for all  $b \notin A$ . We will prove, in the next lemma, that if the size of  $A$  is carefully chosen, then, independently on the structure of  $A$ ,  $G(f^A)$  has at least  $n^2/9$  arcs. It is then easy to prove that this lower bound holds for every digraph in  $\mathcal{G}(f^A)$ .

**Lemma 6.** *Let  $n \geq 9$  and  $A \subseteq \{0, 1\}^n$  of size  $\lceil 2^{n/4} \rceil$  with  $\mathbf{0} \in A$ . Then  $G(f^A)$  has at least  $n^2/9$  arcs.*

The key tool is the following lemma, which can be easily deduced from the well-known Harper's isoperimetric inequality in the hypercube (see the appendix). Let  $Q_n$  be the hypercube graph, with vertex set  $\{0, 1\}^n$  and an edge between  $x$  and  $y$  if and only if  $x$  and  $y$  differ in exactly one component.

**Lemma 7.** *Let  $A \subseteq \{0, 1\}^n$  be non-empty, and let  $d$  be the average degree of the subgraph of  $Q_n$  induced by  $A$ . Then  $|A| \geq 2^d$ .*

**Proof of Lemma 6.** Let  $f = f^A$  and let  $m$  be the number of arcs in  $G(f)$ . Suppose, for a contradiction that  $m < n^2/9$ . Let  $I$  be the set of  $i \in [n]$  with  $a_i = 1$  for some  $a \in A$ . Note that  $|I| \geq n/4$ . Indeed, let  $B$  be the set of  $b \in \{0, 1\}^n$  such that  $b_i = 0$  for all  $i \notin I$ . Then  $A \subseteq B$ , and  $|B| = 2^{|I|}$ , thus  $|I| = \log_2 |B| \geq \log_2 |A| \geq n/4$ . From that observation, we can say something stronger:

$$(1) \quad |I| > n/2.$$

Let  $i \in I$  and  $j \notin I$ . Let  $a \in A$  such that  $a_i = 1$ . Since  $j \notin I$ ,  $a + e_j \notin A$ , thus  $f(a) = \mathbf{0}$  and  $f(a + e_j) = a + e_j$ , and  $f_i(a + e_j) = a_i = 1$  since  $j \neq i$ . Hence  $G(f)$  has an arc from  $j$  to  $i$ . We deduce that  $|I| \cdot (n - |I|) \leq m < n^2/9$ , so  $n/4 \cdot (n - |I|) < n^2/9$ . Thus  $|I| > 5n/9 > n/2$ . This proves (1).

We deduce that some  $a \in A$  has a large weight.

$$(2) \quad \text{There is } a \in A \text{ with } w(a) > 3n/4.$$

Suppose that  $w(a) \leq 3n/4$  for all  $a \in A$ . Let  $i \in I$ . Let  $a \in A$  with  $a_i = 1$ , and  $w(a)$  maximal for that property. Then, for all  $j \in \mathbf{0}(a)$ , we have  $a + e_j \notin A$ . Hence  $f(a) = \mathbf{0}$  and  $f(a + e_j) = a + e_j$  so  $f_i(a + e_j) = a_i = 1$  since  $i \neq j$ . Thus  $G(f)$  has an arc from  $j$  to  $i$ . We deduce that the in-degree of  $i$  in  $G(f)$  is at least  $|\mathbf{0}(a)| = n - w(a) \geq n/4$ . Hence  $m \geq |I| \cdot n/4$  and since  $|I| > n/2$  by (1) we obtain  $m > n^2/8 > n^2/9$ , a contradiction. This proves (2).

Let  $A'$  be the set of  $a \in A$  with  $w(a) \geq n/3 - 1$ , which is not empty by (2). For  $a \in A$ , let  $J(a)$  be the set of  $j \in [n]$  such that  $a + e_j \in A$ , and let  $J'(a)$  be the set of  $j \in [n]$  such that  $a + e_j \in A'$ . Let  $H$  be the subgraphs of  $Q_n$  induced by  $A'$ . If  $a \in A'$ , then  $d(a) = |J'(a)|$  is the degree of  $a$  in  $H$ . We will prove in (3) and (4) below that each vertex in  $H$  has large degree.



(3) If  $a \in A'$  and  $w(a) \geq n/3$  then  $d(a) > n/3$ .

