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# Invariant measures for multilane exclusion process

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## Abstract

We consider the simple exclusion process on  $\mathbb{Z} \times \{0, 1\}$ , that is, an “horizontal ladder” composed of 2 lanes. Particles can jump according to a lane-dependent translation-invariant nearest neighbour jump kernel, i.e. “horizontally” along each lane, and “vertically” along the scales of the ladder. We prove that generically, the set of extremal invariant measures consists of (i) translation-invariant product Bernoulli measures; and, modulo translations along  $\mathbb{Z}$ : (ii) at most two shock measures (i.e. asymptotic to Bernoulli measures at  $\pm\infty$ ) with asymptotic densities 0 and 2; (iii) at most three shock measures with a density jump of magnitude 1. We fully determine this set for certain parameter values. In fact, outside degenerate cases, there is at most one shock measure of type (iii). The result can be partially generalized to vertically cyclic ladders with arbitrarily many lanes. For the latter, we answer an open question of [5] about rotational invariance of stationary measures.

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# 1 Introduction

The simple exclusion process, introduced in [11], is a fundamental model in statistical mechanics. In this markovian process, particles hop on a countable lattice following a certain random walk kernel subject to the exclusion rule, that allows at most one particle per site. As usual for Markov processes, the characterization of its invariant measures is one of the basic questions to address. Still today, outside the case of a symmetric kernel ([10]), the problem is far from being completely solved. In fact, it has been mostly studied for translation invariant kernels. We briefly recall known results in this situation.

For the exclusion process on  $\mathbb{Z}^d$ , the set of extremal *translation invariant* stationary probability measures consists ([9]) of homogeneous Bernoulli product measures. However, for a non-symmetric kernel, there may exist extremal invariant probability measures that are *not* translation invariant. These are fairly well (though not completely) understood in one-space dimension ([8, 6, 4]): under suitable assumptions, there is a unique (up to translations) such extremal probability measure, called either a *blocking* or a *profile* measure (the latter being a weakened version of the former); its main feature is that the asymptotic particle density is 0 to the left and 1 to the right. In several space dimensions, although blocking or profile measures can be exhibited ([5]), the complete characterization of invariant probability measures remains an open question.

In this paper, we obtain more complete results for an intermediate model. We consider the simple exclusion process on  $\mathbb{Z} \times \{0, 1\}$ , that is an “horizontal ladder” composed of 2 lanes. Particles can jump “horizontally” along each lane according to a lane-dependent translation-invariant jump kernel, and “vertically” along the scales of the ladder according to another kernel. In the totally asymmetric case, this can be interpreted as traffic-flow on a highway, with two lanes on which cars have different speeds and different directions.

We next describe our results. Let  $\gamma_0, \gamma_1$  denote mean drifts on each lane,  $p$  the jump rate from lane 0 to lane 1 and  $q$  the jump rate from lane 1 to lane 0. The drifts may be of equal or opposite signs; one or both of them may also vanish. We assume that  $p + q > 0$ , so that both lanes are indeed connected. We prove that the set  $\mathcal{I}_e$  of extremal invariant probability measures can be decomposed as a disjoint union

$$\mathcal{I}_e = \mathcal{I}_0 \cup \mathcal{I}_1 \cup \mathcal{I}_2$$

In this decomposition,  $\mathcal{I}_0 := \{\nu_\rho, \rho \in [0, 2]\}$  is the set of extremal invariant probability measures that are *translation invariant* along lanes. The parameter  $\rho$  represents the total density over the two lanes. Under  $\nu_\rho$ , the mean densities  $\rho_0, \rho_1$  on each lane are functions of  $\rho$ , and they are different when  $p \neq q$ . In the sequel, we refer to these probability measures as “Bernoulli measures”. For  $k \in \{1, 2\}$ ,  $\mathcal{I}_k$  denotes a (possibly empty) set of extremal invariant probability measures that are *shock measures* of *amplitude*  $k$ . By a shock measure, we mean

a probability measure that is asymptotic to two Bernoulli measures of different densities  $\rho^-$ , resp.  $\rho^+$ , when viewed from faraway left, resp. right. The amplitude of the shock is by definition  $k := |\rho^+ - \rho^-|$ . The set  $\mathcal{I}_2$  contains only shocks such that  $(\rho^-, \rho^+) = (0, 2)$  or  $(\rho^-, \rho^+) = (2, 0)$ . These measures are the analogue in our context of blocking measures or profile measures. In some cases,  $\mathcal{I}_1$  may contain *partial* blocking measures, i.e., measures whose restriction to one lane is a blocking measure, and whose restriction to the other lane is either full or empty.

