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Abstract

We consider the simple exclusion process on $\mathbb{Z}\times\{0,1\}$, that is, an "horizontal ladder" composed of 2 lanes, depending on 6 parameters. Particles can jump according to a lane-dependent translation-invariant nearest neighbour jump kernel, i.e. "horizontally" along each lane, and "vertically" along the scales of the ladder. We prove that generically, the set of extremal invariant measures consists of *(i)* translation-invariant product Bernoulli measures; and, modulo translations along $\mathbb{Z}$: *(ii)* at most two shock measures (i.e. asymptotic to Bernoulli measures at $\pm\infty$) with asymptotic densities 0 and 2; *(iii)* at most one (outside degenerate cases) shock measure with a density jump of magnitude 1. We fully determine this set for a range of parameter values. Our results can be generalized in several directions using the same approach and answer certain open questions formulated in [6] as a step towards the process on $\mathbb{Z}^2$.

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1 Introduction

The simple exclusion process, introduced in [17], is a fundamental model in statistical mechanics. In this markovian process, particles hop on a countable

lattice following a certain random walk kernel subject to the exclusion rule, that allows at most one particle per site. As usual for Markov processes, the characterization of its invariant measures is one of the basic questions to address. Still today, outside the case of a symmetric kernel ([15]), the problem is far from being completely solved. In fact, it has been mostly studied for translation invariant kernels. We briefly recall known results in this situation.

For the exclusion process on $\mathbb{Z}^d$, the set of extremal *translation invariant* (also called *homogeneous*) stationary probability measures consists ([14]) of homogeneous Bernoulli product measures. However, for a non-symmetric kernel, there may exist extremal invariant probability measures that are *not* translation invariant. These are fairly well (though not completely) understood in one-space dimension ([13, 11, 7, 5]; see also [6] for open questions): under suitable assumptions, there is a unique (up to translations) such extremal probability measure, called either a *blocking* or a *profile* measure (the latter being a weakened version of the former); its main feature (for a kernel with, say, a positive drift) is that the asymptotic particle density is 0 to the left and 1 to the right of the origin.

In several space dimensions, although analogues of blocking or profile measures can be exhibited ([6]), the complete characterization of invariant probability measures remains an open question. The paper [6] initiated a program in this direction. The authors introduced so-called *v-homogeneous* measures, that is, measures invariant by translations in directions orthogonal to a given vector v, and *v-profile* measures, that is v-homogeneous measures with asymptotic density 0 at $-\infty$ and 1 at $+\infty$ parallel to v. They showed that when v is orthogonal to the drift, extremal stationary v-homogeneous measures are homogeneous Bernoulli measures. They proved that under some conditions on the jump kernel and vector v, extremal v-profile measures are given by an explicit family of product measures analogous to those in [13]. Finally, they decomposed the problem of characterizing all invariant measures into a series of open questions. The first of these are (BL1) whether any non-homogeneous extremal stationary measure is v-homogeneous for some v, and (BL2) whether it is v-profile for some v. These questions were also formulated for the so-called *ladder process*, where one among two dimensions is cyclic, mentioned in [6] as an interesting step towards the process on $\mathbb{Z}^2$. In this context, v-homogeneity is interpreted as cyclic rotational invariance.

In the present paper, we obtain characterization results (Theorems 2.1 to 2.4), for intermediate models between dimensions 1 and 2 containing the above ladder process. As explained below, we exhibit new phenomena and a richer behaviour as compared to the one dimensional single-lane exclusion process. We consider first the simple exclusion process on $\mathbb{Z} \times \{0, 1\}$, that is an "horizontal ladder" composed of 2 lanes. Particles can jump "horizontally" to nearest neighbour sites along each lane according to a lane-dependent translation-invariant jump kernel, and "vertically" along the scales of the ladder according to another kernel. In the totally asymmetric case, this can be interpreted as traffic-flow on a

highway, with two lanes on which cars have different speeds and different directions.

We now describe our results for the two-lane model. Let γ_0, γ_1 denote mean drifts on each lane, p the jump rate from lane 0 to lane 1 and q the jump rate from lane 1 to lane 0. The drifts may be of equal or opposite signs; one or both of them may also vanish. We assume that $p + q > 0$, so that both lanes are indeed connected. We prove that the set $\mathcal{I}_e$ of extremal invariant probability measures can be decomposed as a disjoint union

$$\mathcal{I}_e = \mathcal{I}_0 \cup \mathcal{I}_1 \cup \mathcal{I}_2 \tag{1}$$

In this decomposition, $\mathcal{I}_0 := \{\nu_\rho, \rho \in [0, 2]\}$ is the set of extremal invariant probability measures that are *translation invariant* along lanes. The parameter ρ represents the total density over the two lanes. Under ν_ρ, the mean densities ρ_0, ρ_1 on each lane are functions of ρ, and they are different when $p \neq q$. In the sequel, we refer to these probability measures as "Bernoulli measures". For $k \in \{1, 2\}$, $\mathcal{I}_k$ denotes a (possibly empty) set of extremal invariant probability measures that we call *shock measures* of *amplitude* k. By a shock measure, we mean a probability measure that is asymptotic to two Bernoulli measures of different densities ρ^-, resp. ρ^+, when viewed from faraway left, resp. right (w.r.t. the origin). We define the amplitude of the shock to be $k := |\rho^+ - \rho^-|$. The set $\mathcal{I}_2$ contains only shocks such that $(\rho^-, \rho^+) = (0, 2)$ or $(\rho^-, \rho^+) = (2, 0)$. These measures are the analogue in our context of blocking measures or profile measures. In some cases, $\mathcal{I}_1$ may contain what we call *partial* blocking measures, i.e., measures whose restriction to one lane is a blocking measure, and whose restriction to the other lane is a Dirac measure concentrated either on the full configuration or on the empty one.

We show that the following generic picture holds outside some degenerate cases: up to translations along $\mathbb{Z}$, *(i)* the set $\mathcal{I}_1$ contains at most one probability measure; *(ii)* the set $\mathcal{I}_2$ contains at most two probability measures. In particular, these sets are at most countable. We can fully determine $\mathcal{I}_1$ and $\mathcal{I}_2$, and thus obtain a complete characterization of invariant probability measures, for a subset of parameter values including the following situations: (a) when γ_0, γ_1 are close enough, the ratio q/p small enough or large enough, and

$$d_0/l_0 = d_1/l_1 \neq 1 \tag{2}$$

where d_i, resp. l_i, denotes the jump rate to the right, resp. left, on lane $i \in \{0, 1\}$; (b) when p or q vanishes and $\gamma_0 \neq \gamma_1$; (c) when $\gamma_0 = \gamma_1 = 0$ and p, q are arbitrary.

In case (b), we exhibit *partial* blocking measures (where only one lane has a blocking measure), a new phenomenon with respect to single-lane asymmetric simple exclusion process (ASEP). Another result in sharp contrast with the one-dimensional case is that $\mathcal{I}_2$ may be empty when $pq = 0$ even if the drifts are both

strictly positive (or both strictly negative); and when it is not, it is described by two integer parameters representing two independent shock locations instead of a single parameter in the usual ASEP. In case (a), our characterization can be viewed in this context as a positive answer to open question (BL2) above from [6]. The set $\mathcal{I}_2$ is then derived from a family of two-dimensional product blocking measures that are analogues in this context (see Remark 2.7) of certain v-profile measures constructed on $\mathbb{Z}^d$ in [6]. We observe here some structural similarity between elements of $\mathcal{I}_2$ and extremal blocking measures constructed in [4] for the single-lane Misanthrope's process. It would be interesting to know if two-dimensional blocking measures can lead to remarkable combinatorial identities as in [4].

The following questions are left open. First, we can show that $\mathcal{I}_1$ is indeed nonempty in cases where it contains *only* partial blocking measures, and that it is empty on a set of parameter values for which γ_0 and γ_1 are close enough, and the ratio between p and q small enough (or large enough). We do not know if for certain parameter values it is possible to have $\mathcal{I}_1$ nonempty with a shock of amplitude 1 that is not a partial blocking measure. In the case $p = q$ (and more generally for the vertically cyclic ladder process, see below), it is believed in [6] that this probably does not occur. Next, we conjecture that when $pq > 0$, under a suitable assumption (see Remark 2.6) verified in particular when both drifts are strictly positive, $\mathcal{I}_2$ is nonempty, even without the assumption (2). We believe that this could be proved in the spirit of [7] by means of the hydrodynamic limit. We shall investigate the hydrodynamic behaviour of our model and extensions thereof (see below) in [1].

In the assumptions of Theorems 2.1–2.3, to avoid cumbersome statements and proofs, we have not aimed at fullest possible generality. Nevertheless, we stress that our approach is robust enough to handle more general or related situations without substantial changes. In Appendix A, we provide a detailed discussion of such extensions with precise assumptions and conclusions, and explain why the ideas of proofs developed in the body of the paper carry over to such situations. These include non-nearest neighbour jump kernels, multilane processes with more than two lanes, and Misanthrope's processes. We point out that although the latter are single-lane generalizations of the simple exclusion process, the characterization of their invariant measures (outside translation invariant ones) is still an open problem. We realized along the way that, though this question was not our initial motivation, it could be partly solved by our methods. Among the above extensions, the vertically cyclic multilane ladder process from [6] is however treated in Subsection 2.5 rather than in the appendix, because our corresponding Theorem 2.4 answers question (BL1) above from [6], namely all invariant measures are rotationally invariant.

One of the difficulties of our models is that available approaches ([13, 5]) to classify invariant measures for the one-dimensional single-lane asymmetric simple exclusion process rely heavily on the fact that at most one particle is allowed

on each site. In the aforementioned works, the line of argument is to show that for a non translation-invariant stationary measure, the *mean* density difference between $-\infty$ and $+\infty$ is at least 1. Since the possible density range is $[0, 1]$, this *automatically* implies that the measure is a shock with asymptotic densities 0 and 1 at $\pm\infty$ (see Remark 4.1). In our case, the range of global densities is no longer restricted to 1 but to the number of lanes. A different and more complex scheme of proof (see outline in Subsection 4.1) is imposed by this, but also by the interplay of several parameters leading to a wider variety of behaviours. One key point is to show *a priori* that an invariant measure is a shock. This is done thanks to a novel and robust argument (Proposition 4.2) using extremality and attractiveness, which can be transposed to other attractive models. *Then* we carry out an analysis of possible shocks based on the macroscopic flux function of the model. Note that this density range problem arises also for the Misanthrope's process and similarly makes the characterization problem for this model different than for the simple exclusion process.

Another difficulty that occurs when interlane jumps are possible only in one direction is the lack of irreducibility for the jump kernel. Usual arguments (in the line of [13]) based on attractiveness and irreducibility, showing that discrepancies between two coupled processes eventually disappear (see e.g. [13]), are not sufficient in this case.

We finally mention that while revising this manuscript, we became aware that the case where all lanes are symmetric (corresponding to $\gamma_0 = \gamma_1 = 0$ in the basic two-lane model) had been recently studied in [16] by different duality methods.

This paper is organized as follows. Models are introduced in Section 2. We then state our results on invariant measures for the two-lane simple exclusion process: Theorem 2.1 for the invariant and translation invariant probability measures, Theorems 2.2 and 2.3 for the invariant probability measures; finally Theorem 2.4 deals with the multilane simple exclusion process, and in particular with the ladder process from [6]. Section 3 is devoted to the proof of Theorem 2.1, and Section 4 to the proofs of Theorems 2.2, 2.3 and 2.4. In order to make the general schemes of proofs more visible, the main ideas are first explained in Subsection 4.1, and most intermediate results used to establish Theorems 2.2 and 2.3 are proved in the separate Section 5. Extensions of our results are discussed in Appendix A.

2 Models and results

In this section, we present and state our results for our basic model, the *two-lane* SEP (motivated by traffic-flow considerations), and for its generalization to a *multilane* SEP. Before that, we first recall the definition of the simple exclusion process on a countable set V. The two-lane and the multilane SEP indeed

belong to this class, but they have specific properties due to the structure of the set V.

2.1 Simple exclusion process

Throughout the paper, $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ the set of nonnegative integers. Let V be a nonempty countable set. The state space of the process is

$$\mathcal{X} := \{0,1\}^V \tag{3}$$

that is a compact polish space with respect to product topology. One can think of $\eta \in \mathcal{X}$ as a configuration of particles on V, i.e. for which a site $x \in V$ is occupied by a particle if and only if $\eta(x) = 1$.

We call *kernel* on V a function $p : V \times V \to [0, +\infty)$ such that

$$\sup_{x \in V} \left\{ \sum_{y \in V} p(x,y) + \sum_{y \in V} p(y,x) \right\} < +\infty \tag{4}$$

The (V,p)-simple exclusion process (in short: SEP) is a Markov process $(\eta_t)_{t \geq 0}$ on $\mathcal{X}$ (see [14, Chapter VIII]) with generator

$$Lf(\eta) = \sum_{x,y \in V} p(x,y)\, \eta(x)\, (1 - \eta(y))\, (f(\eta^{x,y}) - f(\eta)), \tag{5}$$

where $\eta^{x,y}$, given by

$$\eta^{x,y}(w) = \begin{cases} \eta(w) & w \neq x, y \\ \eta(x) - 1 & w = x \\ \eta(y) + 1 & w = y \end{cases},$$

is the new configuration after a particle has jumped from x to y, and f is a cylinder (or local) function, that is, a function that depends only on the value of η on a finite number of sites in V. We denote by $(S_t)_{t \geq 0}$ the semigroup generated by (5), and by $\mathbb{E}_\mu$, resp. $\mathbb{E}_\eta$, the expectation for the process with initial distribution a probability measure μ on $\mathcal{X}$, resp. with initial configuration $\eta \in \mathcal{X}$.

The (nearest-neighbour) SEP on $\mathbb{Z}$ is the particular case of (5) with (V,p) given by

$$V = \mathbb{Z}, \quad p(x,y) = d\mathbf{1}_{\{y-x=1\}} + l\mathbf{1}_{\{y-x=-1\}}; \quad d, l \geq 0,\ d + l > 0 \tag{6}$$

Within this category we distinguish the *symmetric*, resp. *asymmetric* exclusion process (SSEP, resp. ASEP), for which $d = l$, resp. $d \neq l$; and the *totally asymmetric* simple exclusion process (TASEP) on $\mathbb{Z}$, for which $dl = 0 < d + l$.

A probability measure μ on $\mathcal{X}$ is said to be *invariant* for the Markov process

generated by (5) if it is invariant with respect to the semigroup $(S_t)_{t\geq 0}$, which is equivalent to

$$\int Lf(\eta)d\mu(\eta) = 0 \tag{7}$$

for every cylinder function f. The set of invariant probability measures is denoted by $\mathcal{I}$. Since $\mathcal{I}$ is convex, by Choquet-Deny Theorem, in order to know $\mathcal{I}$, it is enough to determine the subset of its extremal elements, denoted by $\mathcal{I}_e$.

A probability measure μ is said to be *reversible* if L is a self-adjoint operator in $L^2(\mathcal{X}, \mu)$. Reversible measures are invariant; when they exist, they are usually easier to compute explicitely than non-reversible invariant measures. For instance, the following general result, which will be helpful, can be found (in a slightly different formulation) in [14, Chapter VIII].

Proposition 2.1. *Let S be a countable subset and $\pi(.,.)$ a kernel on S satisfying (4). Let $\rho_. = (\rho_i)_{i\in S}$ be a $[0,1]$-valued family such that, for every $i, j \in S$, the following condition holds:*

$$\rho_i(1-\rho_j)\pi(i,j) = \rho_j(1-\rho_i)\pi(j,i) \tag{8}$$

Define the product measure $\mu_{S,\rho_.}$ on $\{0,1\}^S$ by

$$\mu_{S,\rho_.}(d\eta) = \bigotimes_{i\in S} \mathcal{B}(\rho_i)(d\eta_i), \tag{9}$$

where $\mathcal{B}(\rho)$ denotes the Bernoulli measure with parameter ρ. Then $\mu_{S,\rho_.}$ is reversible with respect to the (S,π) simple exclusion process.

Remark 2.1. *When the family $\rho_.$ has constant value $\rho \in [0,1]$, the product measure defined by (9) will be denoted by $\mu_{S,\rho}$. The subscript S will be dropped whenever there is no ambiguity.*

2.2 The general setup

In the sequel, we shall focus on special choices of V and $p(.,.)$ for which the model has an interesting structure. First, we consider a lattice V of the form

$$V = \mathbb{Z} \times W \tag{10}$$

for some nonempty finite set W. An element x of V will be generically written in the form $x = (x(0), x(1))$, with $x(0) \in \mathbb{Z}$ and $x(1) \in W$. In traffic-flow modeling, we may think of V as a highway, of $\mathbb{Z}$ as a lane, and of x as site $x(0)$ on lane $x(1)$. For $i \in W$,

$$\mathbb{L}_i := \{x \in V : x(0) \in \mathbb{Z},\, x(1) = i\} \tag{11}$$

denotes the i'th lane of V, and η^i the particle configuration on $\mathbb{Z}$, defined by

$$\eta^i(z) = \eta(z,i) \tag{12}$$

for $z \in \mathbb{Z}$. We can view η^i as the configuration on lane i. Another interpretation is that $i \in W$ represents a particle species, then $\eta(z,i) = \eta^i(z)$ is the number of particles of species i at site $z \in \mathbb{Z}$. We also denote by

$$\overline{\eta}(z) = \sum_{i \in W} \eta^i(z) \tag{13}$$

the total number of particles at $z \in \mathbb{Z}$.

Next, we consider kernels $p(.,.)$ of the form

$$p(x,y) = \begin{cases} 0 \text{ if } x(0) \neq y(0) \text{ and } x(1) \neq y(1) & \\ q_i(x(0), y(0)) =: Q_i[y(0) - x(0)] & \text{if} \quad x(1) = y(1) = i \\ q(x(1), y(1)) & \text{if} \quad x(0) = y(0) \end{cases} \tag{14}$$

for $x, y \in V$, where $q(.,.)$ is a kernel on W (that will be given afterwards for the W of interest), and for each $i \in W$, $q_i(.,.)$ is a translation invariant kernel on $\mathbb{Z}$ given by

$$q_i(u,v) = d_i \mathbf{1}_{\{v-u=1\}} + l_i \mathbf{1}_{\{v-u=-1\}}, \quad Q_i(z) = d_i \mathbf{1}_{\{z=1\}} + l_i \mathbf{1}_{\{z=-1\}} \tag{15}$$

for $u, v \in \mathbb{Z}$, where $d_i \geq 0$ and $l_i \geq 0$ are such that $d_i + l_i > 0$.

We shall be interested in translations along $\mathbb{Z}$, but the set W is in general not endowed with a translation operator. We denote by $(\tau_k)_{k \in \mathbb{Z}}$ the group of space shifts on $\mathbb{Z}$. The shift operator τ_k acts on a particle configuration $\eta \in \mathcal{X}$ through

$$(\tau_k \eta)(z,w) := \eta(z+k, w), \quad \forall (z,w) \in \mathbb{Z} \times W \tag{16}$$

It acts on a function $f : \mathcal{X} \to \mathbb{R}$ via

$$(\tau_k f)(\eta) := f(\tau_k \eta), \quad \forall \eta \in \mathcal{X} \tag{17}$$

If μ is a probability measure on $\mathcal{X}$, then τ_k acts on μ via

$$\int_{\mathcal{X}} f(\eta) d(\tau_k \mu)(\eta) := \int_{\mathcal{X}} (\tau_k f)(\eta) d\mu(\eta) \tag{18}$$

for every bounded continuous function $f : \mathcal{X} \to \mathbb{R}$. Last, if $\mathcal{L}$ is a linear operator acting on functions $f : \mathcal{X} \to \mathbb{R}$, then τ_k acts on $\mathcal{L}$ via

$$(\tau_k \mathcal{L}) f := \mathcal{L}(\tau_k f) \tag{19}$$

By an abuse of notation, in what follows, we write τ instead of τ_1. We define $\mathcal{S}$ to be the set of all probability measures on $\mathcal{X}$ that are invariant under the translations τ_k, $k \in \mathbb{Z}$.

2.3 The two-lane SEP

In the sequel, we shall sometimes refer to SEP (resp. SSEP, ASEP, TASEP) as *single-lane* or *one-dimensional* SEP (resp. SSEP, ASEP, TASEP). Our basic model is *the two-lane* SEP, which corresponds to

$$W = \{0, 1\} \tag{20}$$

We can view this model as a dynamics on an infinite horizontal ladder, with vertical steps separating its two bars $\mathbb{L}_0$ and $\mathbb{L}_1$, namely:

$$\begin{array}{rcl} \mathbb{L}_0 & = & \{x \in V : x = (z, 0), z \in \mathbb{Z}\} \\ \mathbb{L}_1 & = & \{x \in V : x = (z, 1), z \in \mathbb{Z}\} \end{array} \tag{21}$$

In the traffic interpretation, we call $\mathbb{L}_0$ and $\mathbb{L}_1$ respectively the *upper* and *lower* lane, and the steps between them the direction a car can follow to change lane. Thus we shall henceforth call *downward* jump a jump from $\mathbb{L}_0$ to $\mathbb{L}_1$, and *upward* jump a jump from $\mathbb{L}_1$ to $\mathbb{L}_0$.

Let $p, q \geq 0$ and $d_0, l_0, d_1, l_1 \geq 0$. The two-lane SEP is the dynamics on $\mathcal{X}$ defined by the generator (5) with kernel (14)–(15), in which $q(.,.)$ is given by

$$q(0, 1) = p, \quad q(1, 0) = q \tag{22}$$

This means that, for $x, y \in V$,

$$p(x, y) = \begin{cases} d_0 & \text{if} \quad x, y \in \mathbb{L}_0, y(0) - x(0) = 1 \\ l_0 & \text{if} \quad x, y \in \mathbb{L}_0, y(0) - x(0) = -1 \\ d_1 & \text{if} \quad x, y \in \mathbb{L}_1, y(0) - x(0) = 1 \\ l_1 & \text{if} \quad x, y \in \mathbb{L}_1, y(0) - x(0) = -1 \\ p & \text{if} \quad x \in \mathbb{L}_0, y \in \mathbb{L}_1, x(0) = y(0) \\ q & \text{if} \quad x \in \mathbb{L}_1, y \in \mathbb{L}_0, x(0) = y(0) \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

In other words, particles move one step to the right or to the left on each lane at a rate depending on the lane, and we allow the rate p at which particles go down to be different than the rate q of going up. We shall assume in the sequel that (cf. (15))

$$(d_0 + l_0)(d_1 + l_1) > 0 \tag{24}$$

so that particles can always move on both lanes. However they cannot go from $\mathbb{L}_0$ to $\mathbb{L}_1$ if $p = 0$, nor from $\mathbb{L}_1$ to $\mathbb{L}_0$ if $q = 0$. If $p = q = 0$, the dynamics reduces to two independent SEP's on each lane. Thus, $p + q \neq 0$ introduces interaction between the two lanes. For $i \in W$, we let

$$\gamma_i := d_i - l_i \tag{25}$$

denote the mean drift on lane i. The following symmetry properties of the two-lane SEP will be useful. Define the *lane symmetry* operator $\sigma : \mathcal{X} \to \mathcal{X}$, the

lane exchange operator $\sigma' : \mathcal{X} \to \mathcal{X}$, and the *particle-hole* symmetry operator $\sigma'' : \mathcal{X} \to \mathcal{X}$ by

$$(\sigma\eta)(z,i) = \eta(-z,i);\ (\sigma'\eta)(z,i) = \eta(z,1-i);\ (\sigma''\eta)(z,i) = 1 - \eta(z,i) \tag{26}$$

for $\eta \in \mathcal{X}$ and $(z,i) \in V$. Let us call the process defined by the generator (5) with transition kernel (23) the $(d_0,l_0);(d_1,l_1);(p,q)$-two-lane SEP. The definition of the two-lane SEP dynamics implies the following.

Lemma 2.1. *Let $(\eta_t)_{t\geq 0}$ be a $(d_0,l_0);(d_1,l_1);(p,q)$-two-lane SEP.*
Then the image of this process by σ, resp. σ', σ'', is a $(l_0,d_0);(l_1,d_1);(p,q)$, resp. $(d_1,l_1);(d_0,l_0);(q,p)$, resp. $(l_0,d_0);(l_1,d_1);(q,p)$-two-lane SEP.

Thus, without loss of generality, we shall assume in the sequel that

$$\gamma_0 \geq 0, \quad \gamma_0 + \gamma_1 \geq 0, \quad p \geq q, \quad p > 0 \tag{27}$$

If we view $i \in \{0,1\}$ as a species rather than a lane, the interpretation is as follows: the dynamics within each species is a SEP of $\mathbb{Z}$, and a lane change becomes a spin flip whereby a particle may change its species. The exclusion rule within species implies that a particle cannot change its species if there is already a particle of the other species sitting at the same site. This is the only point where an interaction occurs between the two species.

2.4 Invariant measures for two-lane SEP

Let us start with translation invariant measures. Recalling (1), this corresponds to $\mathcal{I}_0$; the complete description of $\mathcal{I}_e$ will be given in Subsection 2.4.2.

2.4.1 Translation invariant stationary measures for two-lane SEP

The following two-parameter Bernoulli product probability measures will be central. Let us define ν^{ρ_0,ρ_1} for $(\rho_0,\rho_1) \in [0,1]^2$, as the product probability measure on $\mathcal{X}$ such that

$$\nu^{\rho_0,\rho_1}(\eta(x) = 1) = \begin{cases} \rho_0 & x \in \mathbb{L}_0 \\ \rho_1 & x \in \mathbb{L}_1 \end{cases}. \tag{28}$$

In words, the two lanes are independent, and for $i \in \{0,1\}$, the projection of ν^{ρ_0,ρ_1} on lane $\mathbb{L}_i$ is the product Bernoulli measure $\mu_{\mathbb{L}_i,\rho_i}$ with parameter ρ_i, see (9) and Remark 2.1.

When $p = q = 0$, as mentioned after (24), the two lanes evolve as independent SEP's, hence ν^{ρ_0,ρ_1} is an invariant measure for every $(\rho_0,\rho_1) \in [0,1]^2$. We look for a relation between ρ_0 and ρ_1 under which we could have $\nu^{\rho_0,\rho_1} \in \mathcal{I}$ when $p + q \neq 0$. To this end, we define the following subset $\mathcal{F}$ of $[0,1]^2$:

$$\mathcal{F} := \left\{(\rho_0,\rho_1) \in [0,1]^2 : p\rho_0(1-\rho_1) - q\rho_1(1-\rho_0) = 0\right\} \tag{29}$$

The set $\mathcal{F}$ expresses an equilibrium detailed balance relation for vertical jumps: it states that under ν^{ρ_0,ρ_1}, the mean algebraic "creation rate" on each lane (i.e. resulting from jumps from/to the other lane) has to be 0. Similarly to the single-lane SEP, we have the following theorem, proved in Section 3.

Theorem 2.1. *We have that*

$$(\mathcal{I}\cap\mathcal{S})_e = \{\nu^{\rho_0,\rho_1} : (\rho_0,\rho_1)\in\mathcal{F}\} \quad (30)$$
$$= \{\nu_\rho : \rho\in[0,2]\} \quad (31)$$

for a one-parameter family $\{\nu_\rho : 0\le\rho\le 2\}$ of probability measures on $\mathcal{X}$, where the parameter ρ represents the total mean density over the two lanes:

$$\mathbb{E}_{\nu_\rho}\left\{\eta^0(0)+\eta^1(0)\right\} = \rho \quad (32)$$

Remark 2.2. *When $q=0$, the invariant measures ν_ρ can be guessed naturally. Indeed in this case, particles cannot move upwards from $\mathbb{L}_1$ to $\mathbb{L}_0$. Thus if lane 0 is empty, it remains empty and lane 1 behaves as an autonomous SEP. Hence, for $\rho\in[0,1]$, the measure $\nu^{0,\rho}$ (which has global density ρ over the two lanes) is invariant for the two-lane SEP, because its restriction to lane 1 is invariant for the SEP on this lane. Similarly, if lane 1 is full, it remains full and lane 0 evolves as an autonomous SEP. Hence, for $\rho\in[1,2]$, the measure $\nu^{\rho-1,1}$ (which has global density ρ over the two lanes) is invariant for the two-lane SEP. This is consistent with the fact that for $q=0$, (29) yields (see (71) later on)*

$$\mathcal{F} = \{(0,\rho) : \rho\in[0,1]\}\cup\{(\rho-1,1) : \rho\in[1,2]\}$$

Remark 2.3. *Definition (29) was interpreted above as a detailed balance condition, but this is related only to the vertical part of the dynamics. The measure ν^{ρ_0,ρ_1} is in general not reversible, unless $d_i=l_i$ for every $i\in W$, in which case the relations (8) hold.*

2.4.2 Structure of invariant measures for two-lane SEP

We are now interested in $\mathcal{I}_e$ rather than $(\mathcal{I}\cap\mathcal{S})_e$, and need to consider *blocking-type* configurations adapted to our setting. Blocking configurations for simple exclusion on a general countable set of sites S were defined in [13]. There, for $S=\mathbb{Z}$, the set of blocking configurations is given by

$$\mathcal{X}_1 := \left\{\eta\in\{0,1\}^{\mathbb{Z}} : \sum_{x>0}[1-\eta(x)]+\sum_{x\le 0}\eta(x)<+\infty\right\} \quad (33)$$

and an invariant probability measure supported on $\mathcal{X}_1$ is called a *blocking measure*. For the two-lane model, we must define the following set:

$$\mathcal{X}_2 := \left\{\eta\in\mathcal{X} : \sum_{x\in V:\,x(0)>0}[1-\eta(x)]+\sum_{x\in V:\,x(0)\le 0}\eta(x)<+\infty\right\} \quad (34)$$

In our model, a *blocking measure* will be an invariant probability measure supported on $\mathcal{X}_2$. In Appendix A, we discuss other settings where our approach also yields characterization results. The set of blocking configurations has to be adapted to each model. Among these models are the Misanthrope's process, a *single-lane* particle system with several possible particles per site, for which the definition of blocking configurations can be found in [4]. Let

$$\begin{array}{ll} \mathcal{B}_1 := \{(0,1),(1,0),(1,2),(2,1)\}, & \mathcal{B}_2 := \{(0,2)\} \\ \mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2, & \mathcal{D} := \{(\rho,\rho) : \rho \in [0,2]\} \end{array} \tag{35}$$

Let $(\rho^-,\rho^+) \in [0,2]^2 \setminus \mathcal{D}$, that we call a *shock*. A probability measure μ on $\mathcal{X}$ is called a (ρ^-,ρ^+)-*shock measure* if

$$\lim_{n\to-\infty} \tau_n\mu = \nu_{\rho^-}, \quad \lim_{n\to+\infty} \tau_n\mu = \nu_{\rho^+} \tag{36}$$

in the sense of weak convergence, for ν_ρ defined in (31). The *amplitude* of the shock (or of the shock measure) is by definition $|\rho^+ - \rho^-|$.

We can now state the results of this section. Since they include many different cases, for the sake of readability, we will state them in several steps. The following theorem is proved in Section 4.