Indeed, since  $w(a) \geq n/3$ , for  $j \in J(a)$  we have  $w(a + e_j) \geq n/3 - 1$  thus  $j \in J'(a)$ . Hence  $J(a) = J'(a)$  so  $d(a) = |J(a)|$ . For all  $i \notin J(a)$  we have  $f(a) = \mathbf{0}$  and  $f(a + e_i) = a + e_i$ , thus  $G(f)$  has an arc from  $i$  to each component in  $\mathbf{1}(a + e_i)$ . Thus the out-degree of  $i$  in  $G(f)$  has at least  $w(a + e_i) \geq n/3 - 1$ . We deduce that  $(n - |J(a)|) \cdot (n/3 - 1) \leq m < n^2/9$ , so  $d(a) = |J(a)| > 2n/3 - 3$  and since  $n \geq 9$  we obtain  $d(a) > n/3$ . This proves (3).

(4) If  $a \in A'$  and  $w(a) < n/3$  then  $d(a) > n/3$ .

Let  $K = \mathbf{0}(a) \setminus J'(a)$  and  $i \in K$ . We have  $w(a + e_i) = w(a) + 1 \geq n/3$ , thus if  $a + e_i \in A$  then  $a + e_i \in A'$ ; but then  $i \in J'(a)$ , a contradiction. Thus  $a + e_i \notin A$ , so  $f(a) = \mathbf{0}$  and  $f(a + e_i) = a + e_i$ . We deduce that the out-degree of  $i$  in  $G(f)$  is at least  $w(a + e_i) \geq n/3$ . Thus  $|K| \cdot n/3 \leq m < n^2/9$  so  $|K| < n/3$ . Since  $w(a) < n/3$  we have  $|\mathbf{0}(a)| > 2n/3$ . So  $|K| \geq 2n/3 - |J'(a)|$  and thus  $d(a) = |J'(a)| > n/3$ . This proves (4).

We are now in position to finish the proof. Let  $d$  be the average degree of  $H$ . By (3) and (4), we have  $d > n/3$ . Using Lemma 7 we obtain  $2^{n/3} < 2^d \leq |A'| \leq |A| = \lceil 2^{n/4} \rceil$ , which is false because  $n \geq 9$ .  $\square$

We are now in position to conclude. Let  $n \geq 9$  and  $A \subseteq \{0, 1\}^n$  of size  $\lceil 2^{n/4} \rceil$  with  $\mathbf{0} \in A$ , and let  $f = f^A$ . Let  $\pi$  be any permutation of  $\{0, 1\}^n$  and  $h = \pi \circ f \circ \pi^{-1}$ . We will prove that  $G(h)$  has at least  $n^2/9$  arcs. Let  $h'$  be the  $n$ -component network defined by  $h'(x) = h(x + \pi(\mathbf{0})) + \pi(\mathbf{0})$  for all  $x \in \{0, 1\}^n$ . We have  $G(h) = G(h')$ . Let  $A' = \pi(A) + \pi(\mathbf{0})$ . We have  $\mathbf{0} \in A'$  and one easily check that  $h' = f^{A'}$ . Since  $|A'| = |A| = \lceil 2^{n/4} \rceil$ , by Lemma 6,  $G(h') = G(h)$  has at least  $n^2/9$  arcs.

## References

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## A Examples

Consider the 6 following networks with two components, denoted from  $f^1$  to  $f^6$  (each is given under three forms: a table, a graph, and formulas).

$x$	$f^1(x)$		
00	10	01 ← 11	$f_1^1(x) = x_2 + 1$
01	00	↓     ↑	$f_2^1(x) = x_1$
10	11	00 → 10	
11	01		

$x$	$f^2(x)$		
00	10	01 ↔ 11	$f_1^2(x) = x_1 + 1$
01	11		$f_2^2(x) = x_2$
10	00	00 ↔ 10	
11	01		

$x$	$f^3(x)$		
00	00	01 ← 11	$f_1^3(x) = x_1 + x_2$
01	10	↘     ↑	$f_2^3(x) = x_1$
10	11	00     10	
11	01	↻	



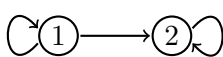
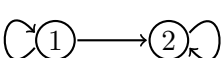
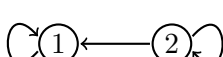
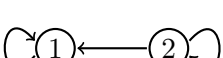
$x$	$f^4(x)$		
00	00	01 ↔ 11	$f_1^4(x) = x_1 + x_2$
01	11		$f_2^4(x) = x_2$
10	10	00     10	
11	01	↻     ↻	