We show that the following generic picture holds outside some degenerate cases: up to translations along  $\mathbb{Z}$ , (i) the set  $\mathcal{I}_1$  contains at most three probability measures; (ii) the set  $\mathcal{I}_2$  contains at most two probability measures. In particular, these sets are at most countable. In fact, outside degenerate cases,  $\mathcal{I}_1$  contains at most one measure. For a subset of parameter values, we can fully determine  $\mathcal{I}_1$  and  $\mathcal{I}_2$ , and thus obtain a complete characterization of invariant probability measures. This includes the following situations: when  $\gamma_0$  and  $\gamma_1$  are close enough and the ratio  $q/p$  small enough or large enough; when  $p$  or  $q$  vanishes and  $\gamma_0 \neq \gamma_1$ ; or when  $\gamma_0 = \gamma_1 = 0$  and  $p, q$  are arbitrary. The following questions are left open. First, we can only show that  $\mathcal{I}_1$  is indeed nonempty in cases where it contains *only* partial blocking measures, and that it is empty on a set of parameter values for which  $\gamma_0$  and  $\gamma_1$  are close enough, and the ratio between  $p$  and  $q$  small enough (or large enough). We do not know if for certain parameter values it is possible to have  $\mathcal{I}_1$  nonempty with a shock of amplitude 1 that is not a partial blocking measure. In the case  $p = q$  (and more generally for the vertically cyclic ladder process, see below), it is believed in [5] that this probably does not occur. Next, we only know that  $\mathcal{I}_2$  is nonempty if

$$d_0/l_0 = d_1/l_1 \neq 1$$

where  $d_i$ , resp.  $l_i$ , denotes the jump rate to the right, resp. left, on lane  $i \in \{0, 1\}$ . We also show that  $\mathcal{I}_2$  is empty when  $pq = 0$ , *even* if the drifts are both strictly positive (or both strictly negative), which is in sharp contrast with the one-dimensional case. However, we conjecture that when  $pq > 0$  and both drifts are strictly positive,  $\mathcal{I}_2$  is nonempty, also without the assumption (1). We believe that this could be proved in the spirit of [6] by means of the hydrodynamic limit. We shall investigate the hydrodynamic behaviour of our model and extensions thereof (see below) in [1].

Our model and approach extend to more general *multi-lane* exclusion processes with an arbitrary (finite) number of lanes. The two-lane model can be generalized in two different ways. When  $p = q$ , a natural extension is to view the vertical direction as a torus, and define a multi-lane model with a vertical (inter-lane) jump kernel that is invariant with respect to rotations of this torus. This corresponds to the so-called *ladder process* mentioned in [5] as a possible direction towards  $\mathbb{Z}^2$ . A byproduct of our results is a positive answer to one of the open questions formulated in [5] for this model: namely, that all invariant measures are rotationally invariant. Another way to extend to the two-lane

model, including the case  $p \neq q$ , is to consider vertical kernels satisfying certain reversibility conditions; see e.g. [12], which studies a tagged particle in such models. Our methods and results could also be extended to this situation.

One of the difficulties of our model is that available approaches ([8, 4]) to classify invariant measures for the one-dimensional single-lane asymmetric simple exclusion process rely heavily on the fact that at most one particle is allowed on each site. A common idea in these proofs is to exhibit an ordered coupling of a supposed invariant measure  $\nu$  with its translate. When the inequality is strict, the difference of mean densities at  $\pm\infty$  for  $\nu$  can be related to the number of discrepancies between  $\nu$  and its translate. Since this must be at least 1, there is no choice but 0 and 1 as extreme densities. Because these are the lowest and highest possible densities, this automatically implies that the measure is asymptotic to Bernoulli measures with densities 0 and 1. Since we do not have this simplifying feature, an important ingredient of our analysis is an *a priori* proof that any invariant measure is asymptotic to Bernoulli measures at  $\pm\infty$ . Another difficulty that occurs when interlane jumps are possible only in one direction is the lack of irreducibility for the jump kernel. Usual arguments based on attractiveness and irreducibility, showing that discrepancies between two coupled processes eventually disappear, are not sufficient in this case.

This paper is organized as follows. In Section 2, we introduce our models, after a detailed reminder for general simple exclusion processes; we then state our results on invariant measures for the two-lane simple exclusion process: Theorem 2.1 for the invariant and translation invariant probability measures, Theorems 2.2 and 2.3 for the invariant probability measures; finally Theorem 2.4 deals with the multi-lane simple exclusion process, and in particular with the ladder process from [5]. Section 3 is devoted to the proof of Theorem 2.1, and Section 4 to the proofs of Theorems 2.2, 2.3 and 2.4. In order to make the general schemes of proofs more visible, most intermediate results used to establish Theorems 2.2 and 2.3 are proved in the separate Section 5. Some additional details of proofs are given in the Appendix.

## 2 Models and results

In this section, we present and state our results for our basic model, the *two-lane* SEP (motivated by traffic-flow considerations), and for its generalization to a *multi-lane* SEP. Before that, we first recall the definition of the simple exclusion process on a countable set  $V$ . The two-lane and the multi-lane SEP indeed belong to this class, but they have specific properties due to the structure of the set  $V$ .