Theorem 2.2. *(i) There exist a (possibly empty) subset $\mathcal{R}$ of $[0,2]^2 \setminus (\mathcal{D} \cup \mathcal{B})$ containing only shocks of amplitude 1, a (possibly empty) subset $\mathcal{R}'$ of $\mathcal{B}_1$, and for each $(\rho^-,\rho^+) \in \mathcal{R} \cup \mathcal{R}'$, a (ρ^-,ρ^+)-shock measure denoted ν_{ρ^-,ρ^+}, such that*

$$\mathcal{I}_e = \{\nu_\rho : 0 \le \rho \le 2\} \cup (Bl_1 \cup Bl_2) \cup \{\tau_z\nu_{\rho^-,\rho^+} : z \in \mathbb{Z},\ (\rho^-,\rho^+) \in \mathcal{R}\} \tag{37}$$

$$Bl_1 = \{\tau_z\nu_{\rho^-,\rho^+} : (\rho^-,\rho^+) \in \mathcal{R}',\ z \in \mathbb{Z}\} \tag{38}$$

$$Bl_2 = \{\nu \in \mathcal{I}_e : \nu \text{ is a } (0,2)\text{-shock measure}\} \tag{39}$$

(ii) The sets $\mathcal{R}$, $\mathcal{R}'$ and Bl_2 enjoy the following properties:

(a) The set Bl_2 is stable by translations, and outside the case

$$l_0 = l_1 = q = 0, \tag{40}$$

it contains at most (up to translations) two elements. If Bl_2 contains at least one blocking measure, then it consists exactly (up to translations) of two blocking measures.

(b) The set Bl_2 is empty if

$$q\gamma_0 + p\gamma_1 < 0 \tag{41}$$

(c) Outside the cases

$$p = q \text{ and } \gamma_0 + \gamma_1 = 0, \tag{42}$$

$$\gamma_0 = \gamma_1 = 0, \tag{43}$$

$$q = 0 = \gamma_0\gamma_1, \tag{44}$$

the set $\mathcal{R}$ contains at most one element and $\mathcal{R}'$ at most two elements (up to translations).

(d) Outside (42)–(44), the following holds. Unless $q = 0$ and $\gamma_0 = \gamma_1 > 0$, the set $\mathcal{R} \cup \mathcal{R}'$ contains at most two elements (up to translations). If $q > 0$ and $\gamma_0 + \gamma_1 \neq 0$, the set $\mathcal{R}'$ is empty. If $q > 0$, $q \neq p$ and $\gamma_0 + \gamma_1 = 0 \neq \gamma_0\gamma_1$, the set $\mathcal{R}$ is empty.

Remark 2.4. *In view of (27), condition (41) implies $p > q$ and $\gamma_1 < 0 < \gamma_0$.*

Theorem 2.2 yields the following information. The decomposition (37)–(39) says that every element of $\mathcal{I}_e$ that is a not a product Bernoulli measure is a shock measure of amplitude 1 or 2, and that for a given shock of amplitude 1, an associated shock measure is unique up to translations. Outside the case (40) (which will be further studied in the next theorem), up to translations, we can have *at most* two shock measures of amplitude 2. This case is special because the kernel (23) lacks standard irreducibility assumptions (see Definition 3.1), so usual ordering properties must be weakened (see Definitions 3.2 and 4.1). The only possible shock of amplitude 2 is $(0, 2)$. The $(0, 2)$-shock measures are analogues of *blocking* or *profile* measures in [5]. We shall see below that when both drifts are positive, shocks of amplitude 2 are blocking measures, and under additional assumptions, there are *exactly* two of them modulo translations. Shock measures of amplitude 1 can be divided into two classes with a different meaning. The first one, namely Bl_1, contains measures associated to shocks in $\mathcal{B}_1$. The second one, namely $\mathcal{R}$, is associated to other shock measures of amplitude 1. There cannot exist measures in $\mathcal{B}_1$ outside cases $q = 0$ or $\gamma_0 + \gamma_1 = 0$; they are then zero-flux measures (see Proposition 4.8, *(o)* and *(ii)*). Among elements of Bl_1 are *partial* blocking measures: we shall see below (in Theorem 2.3) that these may only (and do indeed) arise if $q = 0$. Under such measures, one lane carries a $(0, 1)$-shock and the other is either empty or full. The set $\mathcal{R}$ is associated to other shock measures of amplitude 1. We believe that $\mathcal{R}$ is empty and prove that it contains at most one element outside the case (42). This conjecture comes from the belief that the variance of the shock is of order t with a positive diffusion coefficient, as follows from extrapolating the results of [9] for single-lane ASEP. This property is incompatible with a shock stationary state, but suggests (as in [10] for single-lane ASEP) existence of a stationary state for the process seen from a proper random location. In contrast, based on this extrapolation, we expect the diffusion coefficient to vanish in the last case of Theorem 2.2, *(c)*; we have no clear conjecture whether $\mathcal{R}'$ is empty in this case. Under (42), the model is diffusive and nongradient, and we conjecture that the only invariant measures are Bernoulli. We leave the above conjectures for future investigation, as the methods involved to prove them are presumably quite different from those used here.

Next, we provide more information on the sets $\mathcal{R}$, $\mathcal{R}'$, Bl_1 and Bl_2, and obtain a full description of $\mathcal{I}_e$ for a set of parameter values including (40) and (43)–(44). This is the content of Theorem 2.3 below. Its statement will be completed in

Section 2.4.3 by the explicit description of the sets Bl_1 and Bl_2 referred to in the following statements. Recall (27). We define the reduced parameters

$$r := \frac{q}{p}, \quad \mathfrak{a} := \frac{\gamma_0}{\gamma_0 + \gamma_1} \text{ if } \gamma_0 + \gamma_1 \neq 0 \tag{45}$$

and set

$$r_0 := \frac{1 - 2\sqrt{-7 + \sqrt{52}}}{1 + 2\sqrt{-7 + \sqrt{52}}} = 0,042\cdots \tag{46}$$

Due to (27), we have $(\mathfrak{a}, r) \in [0,1] \times [0,1]$.

Theorem 2.3. *(o) If $\gamma_0 > 0$ and $\gamma_1 > 0$, elements of Bl_2 are supported on the set $\mathcal{X}_2$.*

(i) Assume (43). Then $\mathcal{R} = \mathcal{R}' = Bl_2 = \emptyset$, hence

$$\mathcal{I}_e = \{\nu_\rho : \rho \in [0,2]\} \tag{47}$$

• *Assume $q > 0$. Then:*

(ii) Assume either: (a) $d_0/l_0 = d_1/l_1 > 1$; or (b) $l_0 = l_1 = 0$ and $d_0, d_1 > 0$. Then Bl_2 is nonempty and given by (52).

(iii) There exists an open subset $\mathcal{Z}$ of $[0,1] \times [0,1]$, containing $\{1/2\} \times (0, r_0)$, such that $\mathcal{R} = \mathcal{R}' = \emptyset$ for every $(\mathfrak{a}, r) \in \mathcal{Z}$. In particular, if $r \in (0, r_0)$, $d_1 = \lambda d_0$ and $l_1 = \lambda l_0$ with $\lambda \in \mathbb{R}$ close enough to 1, then (37) holds with Bl_2 as in (ii); this yields a complete description of $\mathcal{I}_e$.

• *Assume now $q = 0 < p$. Then a complete description of $\mathcal{I}_e$ can be obtained whenever $\gamma_0 \neq \gamma_1$. More precisely:*

(iv) (a) If $\gamma_0 > 0$ and $\gamma_1 > 0$, then $\mathcal{R}' = \{(0,1);(1,2)\}$; $\mathcal{R}$ is empty if $\gamma_0 \neq \gamma_1$, or contained in $\{(3/2, 1/2)\}$ if $\gamma_0 = \gamma_1$. The set Bl_1 is given by (58). The set Bl_2 is empty unless $l_0 = l_1 = 0$. (b) If $l_0 = l_1 = 0$, Bl_2 is given by (60).

(v) If $\gamma_1 < 0 < \gamma_0$, then $\mathcal{R}' = \{(1,0),(1,2)\}$, $\mathcal{R} = Bl_2 = \emptyset$. The set Bl_1 is given by (61).

(vi) If $\gamma_0 = 0 < \gamma_1$, then $\mathcal{R}' = \{(0,1)\}$, $\mathcal{R} = Bl_2 = \emptyset$. The set Bl_1 is given by (62).

In *(ii)–(vi)* above, the measures in the sets Bl_1 and Bl_2 defined by (52), (58) and (60)–(62) are reversible.

Remark 2.5. *In case (i), when $p = q$, the kernel defined by (23) is symmetric. The result is then a particular case of the general picture ([15, 14]) for symmetric exclusion processes, although our method of proof is different. However when in case (i) we have $p \neq q$, the two-lane SEP is not a symmetric exclusion process, and our result is new.*

Remark 2.6. *For reader's convenience, we summarize here the questions left open by Theorems 2.2 and 2.3.*

1. *When $p = q$ and $\gamma_0 + \gamma_1 = 0$, we can prove that outside the measures ν_ρ, $\mathcal{I}_e$ may only contain shock measures, but we conjecture that there are no shock measures in $\mathcal{I}_e$.*

2. *When $q > 0$ and $\gamma_0 + \gamma_1 \neq 0$, we conjecture that there are no shock measures of amplitude 1. We prove this when q/p is small enough and $\gamma_0/(\gamma_0 + \gamma_1)$ close enough to $1/2$.*

3. *When $q > 0$, $p \neq q$ and $\gamma_0 + \gamma_1 = 0$, we do not know if there are shock measures of amplitude 1.*

4. *When $q = 0$ and $\gamma_0 = \gamma_1 > 0$, we believe that $\mathcal{R}$ is empty but can only prove that it is contained in $\{(3/2, 1/2)\}$*

5. *We conjecture that outside cases (42)– (43), if $q > 0$ and (41) fails, Bl_2 consists (up to horizontal translations) of two blocking measures. We can prove this when $q > 0$ and $d_0/l_0 = d_1/l_1 \in [1; +\infty]$. This conjecture is related to property (vii) of Proposition 4.8.*

2.4.3 Explicit blocking measures in Theorem 2.3

We here complete the statement of Theorem 2.3 by giving the explicit description of Bl_1 and Bl_2 in each case. For this, we need to recall blocking measures denoted hereafter by $\{\widehat{\mu}_n : n \in \mathbb{Z}\}$, which are reversible ([13]) for single-lane ASEP with jump rate d to the right and l to the left, cf. (5)–(6), where $d+l > 0$ and $d - l > 0$. These measures will be building blocks for certain elements of $\mathcal{I}_e$. For $l = 0$, $\widehat{\mu}_n$ is defined by

$$\widehat{\mu}_n := \delta_{\eta_n^*} \quad \text{where} \quad \eta_n^*(x) := \mathbf{1}_{\{x>n\}} \tag{48}$$

where δ denotes the Dirac measure. In the sequel, we shall also use notations (48) by extension for $n = \pm\infty$. In (48), $\eta^*_{-\infty}$ and $\eta^*_{+\infty}$ are respectively understood as the configuration with all 1's and the one with all 0's. When $l > 0$, $\widehat{\mu}_n$ is defined as follows. First, set

$$\rho_i^c := \frac{c\left(\frac{d}{l}\right)^i}{1 + c\left(\frac{d}{l}\right)^i} \tag{49}$$

where $c > 0$. The measure $\mu_{\rho_\cdot} = \mu_{\mathbb{Z},\rho_\cdot}$ (cf. Definition (9) and Remark 2.1) is supported on the set $\mathcal{X}_1$ defined by (33). On this set, a function H_1 can be defined by

$$H_1(\eta) := \sum_{x \leq 0} \eta(x) - \sum_{x>0} [1 - \eta(x)] \tag{50}$$

One can then define the probability measure (which does not depend on the choice of c)

$$\widehat{\mu}_n := \mu_{\rho^c_\cdot}\left(. \,|H_1(\eta) = n\right) \tag{51}$$

We can now give details for Theorem 2.3.

Case *(ii)*. We set

$$Bl_2 := \{\check{\nu}_z : z \in \mathbb{Z}\} \cup \{\widehat{\nu}_z : z \in \mathbb{Z}\} \tag{52}$$

where the measures $\check{\nu}_z$ and $\widehat{\nu}_z$ are defined below distinguishing cases *(ii), (a)* and *(ii), (b)*:

Case (ii), (a). Let $\theta = d_0/l_0 = d_1/l_1$. Define

$$\rho^c_{z,i} := \frac{c\theta^z \left(\frac{p}{q}\right)^i}{1 + c\theta^z \left(\frac{p}{q}\right)^i}, \quad (z,i) \in \mathbb{Z} \times W, c > 0 \tag{53}$$

We consider the probability measure μ_{ρ^c} on $\mathcal{X}$ under which the random variables $\{\eta(z,i) : x \in \mathbb{Z}, i \in W\}$ are independent, and $\eta(z,i)$ is Bernoulli distributed with parameter $\rho^c_{z,i}$.

Remark 2.7. *The measures μ_{ρ^c} are analogues in this context of the 2-dimensional blocking measures constructed in [6, Theorem 2], which are v-profile measures (for any $i \in \{0,1\}$) where $v = \left(\ln \frac{d_i}{l_i}, \ln \frac{p}{q}\right)$.*

We define the following function on $\mathcal{X}_2$ (cf. (34)):

$$H_2(\eta) := \sum_{x \in V\colon x(0) \le 0} \eta(x) - \sum_{x \in V\colon x(0) > 0} [1 - \eta(x)] \tag{54}$$

Note that $\mathcal{X}_2$ is stable by the dynamics, and H_2 is a conserved quantity for the process on $\mathcal{X}_2$. The measures involved in (52) are defined in the following lemma, proved in Subsection 4.4 along with Theorem 2.3.

Lemma 2.2. *The measure μ_{ρ^c} is supported on $\mathcal{X}_2$, and the measures defined below do not depend on $c > 0$:*

$$\check{\nu}_n := \mu_{\rho^c}(.\,|H_2(\eta) = 2n), \quad \widehat{\nu}_n := \mu_{\rho^c}(.\,|H_2(\eta) = 2n+1), \quad n \in \mathbb{Z} \tag{55}$$

These measures satisfy the relations

$$\check{\nu}_n = \tau_n \check{\nu}_0, \quad \widehat{\nu}_n = \tau_n \widehat{\nu}_0, \quad n \in \mathbb{Z} \tag{56}$$

Case (ii), (b). Let, for $x \in V$,

$$\begin{aligned} \check{\eta}(x) &= 1_{\{x(0)>0\}} \\ \widehat{\eta}^0(x) &= 1_{\{x(0)>0\}} + 1_{\{x=(0,0)\}} \\ \widehat{\eta}^1(x) &= 1_{\{x(0)>0\}} + 1_{\{x=(0,1)\}}. \end{aligned}$$

We define the measures $\check{\nu}_0$ and $\widehat{\nu}_0$ through

$$\check{\nu}_0 = \delta_{\check{\eta}}, \quad \widehat{\nu}_0 = \frac{q}{p+q}\delta_{\widehat{\eta}^0} + \frac{p}{p+q}\delta_{\widehat{\eta}^1} \tag{57}$$

We define also $\check{\nu}_z = \tau_{-z}\check{\nu}_0$ and $\widehat{\nu}_z = \tau_{-z}\widehat{\nu}_0$ for every $z \in \mathbb{Z}$.

Cases *(iv)–(vi)*. Using the blocking measures for single lane ASEP, we define the following two-lane measures. For $n \in \mathbb{Z}$, we denote by $\nu^{\perp,+\infty,n}$ and $\nu^{\perp,n,-\infty}$ the probability measures on $\mathcal{X}$ defined as follows. Under $\nu^{\perp,+\infty,n}$, $\eta^0 = \eta^*_{+\infty}$, see definition (48) (i.e. lane 0 is empty), and $\eta^1 \sim \widehat{\mu}_n$, where $\widehat{\mu}_n$ is given by (51) with $l = l_1$ and $d = d_1$ if $l_1 > 0$, or by (48) if $l_1 = 0$ (where $\sim$ means equality in distribution). Under $\nu^{\perp,n,-\infty}$, $\eta^1 = \eta^*_{-\infty}$ (i.e. lane 1 is full) and $\eta^0 \sim \widehat{\mu}_n$, where $\widehat{\mu}_n$ is given by (51) with $l = l_0$ and $d = d_0$ if $l_0 > 0$, or by (48) if $l_0 = 0$. In the following cases, whenever it is not empty, the set Bl_1 reduces to partial blocking measures; and whenever it is not empty, the set Bl_2 consists of blocking measures.

Case *(iv)*. *(a)* The set Bl_1 is given by

$$Bl_1 \quad := \quad \left\{\nu^{\perp,+\infty,n} : n \in \mathbb{Z}\right\} \cup \left\{\nu^{\perp,n,-\infty} : n \in \mathbb{Z}\right\} \tag{58}$$

(b). Let $\mathbb{B}$ denote the set of $(i,j) \in \mathbb{Z}^2$ such that $i \geq j$, and set $\overline{\mathbb{B}} := \mathbb{B} \cup \{(+\infty, n), (n, -\infty) : n \in \mathbb{Z}\}$. For $(i,j) \in \overline{\mathbb{B}}$, let $\nu^{\perp,i,j}$ denote the Dirac measure supported on the configuration $\eta^{\perp,i,j}$ defined by (recalling (48))

$$\eta^{\perp,i,j}(z,0) = \eta^*_i(z), \quad \eta^{\perp,i,j}(z,1) = \eta^*_j(z) \tag{59}$$

for every $z \in \mathbb{Z}$. The set Bl_2 is given by

$$Bl_2 \quad := \quad \left\{\nu^{\perp,i,j} : (i,j) \in \mathbb{B}\right\} \tag{60}$$

Case *(v)*. For $n \in \mathbb{Z}$, we denote by $\nu^{\perp,+\infty,n\leftarrow}$ the probability measure on $\mathcal{X}$ defined as follows. Recall the lane symmetry operator σ defined by (26). Under $\nu^{\perp,+\infty,n\leftarrow}$, $\eta^0 = \eta^*_{+\infty}$ and $\sigma\eta^1 \sim \widehat{\mu}_n$, where $\widehat{\mu}_n$ is given by (51) with $l = l_1$ and $d = d_1$ if $l_1 > 0$, or by (48) if $l_1 = 0$. The set Bl_1 is then given by

$$Bl_1 := \left\{\nu^{\perp,+\infty,n\leftarrow} : n \in \mathbb{Z}\right\} \cup \left\{\nu^{\perp,n,-\infty} : n \in \mathbb{Z}\right\} \tag{61}$$

Case *(vi)*. The set Bl_1 is given by

$$Bl_1 := \left\{\nu^{\perp,+\infty,n} : n \in \mathbb{Z}\right\} \tag{62}$$

2.5 Multilane SEP and rotational invariance

In this section, we consider the general model defined by (5) with (10), (14) and (15). Without loss of generality, we may consider $W = \{0, \ldots, n-1\}$. We are interested in a generalization of the two-lane model with $p = q$ (cf. (22)). To this end, we introduce the following assumption.

Assumption 2.1. *$W = \mathbb{T}_n$ is a torus, and $q(.,.)$ is an irreducible translation-invariant kernel, that is $q(i,j) = Q(j-i)$ for some function $Q : \mathbb{T}_n \to [0, +\infty)$.*

For $\rho \in [0, n]$, we denote by ν_ρ the product measure on $\mathcal{X}$ such that

$$\nu_\rho \{\eta(z,i) = 1\} = \frac{\rho}{n} \tag{63}$$

for every $(z,i) \in \mathbb{Z} \times W$. For Theorems 2.1, 2.2 and 2.3, the scheme of proof laid out in Sections 3 to 5 carries over to the multilane model. Here, since $W = \mathbb{T}_n$, in addition to the shift operator τ along $\mathbb{Z}$ already considered, we can consider the translation operator τ' along W. Following [6, page 2309], we shall call a probability measure on $\mathcal{X}$ *rotationally invariant* if it is invariant by τ'. The *open question 1. for the ladder process* raised in [6] is whether, when d_i and l_i are independent of i (i.e. the horizontal dynamics is the same on each lane), *all* invariant measures are rotationally invariant. We give a positive answer to this question in item *(iii)* of the following theorem.

Theorem 2.4. *Under Assumption 2.1, the following hold:*

(o) We have $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho,\ \rho \in [0, n]\}$.

(i) For $k = 1, \dots, n$, let $(\rho_k^-, \rho_k^+) = \left(\frac{n-k}{2}, \frac{n+k}{2} = n - \rho_k^-\right)$. Then: (a)

$$\mathcal{I}_e = \{\nu_\rho :\ \rho \in [0, n]\} \cup \bigcup_{k=1}^{n} \mathcal{I}_k \tag{64}$$

where $\mathcal{I}_k$ is a (possibly empty) set of (ρ_k^-, ρ_k^+)-shock measures of amplitude k, which contains at most (up to horizontal translations) k measures. (b) If $\gamma_i > 0$ for all i, $\mathcal{I}_n$ is supported on the set $\mathcal{X}_n$ defined by the right-hand side of (34). (c) If d_i/l_i does not depend on i, $\mathcal{I}_n$ consists (up to horizontal translations) of n explicit blocking measures ν_i defined below for $i = 0, \dots, n-1$.

(ii) If $\gamma_i := d_i - l_i = 0$ for all $i \in W$, then $\mathcal{I}_e = \{\nu_\rho :\ \rho \in [0, n]\}$.

(iii) If d_i and l_i do not depend on i, any invariant measure is rotationally invariant.

The blocking measures in *(i), (c)* are defined as in cases *(ii), (a)* and *(ii), (b)* of Theorem 2.3:

First case. If $l_i > 0$ for all i, we define $\rho^c_{z,i}$ as in (53), with $\theta = d_i/l_i$ and p/q replaced by 1. For $i = 0, \dots, n-1$, we define the conditioned measures (independent of the choice of $c > 0$ as in (55))

$$\nu_i \quad := \quad \mu_{\rho^c_\cdot}\left(.\,|H_n(\eta) = i\right) \tag{65}$$

where $H_n(\eta)$ is defined as the right-hand side of (54).

Second case. If $l_i = 0 < d_i$ for all i, we define the configurations

$$\eta_A := \mathbf{1}_{\{x(0)\geq 0\}} + \mathbf{1}_{\{x(0)=-1,\, x(1)\in A\}}, \quad A \subset \{0,\dots,n-1\} \tag{66}$$

Then ν_i is the law of a random configuration η_A, where A is a uniformly chosen random subset of $\{0,\dots,n-1\}$ such that $|A| = i$:

$$\nu_i := \binom{n}{i}^{-1} \sum_{A\subset\{0,\dots,n-1\}:\,|A|=i} \delta_{\eta_A} \tag{67}$$

3 Proof of Theorem 2.1

The proof of Theorem 2.1 mainly adapts the scheme of [13, Theorem 1.1] to our model. However when $q = l_0 = l_1 = 0$, additional arguments are required because the kernel (23) does not satisfy usual irreducibility assumptions.
First, in Subsection 3.1, we show how to parametrize the set $\mathcal{F}$ in (29) by the global density over the two lanes and establish invariance of the associated product measures given in (30). Next, we introduce coupling prerequisites in Subsection 3.2, and complete the proof of characterization in Subsection 3.4.

3.1 Parametrization and proof of invariance

The following lemma will lead to the parametrization (31). Note that to be exhaustive we choose here to include case 2 in Lemma 3.1 below although it could be omitted thanks to the symmetry restriction (27).

Lemma 3.1. *(i) The mapping $\psi : \mathcal{F} \to [0,2]$; $(\rho_0,\rho_1) \mapsto \psi(\rho_0,\rho_1) := \rho_0 + \rho_1$, is a bijection.*

(ii) Its inverse is of the form $\psi^{-1}(\rho) = (\widetilde{\rho}_0(\rho), \widetilde{\rho}_1(\rho))$, where $\widetilde{\rho}_1(\rho) := \rho - \widetilde{\rho}_0(\rho)$, and the function $\rho \mapsto \widetilde{\rho}_0(\rho)$ is given by the following formulae:

Case 1a. $pq \neq 0$, $p \neq q$. Then, for $r = q/p$, cf. (45),

$$\widetilde{\rho}_0\,(\rho) := \frac{\rho}{2} + \frac{r+1-\sqrt{(r+1)^2+\rho(r-1)^2(\rho-2)}}{2(r-1)}, \tag{68}$$

Case 1b. $p = q \neq 0$. Then

$$\widetilde{\rho}_0(\rho) := \frac{\rho}{2} \tag{69}$$

Case 2. $p = 0 < q$. Then

$$\widetilde{\rho}_0(\rho) := \min(\rho, 1) \tag{70}$$

Case 3. $q = 0 < p$. Then

$$\widetilde{\rho}_0(\rho) := \max(\rho - 1, 0) \tag{71}$$

Remark 3.1. *In Lemma 3.1, the formulae in (ii) imply that for $pq > 0$, $\widetilde{\rho}_i(\rho)$ strictly increases from 0 to 1 as ρ increases from 0 to 2, and $\widetilde{\rho}_i \in C^1([0,2])$.*

Next we define

$$\nu_\rho := \nu^{\widetilde{\rho}_0(\rho),\widetilde{\rho}_1(\rho)} \tag{72}$$

By (72) and (28), we have (recalling definition (12)), for every $i \in \{0,1\}$,

$$\mathbb{E}_{\nu_\rho}[\eta^i(0)] = \widetilde{\rho}_i(\rho) \tag{73}$$

which implies (32).

Remark 3.2. *It follows from (ii) of Lemma 3.1 that the measure ν_ρ is weakly continuous and stochastically nondecreasing with respect to ρ. Namely, if $\rho < \rho'$ then $\nu_\rho \le \nu_{\rho'}$.*

Proof of Lemma 3.1. We have to prove that, for $\rho \in [0,2]$, the equation $\rho_0+\rho_1 = \rho$ has a unique solution $(\rho_0,\rho_1) \in \mathcal{F}$; then we define $\rho_i = \widetilde{\rho}_i(\rho)$ for $i \in \{0,1\}$. For $s > 0$, we define a mapping ϕ_s from $[0,1]$ to $[0,1]$ by

$$\phi_s(\rho_0) := \frac{s\rho_0}{1-\rho_0+s\rho_0}, \quad \forall \rho_0 \in [0,1] \tag{74}$$

One can then distinguish the following cases for $\mathcal{F}$:

Case 1. $pq \neq 0$. Then

$$\mathcal{F} := \left\{(\rho_0,\rho_1) \in [0,1]^2 : \rho_1 = \phi_{p/q}(\rho_0)\right\} \tag{75}$$

Case 2. $p = 0 < q$. Then

$$\mathcal{F} := ([0,1] \times \{0\}) \cup (\{1\} \times [0,1]) \tag{76}$$

Case 3. $q = 0 < p$. Then

$$\mathcal{F} := ([0,1] \times \{1\}) \cup (\{0\} \times [0,1]) \tag{77}$$

Equalities (70)–(71) follow from (76)–(77). For (68)–(69), using (75), we equivalently show that, for $\rho \in [0,2]$, the equation

$$\rho_0 + \phi_{p/q}(\rho_0) = \rho \tag{78}$$

has a unique solution $\rho_0 =: \widetilde{\rho}_0(\rho) \in [0,1]$ and deduce $\widetilde{\rho}_1(\rho)$. If $p = q > 0$, (75) with $s = 1$ yields $\phi_1(\rho_0) = \rho_0$, whence (69). If $p \neq q$, $p > 0$ and $q > 0$, (75) and (78) yield a quadratic equation for ρ_0, and (68) is its unique solution in $[0,1]$. □

The following lemma shows that the measures in (30) of Theorem 2.1 are indeed extremal translation invariant and invariant probability measures. Stationarity can be derived from [6, Theorem 1], but we give an independent proof based on prior knowledge of invariance along horizontal and vertical layers.

Lemma 3.2. *Let* $(\rho_0, \rho_1) \in \mathcal{F}$. *Then* $\nu^{\rho_0,\rho_1} \in (\mathcal{I} \cap \mathcal{S})_e$.

Proof of Lemma 3.2. Let f be a cylinder function on $\mathcal{X}$. Note that the generator (5) has the following structure,

$$L = \sum_{i \in W} L_h^i + \sum_{z \in \mathbb{Z}} L_v^z \tag{79}$$

where, for $i \in W$ and $z \in \mathbb{Z}$,

$$\begin{aligned} L_h^i f(\eta) = \sum_{z \in \mathbb{Z}} \sum_{z' \in \{z+1, z-1\}} & p((z,i),(z',i))\, \eta((z,i))\, (1 - \eta((z',i))) \\ & \times \left(f\left(\eta^{(z,i),(z',i)}\right) - f(\eta) \right) \\ L_v^z f(\eta) = \sum_{i,j \in W} & p((z,i),(z,j))\, \eta((z,i))\, (1 - \eta((z,j))) \left(f\left(\eta^{(z,i),(z,j)}\right) - f(\eta) \right) \end{aligned}$$

In other words, L_h^i, acting only on η^i, and being translation invariant along the $\mathbb{Z}$ direction, describes the evolution of the process on $\mathbb{L}_i$, which is the one of a (single-lane) SEP; while L_v^z, acting only on $\{z\} \times W$, describes the motion of particles along $\{z\} \times W$, that is, the displacements from one lane to another at a fixed spatial location z.

The statement $\nu^{\rho_0,\rho_1} \in \mathcal{S}$ holds because ν^{ρ_0,ρ_1} is a product Bernoulli measure whose parameters are uniform in the $\mathbb{Z}$-direction. Considering (79), to prove that ν^{ρ_0,ρ_1} belongs to $\mathcal{I}$, it is enough to show that, for $i \in W$ and $z \in \mathbb{Z}$,

$$\int L_h^i f(\eta)\, d\nu^{\rho_0,\rho_1}(\eta) = 0 \tag{80}$$

and

$$\int L_v^z f(\eta)\, d\nu^{\rho_0,\rho_1}(\eta) = 0. \tag{81}$$

Let us write, for a fixed $i \in W$,

$$\eta = (\eta^i, \eta'), \quad \nu^{\rho_0,\rho_1}(d\eta) = \nu^i(d\eta^i) \otimes \nu'(d\eta') \tag{82}$$

where η' denotes the restriction of η to lanes other than i. Note that ν^i is invariant for L_h^i because L_h^i is the generator of a single-lane SEP on $\mathbb{L}_i$ and ν^i is a homogeneous product Bernoulli measure. Since L_h^i acts only on η^i, we have

$$\int L_h^i f(\eta)\, d\nu^{\rho_0,\rho_1}(\eta) = \int \left(\int L_h^i [f(.,\eta')](\eta^i,\eta') d\nu^i(\eta^i) \right) d\nu'(\eta') = 0 \tag{83}$$

This establishes (80).