$x$	$f^5(x)$		
00	01	01 ↔ 11	$f_1^5(x) = x_1 + x_2$
01	11	↑     ↑	$f_2^5(x) = 1$
10	11	00     10	
11	01		

$x$	$f^6(x)$		
00	00	01     11	$f_1^6(x) = x_1$
01	00	↓     ↓	$f_2^6(x) = 0$
10	10	00     10	
11	10	↻     ↻	



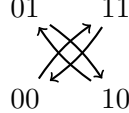

One can check that, given a network  $f \neq \text{cst}, \text{id}$  with 2 components:  $K_2 \notin \mathcal{G}(f)$  if and only if  $f$  is isomorphic to one of the networks given above. Thus

these 6 networks are the counter examples of Theorem 1 for  $n = 2$ . For instance, there are 6 networks isomorphic to  $f^1$ , denoted from  $h^1$  to  $h^6$  (with  $h^1 = f^1$ , given below with their interactions graphs. We deduce that  $\mathcal{G}(f^1)$  contains three digraphs, each distinct from  $K_2$ .

$x$	$h^1(x)$	$\begin{array}{ccc} 01 & \leftarrow & 11 \\ \downarrow & & \uparrow \\ 00 & \longrightarrow & 10 \end{array}$	$h_1^1(x) = x_2 + 1$ $h_2^1(x) = x_1$	$G(h^1)$ 
$x$	$h^2(x)$	$\begin{array}{ccc} 01 & \longrightarrow & 11 \\ \uparrow & & \downarrow \\ 00 & \longleftarrow & 10 \end{array}$	$h_1^2(x) = x_2$ $h_2^2(x) = x_1 + 1$	$G(h^2)$ 
$x$	$h^3(x)$	$\begin{array}{ccc} 01 & \longrightarrow & 11 \\ \swarrow & & \searrow \\ & & \\ \nwarrow & & \nearrow \\ 00 & \longrightarrow & 10 \end{array}$	$h_1^3(x) = x_1 + 1$ $h_2^3(x) = x_1 + x_2$	$G(h^3)$ 
$x$	$h^4(x)$	$\begin{array}{ccc} 01 & \longleftarrow & 11 \\ \swarrow & & \searrow \\ & & \\ \nwarrow & & \nearrow \\ 00 & \longleftarrow & 10 \end{array}$	$h_1^4(x) = x_1 + 1$ $h_2^4(x) = x_1 + x_2 + 1$	$G(h^4)$ 
$x$	$h^5(x)$	$\begin{array}{ccc} 01 & & 11 \\ \uparrow & \swarrow & \uparrow \\ & & \\ \downarrow & \searrow & \downarrow \\ 00 & & 10 \end{array}$	$h_1^5(x) = x_1 + x_2$ $h_2^5(x) = x_2 + 1$	$G(h^5)$ 
$x$	$h^6(x)$	$\begin{array}{ccc} 01 & & 11 \\ \downarrow & \swarrow & \downarrow \\ & & \\ \uparrow & \searrow & \uparrow \\ 00 & & 10 \end{array}$	$h_1^6(x) = x_1 + x_2 + 1$ $h_2^6(x) = x_2 + 1$	$G(h^6)$ 

A corollary of Theorems 1 and 2, stated in the introduction, is that,

for any network  $f$  with  $n \geq 5$  components, we have  $|\mathcal{G}(f)| = 1$  if and only if  $f = \text{id}$  or  $f = \text{cst}$ . This is also true for  $n = 3$  (checked by computer) and open for  $n = 4$ . But for  $n = 2$ ,  $f^2$  is a counter example, the only one with two components. Indeed, there are 3 networks isomorphic to  $f^2$ , denoted from  $g^1$  to  $g^3$  (with  $g^1 = f^2$ ), given below, and they have the same interaction graph, with a loop on each vertex, as for the identity.