## 2.1 Simple exclusion process

Throughout the paper,  $\mathbb{Z}$  denotes the set of integers and  $\mathbb{N}$  the set of nonnegative integers. Let  $V$  be a nonempty countable set. The state space of the process is

$$\mathcal{X} := \{0, 1\}^V \quad (1)$$

that is a compact polish space with respect to product topology. One can think of  $\eta \in \mathcal{X}$  as a configuration of particles on  $V$ , i.e. for which a site  $x \in V$  is occupied by a particle if and only if  $\eta(x) = 1$ .

We call *kernel* on  $V$  a function  $p : V \times V \rightarrow [0, +\infty)$  such that

$$\sup_{x \in V} \left\{ \sum_{y \in V} p(x, y) + \sum_{y \in V} p(y, x) \right\} < +\infty \quad (2)$$

The  $(V, p)$ -simple exclusion process (in short: SEP) is a Markov process  $(\eta_t)_{t \geq 0}$  on  $\mathcal{X}$  (see Chapter VIII in [9]) with generator

$$Lf(\eta) = \sum_{x, y \in V} p(x, y) \eta(x) (1 - \eta(y)) (f(\eta^{x,y}) - f(\eta)), \quad (3)$$

where  $\eta^{x,y}$ , given by

$$\eta^{x,y}(w) = \begin{cases} \eta(w) & w \neq x, y \\ \eta(x) - 1 & w = x \\ \eta(y) + 1 & w = y \end{cases},$$

is the new configuration after a particle has jumped from  $x$  to  $y$ , and  $f$  is a cylinder (or local) function, that is, a function that depends only on the value of  $\eta$  on a finite number of sites in  $V$ . We denote by  $(S_t)_{t \geq 0}$  the semigroup generated by (3), and by  $\mathbb{E}_\mu$ , resp.  $\mathbb{E}_\eta$ , the expectation for the process with initial distribution a probability measure  $\mu$  on  $\mathcal{X}$ , resp. with initial configuration  $\eta \in \mathcal{X}$ .

The (nearest-neighbour) SEP on  $\mathbb{Z}$  is the particular case of (3) with  $(V, p)$  given by

$$V = \mathbb{Z}, \quad p(x, y) = d\mathbf{1}_{\{y-x=1\}} + l\mathbf{1}_{\{y-x=-1\}}; \quad d, l \geq 0, d + l > 0 \quad (4)$$

Within this category we distinguish the *symmetric*, resp. *asymmetric* exclusion process (SSEP, resp. ASEP), for which  $d = l$ , resp.  $d \neq l$ ; and the *totally asymmetric* simple exclusion process (TASEP) on  $\mathbb{Z}$ , for which  $dl = 0 < d + l$ .

A probability measure  $\mu$  on  $\mathcal{X}$  is said to be *invariant* for the Markov process generated by (3) if it is invariant with respect to the semigroup  $(S_t)_{t \geq 0}$ , which is equivalent to

$$\int Lf(\eta) d\mu(\eta) = 0 \quad (5)$$

for every cylinder function  $f$ . The set of invariant probability measures is denoted by  $\mathcal{I}$ . Since  $\mathcal{I}$  is convex, by Choquet-Deny Theorem, in order to know  $\mathcal{I}$ , it is enough to determine the subset of its extremal elements, which we denote by  $\mathcal{I}_e$ .

## 2.2 The general setup

In the sequel, we shall focus on special choices of  $V$  and  $p(.,.)$  for which the model has an interesting structure. First, we consider a lattice  $V$  of the form

$$V = \mathbb{Z} \times W \quad (6)$$

for some nonempty finite set  $W$ . An element  $x$  of  $V$  will be generically written in the form  $x = (x(0), x(1))$ , with  $x(0) \in \mathbb{Z}$  and  $x(1) \in W$ . In traffic-flow modeling, we may think of  $V$  as a highway, of  $\mathbb{Z}$  as a lane, and of  $x$  as site  $x(0)$  on lane  $x(1)$ . For  $i \in W$ ,

$$\mathbb{L}_i := \{x \in V : x(0) \in \mathbb{Z}, x(1) = i\} \quad (7)$$

denotes the  $i$ 'th lane of  $V$ , and  $\eta^i$  the particle configuration on  $\mathbb{Z}$ , defined by

$$\eta^i(z) = \eta(z, i) \quad (8)$$

for  $z \in \mathbb{Z}$ . We can view  $\eta^i$  as the configuration on lane  $i$ . Another interpretation is that  $i \in W$  represents a particle species, then  $\eta(z, i) = \eta^i(z)$  is the number of particles of species  $i$  at site  $z \in \mathbb{Z}$ . We also denote by

$$\bar{\eta}(z) = \sum_{i \in W} \eta^i(z) \quad (9)$$

the total number of particles at  $z \in \mathbb{Z}$ .