Remark 3.3. *In (83), the notation* $L_h^i[f(.,\eta')](\eta^i,\eta')$ *means that we apply the generator* L_h^i *to the function* $\eta^i \mapsto f(\eta^i,\eta')$ *for constant* η'. *Though* L_h^i *acts only on* η^i, *the resulting function depends both on* η^i *and* η' *(in the same way as a partial derivative operator acts only on one variable but yields a function depending on all variables: here* η^i *plays a role similar to the variable with respect to which one differentiates).*

We can similarly write, for a fixed $z \in \mathbb{Z}$,

$$\eta = ({}^z\eta, \eta''), \quad \nu_\rho(d\eta) = {}^z\nu(d\,{}^z\eta) \otimes \nu''(d\eta'') \tag{84}$$

where ${}^z\eta$ is the restriction of η to $\{z\} \times W$, and η'' its restriction to the complement of $\{z\} \times W$. So, to prove (81), it is enough to prove that ${}^z\nu$ is invariant for L_v^z. The latter is the generator of a simple exclusion process on $\{z\} \times W$. The invariance of ${}^z\nu$ follows from Proposition 2.1 applied to $S = \{z\} \times W$, $\pi((z,0),(z,1)) = p$, $\pi((z,1),(z,0)) = q$, and definition (29) of $\mathcal{F}$. Finally, since $\nu^{\rho_0,\rho_1} \in \mathcal{S}$ is a homogeneous product measure, it is spatially ergodic, that is extremal in $\mathcal{S}$, and thus also in $\mathcal{I} \cap \mathcal{S}$. □

3.2 Coupling, attractiveness and discrepancies

Let us first recall these properties for a general SEP; we refer to [14, Chapter VIII, Section 2] for details.

Coupling. We recall the so-called *Harris graphical representation* ([12]). Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space that supports a family of independent Poisson processes $\mathcal{N} = \{\mathcal{N}_{(x,y)} : (x,y) \in V \times V\}$ where $\mathcal{N}_{(x,y)}$ has intensity $p(x,y)$. For a given $\omega \in \Omega$, we let the process evolve according to the following rule: if there is a particle at site $x \in V$ at time t^- where $t \in \mathcal{N}_{(x,y)}$, it will attempt to jump to site y. The attempt is suppressed if at time t^- site y is occupied.
The graphical construction allows to couple the evolutions from different initial configurations through *basic coupling*, that is, by using the same Poisson processes for them. In particular, if $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ are two processes coupled in this way, $(\eta_t, \xi_t)_{t\geq 0}$ is a Markov process on $\mathcal{X} \times \mathcal{X}$ whose generator $\widetilde{L}$ is given by

$$\begin{aligned}
\widetilde{L}f(\eta,\xi) \quad := \quad & \sum_{x,y\in V} p(x,y)[\eta(x)(1-\eta(y)) \wedge \xi(x)(1-\xi(y))]\,[f(\eta^{x,y},\xi^{x,y}) - f(\eta,\xi)] \\
+ \quad & \sum_{x,y\in V} p(x,y)[\eta(x)(1-\eta(y)) - \xi(x)(1-\xi(y))]^+\,[f(\eta^{x,y},\xi) - f(\eta,\xi)] \\
+ \quad & \sum_{x,y\in V} p(x,y)[\eta(x)(1-\eta(y)) - \xi(x)(1-\xi(y))]^-\,[f(\eta,\xi^{x,y}) - f(\eta,\xi)]
\end{aligned} \tag{85}$$

We shall denote by $(\widetilde{S}_t)_{t\geq 0}$ the semigroup generated by $\widetilde{L}$, by $\widetilde{\mathcal{I}}$ the set of invariant probability measures for $\widetilde{L}$, by $\widetilde{\mathcal{S}}$ the set of probability measures on $\mathcal{X} \times \mathcal{X}$ that are invariant with respect to translations along $\mathbb{Z}$, and by $\widetilde{\mathbb{E}}_{\widetilde{\mu}}$ the expectation for the coupled process with initial distribution a probability measure $\widetilde{\mu}$ on $\mathcal{X} \times \mathcal{X}$.

Attractiveness. There is a natural partial order on $\mathcal{X}$, namely, for $\eta, \xi \in \mathcal{X}$,

$$\eta \leq \xi \quad \text{ if and only if } \quad \forall x \in V,\, \eta(x) \leq \xi(x) \tag{86}$$

We shall write $\eta < \xi$ if $\eta \leq \xi$ and $\eta \neq \xi$.
If $\eta \leq \xi$ or $\eta \geq \xi$, we say that η and ξ are *ordered* configurations.

The order (86) endows an order on the set $\mathcal{M}_1$ in the following way. A function f on $\mathcal{X}$ is said to be *increasing* if and only if $\eta \leq \xi$ implies $f(\eta) \leq f(\xi)$. For two probability measures μ_0, μ_1 on $\mathcal{X}$, we write $\mu_0 \leq \mu_1$ if and only if for every increasing function f on $\mathcal{X}$ we have $\int f d\mu_0(\eta) \leq \int f d\mu_1(\eta)$. We shall write $\mu_1 < \mu_2$ if $\mu_1 \leq \mu_2$ and $\mu_1 \neq \mu_2$. We say μ_1 and μ_2 are *ordered* if $\mu_1 \leq \mu_2$ or $\mu_2 \leq \mu_1$. In particular, $\mu_1 \leq \mu_2$ if there exists a measure $\widetilde{\mu}(d\eta, d\xi)$ with marginals $\mu_1(d\eta)$ and $\mu_2(d\xi)$ (that is a *coupling* of μ_1 and μ_2) supported on $\{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta \leq \xi\}$; such a coupling is called an *ordered coupling*.

The basic coupling shows that the simple exclusion process is *attractive*, that is, the partial order (86) is conserved by the dynamics. In other words,

$$\forall \eta_0, \xi_0 \in \mathcal{X},\ \eta_0 \leq \xi_0 \Rightarrow \forall t \geq 0,\ \eta_t \leq \xi_t \ \text{ a.s.} \tag{87}$$

This implies, for two probability measures μ, ν on $\mathcal{X}$,

$$\mu \leq \nu \Rightarrow \mu S_t \leq \nu S_t \tag{88}$$

Discrepancies. If $(\eta, \xi) \in \mathcal{X} \times \mathcal{X}$, we say that at $x \in V$ there is an η *discrepancy* if $\eta(x) > \xi(x)$, a ξ discrepancy if $\eta(x) < \xi(x)$, a coupled particle if $\eta(x) = \xi(x) = 1$, a hole if $\eta(x) = \xi(x) = 0$. An η and a ξ discrepancy are called *opposite discrepancies*. The evolution of the coupled process can be formulated as follows. At a time $t \in \mathcal{N}_{(x,y)}$, a discrepancy or a coupled particle at x exchanges with a hole at y; a coupled particle at x exchanges with a discrepancy at y; if there is a pair of opposite discrepancies at x and y, they are replaced by a hole at x and a coupled particle at y. We call this a *coalescence*. This shows that no new discrepancy can ever be created.

Given an initial tagged discrepancy, we may follow its motion over time. We state in this context a classical *finite propagation* property for discrepancies. Single-lane versions of this statement can be found e.g. in [5, Lemma 3.1] or [2, Lemma 3.1, Lemma 3.2]. Proofs are similar for the two-lane model.

Proposition 3.1. *There exist constants $\sigma, C, C' > 0$ such that the following holds. Assume $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ are two coupled two-lane SEP's with at least one discrepancy at time 0. Let $X_t = (X_t(0), X_t(1)) \in \mathbb{Z} \times W$ denote the position of a tagged discrepancy at time t. Then:*
(i) Outside probability e^{-Ct}, it holds that $|X_t(0) - X_0(0)| \leq (1+\sigma)t$.
(ii) Similarly, if we assume $\eta_0(z,i) = \xi_0(z,i)$ for all $z \in [a,b]$ and $i \in \{0,1\}$, where $a, b \in \mathbb{Z}$ and $a < b$, then outside probability $e^{-C't}$, $\eta_t(z,i) = \xi_t(z,i)$ for all $z \in [a+\sigma t, b-\sigma t]$ and $i \in \{0,1\}$.

3.3 Irreducibility and discrepancies

As for general SEP, a crucial tool to prove Theorem 2.1 is an irreducibility property. We thus begin with the following definitions and properties.

For $x, y \in V$ such that $x \neq y$, and $n \in \mathbb{N}$, we write $x \overset{n}{\to}_p y$ if there exists a path $(x = x_0, \dots, x_{n-1} = y)$ of length n such that $p(x_k, x_{k+1}) > 0$ for $k = 0, \dots, n-1$. We write $x \to_p y$ if there exists $n \in \mathbb{N}$ such that $x \overset{n}{\to}_p y$. We omit mention of p whenever there is no ambiguity on the kernel. We say x and y are p-connected if $x \to_p y$ or $y \to_p x$. We say two configurations η, ξ in $\mathcal{X}$ are *p-ordered* if there exists no $(x, y) \in V \times V$ such that x and y are p-connected and (η, ξ) has opposite discrepancies at x and y.

Definition 3.1. *The kernel $p(.,.)$ is weakly irreducible if, for every $(x, y) \in V \times V$ such that $x \neq y$, x and y are p-connected.*

The above notion is weaker than the *usual* irreducibility property, for which a stronger notion of p-connection is required, namely $x \to_p y$ *and* $y \to_p x$. For instance, the kernel (6) is irreducible if and only if $dl > 0$; if $dl = 0$, it is weakly irreducible but not irreducible. For our two-lane and multilane models, we need the following lemma.

Lemma 3.3. *(i) The two-lane kernel $p(.,.)$ given by (23) is weakly irreducible except when $q = 0$ and both lanes are totally asymmetric in the same direction, that is*

$$d_0 l_0 + d_1 l_1 = 0 < d_0 d_1 + l_0 l_1 \tag{89}$$

(ii) The multilane kernel $p(.,.)$ given by (14) is weakly irreducible under assumption 2.1.

Proof of Lemma 3.3.

Proof of (i). Let $x, y \in \mathbb{Z}$ such that $x \neq y$. We need to go either from $(x, 0)$ to $(y, 1)$, or from $(y, 1)$ to $(x, 0)$, with the kernel $p(.,.)$.
(a) Assume first $q > 0$. Since the kernel (6) is weakly irreducible, the horizontal kernel on lane 0 can either go from $(x, 0)$ to $(y, 0)$ or from $(y, 0)$ to $(x, 0)$. In the former case, since $p > 0$, we go from $(y, 0)$ to $(y, 1)$ with the vertical kernel. In the latter, since $q > 0$, we can go from $(y, 1)$ to $(y, 0)$ vertically and then from $(y, 0)$ to $(x, 0)$ horizontally.
(b) Assume now $q = 0$. If the two lanes are totally asymmetric in the same direction, say e.g. $l_0 = l_1 = 0 < d_0 d_1$, and $x > y$, we can neither go from $(x, 0)$ to $(y, 1)$ (because $l_0 + l_1 = 0$), nor from $(y, 1)$ to $(x, 0)$ (because $q = 0$); otherwise, we have either $d_0 l_1 > 0$ or $d_1 l_0 > 0$, say for instance the former. Then we can go from $(x, 0)$ to $(y, 1)$ via $(y, 0)$ if $x < y$, or via $(x, 1)$ if $x > y$.

Proof of (ii). Let $x, y \in \mathbb{Z}$ such that $x \neq y$ and $i, j \in W$ such that $i \neq j$. Since the vertical kernel $q(.,.)$ is irreducible, the same argument as in case *(a)* of *(i)* shows that we can either go from (x, i) to (y, j) or from (y, j) to (x, i). □

The next lemma gives a characterization of p-ordered configurations. This requires the following definition. Without loss of generality we assume in the sequel that $d_0 d_1 > 0$ and $l_0 l_1 \geq 0$. We leave the reader symmetrically formulate Definition 3.2 and Lemma 3.4 in the case $d_0 d_1 \geq 0$ and $l_0 l_1 > 0$.

Definition 3.2. *For $(\eta,\xi) \in \mathcal{X}\times\mathcal{X}$, we write $\eta >\!< \xi$ if and only if there exist $x,y\in\mathbb{Z}$ such that $x<y$ and the following hold: (a) there are opposite discrepancies at $(x,1)$ and $(y,0)$; (b) $\eta^0 \le \xi^0$ and $\eta^1 \ge \xi^1$ if the discrepancy at $(x,1)$ is an η discrepancy; or $\eta^0 \ge \xi^0$ and $\eta^1 \le \xi^1$ if the discrepancy at $(x,1)$ is a ξ discrepancy; (c) There is no discrepancy at $(z,1)$ if $z>x$, nor any discrepancy at $(z,0)$ if $z<y$.*
We define

$$E_{>\!<} := \{(\eta,\xi)\in\mathcal{X}\times\mathcal{X} : \eta >\!< \xi\} \tag{90}$$

Lemma 3.4. *For the kernel $p(.,.)$ in (23), under (27), we have the following:*
(i) Unless $q=0$ and $l_0=l_1=0$, two configurations η and ξ are p-ordered if and only if they are ordered, i.e. $\eta\le\xi$ or $\xi\le\eta$.
(ii) If $q=0$ and $l_0=l_1=0$, two configurations η and ξ are p-ordered if and only if either they are ordered, or $\eta >\!< \xi$.

Proof of Lemma 3.4. Let η and ξ be two p-ordered configurations. Note that two configurations are ordered (see (86)) if and only if they have no pair of opposite discrepancies. If $pq\neq 0$ or $l_0+l_1>0$, because of (27), any two distinct points of V are p-connected, hence η and ξ are ordered. Assume $q=0<p$. First we note that for all $x,y\in\mathbb{Z}$, $(x,0)$ and $(y,0)$ are p-connected, and so are $(x,1)$ and $(y,1)$. Thus η^i and ξ^i are ordered. If the ordering is the same, then η and ξ are ordered. Otherwise, there exists a pair of opposite discrepancies, one at $(\mathbf{x},1)$ and one at $(\mathbf{y},0)$ for $\mathbf{x},\mathbf{y}\in\mathbb{Z}$. We must have $\mathbf{x}<\mathbf{y}$, otherwise $(\mathbf{x},1)$ and $(\mathbf{y},0)$ are p-connected. The ordering on each lane is imposed by the nature of the discrepancies at $(\mathbf{x},1)$ and $(\mathbf{y},0)$. Assume for instance that there is an η discrepancy at $(\mathbf{x},1)$ and a ξ discrepancy at $(\mathbf{y},0)$. Then $\eta^0\le\xi^0$ and $\eta^1\ge\xi^1$. For every $z>\mathbf{y}$, since $\eta^1\ge\xi^1$, we have $\eta^1(z)=\xi^1(z)$ or an η discrepancy at $(z,1)$. The latter is ruled out because $(\mathbf{y},0)$ and $(z,1)$ are p-connected. Similarly there can be no discrepancy at $(z,0)$ if $z<\mathbf{x}$. We can then redefine x as the location of the rightmost η discrepancy on lane 1, and y denotes the location of the leftmost ξ discrepancy on lane 0. □

3.4 Proof of characterization

The next two results will enable us to deal with discrepancies in the proof of Theorem 2.1.

Lemma 3.5. *Let $\widetilde{\nu}\in(\widetilde{\mathcal{I}}\cap\widetilde{\mathcal{S}})$. If $l_0=l_1=q=0$, then $\widetilde{\nu}(E_{>\!<})=0$.*

Proof of Lemma 3.5. We define the following random variables taking values in $\mathbb{Z}\cup\{\pm\infty\}$:

$$X = X(\eta,\xi) := \sup\{x\in\mathbb{Z} : \eta^1(x)\neq\xi^1(x)\} \tag{91}$$
$$Y = Y(\eta,\xi) := \inf\{x\in\mathbb{Z} : \eta^0(x)\neq\xi^0(x)\} \tag{92}$$

with the convention $\sup\emptyset = -\infty = -\inf\emptyset$. That is, X is the location (if it exists) of the rightmost discrepancy on lane 1. Indeed on $E_{>\!<}$, we have

$X(\eta,\xi) \in \mathbb{Z}$ by Lemma 3.4. Hence

$$\widetilde{\nu}(E_{><}) \le \sum_{k\in\mathbb{Z}} \widetilde{\nu}(X=k) \le 1 \tag{93}$$

Since $\widetilde{\nu} \in \widetilde{\mathcal{S}}$, $\widetilde{\nu}(X=k)$ does not depend on $k \in \mathbb{Z}$. This quantity must vanish by the second inequality in (93), hence the result follows from the first one. □

For $m, n \in \mathbb{Z} \cup \{\pm\infty\}$, where $m \le n$, let

$$D_{m,n}(\eta,\xi) := \sum_{x\in V:\, m\le x(0)\le n} |\eta(x) - \xi(x)| \tag{94}$$

denote the number of discrepancies in the space interval $[m,n] \cap \mathbb{Z}$. We simply write $D(\eta,\xi)$ when $(m,n) = (-\infty,+\infty)$.

Proposition 3.2. *Let $\widetilde{\lambda} \in \widetilde{\mathcal{I}}$. Assume either $\widetilde{\lambda} \in \widetilde{\mathcal{S}}$, or*

$$\int_{\mathcal{X}\times\mathcal{X}} D(\eta,\xi)\widetilde{\lambda}(d\eta, d\xi) < +\infty \tag{95}$$

Then, for every $(x,y) \in V \times V$ such that x and y are p-connected,

$$\widetilde{\lambda}\,(E_{x,y}) = 0 \tag{96}$$

where

$$E_{x,y} := \{(\eta,\xi) \in \mathcal{X}\times\mathcal{X} :\ \textit{there are opposite discrepancies at } x \textit{ and } y\} \tag{97}$$

An equivalent formulation of Proposition 3.2 is

$$\widetilde{\lambda}\,\{(\eta,\xi) \in \mathcal{X}\times\mathcal{X} :\ \eta \text{ and } \xi \text{ are } p\text{-ordered}\} = 1 \tag{98}$$

In [13, Theorem 1.1] it is proved that if $\widetilde{\lambda}$ is a translation invariant and invariant probability measure for a one-dimensional translation invariant SEP (coupled via basic coupling), then (96) holds whenever x and y are p-connected. The argument carries over to our setting by using only translation invariance in the $\mathbb{Z}$ direction. For the sake of completeness, details of the proof of Proposition 3.2 are given in Appendix C.

We are now in a position to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\mu \in (\mathcal{I}\cap\mathcal{S})_e$ and $\rho \in [0,2]$. Since $\nu_\rho \in (\mathcal{I}\cap\mathcal{S})_e$ (cf. Lemma 3.2), using [14, Proposition 2.14 in Chapter VIII], we obtain a measure $\widetilde{\lambda}$ on $\mathcal{X}\times\mathcal{X}$, which belongs to $(\widetilde{\mathcal{I}}\cap\widetilde{\mathcal{S}})_e$ and whose marginals are μ and ν_ρ. The events

$$E_- := \{(\eta,\xi) \in \mathcal{X}\times\mathcal{X} :\ \eta \le \xi\} \quad \text{and} \tag{99}$$

$$E_+ := \{(\eta,\xi) \in \mathcal{X}\times\mathcal{X} :\ \eta \ge \xi\} \tag{100}$$

are invariant with respect to spatial translations, and (by attractiveness) they are conserved by the coupled dynamics. Since $\widetilde{\lambda} \in (\widetilde{\mathcal{I}} \cap \widetilde{\mathcal{S}})_e$, E_+ and E_- both have $\widetilde{\lambda}$-probability 0 or 1. The main step is to prove that

$$\widetilde{\lambda}(E_+ \cup E_-) = \widetilde{\lambda}\{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta \leq \xi \text{ or } \xi \leq \eta\} = 1 \tag{101}$$

implying that one of the events E_+ and E_- has probability 1. It follows that for every $0 < \rho < 2$ we either have $\mu \leq \nu_\rho$ or $\mu \geq \nu_\rho$. By Remark 3.2 we conclude that there exists some $r \in [0, 2]$ such that $\mu = \nu_r$.

We now turn to the proof of (101). Outside the case $q = 0 = l_0 = l_1 < p$, the kernel $p(.,.)$ in (23) is weakly irreducible; thus (101) follows from (98) and *(i)* of Lemma 3.4. Now assume $q = 0 = l_0 = l_1 < p$. By *(ii)* of Lemma 3.4, we obtain

$$\widetilde{\lambda}(E_- \cup E_+ \cup E_{><}) = 1 \tag{102}$$

and the conclusion follows from Lemma 3.5. □

4 Proofs of Theorems 2.2, 2.3 and 2.4

The proofs of Theorems 2.2 and 2.3 are developed respectively in Subsections 4.2 and 4.4. They are decomposed into six steps, summarized in Subsection 4.1. These intermediate results are all established in Section 5, except Proposition 4.3, established in Subsection 4.3. Indeed this proposition is necessary for Theorem 2.2, but its proof introduces material (namely current and flux function) also used for Theorem 2.3. Finally, Theorem 2.4 is proved in Subsection 4.5.

4.1 Main ideas to prove Theorems 2.2, 2.3

We summarize the scheme of proof in six steps described here informally, whose precise statements are given in the next subsection.

Step one: shifting an invariant measure. Let $\mu \in \mathcal{I}_e$. We want to compare the measure μ with its shift $\tau\mu$. This will follow from construction of a coupling of these two measures satisfying some ordering or pseudo-ordering relation. Two very different subcases must be considered:

Weakly irreducible case. This is when either $q > 0$, or $q = 0$ but (89) does not hold. We then prove that $\mu \leq \tau\mu$ or $\tau\mu \leq \mu$ (stochastic order). This follows from construction of a coupling $\overline{\lambda}(d\eta, d\xi)$ of $\mu(d\eta)$ and $\tau\mu(d\xi)$ under which $\eta \leq \xi$ or $\xi \leq \eta$ a.s. This construction, performed in Proposition 4.1, is an adaptation to our model of [5, Proposition 3.2].

Non-weakly irreducible case. When $q = 0$ and (89) holds, as in the proof of Theorem 2.1, the above arguments do not lead to $\eta \leq \xi$ or $\xi \leq \eta$, but only to

$\eta >< \xi$. Unlike in Theorem 2.1, we cannot use translation invariance to eventually obtain $\eta \leq \xi$ or $\xi \leq \eta$. We introduce an intermediate relation denoted by $\eta \bowtie \xi$, that is a strenghtening of $\eta >< \xi$, see Definition 4.1, and obtain a coupling under which $\eta \bowtie \xi$. This is also contained in Proposition 4.1. But unlike in [5], we cannot next prove that μ or $\tau\mu$ are ordered. A different type of argument is required to conclude in this case that μ can only be a blocking or a partial blocking measure. This is the object of Proposition 4.6.

Steps two to five below apply to the weakly irreducible case, whereas the conclusion of the non-weakly irreducible case in contained in Step six.

Step two: getting a "mean" shock. It is shown in Proposition 4.1 that the total number of discrepancies $D(\eta, \xi)$ (see (94)) is a.s. a finite constant k under $\widetilde{\lambda}$. If $k = 0$, then $\tau\mu = \mu$, and we are back to Theorem 2.1. Otherwise, along the proof of Proposition 4.1, we show that the expectation of $D(\eta, \xi)$ under $\widetilde{\lambda}$ yields (since we have an ordered coupling of μ with its shift) a telescoping sum equal to the (positive) difference of mean densities at $\pm\infty$, $\rho^{\pm} = \lim_{x\to\pm\infty} \mu[\eta(x)]$. We call this a "mean" shock. Since $D(\eta, \xi)$ is an integer and this difference cannot exceed 2, $k = \rho^+ - \rho^- \in \{1, 2\}$.

Step three: mean shock implies shock. Since μ and $\tau\mu$ are ordered, the limits $\mu_\pm := \lim_{x\to\pm\infty} \tau_x\mu$ exist, and an averaging argument shows that $\mu_\pm \in (\mathcal{I} \cap \mathcal{S})$. This is Corollary 4.1. At this stage a crucial step appears, that is *not* required for single-lane ASEP because the latter model has the simplifying feature that densities are restricted to 1 (see Remark 4.1 for more details on this). The problem is to show that $\mu_\pm \in (\mathcal{I} \cap \mathcal{S})_e$, implying that $\mu_\pm = \mu_{\rho^\pm}$, hence that μ is a (ρ^-, ρ^+)-*shock measure*. This is done in Proposition 4.2. Thus we know that if $\mu \in \mathcal{I}_e \setminus (\mathcal{I} \cap \mathcal{S})$, then μ is a shock measure of amplitude $|\rho^+ - \rho^-| \in \{1, 2\}$. If $|\rho^+ - \rho^-| = 2$ we have a $(0, 2)$ or a $(2, 0)$ shock that are analogous to profile measures in [5]. The choice (27) implies that only $(0, 2)$ is possible, see Lemma 4.2. If $|\rho^+ - \rho^-| = 1$, we need to restrict possible shocks (ρ^-, ρ^+).

Step four: restricting possible shocks. The relevant object is the (microscopic and macroscopic) *flux function* of our model, introduced in (113)–(117). In Proposition 4.9, we show that a (ρ^-, ρ^+)-flux function cannot exist unless (ρ^-, ρ^+) is an entropy shock for the macroscopic flux function G, see Definition 4.3 and Remark 4.3 below. In Proposition 4.8 and Lemmas 4.1–4.2, explicit computations on the macroscopic flux $\rho \in [0, 2] \mapsto G(\rho)$ allow us to disqualify most shocks and prove (in Proposition 4.3) statement *(ii)* of Theorem 2.2. These computations further show that in a certain parameter range (see statement *(iii)* of Theorem 2.3), no entropy shock, hence no shock measure of amplitude 1 exists. Condition (41) in *(b)* of Theorem 2.2, which excludes blocking measures, expresses the fact that the graph of the flux function $\rho \mapsto G(\rho)$ crosses the ρ-axis.

Special situations are (42)–(43). In these cases the function $G(\rho)$ is identically 0 and does not help to eliminate shocks. In the latter case we show (statement *(i)* of Theorem 2.3) that the system is of diffusive gradient type, i.e. the microscopic flux is a gradient, which leads to non-existence of shocks. In the former, as mentioned in the comments following Theorem 2.2, the model is presumably diffusive but non-gradient, and specific techniques would be required.

Step five: uniqueness of a (ρ^-, ρ^+)-shock measure. We next show in Proposition 4.5 that if $|\rho^+ - \rho^-| = k \in \{1, 2\}$, there are (up to shifts) at most k (ρ^-, ρ^+)-shock measures in $\mathcal{I}_e$, except for $k = 2$ and $q = 0$. Recall indeed from Subsection 2.4.3 that in the statement of Theorem 2.3 we may have a family of two (up to shifts) $(0, 2)$-shock measures when $q > 0$, and infinitely many when $q = 0$. To prove Proposition 4.5, a key step is showing that two (ρ^-, ρ^+)-shock measures μ and ν are comparable (Proposition 4.4). Relying on this, we can extend to arbitrary shocks of any amplitude an argument of [5] for ASEP $(0, 1)$-shock measures, whose idea is to squeeze ν between successive translates of μ. Note that the prior comparability step is not necessary in the single-lane case.

Step six: the case $q = 0$. In this case, the flux function $G(\rho)$ is very explicit, cf. (119) in Example 4.1. This allows more precise shock selection in Step four: in particular $\mathcal{R} = \emptyset$ if $\gamma_0 \neq \gamma_1$.

Next, thanks to the condition $q = 0$, one can compare each lane with an ASEP and use convergence and characterization results for single-lane ASEP ([13, Theorem 1.4], [14, Chapter VIII], [3, Theorem 1]).

In the non-weakly irreducible case (89), starting from the partial conclusion $\eta \bowtie \xi$ of Step one, Proposition 4.6 further concludes that the invariant measure μ must be a blocking or a partial blocking measure as in Theorem 2.3, *(iv) (b)*.

In the weakly irreducible case, that is cases *(iv)–(vi)* of Theorem 2.3, statement *(iii)* of Proposition 4.9 shows that a shock measure of amplitude 1 with a profile outside $\mathcal{R}$ must belong to (58), (61) or (62).

Finally, to show that Bl_2 is empty outside case (89), assuming that one lane carries a blocking measure, we exhibit (see (136)) a Lyapunov functional on one lane that has a positive probability of decreasing unless the other lane is empty.

Remark 4.1. *As mentioned in the introduction, since for the single-lane ASEP the maximal density is 1, for a mean shock of amplitude 1, we must have $\{\rho^+, \rho^-\} = \{0, 1\}$; this automatically implies that μ is asymptotic at $\pm\infty$ to the corresponding (deterministic) Bernoulli measures, i.e., μ is a $(0, 1)$ or a $(1, 0)$-shock measure. Further analysis shows that it cannot be a $(1, 0)$-shock measure. Thus for single-lane ASEP, $\mathcal{I}_e$ contains only profile measures, and there is no need for Steps 3 and 4, namely, proving that μ is a shock measure and analyzing possible shocks.*

4.2 Proof of Theorem 2.2

We have to distinguish the case (40), where the kernel $p(.,.)$ in (23) is not weakly irreducible, cf. Lemma 3.3. In this case, we introduce the following definition.

Definition 4.1. *For $(\eta,\xi)\in\mathcal{X}\times\mathcal{X}$, we write $\eta\bowtie\xi$ if and only if the following hold: (i) $\eta><\xi$ (cf. Definition 3.2); (ii) the number of locations $z\in\mathbb{Z}^+$ on lane 1 that are not occupied by a coupled particle is finite; (iii) the number of locations $z\in\mathbb{Z}^-$ on lane 0 that are not occupied by a hole is finite.*
We define

$$E_{\bowtie}:=\{(\eta,\xi)\in\mathcal{X}\times\mathcal{X}:\ \eta\bowtie\xi\} \tag{103}$$

Following the steps described in Subsection 4.1 (that we recall below), the main results for the proof of Theorem 2.2 are Propositions 4.1–4.6 and Corollary 4.1, stated below. Among these, Proposition 4.3 is proved in Subsection 4.3, and other results in Subsection 5.1.

Step one. Let $\mu\in\mathcal{I}_e$. We prove the following proposition.

Proposition 4.1. *(i) There exists a measure $\widetilde{\lambda}(d\eta,d\xi)$ on $\mathcal{X}\times\mathcal{X}$ with marginals $\mu(d\eta)$ and $\tau_1\mu(d\xi)$, satisfying one of (104)–(106) below (if $q>0$), or one of (104)–(107) below (if $l_0=l_1=q=0<p$, that is (40), the non-weakly irreducible case):*

$$\widetilde{\lambda}(E_1) = 1 \quad \text{where} \quad E_1:=((\eta,\xi)\in\mathcal{X}\times\mathcal{X}:\ \eta<\xi) \tag{104}$$

$$\widetilde{\lambda}(E_2) = 1 \quad \text{where} \quad E_2:=((\eta,\xi)\in\mathcal{X}\times\mathcal{X}:\ \xi<\eta) \tag{105}$$

$$\widetilde{\lambda}(E_3) = 1 \quad \text{where} \quad E_3:=((\eta,\xi)\in\mathcal{X}\times\mathcal{X}:\ \eta=\xi) \tag{106}$$

$$\widetilde{\lambda}(E_{\bowtie}) = 1 \tag{107}$$

(ii) For any measure $\widetilde{\lambda}(d\eta,d\xi)$ with marginals $\mu(d\eta)$ and $\tau_1\mu(d\xi)$ satisfying (104) or (105), there exists $k\in\{1,2\}$ such that (cf. definition of $D(\eta,\xi)$ below (94))

$$\widetilde{\lambda}((\eta,\xi)\in\mathcal{X}\times\mathcal{X}:\ D(\eta,\xi)=k)=1 \tag{108}$$

Step two. Proposition 4.1 has the following consequences.

Corollary 4.1. *(i) In cases (104)–(106), the family $(\tau_n\mu)_{n\in\mathbb{Z}}$ is stochastically monotone.*

(ii) If a probability measure $\widehat{\mu}$ on $\mathcal{X}$ is such that $\widehat{\mu}\in\mathcal{I}$ and $(\tau_n\widehat{\mu})_{n\in\mathbb{Z}}$ is stochastically monotone, then there exist probability measures $\gamma^-(d\rho)$ and $\gamma^+(d\rho)$ on $[0,2]$ such that the limits

$$\widehat{\mu}_{\pm}:=\lim_{n\to\pm\infty}\tau_n\widehat{\mu}=\int_{[0,2]}\nu_\rho\gamma^{\pm}(d\rho) \tag{109}$$

hold in the sense of weak convergence.