$x$	$g^1(x)$	$01 \rightleftarrows 11$	$g_1^1(x) = x_1 + 1$	$G(g^1)$
00	10			
01	11		$g_2^1(x) = x_2$	
10	00	$00 \rightleftarrows 10$		
11	01			
$x$	$g^2(x)$	$01 \quad 11$	$g_1^2(x) = x_1$	$G(g^2)$
00	01	$\updownarrow \quad \updownarrow$	$g_2^2(x) = x_2 + 1$	
01	00			
10	11	$00 \quad 10$		
11	10			
$x$	$g^3(x)$	$01 \quad 11$	$g_1^3(x) = x_1 + 1$	$G(g^3)$
00	11		$g_2^3(x) = x_2 + 1$	
01	10			
10	01	$00 \quad 10$		
11	00			

## B Lemma for the proof of Theorem 1

**Lemma 8.** *Let  $1 \leq k \leq n$  and let  $s_1, \dots, s_{2k}$  be positive integers with sum equal to  $n+k$ . If  $n \geq 5$ , then there are disjoint subsets  $X_1, \dots, X_{2k} \subseteq \{0, 1\}^n$  of size  $s_1, \dots, s_{2k}$  such that, for every  $i \in [n]$ , there is  $p \in [k]$  and  $x \in X_{2p-1}$  with  $x + e_i \in X_{2p}$ .*

*Proof.* Let  $I_1, \dots, I_{2k}$  be a partition of  $[n]$  (with some members possibly empty) such that, for  $1 \leq \ell \leq 2k$ , the size of  $I_\ell$  is  $s_\ell - 1$  if  $\ell$  is odd, and  $s_\ell$  otherwise; it exists since the sum of the  $s_\ell$  is  $n+k$ . For  $1 \leq p \leq k$ , select a configuration  $x^{2p-1} \in \{0, 1\}^n$ , a component  $j_{2p} \in I_{2p}$ , and let

$$\begin{aligned} X_{2p-1} &= \{x^{2p-1}\} \cup \{x^{2p-1} + e_{j_{2p}} + e_i \mid i \in I_{2p-1}\}, \\ X_{2p} &= \{x^{2p-1} + e_i \mid i \in I_{2p}\}. \end{aligned}$$

Clearly,  $|X_{2p-1}| = |I_{2p-1}| + 1 = s_{2p-1}$  and  $|X_{2p}| = |I_{2p}| = s_{2p}$ . Let  $i \in [n]$ . Then  $i \in I_\ell$  for some  $1 \leq \ell \leq 2k$ . If  $\ell = 2p - 1$  then, setting  $x = x^{2p-1} + e_{j_{2p}} + e_i$ , we have  $x \in X_{2p-1}$  and  $x + e_i = x^{2p-1} + e_{j_{2p}} \in X_{2p}$ . If  $\ell = 2p$  then, setting  $x = x^{2p-1}$ , we have  $x \in X_{2p-1}$  and  $x + e_i \in X_{2p}$ . Thus we

only have to prove that we can choose the configurations  $x^{2p-1}$  so that the sets  $X_1, \dots, X_{2k}$  are pairwise disjoint. This is obvious if  $k = 1$  (and it works for any choice of  $x^1$ ). If  $k = 2$ , then, since  $n \geq 5$ , one easily check that  $X_1, \dots, X_{2k}$  are disjoint by taking  $x^1 = \mathbf{0}$  and  $x^3 = \mathbf{1}$ . Suppose now that  $k \geq 3$  and choose  $x^{2p-1} = e_{j_{2p-2}}$  for all  $p \in [k]$ , where  $j_0$  means  $j_{2k}$ . Then each  $X_{2p-1}$  contains configurations of weight 1 or 3 and each  $X_{2p}$  contains configurations of weight 2. Hence, given  $1 \leq p < q \leq k$ , we have to prove that  $X_{2p-1} \cap X_{2q-1} = \emptyset$  and  $X_{2p} \cap X_{2q} = \emptyset$ . Suppose that  $x \in X_{2p-1} \cap X_{2q-1}$ . If  $w(x) = 1$  then we deduce that  $x = e_{j_{2p-2}} = e_{j_{2q-2}}$  which is false since  $p \neq q$ . If  $w(x) = 3$  then  $x_i = 1$  for some  $i \in I_{2p-1}$  while  $y_i = 0$  for all  $y \in X_{2q-1}$ , a contradiction. Thus  $X_{2p-1} \cap X_{2q-1} = \emptyset$ . Suppose now that  $x \in X_{2p} \cap X_{2q}$ . Then  $x = e_{j_{2p-2}} + e_{i_{2p}} = e_{j_{2q-2}} + e_{i_{2q}}$  for some  $i_{2p} \in I_{2p}$  and  $i_{2q} \in I_{2q}$ . Thus  $i_{2p} = j_{2q-2}$  and  $i_{2q} = j_{2p-2}$ . Hence  $j_{2q-2} \in I_{2p}$ , and since  $p < q$  this implies  $q = p + 1$ . Also,  $j_{2p-1} \in I_{2q}$  and since  $p < q$  this implies  $p = 1$  and  $q = k$ , but then  $q \neq p + 1$  since  $k \geq 3$ . Thus indeed  $X_{2p} \cap X_{2q} = \emptyset$ . Hence the sets  $X_1, \dots, X_{2k}$  are indeed pairwise disjoint.  $\square$