Next, we consider kernels  $p(.,.)$  of the form

$$p(x, y) = \begin{cases} 0 & \text{if } x(0) \neq y(0) \text{ and } x(1) \neq y(1) \\ q_i(x(0), y(0)) =: Q_i[y(0) - x(0)] & \text{if } x(1) = y(1) = i \\ q(x(1), y(1)) & \text{if } x(0) = y(0) \end{cases} \quad (10)$$

for  $x, y \in V$ , where  $q(.,.)$  is a kernel on  $W$ , and for each  $i \in W$ ,  $q_i(.,.)$  is a translation invariant kernel on  $\mathbb{Z}$  given by

$$q_i(u, v) = d_i \mathbf{1}_{\{v-u=1\}} + l_i \mathbf{1}_{\{v-u=-1\}}, \quad Q_i(z) = d_i \mathbf{1}_{\{z=1\}} + l_i \mathbf{1}_{\{z=-1\}} \quad (11)$$

for  $u, v \in \mathbb{Z}$ , where  $d_i \geq 0$  and  $l_i \geq 0$  are such that  $d_i + l_i > 0$ .

We shall be interested in translations along  $\mathbb{Z}$ , but the set  $W$  is in general not endowed with a translation operator. We denote by  $(\tau_k)_{k \in \mathbb{Z}}$  the group of space shifts on  $\mathbb{Z}$ . The shift operator  $\tau_k$  acts on a particle configuration  $\eta \in \mathcal{X}$  through

$$(\tau_k \eta)(z, w) := \eta(z + k, w), \quad \forall (z, w) \in \mathbb{Z} \times W \quad (12)$$

It acts on a function  $f : \mathcal{X} \rightarrow \mathbb{R}$  via

$$(\tau_k f)(\eta) := f(\tau_k \eta), \quad \forall \eta \in \mathcal{X} \quad (13)$$

If  $\mu$  is a probability measure on  $\mathcal{X}$ , then  $\tau_k$  acts on  $\mu$  via

$$\int_{\mathcal{X}} f(\eta) d(\tau_k \mu)(\eta) := \int_{\mathcal{X}} (\tau_k f)(\eta) d\mu(\eta) \quad (14)$$

for every bounded continuous function  $f : \mathcal{X} \rightarrow \mathbb{R}$ . Last, if  $\mathcal{L}$  is a linear operator acting on functions  $f : \mathcal{X} \rightarrow \mathbb{R}$ , then  $\tau_k$  acts on  $\mathcal{L}$  via

$$(\tau_k \mathcal{L})f := \mathcal{L}(\tau_k f) \quad (15)$$

By an abuse of notation, in what follows, we write  $\tau$  instead of  $\tau_1$ . We define  $\mathcal{S}$  to be the set of all probability measures on  $\mathcal{X}$  that are invariant under the translations  $\tau_k$ ,  $k \in \mathbb{Z}$ .

### 2.3 The two-lane SEP

In the sequel, we shall sometimes refer to SEP (resp. SSEP, ASEP, TASEP) as *single-lane* or *one-dimensional* SEP (resp. SSEP, ASEP, TASEP). Our basic model is *the two-lane* SEP, which corresponds to

$$W = \{0, 1\} \quad (16)$$

We can view this model as a dynamics on an infinite horizontal ladder, with vertical steps separating its two bars  $\mathbb{L}_0$  and  $\mathbb{L}_1$ , namely:

$$\begin{aligned} \mathbb{L}_0 &= \{x \in V : x = (z, 0), z \in \mathbb{Z}\} \\ \mathbb{L}_1 &= \{x \in V : x = (z, 1), z \in \mathbb{Z}\} \end{aligned} \quad (17)$$

In the traffic interpretation, we call  $\mathbb{L}_0$  and  $\mathbb{L}_1$  the upper and lower lane, and the steps between them the direction a car can follow to change lane.

Let  $p, q \geq 0$  and  $d_0, l_0, d_1, l_1 \geq 0$ . The two-lane SEP is the dynamics on  $\mathcal{X}$  defined by the generator (3) with kernel (10)–(11), in which  $q(\cdot, \cdot)$  is given by

$$q(0, 1) = p, \quad q(1, 0) = q \quad (18)$$

This means that, for  $x, y \in V$ ,

$$p(x, y) = \begin{cases} d_0 & \text{if } x, y \in \mathbb{L}_0, y(0) - x(0) = 1 \\ l_0 & \text{if } x, y \in \mathbb{L}_0, y(0) - x(0) = -1 \\ d_1 & \text{if } x, y \in \mathbb{L}_1, y(0) - x(0) = 1 \\ l_1 & \text{if } x, y \in \mathbb{L}_1, y(0) - x(0) = -1 \\ p & \text{if } x \in \mathbb{L}_0, y \in \mathbb{L}_1, x(0) = y(0) \\ q & \text{if } x \in \mathbb{L}_1, y \in \mathbb{L}_0, x(0) = y(0) \\ 0 & \text{otherwise} \end{cases} \quad (19)$$