Step three. We show that the measures $\gamma^{\pm}$ of Corollary 4.1 are Dirac measures.

Proposition 4.2. *In cases (104)–(105), there exists $(\rho^-, \rho^+) \in [0,2]^2 \setminus \mathcal{D}$ such that (i) $\gamma^{\pm} = \delta_{\rho^{\pm}}$, thus μ is a (ρ^-, ρ^+)-shock measure, cf. (36); (ii) $|\rho^+ - \rho^-| = k$, where k is defined in (ii) of Proposition 4.1.*

Step four. We first introduce the sets $\mathcal{R}$ and $\mathcal{R}'$ involved in Theorem 2.2.

Definition 4.2. *We denote by $\mathcal{R}$ the set of $(\rho^-, \rho^+) \in [0,2]^2 \setminus (\mathcal{D} \cup \mathcal{B})$ such that $\mathcal{I}_e$ contains at least one (ρ^-, ρ^+)-shock measure, and by $\mathcal{R}'$ the set of $(\rho^-, \rho^+) \in \mathcal{B}_1$ such that $\mathcal{I}_e$ contains at least one (ρ^-, ρ^+)-shock measure.*

In Subsection 4.3 below, we prove the following proposition, after introducing the *macroscopic flux function* of our model. Recall that (40) corresponds to the non-weakly ireducible case.

Proposition 4.3. *Outside (42)–(43), the following holds: (i) in cases (104)–(105) with $k = 2$, μ is a $(0,2)$-shock measure; (ii) Statements (ii), (b), (c) and (d) of Theorem 2.2 hold. (iii) Statement (i) of Proposition 4.5 below still holds if we have (40) and $\nu \in Bl_1$.*

Step five. In Proposition 4.5 below, we study the relation between extremal invariant measures that are (ρ^-, ρ^+)-shock measures for a common pair (ρ^-, ρ^+). The proof of Proposition 4.5 requires the following variant of Proposition 4.1.

Proposition 4.4. *Let $(\rho^-, \rho^+) \in [0,2]^2 \setminus \mathcal{D}$, and assume $\nu, \nu' \in \mathcal{I}_e$ are two (ρ^-, ρ^+)-shock measures. Then: (i) there exists a coupling of ν and ν' that satisfies one of the properties (104)–(107), property (107) being possible only under assumption (40); (ii) in case (107), ν and ν' lie in $Bl_1 \cup Bl_2$; (iii) in cases (104)–(105), (108) holds for some $k \in (\mathbb{N} \setminus \{0\}) \cup \{+\infty\}$.*

Proposition 4.5. *Let $\nu, \nu' \in \mathcal{I}_e$ be two (ρ^-, ρ^+)-shock measures. (i) Assume $|\rho^+ - \rho^-| = 1$, and we do not simultaneously have (40) and $\nu \in Bl_1$. Then ν' is a translate of ν, i.e. there exists $n \in \mathbb{Z}$ such that $\nu' = \tau_n \nu$. (ii) Assume $|\rho^+ - \rho^-| = 2$, ν' is not a translate of ν, and we do not have (40). Then every (ρ^-, ρ^+)-shock measure is either a translate of ν, or a translate of ν'.*

Step six. We conclude in case (107) of Proposition 4.1. This step is pursued in Subsection 4.4, in the proof of statements *(iv)–(vi)* of Theorem 2.3.

Proposition 4.6. *In case (107), we have $\mu \in Bl_1 \cup Bl_2$.*

Final step. We assemble the previous steps to conclude the proof of Theorem 2.2.

Proof of Theorem 2.2. First, for $\mu \in \mathcal{I}_e$, we consider the different possibilities in Proposition 4.1. In case (106), we have $\mu \in (\mathcal{I} \cap \mathcal{S})_e$; by Theorem 2.1, $\mu = \nu_\rho$ for some $\rho \in [0,2]$. In case (107) (which may only occur under (40)), Proposition 4.6 implies $\mu \in Bl_1 \cup Bl_2$, with Bl_1 and Bl_2 given by (58)–(60). In cases (104)–(105) with $k = 2$ in (108), Proposition 4.2 and *(i)* of Proposition 4.3 lead

to $\mu \in Bl_2$. In cases (104)–(105) with $k = 1$ in (108), by Proposition 4.2, μ is a shock measure of amplitude 1.

Next, to obtain (37), we consider the structure modulo translations of shock measures. Cardinality bounds for $\mathcal{R}$ and $\mathcal{R} \cup \mathcal{R}'$ are given by Proposition 4.3. By *(i)* of Proposition 4.5 and *(iii)* of Proposition 4.3, for every $(\rho^-, \rho^+) \in \mathcal{R} \cup \mathcal{R}'$, the set of (ρ^-, ρ^+)-shock measures in $\mathcal{I}_e$ consists of translates of a single measure. The set Bl_2 is stable by translation because the generator (5) with transition kernel (23) is translation invariant. This concludes the proof of *(i)*. Statements *(ii), (b), (c)* and *(d)* are contained in statement *(ii)* of Proposition 4.3. We conclude with the proof of *(ii), (a)*. By *(ii)* of Proposition 4.5, outside (40), Bl_2 consists of at most (up to translations) two measures.

We now prove that if Bl_2 is nonempty, outside case (40), it consists of exactly (up to translations) two measures. We already know by *(ii), (a)* of Theorem 2.2 that Bl_2 has at most two elements. Thus we must show that if Bl_2 contains some element ν, it contains another one ν' that is not a shift of ν. Since for $\eta \in \mathcal{X}_2$ and $x \in \mathbb{Z}$, we have

$$H_2(\tau_x \eta) = H(\eta) - 2x \tag{110}$$

for H_2 defined by (54), without loss of generality, we may assume that ν is supported on $\{\eta \in \mathcal{X}_2 : H_2(\eta) = 0\}$ or $\{\eta \in \mathcal{X}_2 : H_2(\eta) = 1\}$. The proof being similar in both cases, we assume the former. Let

$$X_0(\eta) := \inf\{x \in \mathbb{Z} : \eta(x, 0) + \eta(x, 1) = 1\} \tag{111}$$

denote the position of the leftmost η-particle, that is finite on $\mathcal{X}_2$ and thus under ν. At time 0 we consider an initial random configuration $\eta \sim \nu$ and define a random configuration ξ by adding to η a (so-called second-class) particle at $Y_0(\eta, \xi) := (X_0(\eta) - 1, 0)$. We consider the coupled process (η_t, ξ_t) starting from the random initial configuration (η, ξ). We denote t he law of ξ_t by ν'_t and consider

$$M'_t := \frac{1}{t} \int_0^t \nu'_s \, ds$$

The family $(M'_t)_{t>0}$ is tight because it is supported on the compact space $\mathcal{X}$. The following proposition is proved in Appendix B, and with (110), yields the desired conclusion. □

Proposition 4.7. *Any subsequential limit $M'(d\xi)$ of the family $(M'_t)_{t>0}$ is an element of Bl_2 supported on the set*

$$\mathcal{X}_{2,1} := \{\eta \in \mathcal{X}_2 : H_2(\eta) = 1\} \tag{112}$$

4.3 Proof of Proposition 4.3

We begin by defining the flux function, which will also play an important role in the proof of Theorem 2.3, and state some of its properties.

4.3.1 Microscopic current and macroscopic flux

We first define the microscopic current by

$$j(\eta) := \sum_{x(0)\leq 0, y(0)>0} p(x,y)\eta(x)(1-\eta(y)) - \sum_{x(0)\leq 0, y(0)>0} p(y,x)\eta(y)(1-\eta(x)) \tag{113}$$

for $\eta \in \mathcal{X}$. With the kernel defined by (23), this yields

$$j(\eta) = \sum_{i=0}^{1} \left\{ d_i \eta^i(0)[1-\eta^i(1)] - l_i \eta^i(1)[1-\eta^i(0)] \right\} \tag{114}$$

$$= = \sum_{i=0}^{1} \left\{ \gamma_i \eta^i(0)[1-\eta^i(1)] + l_i [\eta^i(0) - \eta^i(1)] \right\} \tag{115}$$

The macroscopic flux is then given by, for $\rho \in [0,2]$,

$$G(\rho) := \int j(\eta) d\nu_\rho(\eta). \tag{116}$$

Using (72) and (28), this yields

$$G(\rho) = \gamma_0 G_0[\widetilde{\rho}_0(\rho)] + \gamma_1 G_0[\widetilde{\rho}_1(\rho)] \tag{117}$$

where G_0 is the flux function of the single-lane TASEP, given by

$$G_0(\alpha) = \alpha(1-\alpha) \quad \forall \alpha \in [0,1]. \tag{118}$$

In the following two special cases, the function G has a simple expression.

Example 4.1. *Assume $q = 0 < p$. Then, by (117) and (71),*

$$G(\rho) = \begin{cases} \gamma_1 \rho(1-\rho) & \text{if} \quad \rho \in [0,1] \\ \gamma_0 (\rho-1)(2-\rho) & \text{if} \quad \rho \in (1,2] \end{cases} \tag{119}$$

In particular, when $\gamma_0 = \gamma_1$, the flux is a function of period 1 whose restriction to $[0,1]$ is the TASEP flux. It exhibits a change of convexity at $\rho = 1$, where it is also non differentiable. Note that the latter property is not seen in usual single-lane models with product invariant measures.

Example 4.2. *Assume $p = q > 0$. Then, by (117) and (69),*

$$G(\rho) = \frac{\gamma_0 + \gamma_1}{4} \rho(2-\rho) \tag{120}$$

Here, the flux has the same shape as the single-lane TASEP flux (from which it is obtained by a scale change in the horizontal and vertical directions). It is in particular strictly concave.

Useful properties of G are gathered in the following proposition. Except for statement *(iii)* below, these properties do not rely on (27).

Proposition 4.8.

(o) $G(0) = G(2) = 0$. *The function* G *is identically* 0 *in cases* (42)– (43).

(i) Outside cases (42), (43) *and* (44), G *has at least one and at most three local extrema.*

(ii) (a) $G(1) = 0$ *if and only if* $q = 0$ *or* $\gamma_0 + \gamma_1 = 0$; *(b) if* $q > 0$, G *is continuously differentiable on* $[0, 2]$, *and* $G'(1) = 0$ *if and only if* $\gamma_0 = \gamma_1$ *or* $p = q$.

(iii) Under (27), $G'(2) \leq 0$. *Besides,* $G'(2) < 0$ *holds unless we have* (42), *or* (43), *or* $q = \gamma_0 = 0$.

(iv) The function G *depends only on the parameters* γ_0, γ_1 *and* r *defined in* (45). *Denoting* $G = G_{\gamma_0,\gamma_1,r}$, *it holds that for every* $\rho \in [0, 2]$,

$$G_{\gamma_0,\gamma_1,r}(2-\rho) = G_{\gamma_1,\gamma_0,r}(\rho) = G_{\gamma_0,\gamma_1,r^{-1}}(\rho) \tag{121}$$

where the last equality holds when $r > 0$. *If* $\gamma_0 + \gamma_1 \neq 0$, *it holds that for every* $\rho \in [0, 2]$,

$$G_{\gamma_0,\gamma_1,r}(\rho) = (\gamma_0 + \gamma_1) G_{\mathfrak{a},1-\mathfrak{a},r}(\rho) \quad \text{with} \quad \mathfrak{a} \text{ defined in (45).} \tag{122}$$

(v) Assume $\gamma_0 = \gamma_1 \neq 0$, *that is* $\mathfrak{a} = 1/2$. *Then: (a)* $G'(1/2) > 0$; *(b) for* $r \in (0, r_0) \cup (1/r_0, +\infty)$, *with* r_0 *given by* (46), *we have* $G(1/2) > G(1)$.

(vi) If $q \neq 0$ *and* $\gamma_0 + \gamma_1 \neq 0$, *the equation* $G(\rho + 1) - G(\rho) = 0$ *has a unique solution in* $[0, 1]$. *If* $q \neq 0$, $p \neq q$ *and* $\gamma_0 + \gamma_1 = 0 \neq \gamma_0\gamma_1$, *the solutions of this equation in* $[0, 1]$ *are* $\rho = 0$ *and* $\rho = 1$.

(vii) Outside cases (42)–(43), *the equation* $G(\rho) = 0$ *has at least one solution in* $(0, 2)$ *if and only if condition* (41) *holds. In this case, the solution is unique and* G *changes sign around this solution.*

Remark 4.2. *Statement (v) of Proposition 4.8 is involved in the proof of statement (iii) of Theorem 2.3. In the latter the condition on* r *is* $r \in (0; r_0)$ *instead of* $r \in (0, r_0) \cup (1/r_0, +\infty)$, *because the theorem is stated under* (27), *hence* $r < 1$.

4.3.2 Proof of Proposition 4.3

The scheme of proof of Proposition 4.3 is the following. We introduce in Definition 4.3 a set denoted by $\mathcal{R}_0$, which depends only on the flux function. Lemma 4.1 (which will be proved using Proposition 4.8) says that $\mathcal{R}_0$ contains at most three elements, in most cases no more than one, and sometimes none. Proposition 4.9 provides information on possible stationary shock measures, implying

that $\mathcal{R}$ is contained in $\mathcal{R}_0$; part of this proposition will be useful for the proof of Theorem 2.3. Lemma 4.2 sets restrictions on possible shock measures of amplitude 2. The proof of Proposition 4.3 is concluded using Lemma 4.1, Lemma 4.2 and Proposition 4.9; these are proved in Subsection 5.2.

Definition 4.3. *Let $\mathcal{R}_0$ denote the set of pairs $(\rho^-, \rho^+) \in [0,2]^2 \setminus \mathcal{D}$ satisfying the following conditions: (i) $|\rho^+ - \rho^-| = 1$; (ii) $G(\rho^+) = G(\rho^-) = \min_{\rho\in[\rho^-,\rho^+]} G(\rho)$ if $\rho^- < \rho^+$; or $G(\rho^+) = G(\rho^-) = \max_{\rho\in[\rho^+,\rho^-]} G(\rho)$ if $\rho^- > \rho^+$, where G is defined by (116)–(117).*

Remark 4.3. *Condition (ii) in Definition 4.3 means that (ρ^-, ρ^+) is an entropy shock for the scalar conservation law with flux function G, that is the expected hydrodynamic equation of our model for the* total *density (that is the sum of densities over all lanes), see [1]. Thus $\mathcal{R}_0$ is exactly the set of entropy shocks of amplitude* 1*.*

Lemma 4.1. *Outside (42)–(44), the set $\mathcal{R}_0$ contains at most three elements. More precisely:*
(i) If $q > 0$ and $\gamma_0 + \gamma_1 \neq 0$, $\mathcal{R}_0$ contains one element, and $\mathcal{B}_1 \cap \mathcal{R}_0 = \emptyset$.
(ii) If $q > 0$, $p \neq q$ and $\gamma_0 + \gamma_1 = 0 \neq \gamma_0\gamma_1$, or if $q = 0$ and $\gamma_0 \neq \gamma_1$, $\mathcal{R}_0$ contains two elements, and $\mathcal{R}_0 \subset \mathcal{B}_1$.
(iii) Assume $\mathfrak{a} = 1/2$, and recall r_0 defined by (46). Then $\mathcal{R}_0 = \{(1/2, 3/2)\}$ if and only if $r \in [r_0, 1]$, $\mathcal{R}_0 = \emptyset$ if and only if

$$r \in (0, r_0), \tag{123}$$

and $\mathcal{R}_0 = \{(3/2, 1/2); (0,1); (1,2)\}$ if and only if $r = 0$.
(iv) There exists an open subset $\mathcal{Z}$ of $[0,1]^2$, containing $\{1/2\} \times (0, r_0)$, such that $\mathcal{R}_0 = \emptyset$ for $(\mathfrak{a}, r) \in \mathcal{Z}$.

Proposition 4.9. *(i) Assume that a measure $\nu \in \mathcal{I}$ is a (ρ^-, ρ^+)-shock measure. Then (ρ^-, ρ^+) satisfies condition (ii) of Definition 4.3.*
(ii) Assume that in Proposition 4.1 we have (104) or (105), and $k = 1$. Then the pair (ρ^-, ρ^+) in Proposition 4.2 satisfies $(\rho^-, \rho^+) \in \mathcal{R}_0$.
(iii) Under the assumptions of (ii), suppose in addition that $(\rho^-, \rho^+) \in \mathcal{B}_1$; then either $\gamma_0 + \gamma_1 = 0$, or $q = 0$. If $q = 0$, we are in one of the cases (iv), resp. (v), (vi) of Theorem 2.3, and ν lies in the set given by (58), resp. (61), (62).

Lemma 4.2. *(i) If $\mu \in \mathcal{I}$ is a (ρ^-, ρ^+)-shock measure of amplitude* 2*, then $(\rho^-, \rho^+) = (0, 2)$; (ii) Under condition (41), no such measure exists.*

Proof of Proposition 4.3.

Proof of (i). This follows from *(i)* of Lemma 4.2.

Proof of (ii). Statement *(ii), (b)* of Theorem 2.2 follows from *(ii)* of Lemma 4.2. We turn to statements *(ii), (c)* and *(d)* of Theorem 2.2. By Definition 4.2 and *(ii)* of Proposition 4.2, $\mathcal{R}$ and $\mathcal{R}'$ contain only shocks of amplitude 1 associated with stationary shock measures. By Proposition 4.9, *(i)*, and Remark 4.3, any

shock associated with a stationary shock measure is an entropy shock. Thus by Definitions 4.2, 4.3 and Remark 4.3, we have $\mathcal{R} \cup \mathcal{R}' \subset \mathcal{R}_0$, $\mathcal{R} \subset \mathcal{R}_0 \setminus \mathcal{B}_1$ and $\mathcal{R}' \subset \mathcal{R}_0 \cap \mathcal{B}_1$. The results then follow from *(i)* and *(ii)* of Lemma 4.1 if $q > 0$. If $q = 0$, $\gamma_0\gamma_1 \neq 0$ and $\gamma_0 \neq \gamma_1$, (119) and Definition 4.3 show that $\mathcal{R}_0$ contains at most two points and $\mathcal{R}_0 \subset \mathcal{B}_1$, thus $\mathcal{R} = \emptyset$ and $\mathcal{R}' \subset \mathcal{R}_0$.

Proof of (iii). In this case, by *(iii)* of Proposition 4.9, Bl_1 is contained in the right-hand side of (58). Each of the two sets on this right-hand side consists of translates of a single measure; the first set contains only $(1, 2)$-shock measures and the second one only $(0, 1)$-shock measures. □

4.4 Proof of Theorem 2.3

We start with the

Proof of Lemma 2.2. We prove the first equality (56), the proof of the second one being similar. Let $\mathcal{X}_n := \{\eta \in \mathcal{X} : H_2(\eta) = 2n\}$, and ξ^n denote the element of $\mathcal{X}_n$ defined by

$$\xi^n(z, i) = \mathbf{1}_{\{z > -n\}}; \quad z \in \mathbb{Z},\ i \in W$$

so that $\xi^n = \tau_n \xi^0$, with $\xi^0 \in \mathcal{X}_0$. For $\eta, \xi \in \mathcal{X}_n$, let $\mathcal{A}$, resp. $\mathcal{A}'$, denote the set of $x = (z, i) \in V$, where $z \in \mathbb{Z}$ and $i \in W$, for which $\eta(x) = 1 - \xi(x) = 0$, resp. $\eta(x) = 1 - \xi(x) = 1$. Then $|\mathcal{A}| = |\mathcal{A}'| < +\infty$, and

$$\begin{aligned} \frac{\nu^{\rho^c_\cdot}(\eta)}{\nu^{\rho^c_\cdot}(\xi)} &= \prod_{x \in \mathcal{A}'} \frac{\rho_x}{1 - \rho_x} \prod_{x \in \mathcal{A}} \frac{1 - \rho_x}{\rho_x} \\ &= \left(\frac{p}{q}\right)^{\sum_{z \in \mathbb{Z},\ i \in W} i[\eta(z,i) - \xi(z,i)]} \theta^{\sum_{z \in \mathbb{Z},\ i \in W} z[\eta(z,i) - \xi(z,i)]} =: r(\eta, \xi) \end{aligned}$$

The second equality above follows from $\rho_{z,i}/(1 - \rho_{z,i}) = c(p/q)^i \theta^z$ for $(z, i) \in V$. We apply this to $\eta \in \mathcal{X}_n$ and ξ^n:

$$\check{\nu}_n(\eta) = \frac{\nu^{\rho^c_\cdot}(\eta)}{\nu^{\rho^c}(\mathcal{X}_n)} = \frac{r(\eta, \xi^n)}{Z^n}$$

where

$$Z^n := \sum_{\xi \in \mathcal{X}_n} r(\xi, \xi^n)$$

Thus $\check{\nu}_n$ does not depend on c. Note that if $\eta \in \mathcal{X}_0$, we have $\tau_n \eta \in \mathcal{X}_n$, and $r(\tau_n \eta, \xi^n) = r(\eta, \xi^0)$. This implies that Z^n does not depend on n and that

$$\check{\nu}_n(\tau_n \eta) = \check{\nu}_0(\eta)$$

□

We will need the following lemma, proved in Section 5.2.

Lemma 4.3. *Assume $\nu^1, \nu^2 \in \mathcal{I}$ are supported on $\mathcal{X}_2$, and $H_2(\eta)$ defined by (54) has the same constant value under ν^1 and ν^2. Then, unless $l_0 = l_1 = q = 0 < p$, that is (40), the non-weakly irreducible case, it holds that $\nu^1 = \nu^2$.*

Proof of Theorem 2.3. We prove here the results stated in Theorem 2.3 as well as their complements given in Subsection 2.4.3.

• *Preliminaries on Bl_2: Proof of (o).* Since $\mu \in \mathcal{I}$, we have (recalling (13))

$$\int L \sum_{x\in\mathbb{Z}:\, m\le x\le n} \overline{\eta}\,(x)\, d\mu(\eta) = 0$$

where

$$L\overline{\eta}(x) = \tau_{x-1} j(\eta) - \tau_x j(\eta) \tag{124}$$

with j defined by (115). Hence, for arbitrary $n, m \in \mathbb{Z}$, we conclude that the quantity $\mu[\tau_x j(\eta)]$ is independent of x. Since μ is a $(0,2)$-shock measure (see (36)), we have

$$\lim_{n\to+\infty} \mu[\eta(n,i)] = 1, \qquad \lim_{n\to-\infty} \mu[\eta(n,i)] = 0$$

Since $0 \le \eta(x,i)(1-\eta(y,i)) \le \min[\eta(x,i), 1-\eta(y,i)]$ for $x, y \in \mathbb{Z}$, this implies

$$\lim_{n\to+\infty} \mu[\eta(x,i)(1-\eta(x+1,i))] = \lim_{n\to+\infty} \mu[\eta(x+1,i)(1-\eta(x,i))] = 0 \tag{125}$$

for $i \in \{0,1\}$. Thus $\mu[\tau_x j(\eta)] = 0$, which can be written

$$\mu\left\{\sum_{i=0}^{1} \gamma_i \eta^i(x)[1-\eta^i(x+1)]\right\} = \mu\left\{\sum_{i=0}^{1} l_i[\eta^i(x) - \eta^i(x+1)]\right\} \tag{126}$$

Summing (126) over $x \in \mathbb{Z}$ and using (125), we obtain

$$\sum_{i=0}^{1} \gamma_i \mu\left\{\sum_{x\in\mathbb{Z}} \eta^i(x)[1-\eta^i(x+1)]\right\} < +\infty \tag{127}$$

For each $i \in \{0,1\}$, $\gamma_i > 0$, hence the series inside braces in (127) converges μ-almost surely. Thus, μ-almost surely, $\eta^i(x)[1-\eta^i(x+1)] \to 0$ as $x \to \pm\infty$ implying $\eta^i(x)[1-\eta^i(x+1)] = 0$ for $|x|$ large enough, and $\eta \in \mathcal{X}_2$.

• *Symmetric case on each lane, $\gamma_0 = \gamma_1 = 0$: Proof of (i).* Let $\varphi \in C_K^0(\mathbb{R})$, that is, a continuous function with compact support. We consider the function $F_N : \mathcal{X} \to \mathbb{R}$ defined by

$$F_N(\eta) := N \sum_{x\in\mathbb{Z}} \varphi\left(\frac{x}{N}\right) \overline{\eta}(x)$$

Since $\gamma_0 = \gamma_1 = 0$, i.e. $l_0 = d_0$ and $l_1 = d_1$, the microscopic current (115) writes

$$j(\eta) = \sum_{i=0}^{1} d_i[\eta^i(0) - \eta^i(1)] \tag{128}$$

Using (124), (128) and two summations by parts, we obtain

$$LF_N(\eta) = N^{-1} \sum_{i=0}^{1} d_i \sum_{x\in\mathbb{Z}} \varphi''\left(\frac{x}{N}\right) \eta^i(x) + o_N(1) \tag{129}$$

where $o_N(1)$ is a quantity bounded in modulus by a deterministic sequence vanishing as $N \to +\infty$. By Theorem 2.2, μ satisfies (36), where either $\rho^+ = \rho^-$ and μ is a product measure given by Theorem 2.1, or $\rho^+ \neq \rho^-$ and μ is a (ρ^-, ρ^+)-shock measure. We show that we are in the first situation. Indeed, (36) implies

$$\lim_{x\to\pm\infty} \int_{\mathcal{X}} \eta^i(x) d\mu(\eta) = \widetilde{\rho}_i(\rho^{\pm}) \tag{130}$$

Thus, taking the expectation of (129) and using stationarity of μ, we have

$$0 = \int_{\mathcal{X}} LF_N(\eta) d\mu(\eta) = \varphi''(0) \sum_{i=0}^{1} d_i \left[\widetilde{\rho}_i(\rho^-) - \widetilde{\rho}_i(\rho^+)\right] + \varepsilon_N \tag{131}$$

where $\varepsilon_N \to 0$ as $N \to +\infty$. Since $\widetilde{\rho}_i$ is increasing and φ arbitrary, it follows that $\rho^+ = \rho^-$.

• *Case $q > 0$: Proof of (ii), (iii).*
Proof of (ii), (a). The product measure μ_{ρ^c} is reversible because $\rho^c_{\cdot}$ satisfies the reversibility equations (8) with respect to the kernel (23). The result follows since the measures in (55) are defined by conditioning the reversible measure $\mu_{\rho_{\cdot}}$ on the conserved quantity H_2. Assume now $\check{\nu}_n = (1-\alpha)\nu^1 + \alpha\nu^2$ with $\nu^1, \nu^2 \in \mathcal{I}$. Since $\check{\nu}_n$ is supported on $\mathcal{X}_2$ and H_2 has constant value $2n$ under $\check{\nu}_n$, the same holds for ν^1 and ν^2. Thus $\nu^1 = \nu^2$ by Lemma 4.3, implying that $\check{\nu}_n$ is extremal. The same argument applies to $\widehat{\nu}_n$.

Proof of (ii), (b). The measure $\check{\nu}_0$ is a product measure of the form (9), with $\rho_{\cdot}$ given by

$$\rho_{x,i} = \check{\rho}_{x,i} := \mathbf{1}_{\{x>0\}}, \quad (x,i) \in \mathbb{Z} \times W, \tag{132}$$

On the other hand, let $\widehat{\mu}_0$ denote the product measure (9) where

$$\rho_{x,i} \quad = \quad \widehat{\rho}_{x,i} := \mathbf{1}_{\{x>0\}} + \rho_0 \mathbf{1}_{\{(0,0)\}}(x,i) + \rho_1 \mathbf{1}_{\{(0,1)\}}(x,i), \quad (x,i) \in \mathbb{Z} \times W$$

with

$$\rho_0 := \frac{c}{1+c}, \quad \rho_1 := \frac{\frac{p}{q}c}{1+\frac{p}{q}c}; \quad c > 0 \tag{133}$$

The functions defined by (132) and (133)–(133) are solutions of (8). Thus, $\widehat{\mu}_0$ and $\check{\nu}_0$ are reversible. Under $\widehat{\mu}_0$, we have a.s. that

$$\eta(x,i) = 1 \text{ for } x > 0, \quad \eta(x,i) = 0 \text{ for } x < 0; \quad i \in \{0,1\} \tag{134}$$

which does not evolve in time. Hence under $\widehat{\mu}_0$, $\eta(0,0) + \eta(0,1)$ is conserved by the evolution, and conditioning $\widehat{\mu}_0$ on $\{\eta(0,0) + \eta(0,1) = 1\}$ yields a reversible measure satisfying (134), under which the vertical layer $\{0\} \times \{0,1\}$ contains a single particle located at $i \in \{0,1\}$ with probability p_i given by

$$\begin{aligned} p_0 &= \frac{\rho_0(1-\rho_1)}{\rho_0(1-\rho_1) + \rho_1(1-\rho_0)} = \frac{q}{p+q} \\ p_1 &= \frac{\rho_1(1-\rho_0)}{\rho_0(1-\rho_1) + \rho_1(1-\rho_0)} = \frac{p}{p+q} \end{aligned}$$

This measure is exactly $\widehat{\nu}_0$. Note that the process starting with (134) and a single particle on $\{0\} \times \{0,1\}$ reduces to the two state Markov process followed by this single particle jumping between lanes 0 and 1, and $\widehat{\nu}_0$ reduces to the unique invariant measure of this process (which is reversible).

For the measures $\check{\nu}_0$ and $\widehat{\nu}_0$, the proof of extremality in *(ii), (a)* also applies here. Finally, by Theorem 2.2, *(ii), (a)*, the above measures are (modulo horizontal translations) the only elements of Bl_2.

Proof of (iii). That $\mathcal{R} = \mathcal{R}' = \emptyset$ follows from Definition 4.2, *(ii)* of Proposition 4.9 and *(ii)–(iv)* of Lemma 4.1. When $d_1 = \lambda d_0$ and $l_1 = \lambda l_0$ with λ close to 1, then $\mathfrak{a}$ is close to 1/2, and (47) follows from (37) and *(ii)* of Theorem 2.3 proven above.

• *Case $q = 0$: Proof of (iv)–(vi)* (end of step six from Subsection 4.1). We first treat Bl_1 with an argument common to the three situations. Indeed in *(iv)*, resp. *(v)*, *(vi)*, by statement *(iii)* of Proposition 4.9, any element of Bl_1 must belong to the set (58), resp. (61), (62). Conversely, elements of these sets are extremal invariant probability measures in each case and they are reversible. Reversibility follows from the fact that the process starting from these measures is isomorphic to an ASEP on the lane that is not full or empty, and its restriction to this lane is a blocking measure for this ASEP, known to be reversible. We detail the extremality argument for $\nu^{\perp,+\infty,n}$ in case *(iv)*, all others are similar. Assume $\nu^{\perp,+\infty,n} = (1-\alpha)\nu^1 + \alpha\nu^2$, with $\nu^1, \nu^2 \in \mathcal{I}$ and $\alpha \in (0,1)$. Since lane 0 is empty under $\nu^{\perp,+\infty,n}$, the same holds under ν^1 and ν^2. Thus under the three measures, lane 1 evolves as an autonomous SEP with jump rate d_1 to the right and l_1 to the left, i.e., (6) with transition kernel (5) with $(d,l) = (d_1, l_1)$. The marginal of each measure on lane 1 is then an invariant measure for this SEP. Since the marginal of $\nu^{\perp,+\infty,n}$ is an extremal (blocking) invariant measure for the SEP on lane 1, we have $\nu^1 = \nu^2 = \nu^{\perp,+\infty,n}$. Since lane 0 remains empty under the evolution, $\nu^{\perp,+\infty,n}$ is indeed an invariant measure for the two-lane SEP.