## C Lemma for the proof of Theorem 2

We prove Lemma 4, restated below. We first give some definitions. Given  $X \subseteq \{0, 1\}^n$  and  $i \in [n]$ , we say that  $X$  is *closed* by  $e_i$  if  $X = X + e_i$ . One easily check that if  $X, Y \subseteq \{0, 1\}^n$  are closed by  $e_i$ , then so is  $X \cap Y$ . Also, if  $|X|$  is closed by  $e_i$  then  $|X|$  is even. Indeed, let  $X_0$  be the set of  $x \in X$  with  $x_i = 0$  and  $X_1 = X \setminus X_0$ . We have  $X_0 + e_i \subseteq X_1$ , thus  $|X_0| \leq |X_1|$  and similarly,  $X_1 + e_i \subseteq X_0$  thus  $|X_1| \leq |X_0|$ . Hence  $|X_0| = |X_1|$  so  $|X|$  is even.

**Lemma 4.** *Let  $f$  be a  $n$ -component network and distinct  $i, j \in [n]$ . Some digraph in  $\mathcal{G}(f)$  has no arc from  $j$  to  $i$  if and only if  $f$  has a  $(2^{n-2})$ -nice set.*

*Proof.* Let  $h = \pi \circ f \circ \pi^{-1}$  for some permutation  $\pi$  of  $\{0, 1\}^n$ , and suppose that  $G(h)$  has no arc from  $j$  to  $i$ . Let  $X$  be the set of  $x \in \{0, 1\}^n$  with  $x_i = 0$ , of size  $2^{n-1}$ , and let  $X^- = h^{-1}(X)$ . If  $x \in X^-$ , that is,  $h_i(x) = 0$ , then  $h_i(x + e_j) = 0$  since otherwise  $G(h)$  has an arc from  $j$  to  $i$ . Thus  $x + e_j \in X^-$ . Hence  $X^-$  is closed by  $e_j$  thus  $|X^-|$  is even. Since  $j \neq i$ ,  $X$  is also closed by  $e_j$ , so  $X^- \cap X$  is closed by  $e_j$  and thus  $|X^- \cap X|$  is even. Let  $A = \pi^{-1}(X)$ . Then one easily check that  $f^{-1}(A) = \pi^{-1}(X^-)$  and that  $f^{-1}(A) \cap A = \pi^{-1}(X^- \cap X)$ . So  $|f^{-1}(A)|$  and  $|f^{-1}(A) \cap A|$  are even, and thus  $A$  is a  $(2^{n-2})$ -nice set of  $f$ .

Conversely, suppose that  $f$  has a  $(2^{n-2})$ -nice set  $A$ , thus of size  $2^{n-1}$ . Let  $A^- = f^{-1}(A)$ . Then  $|A|$ ,  $|A^-|$  and  $|A^- \cap A|$  are even, so  $|A \setminus A^-|$  and  $|A^- \setminus A|$  are also even. Hence there is a balanced partition  $(A_1, A_2)$  of  $A \cap A^-$ , a balanced partition  $(A_3, A_4)$  of  $A \setminus A^-$ , and a balanced partition  $(A_3^-, A_4^-)$  of  $A^- \setminus A$ . Then  $(A_1, A_2, A_3, A_4)$  is a partition of  $A$  and  $(A_1, A_2, A_3^-, A_4^-)$  is a partition of  $A^-$ , and  $A_1, A_2, A_3, A_4, A_3^-, A_4^-$  are disjoint.