In other words, particles move one step to the right or to the left on each lane at a rate depending on the lane, and we allow the rate  $p$  at which particles go down to be different than the rate  $q$  of going up. We shall assume in the sequel that (cf. (11))

$$(d_0 + l_0)(d_1 + l_1) > 0 \quad (20)$$

so that particles can always move on both lanes. However they cannot go from  $\mathbb{L}_0$  to  $\mathbb{L}_1$  if  $p = 0$ , nor from  $\mathbb{L}_1$  to  $\mathbb{L}_0$  if  $q = 0$ . If  $p = q = 0$ , the dynamic reduces to two independent SEP's on each lane. Thus,  $p + q \neq 0$  introduces interaction between the two lanes. For  $i \in W$ , we let

$$\gamma_i := d_i - l_i \quad (21)$$

denote the mean drift on lane  $i$ . The following symmetry properties of the two-lane SEP will be useful. Define the *lane symmetry* operator  $\sigma : \mathcal{X} \rightarrow \mathcal{X}$ , the *lane exchange* operator  $\sigma' : \mathcal{X} \rightarrow \mathcal{X}$ , and the *particle-hole* symmetry operator  $\sigma'' : \mathcal{X} \rightarrow \mathcal{X}$  by

$$(\sigma\eta)(z, i) = \eta(-z, i); (\sigma'\eta)(z, i) = \eta(z, 1 - i); (\sigma''\eta)(z, i) = 1 - \eta(z, i) \quad (22)$$

for  $\eta \in \mathcal{X}$  and  $(z, i) \in V$ . Let us call the process defined by the generator (3) with transition kernel (19) the  $(d_0, l_0); (d_1, l_1); (p, q)$ -two-lane SEP. The definition of the two-lane SEP dynamics implies the following.

**Lemma 2.1.** *Let  $(\eta_t)_{t \geq 0}$  be a  $(d_0, l_0); (d_1, l_1); (p, q)$ -two-lane SEP. Then the image of this process by  $\sigma$ , resp.  $\sigma', \sigma''$ , is a  $(l_0, d_0); (l_1, d_1); (p, q)$ , resp.  $(d_1, l_1); (d_0, l_0); (q, p)$ , resp.  $(l_0, d_0); (l_1, d_1); (q, p)$ -two-lane SEP.*

Thus, without loss of generality, we shall assume in the sequel that

$$\gamma_0 \geq 0, \quad \gamma_0 + \gamma_1 \geq 0, \quad p \geq q, \quad p > 0 \quad (23)$$

If we view  $i \in \{0, 1\}$  as a species rather than a lane, the interpretation is as follows: the dynamics within each species is a SEP of  $\mathbb{Z}$ , and a lane change becomes a spin flip whereby a particle may change its species. The exclusion rule within species implies that a particle cannot change its species if there is already a particle of the other species sitting at the same site. This is the only point where an interaction occurs between the two species.

## 2.4 Invariant measures for two-lane SEP

Let us first recall, in sections 2.4.1 and 2.4.2, the known picture ([10, 8, 9]) for invariant measures of the single-lane SEP defined by (3) and (4), since it will be the building block of the derivation of invariant measures for the two-lane SEP (and for the multi-lane SEP later on). We start with a classical result (Theorem 2.1 in Chapter VIII of [9]) for reversible measures of the *general* SEP defined by (3) for an arbitrary pair  $(V, p)$ .

### 2.4.1 Reversible measures for general SEP

Let  $S$  be a countable set and  $\pi(\cdot, \cdot)$  a kernel on  $S$ . Assume that there exists a  $[0, 1]$ -valued family  $\rho_\cdot = (\rho_i)_{i \in S}$  such that, for every  $i, j \in S$ , the following condition holds:

$$\rho_i(1 - \rho_j)\pi(i, j) = \rho_j(1 - \rho_i)\pi(j, i) \quad (24)$$

Then the product measure  $\mu^\rho$  defined by

$$\mu^\rho \{ \eta(i) = 1 \} = \rho_i \quad (25)$$

for every  $i \in S$ , is reversible with respect to the  $(S, \pi)$  simple exclusion process. If for every  $i, j \in S$ ,  $\rho_i$  and  $\rho_j$  are different from 1, (24) is equivalent to the *detailed balance* condition

$$\lambda_i \pi(i, j) = \lambda_j \pi(j, i) \quad \text{where} \quad \lambda_i := \frac{\rho_i}{1 - \rho_i} \quad (26)$$

### 2.4.2 Invariant measures for single-lane SEP

The single-lane SEP with kernel (4) has two kinds of extremal invariant measures. We refer to [9, Chapter VIII] for details.