We next treat $\mathcal{R}$ and Bl_2 with arguments specific to each situation.

Proof of (iv), (a). This corresponds to (119), that is, example 4.1. Note first that, since $\gamma_0 > 0$ and $\gamma_1 > 0$,

$$(\rho^-, \rho^+) = (0, 1); \quad (\rho^-, \rho^+) = (1, 2) \tag{135}$$

satisfy the conditions of Definition 4.3. These two shocks belong to $\mathcal{B}_1$.

If $\gamma_0 = \gamma_1$, we have $G(\rho+1) = G(\rho)$ for every $\rho \in [0, 1]$; thus $(\rho^-, \rho^+) = (\rho, \rho+1)$ and $(\rho^-, \rho^+) = (\rho + 1, \rho)$ satisfy $G(\rho^+) = G(\rho^-)$ for every $\rho \in [0, 1]$. Among such shocks different from the ones in (135), only $(\rho^-, \rho^+) = (3/2, 1/2)$ satisfies the variational equality in condition *(ii)* of Definition 4.3. Thus $\mathcal{R}_0 \subset \{(3/2, 1/2); (0, 1); (1, 2); (0, 2)\}$, and $\mathcal{R} \subset \{(3/2, 1/2)\}$.

If $\gamma_0 \neq \gamma_1$, then for every $\rho \in (0, 1)$, we have $G(\rho + 1) \neq G(\rho)$. Thus $\mathcal{R}_0$ contains no other shock than those in (135). Hence $\mathcal{R} = \emptyset$.

We finally prove by contradiction that Bl_2 is empty unless $l_0 = l_1 = 0$. Let $\mu \in Bl_2$. The function

$$H_{\mathbb{L}_1}(\eta) := \sum_{z \in \mathbb{Z}:\, z \leq 0} \eta(z, 1) - \sum_{z \in \mathbb{Z}:\, z > 0} [1 - \eta(z, 1)] \tag{136}$$

is well-defined since μ is supported on $\mathcal{X}_2$. It is constant along horizontal jumps and is increased by vertical jumps from lane 0 to lane 1. Let $(\eta_t)_{t \geq 0}$ denote the stationary process such that $\eta_0 \sim \mu$. We claim and prove below that if $l_0 > 0$, there is a positive probability that by time 1, the leftmost particle initially on lane 0 has jumped to lane 1. This implies

$$\mathbb{E}_\mu \left[H_{\mathbb{L}_1}(\eta_1) - H_{\mathbb{L}_1}(\eta_0) \right] > 0 \tag{137}$$

which contradicts stationarity. Similarly, if $l_1 > 0$, there is a positive probability that by time 1, the leftmost particle on lane 1 has jumped to lane 0, which implies the reverse strict inequality in (137).

Remark 4.4. *In (137), η_0 and η_1 denote the two-lane process configurations at respective times 0 and 1. These should not be confused with η^0 and η^1, cf. (12), which denote the restriction of a two-lane configuration η to lane 0 and 1 respectively.*

We now prove the claim when $l_0 > 0$ (the proof in the case $l_1 > 0$ is similar). In the sequel, on each lane, we call *active* those particles initially on the left of the rightmost hole and the next particle to the right of this hole (we also call active those sites where active particles are initially sitting). For $x, y \in V$, we say a Poisson process $\mathcal{N}_{(x,y)}$ of the Harris construction is *attached* to some site $z \in V$ if $z \in \{x, y\}$. We condition μ on the number and positions of active

particles on each lane. Denote respectively by x_0, y_0, x_1 the initial positions of the leftmost particle on lane 0, the next particle on its right, and the leftmost particle on lane 1. We couple our two-lane SEP with a random walk on lane 0 starting from $(x_0, 0)$, that jumps to the right and left with respective rates d_0, l_0 and is reflected at $(y_0, 0)$. The random walk is defined from the Harris system as follows: if its current position is $(x, 0) \in V$, at the first point of a Poisson process $\mathcal{N}_{(x,0),(x+\varepsilon,0)}$ where $\varepsilon \in \{-1, 1\}$, it jumps to $x + \varepsilon$, except if $x = y_0 - 1$ and $\varepsilon = 1$. Let $x_0' := \min(x_0, x_1 - 1, y_0 - 2)$, and E_0 denote the event that the random walk hits $(x_0', 0)$ for the first time before time $1/2$ and stays there at least until time 1 (if $x_0' = x_0$, E_0 corresponds to the return time to $(x_0, 0)$). This event has positive probability and depends only on the Poisson processes $\mathcal{N}_{(x,x+1)}$ and $\mathcal{N}_{(x+1,x)}$ for $x_0' - 1 \le x \le y_0 - 2$. Let T_0 denote the first time among all the following Poisson processes:
(a) $\mathcal{N}_{(y_0,0),(y_0-1,0)}$;
(b) $\mathcal{N}_{(x,0),(x,1)}$ for $x_0' < x \le y_0 - 1$; and
(c) the Poisson processes attached to active sites on lane 1; T_0 is an exponential random variable. Consider the event

$$\begin{aligned} E_0' \quad := \quad & E_0 \cap \{T_0 > 1\} \\ & \cap \{\mathcal{N}_{(x_0',0),(x_0',1)} \text{ has at least one point in the time interval } [1/2, 1]\} \end{aligned}$$

On E_0, in the two-lane SEP starting from the conditioned measure, the particle initially at $(x_0, 0)$ coincides with the random walk until it reaches $(x_0', 0)$; then its next motion is a jump from there to $(x_0', 1)$ before time 1; all this occurs before any particle initially on lane 1 has moved and before the particle initially at $(y_0, 0)$ has moved. The three events of which E_0' is the intersection are independent, because they depend on disjoint sets of Poisson processes. It follows that E_0' has positive probability when starting from the conditioned measure, irrespective of the conditioning.

Proof of (iv), (b). Assume $l_0 = l_1 = 0$. The measure $\nu^{\perp,i,j}$ is shown to be reversible as in *(ii)* above, since it is a product measure of the form (9) with

$$\rho_{x,0} = \mathbf{1}_{\{x>i\}}, \quad \rho_{x,1} = \mathbf{1}_{\{x>j\}}$$

for $x \in \mathbb{Z}$, and the above function $\rho_\cdot$ satisfies the reversibility conditions (8). The measure $\nu^{\perp,i,j}$ is a Dirac measure, hence it is extremal in the set of probability measures on $\mathcal{X}$, thus *a fortiori* extremal in $\mathcal{I}$.

Now we prove that any element μ of Bl_2 is of the form $\nu^{\perp,i,j}$ for $(i, j) \in \mathbb{B}$. Indeed, since μ is supported on $\mathcal{X}_2$, the random variable

$$n_0 := \inf\{z \in \mathbb{Z} : \eta(z, i) = 1, \quad \forall z \ge n_0,\, i \in \{0, 1\}\}$$

is μ-a.s. finite, as well as the number of particles to the left of n_0. Since jumps are totally asymmetric both horizontally and vertically, conditioned on this number and on n_0, the process lives on a finite space, and its irreducible

classes are singletons containing states $\{\eta^{\perp,i,j}\}$ that can be reached from the initial state (indeed, states $\eta^{\perp,i,j}$ are the only ones from which no transition is possible, and no return is possible from a state not belonging to this class). This implies that μ is a mixture of the invariant measures $\nu^{\perp,i,j}$, and by extremality, it must be one of them.

Proof of (v). The flux function G has the form (119), that is, example 4.1. Since $\gamma_1 < 0 < \gamma_0$, the only pairs (ρ^-, ρ^+) satisfying the requirements of Definition 4.3 are $(\rho^-, \rho^+) = (1, 0)$ and $(\rho^-, \rho^+) = (1, 2)$. These shocks belong to $\mathcal{B}_1$, hence $\mathcal{R} = \emptyset$. We next prove that Bl_2 is empty. Indeed by statement *(i)* of Proposition 4.9, a (ρ^-, ρ^+)-shock measure must satisfy condition *(ii)* of Definition 4.3. This is not the case for $(\rho^-, \rho^+) = (0, 2)$ or $(\rho^-, \rho^+) = (2, 0)$, because by (119), $G(0) = G(2)$ and $\gamma_1 < 0 < \gamma_0$ implies that G is negative on $(0, 1)$ and positive on $(1, 2)$.

Proof of (vi). The flux function G has the form (119), that is, example 4.1. The proof that $\mathcal{R} = \emptyset$ is similar to the case $\gamma_0 \neq \gamma_1$ in *(iv), (a)* (the fact that $\gamma_0 = 0$ being irrelevant there).

The proof that Bl_2 is empty can be reduced as follows to the same argument as in *(iv), (a)* above. First, note that $\gamma_0 = 0$ implies $l_0 > 0$. Unlike in *(iv), (a)*, since $\gamma_0 = 0$, we cannot use *(o)*, i.e. we do not know whether μ is supported on $\mathcal{X}_2$. Nevertheless, since $\gamma_1 > 0$, partially repeating the proof of *(o)* shows that μ is supported on the set of configurations η for which $H_{\mathbb{L}_1}(\eta)$, see (136), is finite. Let now η'_0 be the random configuration obtained by removing from $\eta_0 \sim \mu$ all particles at sites $(z, 0)$ such that $z < 0$, and $(\eta'_t)_{t\geq 0}$ be the process starting from η'_0. Then

$$H_{\mathbb{L}_1}(\eta'_0) = H_{\mathbb{L}_1}(\eta_0), \quad H_{\mathbb{L}_1}(\eta'_1) \leq H_{\mathbb{L}_1}(\eta_1) \tag{138}$$

where the inequality follows from attractiveness. We can then repeat the argument in *(iv), (a)* for $(\eta'_t)_{t\geq 0}$ to infer (137) for the latter process. Using (138) we obtain (137) for $(\eta'_t)_{t\geq 0}$. □

4.5 Proof of Theorem 2.4

Proof of (o). The proof of Theorem 2.1 requires only minor changes. First we can repeat the proof of Lemma 3.2. The only difference is that on a vertical layer $\{z\} \times W$, we now use the fact that L_v^z is the generator of a translation-invariant SEP on a torus, and ν_ρ is a homogeneous product Bernoulli measure, which is thus invariant for L_v^z. The rest of the proof is exactly similar to that of Theorem 2.1. Note that here by *(ii)* of Lemma 3.3, a p-ordered pair of configurations is ordered, so we do not need analogues of Lemmas 3.4 and 3.5 in this context.

Proof of (i). (a). We can repeat the following steps of the proof of Theorem 2.2: Proposition 4.1 (leading to (104)–(106) with $k \in \{1, \dots, n\}$ instead

of $k \in \{1,2\}$ in (108), (107) is irrelevant here because $p(.,.)$ is weakly irreducible), Corollary 4.1, Proposition 4.2 and Proposition 4.5. This yields that an extremal invariant measure that is not invariant by horizontal translations is a shock measure whose amplitude lies in $[1,n] \cap \mathbb{Z}$. Similarly to Proposition 4.5, we can prove that there are at most (up to translations) k shock measures of amplitude k. To further characterize possible shocks (ρ^-, ρ^+), we consider the macroscopic flux function G defined by (113) and (116) in this setup. By definition (63) of ν_ρ,

$$G(\rho) = \left(\sum_{i=0}^{n-1} \gamma_i \right) \frac{\rho}{n} \left(1 - \frac{\rho}{n} \right)$$

We can then repeat the proof of statement *(i)* of Proposition 4.9. Since the above function G is strictly concave and symmetric around $\rho = n/2$, shocks satisfying condition *(ii)* of Definition 4.3 are those specified in the theorem.
(b). The proof is similar to Theorem 2.3, *(o)*.
(c) Stationarity of the product measure $\nu^{\rho_\cdot^c}$ is proved as in Lemma 3.2, observing that on vertical layers we have a periodic SEP for which a homogeneous product measure is invariant. Stationarity and extremality of the conditioned measure are proved as in Theorem 2.3, *(ii)*.

Proof of (ii). We can repeat with minor modifications the proof of statement *(i)* of Theorem 2.3. The microscopic current is as in (128). Summation there and in (130)–(131) is now over $i \in W := \{0, \dots, n-1\}$. In the latter two displays, $\widetilde{\rho}_i(\rho)$ is replaced by ρ/n.

Proof of (iii). Let $\mu \in \mathcal{I}_e$. If $\mu \in \mathcal{S}$, the conclusion follows from *(o)*. Otherwise, since the generator (5) is invariant by τ', $\mu' := \tau'\mu \in \mathcal{I}_e$. By *(1)*, μ and μ' are shock measures, and $\mu' = \tau'\mu$ implies that they are (ρ^-, ρ^+)-shock measures for the same pair (ρ^-, ρ^+). The proof of Proposition 4.4 carries over to the multilane model (notice indeed that under condition *(iii')*, the global kernel $p(.,.)$ is weakly irreducible; thus when repeating the part of the proof of Proposition 4.1 that is used to derive Proposition 4.4, we always obtain (147), and do not need an analogue of step three). Hence, we have either $\mu \leq \mu'$ or $\mu' \leq \mu$. Since τ' is a periodic shift, this implies $\mu' = \mu$.

5 Proofs of intermediate results from Subsections 4.2–4.4

5.1 Proofs of intermediate results from Subsection 4.2

Proof of Proposition 4.1.

Proof of (i). Cases (104)–(106) are an adaptation of [5, Proposition 3.2], the main ingredients of which we recall in steps one and two below, whereas an additional argument (step three below) is required for (107). Let us fix $T > 0$.

Step one. Let $\widetilde{\lambda}_0$ denote the distribution on $\mathcal{X} \times \mathcal{X}$ of the coupled configuration (η_0, ξ_0), where $\eta_0 \sim \mu$ and $\xi_0 = \tau\eta_0$. We denote by $(\eta_t, \xi_t)_{t\geq 0}$ the coupled process starting from (η_0, ξ_0). Define

$$\mathcal{R}_T = \{x \in V : -T \leq x(0) \leq T\} \tag{139}$$

Let $\mathcal{N}_T$ be the number of discrepancies of $(\eta_t, \xi_t)_{t\geq 0}$ that visit $\mathcal{R}_T$ at any time in $[\sqrt{T}, T]$, $\mathcal{N}_T^{in}$ the number of these starting from $[-(1+\sigma)T, (1+\sigma)T]$ (where σ is the constant in Proposition 3.1), and $\mathcal{N}_T^{out}$ the number of these starting outside this interval. Adapting the proof in [5, Proposition 2.5] to our model yields

$$\mathbb{E}^{\widetilde{\lambda}_0}(\mathcal{N}_T) = o(T) \qquad \text{when} \quad T \to \infty. \tag{140}$$

The proof of [5, Proposition 2.5] used only the following properties of single-lane SEP, which hold also for our two-lane model.
(a) The finite propagation property (Proposition 3.1) is used to show

$$\mathbb{E}^{\widetilde{\lambda}_0}\left(\mathcal{N}_T^{out}\right) = o(T) \tag{141}$$

(b) the invariance of the generator with respect to horizontal translations, and
(c) the characterization theorem (here Theorem 2.1) for stationary measures invariant with respect to such translations: these are used to show

$$\mathbb{E}^{\widetilde{\lambda}_0}\left(\mathcal{N}_T^{in}\right) = o(T) \tag{142}$$

Step two. For $x, y \in \mathbb{Z}$, let $\mathcal{N}_T^{x,y}$ denote the number of discrepancies that visit either x or y and disappear during the time interval $[\sqrt{T}, T]$. Recall the definition (97) of $E_{x,y}$, and define

$$e_{x,y} := \inf_{(\eta,\xi)\in E_{x,y}} \mathbb{P}_{(\eta,\xi)}\left(\text{one of the discrepancies at } x \text{ and } y \text{ has coalesced by time 1}\right)$$

where $\mathbb{P}_{(\eta,\xi)}$ denotes the law of the coupled process starting from (η, ξ). The same argument as in [5, Lemma 3.1] shows that

$$e_{x,y} > 0 \text{ if } x \to_p y \text{ or } y \to_p x \tag{143}$$

Let

$$\widetilde{\lambda}^T = \frac{1}{T - \sqrt{T}} \int_{\sqrt{T}}^{T} \widetilde{\lambda}_0 \widetilde{S}_t dt \tag{144}$$

and let $\widetilde{\lambda} = \lim_{i\to\infty} \widetilde{\lambda}^{T_i}$ be a subsequential weak limit. Then

$$\widetilde{\lambda} \in \widetilde{\mathcal{I}} \tag{145}$$

Since $\mu \in \mathcal{I}_e$ and the two-lane SEP is translation-invariant in the $\mathbb{Z}$-direction, we have $\tau\mu \in \mathcal{I}_e$. Since $\widetilde{\lambda}_0$ has marginals $\mu \in \mathcal{I}$ and $\tau\mu \in \mathcal{I}$, $\widetilde{\lambda}$ has marginals μ

and $\tau\mu$. As in [5, Proposition 3.2], (140) and the strong Markov property yield respectively the following equality and inequality:

$$0 = \liminf_{T\to+\infty} \frac{1}{T}\mathbb{E}_{\widetilde{\lambda}_0}\left(\mathcal{N}_T^{x,y}\right) \geq e_{x,y}\widetilde{\lambda}(E_{x,y}) \tag{146}$$

Combining (143) and (146), we obtain

$$\widetilde{\lambda}\left\{(\eta,\xi) \text{ is } p-\text{ordered}\right\} = 1 \tag{147}$$

that is, (98). In the case $q > 0$, by *(i)* of Lemma 3.4, (147) implies (101). When $q = 0$, we only arrive at (102).

Step three. Assuming $q = 0$, we prove below that

$$\widetilde{\lambda}(E_{><} \setminus E_{\bowtie}) = 0 \tag{148}$$

This together with (102) implies

$$\widetilde{\lambda}(E_0 \cup E_1 \cup E_2 \cup E_{\bowtie}) = 1 \tag{149}$$

Moreover, each of the events in (149) is invariant under the coupled dynamics. Then using the fact that μ and $\tau\mu$ lie in $\mathcal{I}_e$, we can conclude as in [5, Proposition 3.2] that $\widetilde{\lambda}$ actually satisfies one of the conditions (104)–(107).

We now prove the claim (148). Recall the random variables X, Y defined by (91)–(92). Then, by conditions *(ii)–(iii)* of Definition 4.1,

$$E_{><} \setminus E_{\bowtie} \subset \bigcup_{x,y\in\mathbb{Z}:\, x<y} E'_{\bowtie,x,y} \cup \bigcup_{x,y\in\mathbb{Z}:\, x<y} F'_{\bowtie,x,y} \tag{150}$$

where, for $x < y$,

$$\begin{aligned} E'_{\bowtie,x,y} \quad &:= \quad E_{><} \cap \{X = x,\, Y = y\} \\ &\quad\ \cap \{\text{There are at least } y - x \text{ holes on lane 1 to the right of } x\} \\ F'_{\bowtie,x,y} \quad &:= \quad E_{><} \cap \{X = x,\, Y = y\} \\ &\quad\ \cap \{\text{There are at least } y - x \text{ coupled particles on lane 0 to the} \\ &\qquad \text{left of } y\} \end{aligned}$$

We claim that $\widetilde{\lambda}(E'_{\bowtie,x,y}) = \widetilde{\lambda}(F'_{\bowtie,x,y}) = 0$ which, in view of (150), implies (148). On $E'_{\bowtie,x,y}$, there is a possible sequence of moves with positive probability that brings the discrepancy from $(x, 1)$ to $(y, 1)$ that is p-connected to $(y, 0)$. Indeed one can construct an event on the Harris system prescribing that on the time interval $[0, 1]$, the corresponding Harris clocks will ring in the desired order while no other clock rings. Hence, by stationarity, $\widetilde{\lambda}(E'_{\bowtie,x,y}) > 0$ implies $\widetilde{\lambda}(E_{(y,0),(y,1)}) > 0$, in contradiction with (147). Similarly on $F'_{\bowtie,x,y}$, there is a possible sequence of moves with positive probability that brings the discrepancy

from $(y,0)$ to $(x,0)$ that is p-connected to $(x,1)$.

Proof of (ii). Since the coupled configurations η and ξ are a.s. ordered under $\widetilde{\lambda}$, all discrepancies (if any) are of the same type (that is η or ξ discrepancies), so no coalescence occurs. Thus, recalling the definition of $D(\eta,\xi)$ from (94), the sets $A_k := \{D(\eta,\xi) = k\}$, with $k \in \mathbb{N} \cup \{+\infty\}$, are invariant under the dynamics. Hence,

$$\widetilde{\lambda} = \sum_{k\in\mathbb{N}\cup\{+\infty\}:\, \widetilde{\lambda}(A_k)>0} \widetilde{\lambda}(A_k)\widetilde{\lambda}_k \tag{151}$$

where $\widetilde{\lambda}_k := \widetilde{\lambda}(.|A_k) \in \widetilde{\mathcal{I}}$. Since μ and $\tau\mu$ are extremal elements of $\mathcal{I}$, for each k such that $\widetilde{\lambda}(A_k) > 0$, $\widetilde{\lambda}_k$ has marginals μ and $\tau\mu$. Assume for instance that $\widetilde{\lambda}$ (and thus $\widetilde{\lambda}_k$) satisfies (104). Then

$$\begin{aligned} \widetilde{\lambda}_k[D_{m,n}(\eta,\xi)] &= \widetilde{\lambda}_k \left\{ \sum_{x\in V:\, m\le x(0)\le n} [\xi(x) - \eta(x)] \right\} \\ &= \mu[\overline{\eta}(n+1)] - \mu[\overline{\eta}(m)] \in [0,2] \end{aligned} \tag{152}$$

Letting $m \to -\infty$ and $n \to +\infty$, by monotone convergence, and because $\widetilde{\lambda}_k$ is supported on A_k, we obtain

$$k = \widetilde{\lambda}_k[D(\eta,\xi)] \in \{0,1,2\} \tag{153}$$

Notice that the right-hand side of (152), and thus also its limit, depends only on μ. Hence k depends only on μ. This shows that $\widetilde{\lambda} = \widetilde{\lambda}_k$ for a unique $k \in \{0,1,2\}$. Since we are in case (104), $k = 0$ would yield a contradiction. Thus $k \in \{1,2\}$. Dealing with the case (105) is similar. □

Proof of Corollary 4.1. (i) The marginals of $\widetilde{\lambda}$ are μ and $\tau\mu$, thus $\mu \le \tau\mu$ in case (104), or $\tau\mu \le \mu$ in case (105), or $\tau\mu = \mu$ in case (106).

(ii) Given the assumptions, the limits (109) exist and satisfy $\tau\widehat{\mu}_\pm = \widehat{\mu}_\pm$, that is $\widehat{\mu}_\pm \in \mathcal{S}$. Besides, we have $\widehat{\mu}_\pm \in \mathcal{I}$. Indeed if f is a local function on $\mathcal{X}$,

$$\begin{aligned} \int_{\mathcal{X}} Lf(\eta) d\widehat{\mu}_\pm(\eta) &= \lim_{n\to\pm\infty} \int_{\mathcal{X}} Lf(\eta) d(\tau_n\widehat{\mu})(\eta) \\ &= \lim_{n\to\pm\infty} \int_{\mathcal{X}} L[\tau_n f](\eta) d\widehat{\mu}(\eta) = 0 \end{aligned}$$

where we used that L commutes with the shift and $\widehat{\mu} \in \mathcal{I}$. The last equality in (109) follows from Theorem 2.1 and (31). □

Proof of Proposition 4.6. Recall from Proposition 4.1 that (107) may only occur when $q = l_0 = l_1 = 0$. Hence the dynamics of horizontal jumps on each lane is a TASEP, and these TASEP's interact through vertical jumps from lane 0 to lane 1. Let $(\eta_0,\xi_0) = (\eta,\xi) \sim \widetilde{\lambda}$, where $\widetilde{\lambda}$ is the measure in Proposition 4.1.

We couple the process $\eta_\cdot$ through basic coupling with a process $\zeta_\cdot$ such that for every $i \in \{0,1\}$, ζ_t^i is a TASEP on lane i starting from configuration $\eta_0^i = \eta^i$ (with jumping rates $l_i = 0 < d_i$). Then one has, for every $t \geq 0$,

$$\eta_t^1 \geq \zeta_t^1, \quad \eta_t^0 \leq \zeta_t^0 \tag{154}$$

Indeed, to derive the first inequality in (154), note that at certain random times belonging to one of the Poisson processes $\mathcal{N}_{(z,0),(z,1)}$, a new particle may appear (following a jump of a particle from lane 0) at site $z \in \mathbb{Z}$ in $\eta_\cdot^1$ that does not appear in $\zeta_\cdot^1$. On the other hand, between such times, both processes evolve as coupled TASEP's on lane 1, whose order is preserved by attractiveness property (88). A similar argument holds for the second inequality in (154).

For $t > 0$, define the empirical measures

$$M_t^i := \frac{1}{t}\int_0^t \mu_s^i ds, \quad N_t^i := \frac{1}{t}\int_0^t \nu_s^i ds \tag{155}$$

where μ_t^i denotes the law of η_t^i and ν_t^i that of ζ_t^i. Since $\mu \in \mathcal{I}$, $\mu_t = \mu$ does not depend on t, hence $M_t^i =: \mu^i$ does not depend on t and is the marginal of μ on lane i.

Let $t_n \uparrow +\infty$ be a subsequence along which $N_{t_n}^i \to \nu_\infty^i$, where ν_∞^i is an invariant measure for TASEP. Since $\eta \bowtie \xi$, there is a random variable $N \in \mathbb{Z} \cup \{-\infty\}$ such that $\zeta_0^i(x) = \eta_0^i(x) = 1$ for $i \in \{0,1\}$ and $x \in \mathbb{Z}$ with $x \geq N$. By TASEP dynamics, this remains true at time t for ζ_t^i with the same N. Thus if ζ is a random configuration with distribution ν_∞^i, we a.s. have

$$\zeta(x) = 1, \quad \forall x \geq N' \tag{156}$$

where N' has the same law as N. As ν_∞^i is invariant for TASEP by [13, Theorem 1.4], it is a mixture of Bernoulli and blocking measures. But by (156), the only possible Bernoulli measure is the one with density 1. Thus there exists a random variable $N_i \in \mathbb{Z} \cup \{-\infty\}$ such that the random configuration $\zeta_\infty^i := \eta_{N_i}^*$ has distribution ν_∞^i.

By (154), $\mu^1 \geq \nu_\infty^1$ and $\mu^0 \leq \nu_\infty^0$. It follows from the above that there exist random variables M_0 and M_1 with values in $\mathbb{Z} \cup \{-\infty\}$ such that $\eta_0^1 \geq \eta_{M_1}^*$ and $\eta_0^0 \leq \eta_{M_0}^*$ a.s. Since $\eta \bowtie \xi$, the dynamics of $(\eta_t^1)_{t\geq 0}$ can only possibly create a finite number of particles (from lane 0) to the left of M_1 and move these particles to the right until they pile up and get blocked. The same argument applies to holes in η^0, since the dynamics of holes is a two-lane TASEP with jumps to the left and from lane 1 to lane 0. Thus there exist random variables $-\infty \leq M_1' \leq M_1 < +\infty$ and $-\infty < M_0 \leq M_0' \leq +\infty$ such that

$$\lim_{t\to+\infty} \eta_t^1 = \eta_{M_1'}^*, \quad \lim_{t\to+\infty} \eta_t^0 = \eta_{M_0'}^* \tag{157}$$

Since particles can jump from lane 0 to lane 1 but not the other way, the dynamics imposes

$$M_0' \geq M_1' \tag{158}$$

The limits in (157) imply that η_t converges in law to the distribution of the random configuration η_∞ defined by $\eta_\infty^i = \eta^*_{M_i'}$ for $i \in \{0,1\}$, that is (cf. (59)) $\eta_\infty = \eta^{\perp, M_0', M_1'}$. By stationarity, μ is the distribution of this configuration; hence, recalling the definition of $\overline{\mathbb{B}}$ above (59),

$$\mu = \int_{\overline{\mathbb{B}}} \nu^{\perp,i,j} dm(i,j) \tag{159}$$

where $m(di, dj)$ denotes the law of (M_0', M_1'). This with (58)–(60) implies that μ is a mixture of the measures in $Bl_1 \cup Bl_2$. □

Proof of Proposition 4.2.

Proof of (i). Without loss of generality, we may assume $\mu \leq \tau\mu$. Since $\mu \leq \mu_+$ (where μ_+ is defined as in Corollary 4.1) and $\mu \neq \mu_+$ (because we are not in case (106)), by [14, Proposition 2.14 in Chapter VIII] there exists a coupling measure $\widetilde{\mu}(d\eta, d\xi)$ with marginals $\mu(d\eta)$ and $\mu_+(d\xi)$, such that

$$\widetilde{\mu}\left((\eta,\xi) \in \mathcal{X} \times \mathcal{X} : \eta < \xi\right) = 1 \tag{160}$$

and which is invariant for the coupled process.

For $n, m \in \mathbb{Z}$ such that $m \leq n$, and $\xi \in \mathcal{X}$, we set

$$M_{m,n}(\xi) := \frac{1}{n-m+1} \sum_{x \in V:\, m \leq x(0) \leq n} \xi(x)$$

and simply write $M_n(\xi)$ when $m = 1$. Because μ_+ is a mixture of Bernoulli measures, by the ergodic theorem, the limit

$$M(\xi) := \lim_{n \to +\infty} M_n(\xi) = \lim_{n \to +\infty} M_{-n,n}(\xi) \tag{161}$$

exists $\widetilde{\mu}$-a.s. The distribution of $M(\xi)$ is exactly γ^+. Besides, $M(\xi)$ is a conserved quantity for the dynamics of the stationary coupled process $(\eta_t, \xi_t)_{t \geq 0}$ starting from $\widetilde{\mu}(d\eta_0, d\xi_0)$. Indeed, by the finite propagation property (Proposition 3.1),

$$\begin{aligned} \frac{2n+1-4\lfloor \sigma t \rfloor}{2n+1} M_{-n+2\lfloor \sigma t \rfloor, n-2\lfloor \sigma t \rfloor}(\xi_0) &\leq M_{-n,n}(\xi_t) \\ &\leq \frac{2n+1+4\lfloor \sigma t \rfloor}{2n+1} M_{-n-2\lfloor \sigma t \rfloor, n+2\lfloor \sigma t \rfloor}(\xi_0) \end{aligned}$$

with probability greater than $1 - e^{-Cn}$. Letting $n \to +\infty$ yields

$$M(\xi_t) = M(\xi_0) \tag{162}$$

It follows that for every ρ in the support of γ^+, the conditioned measure $\widetilde{\mu}(\rho)$ defined by

$$\widetilde{\mu}(\rho)(d\eta, d\xi) := \widetilde{\mu}\left((d\eta, d\xi)|M(\xi) = \rho\right) \tag{163}$$

is invariant for the coupled process. Indeed, for every bounded function f on $\mathcal{X} \times \mathcal{X}$ and every bounded measurable function g on $[0, 2]$,

$$\begin{aligned}
\int_{[0,2]} < \widetilde{\mu}_\rho, \widetilde{S}_t f > g(\rho)\gamma^+(d\rho) &= \widetilde{\mathbb{E}}_{\widetilde{\mu}}\left[f(\eta_t, \xi_t)g\left(M(\xi_0)\right)\right] \\
&= \widetilde{\mathbb{E}}_{\widetilde{\mu}}\left[f(\eta_t, \xi_t)g\left(M(\xi_t)\right)\right] \\
&= \widetilde{\mathbb{E}}_{\widetilde{\mu}}\left[f(\eta_0, \xi_0)g\left(M(\xi_0)\right)\right] \\
&= \int_{[0,2]} < \widetilde{\mu}(\rho), f > g(\rho)d\gamma^+(\rho)
\end{aligned}$$

In the above display, the first and last equality follow from definition (163), the second one from (162), and the third one from stationarity.