For  $k = 0, 1$ , let  $Y_k$  be the set of  $x \in \{0, 1\}^n$  with  $x_i = k$  and  $x_j = 0$ . Since  $|A| = 2^{n-1}$  we have  $|A_1| + |A_3| = 2^{n-2}$  thus there is a partition  $(X_1, X_3)$  of  $Y_0$  with  $|X_1| = |A_1|$  and  $|X_3| = |A_3|$ . Since  $|A_3^-| = |A^- \setminus A|/2 \leq (2^n - |A|)/2 = 2^{n-2}$  there is  $X_3^- \subseteq Y_1$  of size  $|A_3^-|$ . Let  $X_2 = X_1 + e_j$ ,  $X_4 = X_3 + e_j$  and  $X_4^- = X_3^- + e_j$ . Then  $X_1, X_2, X_3, X_4, X_3^-, X_4^-$  are disjoint and of size  $|A_1|, |A_2|, |A_3|, |A_4|, |A_3^-|, |A_4^-|$ . Hence there is a permutation  $\pi$  of  $\{0, 1\}^n$  that sends  $A_1, A_2, A_3, A_4, A_3^-, A_4^-$  on  $X_1, X_2, X_3, X_4, X_3^-, X_4^-$ , respectively.

Let  $h = \pi \circ f \circ \pi^{-1}$ . Let  $X = X_1 \cup X_2 \cup X_3 \cup X_4$  and  $X^- = X_1 \cup X_2 \cup X_3^- \cup X_4^-$ . One easily check that:  $X$  is the set of  $x \in \{0, 1\}^n$  with  $x_i = 0$ ;  $X^-$  is closed by  $e_j$ ; and  $X^- = h^{-1}(X)$ . We will prove that  $G(h)$  has no arc from  $j$  to  $i$ . Suppose, for a contradiction, that there is  $x \in \{0, 1\}^n$  with  $h_i(x) \neq h_i(x + e_j)$ . Without loss, we can assume that  $h_i(x) = 0$ , that is,  $h(x) \in X$ . So  $x \in X^-$ , hence  $x + e_j \in X^-$ , thus  $h(x + e_j) \in X$ , that is,  $h_i(x + e_j) = 0$ , a contradiction. Thus  $G(h)$  has no arc from  $j$  to  $i$ .  $\square$

## D Lemma for the proof of Theorem 3

For  $X \subseteq \{0, 1\}^n$ , let  $\partial(X)$  be the number of edges in  $Q_n$  with exactly one end in  $X$ ; equivalently, it is the number of pairs  $(x, i)$  with  $x \in X$  and  $i \in [n]$  such that  $x + e_i \notin X$ . We can regard  $|X|$  as the volume of  $X$  and  $\partial(X)$  as its perimeter and ask: what is the minimum perimeter for a given volume? The answer is Harp's isoperimetric inequality, from 1964 [2]. We denote by  $L_k$  the first  $k$  configurations in  $\{0, 1\}^n$  according to the lexicographic order (e.g. for  $n = 4$  we have  $L_5 = \{0000, 1000, 0100, 1100, 0010\}$ ).

**Theorem 4** ([2]). *If  $n \geq 1$  and  $X \subseteq \{0, 1\}^n$  then  $\partial(X) \geq \partial(L_{|X|})$ .*

In practice, the following approximation is often used; see [3] for instance.

**Corollary 1.** *If  $n \geq 1$  and  $X \subseteq \{0, 1\}^n$  is non-empty, then*

$$\partial(X) \geq |X|(n - \log_2 |X|).$$

From this inequality, we deduce Lemma 7, that we restate.

**Lemma 7.** *Let  $A \subseteq \{0, 1\}^n$  be non-empty, and let  $d$  be the average degree of the subgraph of  $Q_n$  induced by  $A$ . Then  $|A| \geq 2^d$ .*

*Proof.* For  $a \in A$ , let  $d(a)$  be the degree of  $a$  in the subgraph of  $Q_n$  induced by  $A$ , and let  $d$  be the average degree, that is,  $d = \sum_{a \in A} d(a)/|A|$ . Since, for each  $a \in A$ , there are exactly  $n - d(a)$  components  $i \in [n]$  such that  $a + e_i \notin A$ , we have, using Corollary 1,

$$\partial(A) = \sum_{a \in A} n - d(a) = |A|n - |A|d = |A|(n - d) \geq |A|(n - \log_2 |A|),$$

thus  $d \leq \log_2 |A|$  as desired.  $\square$