*Translation invariant measures.* First, we have the homogeneous product Bernoulli probability measures, denoted by  $\{\mu_\rho, \rho \in [0, 1]\}$ , where  $\rho$  represents the average particle density per site; there are two cases under which particles do not move: when either  $\rho = 0$  (there are no particles), or  $\rho = 1$  (each site being occupied, particles cannot move because of the exclusion rule).

*Blocking measures for ASEP.* Assume in addition that  $d \neq l$ . Then one has an additional family of (non translation invariant) probability measures obtained from the reversible measures in Section 2.4.1. When  $l > 0$ , we set

$$\rho_i^c := \frac{c \left(\frac{d}{l}\right)^i}{1 + c \left(\frac{d}{l}\right)^i} \quad (27)$$

where  $c > 0$ . When  $l = 0 < d$ , that is for TASEP, we set

$$\rho_i^{n,c} := \mathbf{1}_{\{i > n\}} + \frac{c}{1 + c} \mathbf{1}_{\{i = n\}}, \quad i \in \mathbb{Z} \quad (28)$$

where  $n \in \mathbb{Z}$  and  $c \geq 0$ . Then  $\rho_\cdot$  is a nonconstant  $[0, 1]$ -valued solution of (24) if and only if it is of the form (27) in the case  $l > 0$ , or (28) in the case  $l = 0 < d$  (the case  $d = 0 < l$  is deduced by symmetry).

For such solutions  $\rho_\cdot$ , the measure  $\mu^\rho$  defined by (25) is reversible. If  $l > 0$ ,  $\mu^\rho$  with  $\rho_\cdot = \rho_\cdot^c$  given by (27) is not an extremal invariant probability measure.

Extremal invariant probability measures are obtained as follows. The measure  $\mu^\rho$  is supported on the set

$$\left\{ \eta \in \{0, 1\}^{\mathbb{Z}} : \sum_{x>0} [1 - \eta(x)] + \sum_{x \leq 0} \eta(x) < +\infty \right\} \quad (29)$$

On this set, the function  $H$  defined by

$$H(\eta) := \sum_{x \leq 0} \eta(x) - \sum_{x > 0} [1 - \eta(x)] \quad (30)$$

is conserved by the SEP dynamics if it is initially finite, and the process restricted to a level set of  $H$  is irreducible. The function  $H$  satisfies

$$H(\tau_n \eta) = H(\eta) + n, \quad \forall n \in \mathbb{Z} \quad (31)$$

Then, for  $c > 0$  and  $n \in \mathbb{Z}$ , the probability measure

$$\hat{\mu}_n := \mu^{\rho^c} (\cdot | H(\eta) = n) \quad (32)$$

does not depend on the choice of  $c > 0$ , and is an extremal invariant probability measure. Because of (31), it satisfies

$$\hat{\mu}_n = \tau_n \hat{\mu}_0 \quad (33)$$

For  $l = 0$ , the measure  $\mu^\rho$  with  $\rho = \rho^{n,c}$  given by (28) is an extremal invariant measure if and only if  $c = 0$ ; we again denote this measure by  $\hat{\mu}_n$ . Note that  $\hat{\mu}_n$  is supported on the single configuration  $\eta_n^*$  that is empty to the left up to (including) site  $n$  and full to the right of site  $n$ :

$$\eta_n^*(x) := \mathbf{1}_{\{x > n\}}, \quad \hat{\mu}_n := \delta_{\eta_n^*} \quad (34)$$

For convenience, by extension, we shall let  $\eta_{-\infty}^*$  and  $\eta_{+\infty}^*$  respectively denote the configurations with all 1's and all 0's.

In both cases, the measures  $\hat{\mu}_n$  are called *blocking measures*.

In [10] it is shown that in the symmetric case  $l = d$ , the set of extremal invariant measures consists of Bernoulli measures  $\mu_\rho$  for  $\rho \in [0, 1]$ . In [8] it is shown that in the asymmetric case  $l \neq d$ , this set consists of Bernoulli measures  $\mu_\rho$  for  $\rho \in [0, 1]$ , plus blocking measures  $\hat{\mu}_n$  for  $n \in \mathbb{Z}$ .

For the two-lane SEP (and later generalizations thereof), we shall define similar families of measures. Let us start with translation invariant measures.