Hence, the η-marginal of $\widetilde{\mu}(\rho)$, that is $\mu(\rho)$ defined by $\mu(\rho)(d\eta) := \widetilde{\mu}\left(d\eta|M(\xi) = \rho\right)$ is invariant for L. Since

$$\mu = \int_{[0,2]} \mu(\rho)d\gamma^+(\rho), \tag{164}$$

by extremality of μ, we must have $\mu(\rho) = \mu$ for γ^+-a.e. $\rho \in [0, 2]$. This means that that under $\widetilde{\mu}(d\eta, d\xi)$, η is independent of $M(\xi)$.

Notational remark. In the above argument, we emphasize the use of notations $\widetilde{\mu}(\rho)$ and $\mu(\rho)$, but *not* μ_ρ, to avoid any confusion with the unrelated product measures μ_ρ defined in (9) and Remark 2.1.

Now we consider $A, B, A', B' \in \mathbb{R}$ such that A lies in the support of γ^+ and $B < B' < A' < A$. Let f, g be nondecreasing continuous functions on $\mathcal{X}$ supported respectively on $[A', +\infty)$ and $(-\infty, B']$, taking constant value 1 respectively on $[A, +\infty)$ and $(-\infty, B]$. By (160), (161), and independence of $M_n(\eta)$ and $M(\xi)$, the following holds under $\widetilde{\mu}$:

$$\begin{aligned}
0 = \widetilde{\mathbb{E}}_{\widetilde{\mu}}\left[f(M_n(\eta))g(M_n(\xi))\right] &= \widetilde{\mathbb{E}}_{\widetilde{\mu}}\left[f(M_n(\eta))g(M(\xi))\right] + \varepsilon_n \\
&= \widetilde{\mathbb{E}}_{\widetilde{\mu}}\left[f(M_n(\eta))\right]\widetilde{\mathbb{E}}_{\widetilde{\mu}}\left[g(M(\xi))\right] + \varepsilon_n
\end{aligned}$$

for some sequence $\varepsilon_n \to 0$. It follows that

$$\lim_{n\to+\infty} \widetilde{\mu}\left(M_n(\eta) > A\right)\widetilde{\mu}\left(M(\xi) < B\right) = 0$$

Choosing B strictly larger than the infimum of the support of γ^+ yields

$$\lim_{n\to+\infty} \widetilde{\mu}\left(M_n(\eta) > A\right) = 0$$

It follows that

$$\limsup_{n\to+\infty} \widetilde{\mathbb{E}}_{\widetilde{\mu}}[M_n(\eta)] \leq A \tag{165}$$

Set

$$\overline{\mu}_n := \frac{1}{n}\sum_{x=1}^{n} \tau_x \mu$$

so that (165) also writes

$$\limsup_{n\to+\infty} \int_{\mathcal{X}} \eta(0) d\overline{\mu}_n(\eta) \leq A \tag{166}$$

On the other hand, by Proposition 4.1, $\overline{\mu}_n \to \mu_+$, thus

$$\int_{\mathcal{X}} \eta(0) d\mu_+(\eta) = \int_{[0,2]} \rho d\gamma^+(\rho) \leq A$$

for every A in the support of γ^+. Hence $\gamma^+ = \delta_{\rho^+}$ for some $\rho^+ \in [0,2]$.

Proof of (ii). Assume for instance $\rho^- < \rho^+$, the other case being similar. The equality (152) yields (recall that $\widetilde{\lambda} = \widetilde{\lambda}_k$ for $k \in \{0,1,2\}$, cf. (153))

$$\widetilde{\lambda}[D(\eta,\xi)] = k = \lim_{n\to+\infty} \mu[\eta(n)] - \lim_{m\to-\infty} \mu[\eta(m)] = \rho^+ - \rho^-$$

□

Proof of Proposition 4.4. The proof of *(ii)* is similar to that of Proposition 4.6. We prove *(i)* and *(iii)* below.

Proof of (i), step one. We show that if $\widetilde{\lambda}_0 \in \widetilde{\mathcal{I}}_e$ is a coupling of ν and ν' (that exists by [14, Proposition 2.14 in Chapter VIII]), then

$$\int_{\mathcal{X}\times\mathcal{X}} \left(\sum_{x\in\mathbb{Z},\, i\in W,\ |x|\leq T} |\eta^i(x) - \xi^i(x)| \right) d\widetilde{\lambda}_0(\eta,\xi) = o(T), \quad \text{as } T \to +\infty \tag{167}$$

Let (recall definition (94))

$$\widetilde{\lambda}_T^{\pm} := \frac{1}{|[-T,T]\cap\mathbb{Z}^{\pm}|} \sum_{x\in\mathbb{Z}^{\pm}:\,|x|\leq T} \tau_x \widetilde{\lambda}_0,$$

$$A_l(\eta,\xi) := \frac{1}{2(2l+1)} \sum_{y\in\mathbb{Z},\, i\in W:\,|y|\leq l} |\eta^i(y) - \xi^i(y)| = \frac{1}{2(2l+1)} D_{-l,l}(\eta,\xi)$$

$$B_l(\eta,\xi) := \left| \frac{1}{2(2l+1)} \sum_{y\in\mathbb{Z}:\ i\in W:\,|y|\leq l} \eta^i(y) - \frac{1}{2(2l+1)} \sum_{y\in\mathbb{Z}:\ i\in W:\,|y|\leq l} \xi^i(y) \right|$$

Every subsequential weak limit $\widetilde{\lambda}_\infty^{\pm}$ of the family $(\widetilde{\lambda}_T^{\pm})_{T\geq 0}$ (which is tight as it lives on a compact space) lies in $\widetilde{\mathcal{I}} \cap \widetilde{S}$. Thus by Proposition 3.2 and Lemma

3.5, it is supported on $E_- \cup E_+$ (see (99)–(100)), where $A_l = B_l$. The desired conclusion (167) is equivalent to having, for any subsequential limit $\widetilde{\lambda}_\infty$,

$$\begin{aligned} 0 &= \lim_{l\to+\infty} \lim_{T\to+\infty} \int_{\mathcal{X}\times\mathcal{X}} \left(\frac{1}{T} \sum_{x\in\mathbb{Z}^\pm,\, |x|\le T} \tau_x A_l(\eta,\xi) \right) d\widetilde{\lambda}_0(\eta,\xi) \\ &= \lim_{l\to+\infty} \int_{\mathcal{X}\times\mathcal{X}} B_l(\eta,\xi) d\widetilde{\lambda}^\pm_\infty(\eta,\xi) \end{aligned} \tag{168}$$

By definition (36) of shock measures, $\widetilde{\lambda}^\pm_\infty$ has marginals $\nu_{\rho^\pm}$. It follows that under $\widetilde{\lambda}^\pm_\infty$, the spatial averages in $B_l(\eta,\xi)$ both converge in probability and (being bounded by 2) in L^1 to $\rho^\pm$, thus implying the limits in (168).

Proof of (i), step two. We now adapt the proof of Proposition 4.1, defining $\mathcal{N}_T$, $\mathcal{N}_T^{in}$ and $\mathcal{N}_T^{out}$ as we did there, and replacing the initial distribution $\widetilde{\lambda}_0$ defined there by the one considered in the first step of the current proof. In the first step of the proof of Proposition 4.1, we similarly derive (141) from Proposition 3.1, whereas we can now obtain (142) as a consequence of (167). Steps two and three are unchanged and yield (147), where the measure $\widetilde{\lambda}$ now coincides with $\widetilde{\lambda}_0$, because the latter is invariant. Hence, we obtain (101) if $q > 0$, or (149) if $q = 0$. By extremality, this implies that $\widetilde{\lambda}_0$ satisfies one of (104)–(107).

Proof of (iii). The proof is similar to that of Proposition 4.1, statement *(ii)*. The only differences lie in the following points, assuming for instance that the conclusion of *(i)* is (104). First, the second line of (152) is now

$$\sum_{x\in V:\, m\le x(0)\le n} [\nu'(\xi(x)) - \nu(\xi(x))] \in [0,+\infty] \tag{169}$$

which depends only on ν, ν'. Next, in (153), k can be a priori any value in $\mathbb{N}\cup\{+\infty\}$ instead of only $0, 1, 2$. □

Proof of Proposition 4.5. For two *ordered* probability measures γ, γ' on $\mathcal{X}$, let

$$\Delta(\gamma,\gamma') := \sum_{x\in V} |\gamma(\eta(x)) - \gamma'(\eta(x))| \in [0,+\infty] \tag{170}$$

Note that $\Delta(\gamma,\gamma')$ satisfies the three following properties:

$$\Delta(\gamma,\gamma') = 0 \text{ if and only if } \gamma = \gamma' \tag{171}$$

If $\widetilde{\gamma}$ is an ordered coupling of γ and γ', we have

$$\Delta(\gamma,\gamma') = \int_{\mathcal{X}\times\mathcal{X}} D(\eta,\xi) d\widetilde{\gamma}(\eta,\xi) \tag{172}$$

If a probability measure γ'' on $\mathcal{X}$ is such that $\gamma \le \gamma' \le \gamma''$ or $\gamma'' \le \gamma' \le \gamma$, then

$$\Delta(\gamma,\gamma'') = \Delta(\gamma,\gamma') + \Delta(\gamma',\gamma'') \tag{173}$$

Proof of (i). Without loss of generality, we assume $\rho^- < \rho^+$. For $n \in \mathbb{Z}$, let us denote $\nu_n := \tau_n \nu$. We can apply Proposition 4.1 to ν and rule out the case (107) by assumption and Proposition 4.6. Thus by *(i)* of Corollary 4.1, $\nu_n \leq \nu_{n+1}$ for all $n \in \mathbb{Z}$. We can also exclude the case $k = 2$ by *(ii)* of Proposition 4.2 because $|\rho^+ - \rho^-| = 1$; and the case (106) because $\rho^- \neq \rho^+$. Thus by (172), *(ii)* of Proposition 4.1 and *(ii)* of Proposition 4.2,

$$\Delta(\nu_{n-1}, \nu_n) = 1, \quad \forall n \in \mathbb{Z} \tag{174}$$

By *(i)* of Proposition 4.4, for $n \in \mathbb{Z}$, there exists a coupling $d\widetilde{\nu}_n(\eta, \xi)$ of $d\nu_n(\eta)$ and $d\nu'(\xi)$ that satisfies one of the properties (104)–(107) of Proposition 4.1. By assumption and *(ii)* of Proposition 4.4, we can rule out (107). Thus ν_n and ν' are ordered. Besides, *(iii)* of Proposition 4.4 and (172) imply

$$\Delta(\nu_n, \nu') \in \mathbb{N}, \quad \forall n \in \mathbb{Z} \tag{175}$$

Let $S := \{n \in \mathbb{Z} : \nu' \leq \nu_n\}$. We claim that S is non-empty and bounded from below. Indeed if S were empty, since ν a (ρ^-, ρ^+)-shock measure (cf. definition (36)), $\nu' \geq \nu_n$ and $n \to +\infty$ would imply $\nu' \geq \nu_{\rho^+}$; if S were not bounded from below, $n \to -\infty$ along a subsequence where $\nu' \leq \nu_n$ would imply $\nu' \leq \nu_{\rho^-}$. Both conclusions would contradict ν' being a (ρ^-, ρ^+)-shock measure. We set $n_0 := \min(S)$, thus

$$\nu_{n_0 - 1} < \nu' \leq \nu_{n_0} \tag{176}$$

By (176) and (173),

$$\Delta(\nu_{n_0-1}, \nu_{n_0}) = \Delta(\nu_{n_0-1}, \nu') + \Delta(\nu', \nu_{n_0}) \tag{177}$$

By (176), (171) and (175), the first term on the right-hand side of (177) is a nonzero integer; thus by (174) for $n = n_0$, the second term is zero, and the conclusion follows from (171).

Proof of (ii). We can consider n_0 and the couplings of ν_{n_0} with ν' and ν_{n_0-1} with ν_{n_0} as in *(i)*. Let ν'' be a (ρ^-, ρ^+)-shock measure. For $n \in \mathbb{Z}$, we can also apply *(i)* of Proposition 4.4 to $\nu''_n := \tau_n \nu''$ and ν, and rule out case (107), since by assumption we exclude (40); thus these measures are ordered. The same holds for ν''_n and ν'. Similarly to n_0, we can then define $n_1 \in \mathbb{Z}$ such that

$$\nu''_{n_1 - 1} < \nu_{n_0 - 1} \leq \nu''_{n_1} \tag{178}$$

Property (175) holds, but instead of (174), *(iii)* of Proposition 4.4 now implies

$$\Delta(\nu_{n-1}, \nu_n) = 2 = \Delta(\nu''_{n-1}, \nu''_n), \quad \forall n \in \mathbb{Z} \tag{179}$$

Since ν' is not a translate of ν, both terms on the right-hand side of (177) are now nonzero integers. The first equality in (179) for $n = n_0$, combined with (177), then yields

$$\Delta(\nu_{n_0-1}, \nu') = \Delta(\nu', \nu_{n_0}) = 1 \tag{180}$$

We now distinguish the following cases.

(1) If $\nu''_{n_1} \geq \nu_{n_0}$, by (178) we have $\nu''_{n_1-1} \leq \nu_{n_0-1} \leq \nu_{n_0} \leq \nu''_{n_1}$; by (173),

$$\Delta(\nu''_{n_1-1}, \nu''_{n_1}) = \Delta(\nu_{n''_1-1}, \nu_{n_0-1}) + \Delta(\nu_{n_0-1}, \nu_{n_0}) + \Delta(\nu_{n_0}, \nu''_{n_1})$$

From (178), (179) with $n = n_1$, and (171), we obtain $\nu''_{n_1} = \nu_{n_0}$.

(2) If $\nu_{n_0-1} \leq \nu''_{n_1} \leq \nu_{n_0}$, we distinguish whether (a) $\nu_{n_0-1} \leq \nu''_{n_1} \leq \nu'$ or (b) $\nu' \leq \nu''_{n_1} \leq \nu_{n_0}$. In the former case, (173) and (180) yield

$$1 = \Delta(\nu_{n_0-1}, \nu') = \Delta(\nu_{n_0-1}, \nu''_{n_1}) + \Delta(\nu''_{n_1}, \nu')$$

and one of the terms on the r.h.s. must be 0. Case (b) is similar. □

5.2 Proofs of intermediate results from Subsections 4.3–4.4

Proof of Proposition 4.8.

Proof of (o). This follows from (117), (118), and Lemma 3.1.

Proof of (i). For the following, we rely on the expression for G given in (117). Therefore $G \geq 0$. In cases (42) and (43), G is identically 0 (the former follows from example 4.2). We henceforth exclude these cases. If $q = 0$, the conclusion follows from example 4.1. If $q > 0$ (that is $r > 0$), which we henceforth assume, G is continuously differentiable. First, G' vanishes at least once because $G(0) = G(2) = 0$, cf. *(o)*. Next,

$$G(\rho) = (\gamma_0 + \gamma_1)\frac{\rho}{2}\left(1 - \frac{\rho}{2}\right) + (\gamma_0 - \gamma_1)(1-\rho)\varphi(\rho) - (\gamma_0 + \gamma_1)\varphi(\rho)^2 \tag{181}$$

with

$$\begin{aligned} \varphi(\rho) &= \frac{1}{2}\left(\frac{r+1}{r-1}\right)\left(1 - \sqrt{\psi(\rho)}\right) \text{ if } r \neq 1 \\ &= 0 \text{ if } r = 1 \end{aligned} \tag{182}$$

$$\psi(\rho) = 1 + \left(\frac{r-1}{r+1}\right)^2 \rho(\rho - 2) \tag{183}$$

Note that $0 < \psi(\rho) \leq 1$, where the first inequality follows from $r > 0$. We then compute

$$\psi'(\rho) = \left(\frac{r-1}{r+1}\right)^2 2(\rho-1) \tag{184}$$

$$\varphi'(\rho) = -\frac{1}{2}\left(\frac{r-1}{r+1}\right)\frac{(\rho-1)}{\sqrt{\psi(\rho)}} \tag{185}$$

$$\varphi''(\rho) = -2r\left(\frac{(r-1)}{(r+1)^3}\right)\psi(\rho)^{-3/2} \tag{186}$$

$$\varphi^{(3)}(\rho) = 6r(\rho-1)\left(\frac{(r-1)^3}{(r+1)^5}\right)\psi(\rho)^{-5/2} \tag{187}$$

$$\begin{aligned} G'(\rho) &= (\gamma_0+\gamma_1)\frac{1}{2}(1-\rho) + (\gamma_0-\gamma_1)\left[-\varphi(\rho) + (1-\rho)\varphi'(\rho)\right] \\ &\quad -2(\gamma_0+\gamma_1)\varphi(\rho)\varphi'(\rho) \end{aligned} \tag{188}$$

$$\begin{aligned} G''(\rho) &= -\frac{1}{2}(\gamma_0+\gamma_1) + (\gamma_0-\gamma_1)\left[-2\varphi'(\rho) + (1-\rho)\varphi''(\rho)\right] \\ &\quad -2(\gamma_0+\gamma_1)[\varphi'(\rho)^2 + \varphi(\rho)\varphi''(\rho)] \end{aligned} \tag{189}$$

$$\begin{aligned} G^{(3)}(\rho) &= +(\gamma_0-\gamma_1)\left[-3\varphi''(\rho) + (1-\rho)\varphi^{(3)}(\rho)\right] \\ &\quad -(\gamma_0+\gamma_1)[6\varphi'(\rho)\varphi''(\rho) + 2\varphi(\rho)\varphi^{(3)}(\rho)] \\ &= 6r\frac{(r-1)^2}{(r+1)^4}\psi(\rho)^{-5/2} \times \\ &\quad \left[(\gamma_0-\gamma_1)\frac{4r}{(r-1)(r+1)} + (\gamma_0+\gamma_1)(1-\rho)\right] \end{aligned} \tag{190}$$

Hence, $G^{(3)}$ is identically 0 if $r = 1$ (that is $p = q$), and has a constant sign if $\gamma_0 + \gamma_1 = 0$ and $r \neq 1$. Whereas if $\gamma_0 + \gamma_1 \neq 0$ and $r \neq 1$, we have that $G^{(3)}(\rho)$ changes sign exactly once, for the value

$$\widetilde{\rho}_0 := 1 + \frac{\gamma_0-\gamma_1}{\gamma_0+\gamma_1}\frac{4r}{(r-1)(r+1)} \tag{191}$$

Therefore G'' is increasing before $\widetilde{\rho}_0$ and decreasing afterwards. Hence G'' changes sign at most twice and G' changes sign at most three times.

Proof of (ii). If $q = 0$, then $G(1) = 0$ by (119). If $q \neq 0$, the functions $\widetilde{\rho}_i$ in Lemma 3.1 are continuously differentiable on $[0, 2]$, thus the same holds for G. By (181), (182)–(183), (185) and (188),

$$G(1) = \frac{(\gamma_0+\gamma_1)\sqrt{r}}{(\sqrt{r}+1)^2}, \quad G'(1) = \frac{\gamma_1-\gamma_0}{2}\left(\frac{\sqrt{r}-1}{\sqrt{r}+1}\right),$$

whence the desired conclusions.

Proof of (iii). Here we obtain

$$G'(2) = -\frac{\gamma_0 + r\gamma_1}{r+1}$$

Under (27), we have $\gamma_0 + r\gamma_1 \geq r(\gamma_0 + \gamma_1) \geq 0$. The lower bound is positive if $r > 0$ and $\gamma_0 + \gamma_1 > 0$. On the other hand, $\gamma_0 + r\gamma_1 = (1-r)\gamma_0 > 0$ if $\gamma_0 + \gamma_1 = 0$ and $\gamma_0 \neq 0$; and $\gamma_0 + r\gamma_1 > 0$ if $r = 0$ and $\gamma_0 > 0$. Finally, $\gamma_0 + r\gamma_1 = 0$ if we have (42), or (43), or $q = \gamma_0 = 0$.

Proof of (iv). This follows from (181), (182) and (183).

Proof of (v). Without loss of generality, we assume $\gamma_0 = \gamma_1 = 1/2$. Then (181) becomes

$$G(\rho) \quad = \quad \frac{\rho}{2}\left(1 - \frac{\rho}{2}\right) - \varphi(\rho)^2 \tag{192}$$

and (188) becomes

$$G'(\rho) \quad = \quad \frac{1}{2}(1-\rho) - 2\varphi(\rho)\varphi'(\rho) = (1-\rho)\left(1 - \frac{1}{2\sqrt{\psi(\rho)}}\right) \tag{193}$$

We have that

$$\begin{aligned} G'(1/2) \quad &= \quad \frac{1}{2}\left(1 - \frac{1}{2\sqrt{\psi(1/2)}}\right) \\ G'(1/2) \quad &> \quad 0 \Leftrightarrow \psi(1/2) > \frac{1}{4} \Leftrightarrow 1 > \left(\frac{r-1}{r+1}\right)^2 \end{aligned}$$

which is true. Then after some computations, one can see that

$$\begin{aligned} G(1/2) > G(1) \quad &\Leftrightarrow \quad 4\psi(\frac{1}{2})\psi(1) < \left[-1 + \frac{7}{4}\left(\frac{r-1}{r+1}\right)^2\right]^2 \\ &\Leftrightarrow \quad 3 - 14R - R^2 < 0 \end{aligned} \tag{194}$$

where

$$R = \frac{1}{4}\left(\frac{r-1}{r+1}\right)^2$$

Indeed, we have that

$$\psi(1/2) = 1 - 3R;\ \psi(1) = 1 - 4R \tag{195}$$

so that $\psi(1/2)+\psi(1)=2-7R$, $\psi(1/2)-\psi(1)=R$, and

$$\begin{aligned} G(1/2)>G(1) &\Leftrightarrow R<\left[\sqrt{\psi(1/2)}-\sqrt{\psi(1)}\right]\times\left[2-\left(\sqrt{\psi(1/2)}+\sqrt{\psi(1)}\right)\right] \\ &\Leftrightarrow R<2\left[\sqrt{\psi(1/2)}-\sqrt{\psi(1)}\right]+[\psi(1)-\psi(1/2)] \\ &\Leftrightarrow R\left[\sqrt{\psi(1/2)}+\sqrt{\psi(1)}\right]<R \\ &\Leftrightarrow 4\psi(1/2)\psi(1)<[1-(\psi(1)+\psi(1/2))]^2 \end{aligned}$$

Solving inequation (194) with respect to r gives the condition in *(b)*.

Proof of (vi). Let $F(\rho):=G(\rho+1)-G(\rho)$. Note that

$$F(\rho) = \mathcal{F}(\rho+1)-\mathcal{F}(\rho) \qquad \text{with} \tag{196}$$

$$\mathcal{F}(\rho) = (\gamma_0+\gamma_1)\left[-\frac{1}{4}(\rho-1)^2-\varphi(\rho)^2\right]-(\gamma_0-\gamma_1)(\rho-1)\varphi(\rho) \tag{197}$$

First case. We assume $\gamma_0+\gamma_1\neq 0$. By (122), without loss of generality, we may consider $\gamma_0=\mathfrak{a}$ and $\gamma_1=1-\mathfrak{a}$ with $\mathfrak{a}\in\mathbb{R}$. We have

$$\begin{aligned} \mathcal{F}'(\rho) &= -\frac{1}{2}(\rho-1)-2\varphi(\rho)\varphi'(\rho)+(2\mathfrak{a}-1)(1-\rho)\varphi'(\rho)-(2\mathfrak{a}-1)\varphi(\rho) \\ F'(\rho) &= -1+\frac{1}{2}\left[\frac{\rho}{\sqrt{\psi(\rho+1)}}-\frac{(\rho-1)}{\sqrt{\psi(\rho)}}\right] \\ &\quad +\frac{(2\mathfrak{a}-1)}{2}\left(\frac{r-1}{r+1}\right)\left[\frac{\rho^2}{\sqrt{\psi(\rho+1)}}-\frac{(\rho-1)^2}{\sqrt{\psi(\rho)}}\right] \\ &\quad +\frac{(2\mathfrak{a}-1)}{2}\left(\frac{r+1}{r-1}\right)\left[\sqrt{\psi(\rho+1)}-\sqrt{\psi(\rho)}\right] \end{aligned}$$

then

$$\begin{aligned} \mathcal{F}''(\rho) &= -\frac{1}{2}-2\varphi'(\rho)^2-2\varphi(\rho)\varphi''(\rho)+(2\mathfrak{a}-1)(1-\rho)\varphi''(\rho)-2(2\mathfrak{a}-1)\varphi'(\rho) \\ &= -\frac{1}{2}-2\left[-\frac{1}{2}\left(\frac{r-1}{r+1}\right)\frac{\rho-1}{\sqrt{\psi(\rho)}}\right]^2 \\ &\quad +2r\left(\frac{r+1}{r-1}\right)\left(1-\sqrt{\psi(\rho)}\right)\left(\frac{r-1}{(r+1)^3}\right)\psi(\rho)^{-3/2} \\ &\quad -2r(2\mathfrak{a}-1)(1-\rho)\left(\frac{r-1}{(r+1)^3}\right)\psi(\rho)^{-3/2} \\ &\quad +(2\mathfrak{a}-1)\left(\frac{r-1}{r+1}\right)\frac{(\rho-1)}{\sqrt{\psi(\rho)}} \end{aligned}$$

so that

$$F''(\rho) = -\frac{1}{2}\left(\frac{r-1}{r+1}\right)^2\left[\frac{\rho^2}{\psi(\rho+1)} - \frac{(\rho-1)^2}{\psi(\rho)}\right] \quad (198)$$

$$+\frac{2r}{(r+1)^2}\left[\frac{1}{\psi(\rho+1)^{3/2}} - \frac{1}{\psi(\rho)^{3/2}} - \frac{1}{\psi(\rho+1)} + \frac{1}{\psi(\rho)}\right] \quad (199)$$

$$+(2\mathfrak{a}-1)\left(\frac{r-1}{r+1}\right)\left[\left(\frac{\rho}{\sqrt{\psi(\rho+1)}} + \frac{2r}{(r+1)^2}\frac{\rho}{\psi(\rho+1)^{3/2}}\right)\right.$$

$$\left. - \left(\frac{(\rho-1)}{\sqrt{\psi(\rho)}} + \frac{2r}{(r+1)^2}\frac{(\rho-1)}{\psi(\rho)^{3/2}}\right)\right] \quad (200)$$

We check the sign of each term.

$$f(\rho) = \frac{(\rho-1)}{\sqrt{\psi(\rho)}}$$

$$f'(\rho) = \frac{1}{\psi(\rho)^{3/2}}\frac{4r}{(r+1)^2} > 0$$

$$g(\rho) = f(\rho)^2$$

$$g'(\rho) = 2f(\rho)f'(\rho) < 0 \quad \text{for } \rho \in [0,1) \quad (201)$$

$$\bar{f}(\rho) = \frac{1}{\psi(\rho)^{3/2}} - \frac{1}{\psi(\rho)}$$

$$\bar{f}'(\rho) = \frac{\psi'(\rho)}{2\psi(\rho)^{5/2}[2\sqrt{\psi(\rho)}+3]}\left[-5 + 4\left(\frac{r-1}{r+1}\right)^2\rho(\rho-2)\right] \geq 0 \quad (202)$$

$$h(\rho) = \frac{(\rho-1)}{\psi(\rho)^{3/2}}\left[\psi(\rho) + \frac{2r}{(r+1)^2}\right]$$

$$h'(\rho) = \frac{1}{\psi(\rho)^{5/2}}\frac{2r}{(r+1)^2}\frac{12r}{(r+1)^2} > 0 \quad (203)$$

(note that $\bar{f}'(\rho) = 0$ if $r = 1$, and $\bar{f}'(\rho) > 0$ if $r \neq 1$). By (201) and (202), the first two terms (198) and (199) of $F''(\rho)$ are non-negative. By (203) the sign of the third term (200) of $F''(\rho)$ depends on the sign of $(2\mathfrak{a}-1)(r-1)$: If $(2\mathfrak{a}-1)(r-1) \geq 0$, then the third term of $F''(\rho)$ will also be non-negative.

Let us assume first that $\mathfrak{a} \geq 1/2$, $r \leq 1$. Hence $F''(\rho) > 0$ for $\rho \in [0,1)$. Then

$$F'(0) = -\frac{1}{2} - \frac{2\mathfrak{a}-1}{2}\left[\left(\frac{r-1}{r+1}\right) + \left(\frac{\sqrt{r}-1}{\sqrt{r}+1}\right)\right] < 0 \quad (204)$$

$$F'(1) = -\frac{1}{2} + \frac{2\mathfrak{a}-1}{2}\left[\left(\frac{r-1}{r+1}\right) + \left(\frac{\sqrt{r}-1}{\sqrt{r}+1}\right)\right] < 0 \quad \text{(see below)} \quad (205)$$

$$F(0) = G(1) = \frac{\sqrt{r}}{(\sqrt{r}+1)^2} > 0 \quad (206)$$

$$F(1) = -G(1) < 0 \quad (207)$$

Thus F being non-increasing the equation $F(\rho) = 0$ has a unique solution. We now show that $F'(1) < 0$. We write $X = \sqrt{r}$, and we consider $X \geq 1$.

$$\begin{aligned} \mathfrak{f}(X) &:= 2(r+1)(\sqrt{r}+1)F'(1) \\ &= (4\mathfrak{a}-3)X^3 - X^2 - X - (4\mathfrak{a}-3) \\ \mathfrak{f}(1) &= -4 < 0 \\ \mathfrak{f}'(X) &= 3(4\mathfrak{a}-3)X^2 - 2X - 1 \end{aligned}$$

If $\mathfrak{a} = 3/4$, $\mathfrak{f}'(X) < 0$. Otherwise we solve $\mathfrak{f}'(X) = 0$.

$$\begin{aligned} \delta &= 4(3\mathfrak{a}-2) > 0 \quad \text{for} \quad \mathfrak{a} > 2/3 \\ X_\pm &= \frac{1 \pm \sqrt{\delta}}{3(4\mathfrak{a}-3)} \quad \text{for} \quad \delta \geq 0 \end{aligned}$$

Then

- if $\mathfrak{a} < 2/3$, $\delta < 0$, $\mathfrak{f}'(X) < 0$, $\mathfrak{f}$ is decreasing hence $F'(1) < 0$.
- if $2/3 \leq \mathfrak{a} < 3/4$, $\mathfrak{f}'(X) > 0$ for $X \in (X_-, X_+)$; but $X_\pm < 0$, hence $\mathfrak{f}'(X) < 0$, $\mathfrak{f}$ is decreasing and $F'(1) < 0$.
- if $\mathfrak{a} = 3/4$, $\mathfrak{f}'(X) < 0$, hence $F'(1) < 0$.
- if $\mathfrak{a} > 5/6$, $X_- < 0 < X_+$ and $X_+ > 1$ because

  $$X_+ < 1 \Leftrightarrow 9(\mathfrak{a}-1)(4\mathfrak{a}-3) > 0 \Leftrightarrow \mathfrak{a} \notin (3/4, 1)$$

  thus $\mathfrak{f}'(X) < 0$, $\mathfrak{f}$ is decreasing hence $F'(1) < 0$.
- if $3/4 < \mathfrak{a} < 5/6$ we also have $X_+ > 1$, thus $F'(1) < 0$.
- if $\mathfrak{a} = 5/6$, then $X_+ = 1 + \frac{2}{\sqrt{2}} > 1$ hence $\mathfrak{f}$ is decreasing and $F'(1) < 0$.