### 2.4.3 Translation invariant stationary measures for two-lane SEP

The following two-parameter Bernoulli product probability measures will be central. Let us define  $\nu^{\rho_0, \rho_1}$  for  $(\rho_0, \rho_1) \in [0, 1]^2$ , as the product probability

measure on  $\mathcal{X}$  such that

$$\nu^{\rho_0, \rho_1}(\eta(x) = 1) = \begin{cases} \rho_0 & x \in \mathbb{L}_0 \\ \rho_1 & x \in \mathbb{L}_1 \end{cases}. \quad (35)$$

In words, the two lanes are independent, and the projection of  $\nu^{\rho_0, \rho_1}$  on the upper lane  $\mathbb{L}_0$  (respectively, the lower lane  $\mathbb{L}_1$ ) is  $\mu_{\rho_0}$  (respectively  $\mu_{\rho_1}$ ).

When  $p = q = 0$ , as mentioned above (after (19)), the two lanes evolve as independent SEP's, hence  $\nu^{\rho_0, \rho_1}$  is an invariant measure for every  $(\rho_0, \rho_1) \in [0, 1]^2$ . We look for a relation between  $\rho_0$  and  $\rho_1$  under which we could have  $\nu^{\rho_0, \rho_1} \in \mathcal{I}$  when  $p + q \neq 0$ . To this end, we define the following subset  $\mathcal{F}$  of  $[0, 1]^2$ :

$$\mathcal{F} := \{(\rho_0, \rho_1) \in [0, 1]^2 : p\rho_0(1 - \rho_1) - q\rho_1(1 - \rho_0) = 0\} \quad (36)$$

The set  $\mathcal{F}$  expresses an equilibrium relation for vertical jumps: it states that under  $\nu^{\rho_0, \rho_1}$ , the mean algebraic “creation rate” on each lane (i.e. resulting from jumps from/to the other lane) has to be 0. Similarly to the single lane SEP, we have the following theorem, proved in Section 3.

**Theorem 2.1.** *We have that*

$$(\mathcal{I} \cap \mathcal{S})_e = \{\nu^{\rho_0, \rho_1} : (\rho_0, \rho_1) \in \mathcal{F}\} \quad (37)$$

$$= \{\nu_\rho : \rho \in [0, 2]\} \quad (38)$$

for a one-parameter family  $\{\nu_\rho : 0 \leq \rho \leq 2\}$  of probability measures on  $\mathcal{X}$ , where the parameter  $\rho$  represents the total mean density over the two lanes:

$$\mathbb{E}_{\nu_\rho} \{\eta^0(0) + \eta^1(0)\} = \rho \quad (39)$$

**Remark 2.1.** *When  $q = 0$ , the invariant measures  $\nu_\rho$  can be guessed naturally. Indeed in this case, particles cannot move upwards. Thus if lane 0 is empty, it remains empty and lane 1 behaves as an autonomous SEP. Hence, for  $\rho \in [0, 1]$ , the measure  $\nu^{0, \rho}$  (which has global density  $\rho$  over the two lanes) is invariant for the two-lane SEP, because its restriction to lane 1 is invariant for the SEP on this lane. Similarly, if lane 1 is full, it remains full and lane 0 evolves as an autonomous SEP. Hence, for  $\rho \in [1, 2]$ , the measure  $\nu^{\rho-1, 1}$  (which has global density  $\rho$  over the two lanes) is invariant for the two-lane SEP. This is consistent with the fact that for  $q = 0$ , (36) yields (see (70) later on)*

$$\mathcal{F} = \{(0, \rho) : \rho \in [0, 1]\} \cup \{(\rho - 1, 1) : \rho \in [1, 2]\}$$

#### 2.4.4 Structure of invariant measures for two-lane SEP

We define the following subset of  $\mathcal{X}$  (analogous to (29)):

$$\mathcal{X}_2 := \left\{ \eta \in \mathcal{X} : \sum_{x \in V : x(0) > 0} [1 - \eta(x)] + \sum_{x \in V : x(0) \leq 0} \eta(x) < +\infty \right\} \quad (40)$$

Let

$$\begin{aligned} \mathcal{B}_1 &:= \{(0, 1), (1, 0), (1, 2), (2, 1)\}, & \mathcal{B}_2 &:= \{(0, 2)\} \\ \mathcal{B} &:= \mathcal{B}_1 \cup \mathcal{B}_2, & \mathcal{D} &:= \{(\rho, \rho) : \rho \in [0, 2]\} \end{aligned} \quad (41)$$

Let  $(\rho^-, \rho^+) \in [0, 2]^2 \setminus \mathcal{D}$ , that we call a *shock*. A probability measure  $\mu$  on  $\mathcal{X}$  is called a  $(\rho^-, \rho^+)$ -*shock measure* if

$$\lim_{n \rightarrow -\infty} \tau_n \mu = \nu_{\rho^-}, \quad \lim_{n \rightarrow +\infty} \tau_n \mu = \nu_{\rho^+} \quad (42)$$

in the sense of weak convergence. The *amplitude* of the shock (or of the shock measure) is by definition  $|\rho^+ - \rho^-|$ . We define

$$Bl_1 = \{\nu \in \mathcal{I}_e : \exists (\rho^-, \rho^+) \in \mathcal{B}_1, \nu \text{ is a } (\rho^-, \rho^+)\text{-shock measure}\} \quad (43)$$

$$Bl_2 = \{\nu \in \mathcal{I}_e : \exists (\rho^-, \rho^+) \in \mathcal{B}_2, \nu \text{ is a } (\rho^-, \rho^+)\text{-shock measure}\} \quad (44)$$

$$Bl = Bl_1 \cup Bl_2 \quad (45)$$

The following theorem is proved in Section 4.