Thanks to (121), the case $\mathfrak{a} \geq 1/2$, $r \leq 1$ is also solved. A similar reasoning enables to deal with $\mathfrak{a} \leq 1/2$, $r \leq 1$, then with $\mathfrak{a} \leq 1/2$, $r \geq 1$.

Second case. We assume $\gamma_0 + \gamma_1 = 0$ and $p \neq q$. Without loss of generality, we can consider $\gamma_0 = 1$. This amounts to repeating the computations of the first case keeping only in $F'(\rho)$ and $F''(\rho)$ those terms with the factor $(2\mathfrak{a}-1)/2$, which we replace by 1. This leads similarly to $F''(\rho) < 0$ for $\rho \in [0, 1)$. However, we now have $F'(0) < 0$ and $F'(1) > 0$. Thus there exists $\rho^* \in (0, 1)$ such that F is decreasing on $[0, \rho^*]$ and increasing on $[\rho^*, 1]$. Besides, (206)–(207) are now replaced by $F(0) = F(1) = 0$, cf. *(o)* and *(ii)* of Proposition 4.8. This implies that 0 and 1 are the only solutions of the equation $G(\rho+1) - G(\rho) = 0$.

Proof of (vii). If $q = 0$, this follows from Remark 2.4 and Example 4.1. We now assume $q > 0$. We write the function $G(\rho)$ in terms of a different variable.

Recall definition (29) of $\mathcal{F}$, definition of $\widetilde{\rho}_0(.)$ in Lemma 3.1, and expression (117) for G. We can then write

$$G(\rho) = \gamma_0\rho_0(1-\rho_0) + \gamma_1\rho_1(1-\rho_1)$$

where (ρ_0, ρ_1) is the unique element of $\mathcal{F}$ such that $\rho_0 + \rho_1 = \rho$. By (29), setting $r = q/p > 0$, there is a unique $\lambda \in [0, +\infty)$ such that

$$\rho_0 = \frac{r\lambda}{1+r\lambda}, \quad \rho_1 = \frac{\lambda}{1+\lambda} \tag{208}$$

It follows that

$$G(\rho) = \widetilde{G}(\lambda) = \lambda\left\{\frac{\gamma_0 r}{(1+r\lambda)^2} + \frac{\gamma_1}{(1+\lambda)^2}\right\} \tag{209}$$

Then, nonzero solutions of the equation $\widetilde{G}(\lambda) = 0$ are solutions of

$$r(\gamma_0 + \gamma_1 r)\lambda^2 + 2r(\gamma_0+\gamma_1)\lambda + \gamma_0 r + \gamma_1 = 0 \tag{210}$$

Positive solutions of (210) correspond to solutions of $G(\rho) = 0$ in $(0, 2)$. If $\gamma_0 + \gamma_1 r = 0$, that is $p\gamma_0 + q\gamma_1 = 0$, then since $p > q$, we have $\gamma_0 + \gamma_1 \neq 0$. The unique solution of (210) is

$$\lambda = -\frac{\gamma_1 + \gamma_0 r}{2r(\gamma_0+\gamma_1)}$$

and $\widetilde{G}$ changes sign around this solution. Recalling (27), we find that $\lambda > 0$ if $q\gamma_0 + p\gamma_1 < 0$, that is (41), otherwise $\lambda \leq 0$. If $p\gamma_0 + q\gamma_1 \neq 0$, then (210) is quadratic with reduced discriminant

$$\Delta' = -r(1-r)^2\gamma_0\gamma_1$$

Under (41), by (27) and $q > 0$, we have $\gamma_0 + \gamma_1 r > 0$. By Remark 2.4, we have $\Delta' > 0$, hence two solutions $\lambda_1 < \lambda_2$ around which $\widetilde{G}(\lambda)$ changes sign. These solutions are such that

$$\lambda_1\lambda_2 = \frac{\gamma_0 r + \gamma_1}{r(\gamma_0+\gamma_1 r)} < 0, \quad \lambda_1 + \lambda_2 = -2\frac{\gamma_0+\gamma_1}{\gamma_0+\gamma_1 r} \leq 0$$

where the first inequality follows from (41) and the second one from (27). By the former, there is a positive solution to (210). If (41) fails and $\Delta' \geq 0$, since $\lambda_1\lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 \leq 0$, there is no positive solution. Finally, if (41) fails and $\Delta' < 0$, there is no solution. □

Proof of Lemma 4.1. In cases *(i)–(ii)* below, we always have $|\mathcal{R}_0| \leq 3$. The only case not covered below is $q = 0 < p$ and $\gamma_0 \neq \gamma_1$. Then (119) and Definition 4.3 show that $\mathcal{R}_0$ is reduced to two elements of $\mathcal{B}_1$.

Proof of (i). By Definition 4.3, for any (ρ^-, ρ^+) in $\mathcal{R}_0$, $\rho = \min(\rho^-, \rho^+)$

must be a solution of the equation $G(\rho+1)-G(\rho)=0$. By *(vi)* of Proposition 4.8, this equation has exactly one solution ρ in $[0,1]$. This implies $\mathcal{R}_0 \subset \{(\rho,\rho+1);(\rho+1,\rho)\}$. But condition *(ii)* of Definition 4.3 implies that $(\rho,\rho+1)$ and $(\rho+1,\rho)$ cannot both lie in $\mathcal{R}_0$. Indeed, G would then be constant on $[\rho,\rho+1]$, and the only situations where G can be constant on a nontrivial interval are (42), (43) and (44), which are excluded here.

Since $G(0)=G(2)=0$ by *(o)* of Proposition 4.8, in order to have $\mathcal{B}_1 \cap \mathcal{R}_0 \neq \emptyset$, it is necessary to have $G(1)=0$. By *(ii)* of Proposition 4.8, this only occurs if $q=0$ or $\gamma_0+\gamma_1=0$.

Proof of (ii). *First case:* $q>0$, $p\neq q$ and $\gamma_0+\gamma_1=0\neq\gamma_0\gamma_1$. Similarly to *(i)*, using *(vi)* of Proposition 4.8, we see that $\mathcal{R}_0\subset\mathcal{B}_1$. By *(o)* and *(ii)* of Proposition 4.8, G only vanishes for $\rho\in\{0,1,2\}$; thus by Definition 4.3, one of the points $(0,1)$ or $(1,0)$, and one of the points $(1,2)$ or $(2,1)$, lie in $\mathcal{R}_0$. And since $(\rho,\rho+1)$ and $(\rho+1,\rho)$ cannot both lie in $\mathcal{R}_0$, $\mathcal{R}_0$ contains two elements. *Second case:* $q=0$ and $\gamma_0\neq\gamma_1$. Then (119) and Definition 4.3 shows that $\mathcal{R}_0$ is reduced to two elements of $\mathcal{B}_1$.

Proof of (iii). Assume first $r>0$. By (121), since $\gamma_0=\gamma_1$, we have $G(2-\rho)=G(\rho)$ for all $\rho\in[0,2]$. Thus $G(1/2)=G(3/2)$ and $G'(1)=0$. Recalling *(i)*, there can be no shock of amplitude 1 other than $(1/2,3/2)$ or $(3/2,1/2)$; and at most one of these lies in $\mathcal{R}_0$. If G has a single extremum (which must be at 1), by *(iii)* of Proposition 4.8, G is bell-shaped and this extremum is a maximum. Thus $\mathcal{R}_0=\{(1/2,3/2)\}$. If G has more than one extremum, by symmetry it must have three. Still by *(iii)* of Proposition 4.8, the extremum at 1 is then a local minimum and the other two are local maxima symmetric with respect to 1. Since $G'(1/2)>0$ by *(v)* of Proposition 4.8, condition *(ii)* of Definition 4.3 cannot hold with $(\rho^-,\rho^+)=(3/2,1/2)$. On the other hand, this condition holds with $(\rho^-,\rho^+)=(1/2,3/2)$ if and only if $G(1/2)\leq G(1)$. The conclusion then follows from *(v)* of Proposition 4.8. Finally, for $r=0$, $\mathcal{R}_0$ follows from (119) and Definition 4.3 (recall (27), implying here that $\gamma_0>0$ and $\gamma_1>0$).

Proof of (iv). For $(\mathfrak{a},r)\in[1/2,1]\times[1,+\infty)$, let us denote by $\rho(\mathfrak{a},r)$ the unique solution given by *(vi)* of Proposition 4.8 of $F_{\mathfrak{a},1-\mathfrak{a},r}(\rho)=0$, where $F_{\mathfrak{a},1-\mathfrak{a},r}(\rho):=G_{\mathfrak{a},1-\mathfrak{a},r}(\rho+1)-G_{\mathfrak{a},1-\mathfrak{a},r}(\rho)$. The proof of Proposition 4.8, *(vi)* showed that $F'_{\mathfrak{a},1-\mathfrak{a},r}(\rho)<0$ for every $\rho\in[0,1]$. Besides, by (181), (182) and (183), $F_{\mathfrak{a},1-\mathfrak{a},r}$ is continuously differentiable with respect to $(\mathfrak{a},r)$. Thus the implicit function theorem implies that $(\mathfrak{a},r)\mapsto\rho(\mathfrak{a},r)$ is continuously differentiable. Let

$$I(\mathfrak{a},r):=\inf_{\rho\in[\rho(\mathfrak{a},r),1+\rho(\mathfrak{a},r)]}G_{\mathfrak{a},1-\mathfrak{a},r}(\rho),\quad S(\mathfrak{a},r):=\sup_{\rho\in[\rho(\mathfrak{a},r),1+\rho(d,r)]}G_{\mathfrak{a},1-\mathfrak{a},r}(\rho)$$

We define

$$\mathcal{Z}:=\{(\mathfrak{a},r)\in[0,1]\times[0,1]:\ I(\mathfrak{a},r)<G_{\mathfrak{a},1-\mathfrak{a},r}[\rho(\mathfrak{a},r)]<S(\mathfrak{a},r)\}\qquad(211)$$

The set $\mathcal{Z}$ is an open subset of $[0,1]^2$ because $(\mathfrak{a},r)\mapsto\rho(\mathfrak{a},r)$ is continuous. By *(iii)*, it contains $\{1/2\}\times(0,r_0)$. Finally, by *(ii)* of Definition 4.3, for $(\mathfrak{a},r)\in\mathcal{Z}$, neither $(\rho(\mathfrak{a},r),1+\rho(\mathfrak{a},r))$ nor $(1+\rho(\mathfrak{a},r),\rho(\mathfrak{a},r))$ lies in $\mathcal{R}_0$, thus $\mathcal{R}_0=\emptyset$. □

Proof of Lemma 4.2.

Proof of (i). By *(i)* of Proposition 4.2, μ is a shock measure of amplitude 2, that is either a $(0,2)$ or a $(2,0)$-shock measure. The second possibility and *(i)* of Proposition 4.9 would imply that $(2,0)$ satisfies condition *(ii)* of Definition 4.3, thus by *(o)* of Proposition 4.8, that the maximum of G is 0; whereas *(iii)* of Proposition 4.8 (when $q>0$) and (119) (when $q=0$) imply that this maximum is positive under (27).

Proof of (ii). By *(vii)* of Proposition 4.8, the equation $G(\rho)=0$ has a solution in $(0;2)$ and changes sign around this solution. Since $G(0)=G(2)=0$, condition *(ii)* of Definition 4.3 cannot hold, and Proposition 4.9 implies that a $(0,2)$-shock measure cannot exist. This and *(i)* above imply the desired conclusion. □

Proof of Proposition 4.9.

Proof of (i). Assume for instance $\rho^-<\rho^+$, the case $\rho^->\rho^+$ being similar. Let $r\in[\rho^-,\rho^+]$. Let $d\widetilde{\nu}(\eta,\xi)$ be a coupling of $d\nu(\eta)$ and $d\nu_r(\xi)$ that is invariant for the coupled generator (85) (it exists by [14, Proposition 2.14 in Chapter VIII]). Since $\widetilde{\nu}$ is supported on a compact space, there exists an increasing $\mathbb{N}$-valued sequence $x_n\to+\infty$ such that $\tau_{-x_n-1}\widetilde{\nu}$ and $\tau_{x_n}\widetilde{\nu}$ have weak limits denoted respectively by $\widetilde{\nu}_{-\infty}$ and $\widetilde{\nu}_{+\infty}$. By (36) and translation invariance of ν_r, $\widetilde{\nu}_{\pm\infty}$ is a coupling of $\nu_{\rho^\pm}$ and ν_r. Since the coupled generator $\widetilde{L}$ given by (85) for the transition kernel (23) is translation invariant in the $\mathbb{Z}$-direction, we have $\widetilde{\nu}\in\widetilde{\mathcal{I}}\cap\widetilde{\mathcal{S}}$. Hence, by (101) in the proof of Theorem 2.1, $\widetilde{\nu}$ is supported on ordered pairs (η,ξ). On the other hand, under $\widetilde{\nu}_{\pm\infty}$, empirical averages (cf. (161)) exist by the law of large numbers and are given by $M(\eta)=\rho^\pm$ and $M(\xi)=r$. These averages must be ordered like η and ξ, hence $\widetilde{\nu}_{-\infty}$ and $\widetilde{\nu}_{+\infty}$ are supported respectively on E_- and E_+.

Let $N\in\mathbb{N}$, $R_N:=(\mathbb{Z}\cap[-N,N])\times W$, and

$$\widetilde{F}_N(\eta,\xi):=D_{-N,N}(\eta,\xi)=\sum_{i\in W}\sum_{z\in\mathbb{Z}\cap[-N,N]}|\eta(z,i)-\xi(z,i)| \tag{212}$$

Since $\widetilde{\nu}\in\widetilde{\mathcal{I}}$, we have

$$\int_{\mathcal{X}\times\mathcal{X}}\widetilde{L}\widetilde{F}_N(\eta,\xi)d\widetilde{\nu}(\eta,\xi)=0 \tag{213}$$

By [13, Lemma 2.4], we have

$$\widetilde{L}\widetilde{F}_N(\eta,\xi) = \sum_{x\notin R_N,\, y\in R_N} p(x,y)J_{x,y}(\eta,\xi) \tag{214}$$

$$- \sum_{x\in R_N,\, y\notin R_N} p(x,y)J_{x,y}(\eta,\xi) \tag{215}$$

$$- \sum_{x\in R_N,\, y\in R_N,\, x\neq y} [p(x,y)+p(y,x)]\mathbf{1}_{E_{x,y}}(\eta,\xi) \tag{216}$$

where $E_{x,y}$ was defined in (97), and

$$J_{x,y}(\eta,\xi) := [\eta(x)(1-\eta(y)) - \xi(x)(1-\xi(y))]\left\{\mathbf{1}_{\{\eta(x)\geq\xi(x),\,\eta(y)\geq\xi(y)\}} - \mathbf{1}_{\{\eta(x)\leq\xi(x),\,\eta(y)\leq\xi(y)\}}\right\} \tag{217}$$

Let

$$\widetilde{j}(\eta,\xi) := \sum_{x(0)\leq 0, y(0)>0} p(x,y)J_{x,y}(\eta,\xi) - \sum_{x(0)\leq 0, y(0)>0} p(y,x)J_{y,x}(\eta,\xi)$$

where $J_{x,y}$ is defined by (217). Then (214)–(215) can be written as $\tau_{-N-1}\widetilde{j}(\eta,\xi) - \tau_N\widetilde{j}(\eta,\xi)$. By (217) and (113)

$$\widetilde{j}(\eta,\xi) = j(\eta) - j(\xi) \text{ if } \eta\leq\xi, \quad \widetilde{j}(\eta,\xi) = j(\xi) - j(\eta) \text{ if } \xi\leq\eta \tag{218}$$

The stationarity relation (213) combined with (214)–(216) yields

$$\widetilde{\nu}(\tau_{-N-1}\widetilde{j}) - \widetilde{\nu}(\tau_N\widetilde{j}) \geq 0 \tag{219}$$

Taking $N = x_n$ and letting $n\to+\infty$ yields

$$\widetilde{\nu}_{-\infty}(\widetilde{j}) - \widetilde{\nu}_{+\infty}(\widetilde{j}) \geq 0 \tag{220}$$

Under $\widetilde{\nu}_{\pm\infty}$, we can use (218) for ordered configurations. The marginals of $\widetilde{\nu}_{\pm\infty}$ then yield

$$G(r) - G(\rho^-) \geq G(\rho^+) - G(r) \tag{221}$$

Since $r\in[\rho^-,\rho^+]$ is arbitrary, we first obtain $G(\rho^+) = G(\rho^-)$ by letting $r = \rho^\pm$, and then $G(\rho^+) = G(\rho^-) = \min_{r\in[\rho^-,\rho^+]} G(r)$.

Proof of (ii). This follows from *(i)* above, and *(ii)* of Proposition 4.2.

Proof of (iii). By Lemma 4.1, $\gamma_0+\gamma_1 = 0$ or $q = 0$. Assume from now on that the latter holds.

(a) We assume first $\gamma_1\geq 0$. Then by (119) and Definition 4.3, if γ_0 and γ_1 are not both 0, we have $\mathcal{R}_0\cap\mathcal{B}_1 = \{(0,1);(1,2)\}$.

We consider first $(\rho^-, \rho^+) = (0, 1)$. We show that this case is impossible if $\gamma_1 = 0$, whereas if $\gamma_1 > 0$, μ is one of the measures $\nu^{\perp,+\infty,j}$ in (58). To this end, observe first that since $q = 0 < p$, ν^0 is the probability measure supported on the empty configuration and ν^1 is supported on the configuration that is empty on lane 0 and full on lane 1. Since μ is a $(0, 1)$-shock measure, we have

$$\lim_{x\to-\infty} \tau_x \eta_0^0 = \mu_0, \quad \lim_{x\to+\infty} \tau_x \eta_0^0 = \mu_0, \tag{222}$$

$$\lim_{x\to-\infty} \tau_x \eta_0^1 = \mu_0, \quad \lim_{x\to+\infty} \tau_x \eta_0^1 = \mu_1 \tag{223}$$

where $\mu_\rho = \mu_{\mathbb{Z},\rho}$ (recall (9) and Remark 2.1) denotes the product Bernoulli measure on $\{0, 1\}^{\mathbb{Z}}$ with parameter ρ; for $\rho \in \{0, 1\}$ as above, these are Dirac measures supported on the empty or full configuration. As in the proof of Proposition 4.6, we couple $\eta_\cdot$ with an ASEP $\zeta_\cdot^0$ on lane 0 starting from $\zeta_0^0 := \eta_0^0$, with jump rate d_0 to the right and l_0 to the left, that is (6)–(5) with $(l, d) = (l_0, d_0)$. The limit (222) implies

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{x=1}^{n} \zeta_0^0(x) = \lim_{n\to+\infty} \frac{1}{n} \sum_{x=-n}^{1} \zeta_0^0(x) = 0 \tag{224}$$

in probability. Since the initial configuration satisfies (224), ζ_t^0 converges in law as $t \to +\infty$ to the Bernoulli invariant measure with zero density, that is the empty configuration; this follows from [3, Theorem 1] when $\gamma_0 > 0$, or [14, Chapter VIII] when $\gamma_0 = 0$. Since $\eta_t^0 \le \zeta_t^0$, the same limit holds for η_t^0. By stationarity of μ, this implies that under μ, lane 0 is almost surely empty. It follows that $(\eta_t^1)_{t\ge0}$ is itself an autonomous SEP. Thus the marginal of μ on lane 1 is an invariant measure for SEP. By [13, Theorem 1.4], it is a mixture of Bernoulli and blocking measures. Because of (223), only blocking measures are present in the mixture. Note that this is only possible if $\gamma_1 > 0$. In this case, μ is a mixture of the invariant measures $\nu^{\perp,+\infty,j}$ in (58) for $j \in \mathbb{Z}$. Since μ is extremal, it is one of them.

Next, we consider $(\rho^-, \rho^+) = (1, 2)$. This can be reduced to the previous case by Lemma 2.1, considering the image of η_t by $\sigma\sigma'\sigma''$. The resulting process has drift $\gamma_0' = \gamma_1$ on lane 0, and $\gamma_1' = \gamma_0$ on lane 1. The image μ'' of μ is a $(0, 1)$-shock measure invariant for the transformed process. It follows from the above that:

- If $\gamma_0 > 0$, $\mu'' = \nu^{\perp,+\infty,j}$, thus $\mu = \nu^{\perp,j,-\infty}$, for some $j \in \mathbb{Z}$.

- If $\gamma_0 = 0$, that is $\gamma_1' = 0$, from the above discussion, it is impossible for μ'' to be a $(0, 1)$-shock measure, and thus for μ to be a $(1, 2)$-shock measure.

Putting together the cases $(\rho^-, \rho^+) = (0, 1)$ and $(\rho^-, \rho^+) = (1, 2)$, we conclude that in case *(iv)* of Theorem 2.3, a (ρ^-, ρ^+)-shock measure with $(\rho^-, \rho^+) \in \mathcal{B}_1$ lies in the set (58); whereas in case *(vi)* it lies in the set (62). In the former

case $\mathcal{R}' = \mathcal{R}_0 \cap \mathcal{B}_1 = \{(0,1);(1,2)\}$, whereas in the latter case $\mathcal{R}' = \{(0,1)\} \neq \mathcal{R}_0 \cap \mathcal{B}_1 = \{(0,1);(1,2)\}$.

(b) We consider now $\gamma_1 < 0 < \gamma_0$. Here, by (119) and Definition 4.3, we have $\mathcal{R}_0 \cap \mathcal{B}_1 = \{(1,0);(2,1)\}$. The case $(\rho^-, \rho^+) = (1,0)$ is treated like $(\rho^-, \rho^+) = (0,1)$ in (a) above; except that on lane 1 we have a $(1,0)$-shock with a negative drift. The case $(\rho^-, \rho^+) = (1,2)$ is deduced by Lemma 2.1 and particle-hole symmetry (recall (26)). □

Proof of Lemma 4.3. Let $\widetilde{\nu}$ denote a coupling of ν^1 and ν^2 such that $\widetilde{\nu} \in \widetilde{\mathcal{I}}$. Since ν^1 and ν^2 are supported on $\mathcal{X}_2$, $\widetilde{\nu}$ satisfies assumption (95) of Proposition 3.2. Since we excluded the case $l_0 = l_1 = q = 0 < p$, by Lemma 3.3, $p(.,.)$ is weakly irreducible. Thus, by (98) and the proof of Theorem 2.1, $\widetilde{\nu}$ is supported on ordered pairs of configurations. Since H_2 is a nondecreasing function on $\mathcal{X}_2$ and has the same value under both marginals of $\widetilde{\nu}$, it follows that $\widetilde{\nu}$ is supported on E_3 (defined in (106)), whence the conclusion. □

A Extensions

A.1 Results

We discuss below situations where our approach should still work to extend parts of our results with minor modifications, or with suitable extensions but without essentially new ideas. Some explanations on the feasability of these extensions are given in Appendix A.2.

A.1.1 Non nearest-neighbour horizontal kernels

We may consider kernels of the form (14) in which the horizontal kernels $Q_i(.)$ are no longer assumed nearest-neighbour, but only weakly irreducible, cf. Definition 3.1. The results of Theorems 2.1 and 2.2 remain valid as such, because their proofs do not require the nearest-neighbour assumption. This is partly true for Theorems 2.3 and 2.4 with the following restrictions or modifications (only statements that do not carry over as such are mentioned).

In Theorem 2.3.

Statement (o). The proof carries over if we assume that $Q_i(z) > Q_i(-z)$ for all $i \in W$ and $z > 0$ such that $Q_i(z) > 0$; this is an intermediate condition between the single-lane conditions in [11, Theorem 5.1] and [5, Theorem 1.4].

Statement (i). The proof carries over under the assumption that the kernel $Q_i(.)$ on each lane is symmetric.

Statement (ii). This may be extended under the following assumption, automatically satisfied for nearest-neighbour kernels (see [11] for a similar condition for single-lane ASEP, or [6] for d-dimensional ASEP): there exists a constant $\theta \in (0, +\infty]$ such that

$$\forall i \in W,\, z \in \mathbb{Z}, \quad \frac{Q_i(z)}{Q_i(-z)} = \theta^z \tag{225}$$

Under this condition blocking measures can still be constructed from (53).

Statement (iii). Under condition (225), the conditions $d_1 = \lambda d_0$ and $l_1 = \lambda l_0$ for some $\lambda \in \mathbb{R}$ should be replaced by $Q_1(z) = \lambda Q_0(z)$ for all $z \in \mathbb{Z}$.

Statements (iv)–(vii). The proof of the description of Bl_1, cf; (58), (61), (62), remains valid as long as the kernels $Q_i(.)$ satisfy assumptions of [7] or [5] ensuring existence of profile measures for the corresponding single-lane ASEP. Then the family of measures $\{\widehat{\mu}_n, n \in \mathbb{Z}\}$ involved in the construction of $\nu^{\perp,i,j}$ is more generally the family of profile measures from [5], instead of being defined by (48), (51). The statement that $Bl_2 = \emptyset$ in *(iv)* and *(vii)* can be generalized under the assumption (similar to *(o)* above) that $Q_1(z) > Q_1(-z)$ for all $z > 0$.

In Theorem 2.4. As above, statement (2) and its proof carry over under the assumption that the kernel on each lane is symmetric, and the description of blocking measures in $\mathcal{I}_n$ based on (53) can be generalized under condition (225).

A.1.2 Multilane models

While the ladder process (that is a vertically cyclic multilane ASEP) was discussed in Subsection 2.5, another natural multilane generalization of the two-lane model is the kernel (14), where $W = \{0, \dots, n-1\}$ and

$$q(i,j) = p\mathbf{1}_{\{i<n-1,\, j=i+1\}} + q\mathbf{1}_{\{i>0,\, j=i-1\}} \tag{226}$$

with a nearest-neighbour kernel $Q_i(.)$ and corresponding drift γ_i on lane i. Note that the cyclic model of Subsection 2.5 contains the vertical kernels (for which tagged particle motion is studied in [18])

$$q(i,j) = p\mathbf{1}_{\{i<n-1,\, j=i+1\}} + p\mathbf{1}_{\{i=n-1,\, j=0\}} + q\mathbf{1}_{\{i>0,\, j=i-1\}} + q\mathbf{1}_{\{i=0,\, j=n-1\}} \tag{227}$$

and (226)–(227) coincide if and only if $n = 2$ and $p = q$.

Theorem 2.1. The result can be proved similarly for the multilane model (226) if we extend the definition of $\mathcal{F}$ as the set of $(\rho_0, \dots, \rho_{n-1}) \in [0,1]^n$ such that

$$p\rho_i(1-\rho_{i+1}) = q\rho_{i+1}(1-\rho_i), \quad \forall i \in \{0, \dots, n-1\} \tag{228}$$

Theorems 2.2–2.3. Outside the case

$$p = q \text{ and } \sum_{i=0}^{n-1} \gamma_i = 0 \tag{229}$$

(that is, the extension of (42)), the proofs of this paper could be extended to show the following statements, some of which are analogous to Theorem 2.4.

1. Elements of $\mathcal{I}_e$ that are not homogeneous Bernoulli measures consist of finitely many (up to shifts) shock measures of amplitude $k \in \{1, \dots, n\}$, with at most k shock measures of amplitude k.

2. Statement *(1), (b)* of Theorem 2.4 holds.

3. If $q > 0$, under the same condition as in statement *(1), (c)*, the set of shock measures with shock $(\rho^- = 0, \rho^+ = n)$ consists of (up to shift) n blocking measures. These are constructed similarly to Lemma 2.2 when $l_i > 0$ for all i; now the conditioning on H_2 depends on the remainder of H_2 modulo n. When $l_i = 0$ for all i, these measures are constructed as in Subsection 2.4.3, case *(ii), (b)* and *1, (c)* of Theorem 2.4. As in (66)–(67), we have a family of blocking measures ν_i for $i = 0, \dots, n-1$. However the weights are no longer uniform as in (67): we now have

$$\nu_i := \sum_{A \subset \{0,\dots,n-1\}:\, |A|=i} w_i(A) \delta_{\eta_A}$$

We may identify a subset A of W with an exclusion configuration on W for which A is the set occupied sites. The weight $w_i(A)$ is then the probability of A under the unique invariant measure with i particles of the SEP on W with jump kernel (226).

4. If $q = 0$, results of Theorem 2.3 could be extended as follows. First, if all lane drifts γ_i are different, the set of non-homogeneous invariant measures contains no shock measure except blocking or partial blocking measures described below.

To construct blocking or partial blocking measures, we must partition lanes into "groups". A group of $k \geq 2$ lanes consists of adjacent lanes that are totally asymmetric in the same direction (if such lanes exist). Other lanes are viewed as singleton groups. For a group containing $k \geq 2$ lanes, we can construct blocking measures similar to those of Theorem 2.3, *(iv), (b)*, cf. (60). On singleton groups we use blocking measures of single-lane ASEP as defined in (48)–(51). Then partial blocking measures are defined by considering blocking measures on one group, setting downstream lanes to 1 and upstream lanes to 0.

To state this precisely, we introduce more notation. Let $i_1 = n - 1$, and for $k \geq 1$, define i_{k+1} as follows: $i_{k+1} = i_k - 1$ unless lanes i_k and $i_k - 1$ are totally asymmetric in the same direction, that is, $0 = d_k l_k + d_{k+1} l_{k+1} < d_k d_{k+1} + l_k l_{k+1}$. In this case we set i_{k+1} to be the smallest $\ell < i_k$ such that lanes ℓ to i_k are totally asymmetric in the same direction. For some $k = k_0$ we eventually reach $i_k = 0$. Let $G_k = \{i_{k+1}, \dots, i_k\}$ be the k-th group, and let $s_k := (j_{i_{k+1}}, \dots, j_{i_k})$ be a "shift vector" taking values in $\mathbb{Z} \cup \{\pm\infty\}$ such that $j_{i_{k+1}} \geq \cdots \geq j_{i_k}$. Similarly to (60), we define $\nu^{\perp, j_{i_{k+1}}, \dots, j_{i_k}}$ to be the Dirac mass on the configuration $\eta^{\perp, j_{i_{k+1}}, \dots, j_{i_k}}$ whose restriction to lane $\ell \in \{i_{k+1}, \dots, i_k\}$ is $\eta^*_{j_\ell}$. We can then

define a family of (generally partial) blocking measures $\nu^{\perp,k,s_k}$ indexed by a group number $k = 1, \dots, k_0$ and a shift vector s_k, but excluding group numbers k such that $|G_k| = 1$ with drift $\gamma_{i_k} = 0$. Under the measure $\nu^{\perp,k,s_k}$:

(A) All lanes with numbers $i > i_k$ are fully occupied, i.e. $\eta(z,i) = 1$ for all $z \in \mathbb{Z}$.
(B) All lanes with numbers $i < i_{k+1}$ are empty, i.e. $\eta(z,i) = 0$ for all $z \in \mathbb{Z}$.
(C) Assume $G_k = \{i_k\}$. Then if $\gamma_{i_k} > 0$, the restriction of $\nu^{\perp,k,s_k}$ to lane i_k is $\widehat{\mu}_n$ with $n = j_{i_k}$, given by (48) or (51) with $l = l_{i_k}, d = d_{i_k}$. If $\gamma_{i_k} < 0$, the restriction of $\nu^{\perp,k,s_k}$ to lane i_k is the image of $\widehat{\mu}_n$ with $n = j_{i_k}$ by the symmetry operator σ (defined in (26)).
(D) Assume $|G_k| \geq 2$. Then if lanes in G_k are totally asymmetric to the right, the restriction of $\nu^{\perp,k,s_k}$ to lanes in G_k is $\nu^{\perp,j_{i_{k+1}},\dots,j_{i_k}}$. If it is to the left, the restriction is the image of $\nu^{\perp,j_{i_{k+1}},\dots,j_{i_k}}$ by σ.

Notice that only one group at a time can carry actual "blocking" measures. The above measures are (ρ^-, ρ^+)-shock measures where $\rho^\pm$ are integers such that $0 \leq \rho^- < \rho^+ \leq n$. If there is a single group, i.e. all lanes are totally asymmetric in the same direction (as in Theorem 2.3, *(iv) (b)*), finite-valued shift vectors yield global blocking measures, i.e. $\rho^- = 0$ and $\rho^+ = n$. If there are at least two groups (as in Theorem 2.3, *(iv) (b)*, *(v)* and *(vi)*) , only partial blocking measures are obtained, i.e. $\rho^+ - \rho^- < n$.

A.1.3 Single-lane models with several particles per site.

We may consider the Misanthrope's process introduced in [8]. Let us recall the definition of this model. Given some $K \in \mathbb{N} \setminus \{0\}$, the state space of this process is $\mathcal{X} = \{0, \dots, K\}^{\mathbb{Z}}$, and its generator of the form

$$Lf(\eta) = \sum_{x,y \in V} p(x,y)\, b[\eta(x), \eta(y)] \left(f\left(\eta^{x,y}\right) - f(\eta)\right), \tag{230}$$

where $p(x,y) = P(y-x)$ is a kernel satisfying (4), and $b(.,.) : \{0, \dots, K\}^2 \to [0, +\infty)$ is a jump rate function satisfying the following assumptions:

(M1) $b(0,.) = b(.,K) = 0$
(M2) For every $n, m \in \{0, \dots, K\}$, $b(.,m)$ and $b(n,.)$ are respectively nondecreasing and nonincreasing.
(M3) For every $n, m \in \{0, \dots, K\}$,

$$\frac{b(n,m)}{b(m+1,n-1)} = \frac{b(n,0)b(1,m)}{b(m+1,0)b(1,n-1)} \tag{231}$$

(M4) For every $n, m \in \{0, \dots, K\}$,

$$b(n,m) - b(m,n) = b(n,0) - b(m,0) \tag{232}$$

Homogeneous product invariant measures. In [8], homogeneous product invariant measures are constructed. The one-site marginal of these invariant measures is an exponential family $(\theta^\lambda)_{\lambda\geq 0}$ of probability measures on $\mathbb{N}$ of the form

$$\theta^\lambda(n) := Z(\lambda)^{-1}\lambda^n\theta^1(n) \tag{233}$$

where λ is the fugacity, $Z(\lambda)$ the normalizing factor, and $\theta^1(.)$ depends explicitely on the jump rate function $b(.,.)$. Homogeneous invariant measures are product measures ν^λ on $\mathcal{X}$ such that

$$\nu^\lambda[\eta(x) = i] = \theta^\lambda(i), \quad i \in \mathbb{N},\, x \in \mathbb{Z} \tag{234}$$

For $\lambda \to +\infty$, θ^λ converges weakly to δ_K and ν^λ to the corresponding product measure under which each site has K particles. We may thus define by extension $\theta^{+\infty}$ and $\nu^{+\infty}$. Then, one can reparametrize the family $(\nu^\lambda)_{\lambda\in[0,+\infty]}$ to get a family $(\nu_\rho)_{\rho\in[0,K]}$ of product invariant measures indexed by the mean density of particles, i.e. such that the expectation of $\eta(x)$ under ν_ρ is ρ, by setting

$$\nu_\rho := \nu^{R^{-1}(\rho)} \tag{235}$$

where $R(\lambda)$ is the mean of θ^λ (which is increasing and continuous with respect to λ). With these measures, a characterization theorem similar to Theorem 2.1 is given in [8] for $\mathcal{I} \cap \mathcal{S}$, under the assumption that the jump kernel $p(.,.)$ is weakly irreducible.

Blocking measures. In the case of nearest-neighbour jumps, $P(1) = d$ and $P(-1) = l$, explicit blocking measures can be obtained by letting the fugacity in (234) depend on the site as follows:

$$\mu^c[\eta(x) = i] = \theta^{\lambda(x)}(i), \quad i \in \mathbb{N},\, x \in \mathbb{Z} \tag{236}$$

with

$$\lambda(x) = c\left(\frac{d}{l}\right)^x, \quad c > 0 \tag{237}$$

Such blocking measures are studied in [4] as a basis for deriving remarkable combinatorial identites. Interestingly, though the Misanthrope's and two-lane exclusion process look quite different, the particular structure (55)–(56) is found in both settings. Namely, the above blocking measures can be decomposed by conditioning on the analogue of (54), that is here the conserved quantity (when initially finite)

$$H(\eta) := \sum_{x\in\mathbb{Z}:\, x\leq 0} \eta(x) - \sum_{x\in\mathbb{Z}:\, x>0} [1-\eta(x)] \tag{238}$$

As in Lemma 2.2, the conditioned measure

$$\mu^c(.|H(\eta) = k) \tag{239}$$

does not depend on $c > 0$.

Characterization of invariant measures. To our knowledge, there exists so far no characterization result for $\mathcal{I}$. As mentioned in the introduction and explained in Subsection 4.1, compared to what is known for ASEP, new problems are induced by the fact that several particles per site are allowed. With a suitable adaptation of our proofs, the following results may be obtained for this model in the line of Theorems 2.4–2.3.

1. Extremal invariant measures that are not homogeneous product measures consist (up to shifts) of a finite set of shock measures with integer amplitude $k \in \{1, \dots, K\}$. For each $k \in \{1, \dots, K\}$, there are at most k shock measures. For $k = K$, there are exactly K shock measures, which are the above conditioned blocking measures (239).

2. For $K = 2$, a function $b(.,.)$ satisfying conditions (M1)–(M4) above is uniquely determined by the parameters $b(1,0)$, $b(2,0) \geq b(1,0)$, and $b(1,1) \leq b(1,0)$; then $b(2,1) = b(2,0) - b(1,0)$. One can then obtain the following result similar to *(iii)* of Theorem 2.3. When

$$|b(2,0) - 2b(1,0)| \text{ and } b(1,1) \text{ are small enough,} \tag{240}$$

all extremal invariant measures are either homogeneous product measures or blocking measures (i.e., there is no shock measure of amplitude 1). An explanation of the link between condition (240) and the set $\mathcal{Z}$ in *(iii)* of Theorem 2.3 (i.e. the conditions that d is close to $1/2$ and r is small enough) is given by Lemma A.1 below.

3. If $p(.,.)$ is weakly irreducible and symmetric, all extremal invariant measures are homogeneous product measures.

A.2 Extensions of main ideas

We now comment on the robustness of the steps of proof outlined in Subsection 4.1 with respect to the extensions mentioned in Appendix A.1. These steps mainly use the following general properties of the model: (i) attractiveness property (87)–(88); (ii) weak irreducibility property (see Definition 3.1) for the global kernel (14) when $q > 0$, see Lemma 3.4; (iii) finite propagation property (Proposition 3.1); (iv) the characterization Theorem 2.1 for $(\mathcal{I} \cap \mathcal{S})_e$; (v) the fact that $(\mathcal{I} \cap \mathcal{S})_e$ consists of product measures.

Besides these properties, we use a fairly explicit expression of the flux function $G(\rho)$ in Step four, and the incomplete ordering relations $\eta >\!< \xi$ and $\eta \bowtie \xi$ introduced in the case $q = 0$. The explicit expression of the flux is allowed by property (v), and its degree of precision is still improved when the number of lanes is 2. For the models in Appendix A.1, we have:

1. *For all models.* The ingredients listed in Step one hold so long as the global kernel $p(.,.)$ (see (14) for multilane models) is weakly irreducible. This is the case for the multilane model (226) when $q > 0$; for the ladder process (227) even

if $q = 0$; for the model with finite-range horizontal kernels $Q_i(.)$ in (14), if $q > 0$ and each $Q_i(.)$ is assumed weakly irreducible; for the Misanthrope's process if the single-lane kernel $p(.,.)$ in (230) is weakly irreducible.

2. *For non-nearest neighbour horizontal kernels.* Expression (117) of the flux function is still valid. When $q = 0$, in Step six above, the use of [13, Theorem 1.4] (which is restricted to nearest-neighbour kernels) can be replaced by the more general [5].

3. *For non-nearest neighbour horizontal kernels and multilane models.* Suitable extensions of the incomplete ordering relations $\eta >< \xi$ and $\eta \bowtie \xi$ can be introduced when $q = 0$.

4. *For multilane models.* When $q = 0$, an explicit expression of the form (119) for the flux when $q = 0$ still holds. When $q > 0$, we do not know how to obtain as detailed information on the flux function $G(\rho)$ as in Proposition 4.8, because its expression is less explicit. However, one can still show that G has finitely many extrema, which implies a weaker form of statement *(vi)* in Proposition 4.8: namely that the equation $G(\rho + k) - G(\rho)$ has finitely many solutions for any integer k. This allows us to infer in Step four above that the number of possible shock profiles is finite.

5. *For Misanthrope's process.* The proof of Theorem 2.3 *(i)* is similar to the two-lane ASEP proof. Indeed using the symmetry of the jump kernel $p(.,.)$ in (230) and the gradient condition (M2), one can write the microscopic current as a gradient as in (128).

We next come to possible shock measures when $\gamma \neq 0$ ($\gamma := \sum_{z\in\mathbb{Z}} zp(z)$ denotes the mean drift of the jump kernel). The flux function expressed as a function of fugacity is a ratio of two polynomials. Indeed, let

$$R(\lambda) := \mu_\lambda[\eta(0)] = \frac{\sum_{k=0}^{K} k\lambda^k\theta_1(k)}{\sum_{k=0}^{K} \lambda^k\theta_1(k)} \tag{241}$$

denote the mean density as a function of fugacity. Then,

$$G(R(\lambda)) = \gamma\mu_\lambda[b(\eta(0),\eta(1)] = \gamma\frac{\sum_{k=0}^{K}\sum_{l=0}^{K} b(k,l)\lambda^{k+l}\theta_1(k)\theta_1(l)}{\left(\sum_{k=0}^{K} \lambda^k\theta_1(k)\right)^2} \tag{242}$$

From this one can show that when $\gamma \neq 0$, $G(.)$ has finitely many extrema. This leads (as above for multilane models) to the conclusion that there are finitely many possible shock profiles. The more complete result under condition (240) can be obtained because for $K = 2$, the misanthrope's flux is as explicit as that of the two-lane ASEP.

More precisely, the following mapping proven below holds between fluxes of two-lane ASEP and two-particle misanthrope's process.

Lemma A.1. *Let $K = 2$. Without loss of generality, assume $\gamma = 1$, $b(1,0) = 1$, $b(2,0) = \alpha \geq 1$, $0 < b(1,1) = \beta \leq 1$, $b(2,1) = \alpha - 1$ (cf. (232)). Let*

$$G^M_{\alpha,\beta}(\rho) := \nu_{\rho,\alpha,\beta}[b(\eta(0),\eta(1))]$$

denote the macroscopic flux function of the corresponding Misanthrope's process, where $\nu_{\rho,\alpha,\beta}$ is the product invariant measure of this process with mean density ρ, see (235), where we added notational dependence on α, β. Denote by $G_{\gamma_0,\gamma_1,r}$ the flux function of the two-lane ASEP, cf. (121)–(122). Then we have

$$G_{\gamma_0,\gamma_1,r} = G^M_{\alpha,\beta} \tag{243}$$

if the following relations hold:

$$\frac{r}{(1+r)^2} = \frac{\beta}{\alpha} \tag{244}$$

and

$$\gamma_0 r + \gamma_1 = 1 + r, \quad \gamma_0 + \gamma_1 = \alpha \tag{245}$$

In particular, for given $0 < \beta \leq 1 \leq \alpha$ such that $\beta \leq \alpha/4$, the system (244)–(245) has a unique solution (r, γ_0, γ_1) such that $r \in (0,1]$ and $\gamma_0, \gamma_1 \geq 0$.

We note that for $\gamma_0 = \gamma_1$ and $r \to 0$, we obtain $\beta \to 0$ and $\alpha \to 2$. Thus the image of the set $\mathcal{Z}$ in *(iii)* of Theorem 2.3 is a neighbourhood of $(\alpha = 2, \beta = 0)$ excluding $\beta = 0$.

Proof of Lemma A.1. For $n \in \{0, \dots, K\}$, let

$$q(n) := \frac{b(n,0)}{b(1,n-1)}, \quad q(n)! := \prod_{i=1}^{n} q(i)$$

where by convention the empty product equals 1. The one-site marginal of ν^1 is then given ([8]) by

$$\theta^1(n) = \frac{1}{q(n)!}$$

Under the assumptions of the lemma, we have

$$q(0) = 0, \quad q(1) = 1, \quad q(2) = \frac{\alpha}{\beta}$$

Plugging this into (242), we obtain the density and flux of the misanthrope's process as functions of fugacity:

$$R^M(\lambda) = \frac{\lambda + 2\frac{\beta}{\alpha}\lambda^2}{1 + \lambda + \frac{\beta}{\alpha}\lambda^2} \tag{246}$$

$$\widetilde{G}^M(\lambda) = \frac{\lambda + 2\beta\lambda^2 + (\alpha - 1)\frac{\beta}{\alpha}\lambda^3}{\left(1 + \lambda + \frac{\beta}{\alpha}\lambda^2\right)^2} \tag{247}$$

We want to match the above expressions with the density and flux of two-lane ASEP as functions of fugacity λ. These respectively correspond to $\rho_0+\rho_1$ in (208) and $\widetilde{G}(\lambda)$ in (209). They can be written as follows, first in λ, and then in $\Lambda := (1+r)\lambda$:

$$\begin{aligned} R(\lambda) &= \frac{(1+r)\lambda+2r\lambda^2}{1+(1+r)\lambda+r\lambda^2} = \frac{\Lambda+\frac{2r}{(1+r)^2}\Lambda^2}{1+\Lambda+\frac{r}{(1+r)^2}\Lambda^2} =: S(\Lambda) \qquad (248)\\ \widetilde{G}(\lambda) &= \frac{(\gamma_0 r+\gamma_1)\lambda+2(\gamma_0+\gamma_1)r\lambda^2+r(\gamma_0+\gamma_1 r)\lambda^3}{\left[1+(1+r)\lambda+r\lambda^2\right]^2} \\ &= \frac{\frac{(\gamma_0 r+\gamma_1)}{1+r}\Lambda+2(\gamma_0+\gamma_1)\frac{r}{(1+r)^2}\Lambda^2+\frac{r}{(1+r)^3}(\gamma_0+\gamma_1 r)\Lambda^3}{\left[1+\Lambda+\frac{r}{(1+r)^2}\Lambda^2\right]^2} \\ &=: \widetilde{H}(\Lambda) \qquad (249)\end{aligned}$$

We see that $R^M = S$ and $\widetilde{G}^M = \widetilde{H}$ if (244) and (245) hold as well as

$$\gamma_0+\gamma_1 r = (1+r)(\alpha-1)$$

But the latter is actually a consequence of (245). Finally, since $G = \widetilde{H}\circ S^{-1}$ and $G^M = \widetilde{G}^M \circ (R^M)^{-1}$, we obtain $G^M = G$. □

B Proof of Proposition 4.7

First it is a standard fact for Markov processes that $M' \in \mathcal{I}$. We must prove that it is supported on the set $\mathcal{X}_{2,1}$ in (112). We consider the coupled process $(\eta_t,\xi_t)_{t\geq 0}$. Its distribution at time t is denoted by $\overline{\nu}_t$, and we set

$$\overline{M}_t := \frac{1}{t}\int_0^t \overline{\nu}_s\, ds$$

The family $(\overline{M}_t)_{t\geq 0}$ is tight because it is supported on the compact set $\mathcal{X}\times\mathcal{X}$. Thus there exists a subsequential weak limit $\overline{M}$ of the family such that M' is the ξ-marginal of $\overline{M}$. By attractiveness and particle conservation, ξ_t is obtained from η_t by adding a second-class particle at some site. This implies for any $t>0$,

$$H_2(\xi_t) = H_2(\eta_t)+1 = 1 \qquad (250)$$

because $H_2(.)$ is a conserved quantity, thus M'_t is supported on $\mathcal{X}_{2,1}$. However since H_2 is not continuous on $\mathcal{X}_2$, it is not *a priori* true that M' is supported on $\mathcal{X}_{2,1}$. However we now prove that it is indeed the case. We couple via a common Harris system (see Subsection 3.2) the process $(\xi_t)_{t\geq 0}$ starting from ξ and the processes $\left(\eta_t^{(n)}\right)_{t\geq 0}$ starting from $\eta^{(n)} := \tau_n\eta$. We denote $\vec{\eta} := \left(\eta^{(n)}\right)_{n\in\mathbb{N}}$, $\vec{\eta}_t := \left(\eta_t^{(n)}\right)_{n\in\mathbb{N}}$ (so $\eta^{(0)}=\eta$ and $\eta_t^{(0)}=\eta_t$). Let $\vec{\mathbb{P}}$ denote the law of the process

$(\xi_t, \vec{\eta}_t)_{t\geq 0}$ starting from the random coupled configuration $(\xi, \vec{\eta})$. For $\eta \in \mathcal{X}_2$, we set

$$X_1(\eta) := \inf\{x \in \mathbb{Z} : \eta(y,0) = \eta(y,1) = 1 \text{ for all } y \geq x\} \in \mathbb{Z}$$

Note that $X_1(\eta) \geq X_0(\eta)$, cf. (111), and

$$\vec{\mathbb{P}}\left\{\xi \leq \eta^{(n)}\right\} \geq \vec{\mathbb{P}}\left\{X_1(\eta) - n \leq X_0(\eta) - 1\right\} \to 1 \text{ as } n \to +\infty \tag{251}$$

The inequality above holds because if we shift η far enough to the left so that its fully occupied region $[X_1(\eta), +\infty) \cap \mathbb{Z}$ comes to the right of the second-class ξ-particle at $X_0(\eta) - 1$, the shifted configuration becomes greater or equal than ξ. Then

$$\vec{\mathbb{P}}\left\{X_0(\xi_t) \geq X_0\left(\eta_t^{(n)}\right)\right\} \geq \vec{\mathbb{P}}\left\{\xi_t \leq \eta_t^{(n)}\right\} \geq \vec{\mathbb{P}}\left\{\xi \leq \eta^{(n)}\right\} \tag{252}$$

where the first inequality follows from the fact that the position of the leftmost particle is a nonincreasing function, and the second one from attractiveness. Since $\left(\eta_t^{(n)}\right)_{t\geq 0}$ is stationary, the family $\left\{X_0\left(\eta_t^{(n)}\right) : t \geq 0\right\}$ is tight. We next write

$$\vec{\mathbb{P}}\left\{X_0(\xi_t) < -A\right\} \leq \vec{\mathbb{P}}\left\{X_0(\xi_t) < X_0\left(\eta_t^{(n)}\right)\right\} + \vec{\mathbb{P}}\left\{X_0\left(\eta_t^{(n)}\right) < -A\right\}, \tag{253}$$

We use (251), (252), (253) and the above mentioned tightness, let $A \to +\infty$ and then $n \to +\infty$, to obtain

$$\liminf_{A\to+\infty} \liminf_{t\to+\infty} \vec{\mathbb{P}}\left\{X_0(\xi_t) \geq -A\right\} = 1 \tag{254}$$

On the other hand, since $\xi_t \geq \eta_t$ by attractiveness, we also have

$$\vec{\mathbb{P}}\left\{X_1(\xi_t) \leq X_1(\eta_t)\right\} = 1 \tag{255}$$

Again using stationarity and thus tightness of the process $(X_1(\eta_t))_{t\geq 0}$, we obtain

$$\liminf_{A\to+\infty} \liminf_{t\to+\infty} \vec{\mathbb{P}}\left\{X_1(\xi_t) \leq A\right\} = 1 \tag{256}$$

Since for $\eta \in \mathcal{X}$,

$$N(\eta) := \sum_{x\in V:\, x(0)\leq 0} \eta(x) + \sum_{x\in V:\, x(0)>0} [1-\eta(x)] \leq 2[\max(0, X_1(\eta)) + \max(0, -X_0(\eta))],$$

by (254) and (256), $(N(\xi_t))_{t\geq 0}$ is a tight family. This implies M' is supported on $\mathcal{X}_2$. For $A \in \mathbb{N}$, let

$$H_2^A(\eta) := H_2(\eta) := \sum_{x\in V:\, x(0)\in[-A,0]} \eta(x) - \sum_{x\in V:\, x(0)\in[1,A]} [1-\eta(x)]$$

Note that

$$H_2 = H_2^A \quad \text{on } \{\eta \in \mathcal{X}_2 : -A \le X_1(\eta) \le X_2(\eta) \le A\} \tag{257}$$

It follows from (254) and (256) that

$$\lim_{A\to+\infty} \liminf_{t\to+\infty} \vec{\mathbb{P}} \left\{ H_2^A(\xi_t) = H_2^A(\eta_t) + 1 \right\} = \lim_{A\to+\infty} \liminf_{t\to+\infty} \overline{\nu}_t \left\{ H_2^A(\xi) = H_2^A(\eta) + 1 \right\} = 1$$

By Cesaro limit along a subsequence of $\overline{M}_t$ converging to $\overline{M}$,

$$\lim_{A\to+\infty} \liminf_{t\to+\infty} \overline{M}_t \left\{ H_2^A(\xi) = H_2^A(\eta) + 1 \right\} = 1 = \lim_{A\to+\infty} \overline{M} \left\{ H_2^A(\xi) = H_2^A(\eta) + 1 \right\}$$

Further, since $\lim_{A\to+\infty} H_2^A(\eta) = H_2(\eta)$ for every $\eta \in \mathcal{X}_2$, and H_2 and H_2^A take integer values, on $\mathcal{X}_2 \times \mathcal{X}_2$, we have

$$\limsup_{A\to+\infty} \left\{ H_2^A(\xi) = H_2^A(\eta) + 1 \right\} = \{ H_2(\xi) = H_2(\eta) + 1 \}$$

It follows by Fatou's lemma that

$$\overline{M} \{ H_2(\xi) = H_2(\eta) + 1 \} = 1$$

so M' is indeed supported on $\mathcal{X}_{2,1}$.

C Proof of Proposition 3.2

Let us rewrite the coupled generator (85) as

$$\overline{L} f(\eta, \xi) = \sum_{(\eta', \xi') \in \mathcal{X} \times \mathcal{X}} a[(\eta, \xi); (\eta', \xi')] \, [f(\eta', \xi') - f(\eta, \xi)] \tag{258}$$

where the rates $a[(\eta, \xi); (\eta', \xi')]$ are defined as follows. First, for any $(x, y) \in V$ such that $x \neq y$, $a[(\eta, \xi); (\eta', \xi')]$ is given by

$$\begin{cases} p(x,y)[\eta(x)(1-\eta(y))] \vee [\xi(x)(1-\xi(y))] & \text{if} \quad (\eta', \xi') = (\eta^{x,y}, \xi^{x,y}) \\ p(x,y)[\eta(x)(1-\eta(y)) - \xi(x)(1-\xi(y))]^+ & \text{if} \quad (\eta', \xi') = (\eta^{x,y}, \xi) \\ p(x,y)[\eta(x)(1-\eta(y)) - \xi(x)(1-\xi(y))]^- & \text{if} \quad (\eta', \xi') = (\eta, \xi^{x,y}) \end{cases} \tag{259}$$

with the kernel $p(.,.)$ given by (23). Next, $a[(\eta, \xi); (\eta', \xi')] = 0$ if there exists no $(x, y) \in V^2$ such that $x \neq y$ and $(\eta', \xi') \in \{(\eta^{x,y}, \xi^{x,y}), (\eta^{x,y}, \xi), (\eta, \xi^{x,y})\}$.

If $a[(\eta, \xi); (\eta', \xi')] \neq 0$, we say there is a transition from (η, ξ) to (η', ξ'). Recalling the notation $x \overset{k}{\to} y$ introduced before Definition 3.1, we shall prove the following.

Lemma C.1. *Let $\widetilde{\nu} \in \widetilde{\mathcal{I}} \cap \widetilde{\mathcal{S}}$. Then (96) holds for every $(x, y) \in V \times V$ such that $x \neq y$, and $x \overset{k}{\to} y$ or $y \overset{k}{\to} x$ for some k.*

Proof of Lemma C.1. We prove by induction on k that (96) holds for every $(x,y) \in V \times V$ such that $x \neq y$ and $x \overset{k}{\to} y$. Applying the statement to (ξ,η) then shows that it holds for (η,ξ) and $y \overset{k}{\to} x$.

We now use the computation done between (212) and (217). The sums in (214)–(215) are boundary contributions, that we denote respectively by $\Gamma^i_N(\eta,\xi)$ and $\Gamma^o_N(\eta,\xi)$. Since $\widetilde{\nu} \in \widetilde{\mathcal{I}}$, we have

$$\int_{\mathcal{X}\times\mathcal{X}} \widetilde{L} F_N(\eta,\xi) d\widetilde{\nu}(\eta,\xi) = 0 \tag{260}$$

We have to exploit (260); for this we distinguish between the two assumptions:

First case. We assume that $\widetilde{\nu} \in \widetilde{\mathcal{S}}$. Since $J_{((u+z,i),(v+z,j))} = \tau_z J_{(u,i),(v,j)}$ for all $u,v,z \in \mathbb{Z}$, we have

$$\int_{\mathcal{X}\times\mathcal{X}} \Gamma^i_N(\eta,\xi) d\widetilde{\nu}(\eta,\xi) - \int_{\mathcal{X}\times\mathcal{X}} \Gamma^o_N(\eta,\xi) d\widetilde{\nu}(\eta,\xi) = 0 \tag{261}$$

Second case. We assume (95). The latter with the inequalities

$$\begin{aligned} |\Gamma^i_N(\eta,\xi)| &\leq \sum_{i\in W} l_i(|\eta(-N-1,i) - \xi(-N-1,i)| + |\eta(-N,i) - \xi(-N,i)|) \\ |\Gamma^o_N(\eta,\xi)| &\leq \sum_{i\in W} d_i(|\eta(N,i) - \xi(N,i)| + |\eta(N+1,i) - \xi(N+1,i)|) \end{aligned}$$

leads to

$$\lim_{N\to+\infty} \left\{ \int_{\mathcal{X}\times\mathcal{X}} \Gamma^i_N(\eta,\xi) d\widetilde{\nu}(\eta,\xi) - \int_{\mathcal{X}\times\mathcal{X}} \Gamma^o_N(\eta,\xi) d\widetilde{\nu}(\eta,\xi) \right\} = 0 \tag{262}$$

Using (261) for all N, we obtain that for every $(x,y) \in V^2$ such that $p(x,y) > 0$,

$$\int_{\mathcal{X}\times\mathcal{X}} D_{x,y}(\eta,\xi) d\widetilde{\nu}(\eta,\xi) = 0$$

This implies (96) for $k = 1$.

Now assume (96) holds for $k-1$. If A is a subset of $\mathcal{X}\times\mathcal{X}$ and $(\eta,\xi) \in \mathcal{X}\times\mathcal{X}$, we write $(\eta,\xi) \overset{n}{\to} A$ if there exists a sequence of coupled configurations, $(\eta_0,\xi_0) = (\eta,\xi), \ldots, (\eta_n,\xi_n) = (\eta',\xi')$, such that $a[(\eta_i,\xi_i);(\eta_{i+1},\xi_{i+1})] > 0$ for every $i = 0,\ldots,n-1$, and $(\eta',\xi') \in A$. Assume $A = A_0$ is a local set (that is, such that its indicator function is a local function) and

$$\begin{aligned} A_n &:= \{(\eta,\xi) \in \mathcal{X}\times\mathcal{X} : (\eta,\xi) \overset{n}{\to} A_0\} \\ A'_n &:= \{(\eta,\xi) \in \mathcal{X}\times\mathcal{X} : (\eta,\xi) \overset{i}{\to} A_0 \text{ for some } i \leq n\} \end{aligned}$$

Then (258)–(259) implies that there exist positive constants a_n, b_n such that

$$\widetilde{L}\mathbf{1}_{A_n} \geq a_n \mathbf{1}_{A_{n+1}} - b_n \mathbf{1}_{A_n} \tag{263}$$

Iterating (263) shows that if $\widetilde{\nu} \in \widetilde{\mathcal{I}}$ and $\widetilde{\nu}(A) = 0$, then $\widetilde{\nu}(A_n) = 0$, hence $\widetilde{\nu}(A'_n) = 0$. For the induction step, we use this as follows. Let E^n denote the set of coupled configurations $(\eta, \xi) \in \mathcal{X} \times \mathcal{X}$ such that there is no pair of opposite discrepancies at sites $x, y \in V$ if $x \xrightarrow{i} y$ or $y \xrightarrow{i} x$ for any $i \leq n$. We choose $A_0 = E^{k-1}$ so that $\widetilde{\nu}(A_0) = 0$ by the induction assumption. Then we claim that E^k is contained in A'_{k-1}. Indeed, assume $x \xrightarrow{k} y$ and $(\eta, \xi) \in E_{x,y}$. Let $(x = x_0, \dots, x_k = y)$ denote a p-path from x to y. By the induction assumption, $\widetilde{\nu}$-almost surely, we have $\eta(x_i) = \xi(x_i)$ for all $i = 1, \dots, k-1$. If $\eta(x_1) = \xi(x_1) = 0$, then $(\eta^{x_0,x_1}, \xi^{x_0,x_1}) \in E_{x_1,y}$. Otherwise let i^* be the maximum index i such that $\eta(x_i) = \xi(x_i) = 1$. Then one can find a sequence of at most $k-1$ transitions leading from (η, ξ) to some $(\eta', \xi') \in E_{x,x_{k-1}}$ as follows: *(i)* if $i^* < k-1$, the coupled particle at x_{i^*} jumps from x_{i^*} to x_{k-1} along the path; *(ii)* the coupled particle at x_{k-1} exchanges with the ξ-discrepancy at y. □

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