**Theorem 2.2.** (i) *There exist a (possibly empty) subset  $\mathcal{R}$  of  $[0, 2]^2 \setminus (\mathcal{D} \cup \mathcal{B})$  containing only shocks of amplitude 1, a (possibly empty) subset  $\mathcal{R}'$  of  $\mathcal{B}_1$ , and for each  $(\rho^-, \rho^+) \in \mathcal{R} \cup \mathcal{R}'$ , a  $(\rho^-, \rho^+)$ -shock measure denoted  $\nu_{\rho^-, \rho^+}$ , such that*

$$\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 2\} \cup Bl \cup \{\tau_z \nu_{\rho^-, \rho^+} : z \in \mathbb{Z}, (\rho^-, \rho^+) \in \mathcal{R}\} \quad (46)$$

$$Bl_1 = \{\tau_z \nu_{\rho^-, \rho^+} : (\rho^-, \rho^+) \in \mathcal{R}', z \in \mathbb{Z}\} \quad (47)$$

(ii) *The sets  $\mathcal{R}$ ,  $\mathcal{R}'$  and  $Bl_2$  enjoy the following properties:*

(a) *The set  $Bl_2$  is stable by translations, and outside the case*

$$l_0 = l_1 = q = 0, \quad (48)$$

*it contains at most (up to translations) two elements.*

(b) *Outside the cases*

$$p = q \text{ and } \gamma_0 + \gamma_1 = 0, \quad (49)$$

$$\gamma_0 = \gamma_1 = 0, \quad (50)$$

$$q = 0 = \gamma_0 \gamma_1, \quad (51)$$

*the set  $\mathcal{R}$  contains at most one element and  $\mathcal{R}'$  at most two elements.*

(c) *Outside (49)–(51), the following holds. Unless  $q = 0$  and  $\gamma_0 = \gamma_1 > 0$ , the set  $\mathcal{R} \cup \mathcal{R}'$  contains at most two elements. If  $q > 0$  and  $\gamma_0 + \gamma_1 \neq 0$ , the set  $\mathcal{R}'$  is empty. If  $q > 0$ ,  $q \neq p$  and  $\gamma_0 + \gamma_1 = 0 \neq \gamma_0 \gamma_1$ , the set  $\mathcal{R}$  is empty.*

Theorem 2.2 yields the following information. The decomposition (46) and statement (47) say that every element of  $\mathcal{I}_e$  that is not a product Bernoulli

measure is a shock measure of amplitude 1 or 2, and that for a given shock of amplitude 1, an associated shock measure is unique up to translations. Outside the case (48) (which will be further studied in the next theorem), up to translations, we can have *at most* two shock measures of amplitude 2. This case is special because the kernel (19) lacks standard irreducibility assumptions (see Definition 3.1), so usual ordering properties must be weakened (see Definitions 3.2 and 4.1). The only possible shock of amplitude 2 is  $(0, 2)$ . The  $(0, 2)$ -shock measures are analogues of *blocking* or *profile* measures in [4]. We shall see below that when both drifts are positive, shocks of amplitude 2 are blocking measures, and under additional assumptions, there are *exactly* two of them modulo translations. Shock measures of amplitude 1 can be divided into two classes with a different meaning. The set  $Bl_1$  contains measures associated to shocks in  $\mathcal{B}_1$ ; these are zero-flux measures. Among elements of  $Bl_1$  are *partial* blocking measures: we shall see below (in Theorem 2.3) that these may only (and do indeed) arise if  $q = 0$ . Under such measures, one lane carries a  $(0, 1)$ -shock and the other is either empty or full. The set  $\mathcal{R}$  is associated to other shock measures of amplitude 1. We believe that  $\mathcal{R}$  is empty and prove that it contains at most one element outside the case (49). This conjecture comes from the belief that the variance of the shock is of order  $t$  with a positive diffusion coefficient, which is incompatible with a stationary state. In contrast, we expect the diffusion coefficient to vanish in the last case of Theorem 2.2, (c); we have no clear conjecture whether  $\mathcal{R}'$  is empty in this case. Under (49), the model is diffusive and nongradient, and we conjecture that the only invariant measures are Bernoulli. We leave the above conjectures for future investigation, as the methods involved to prove them are presumably quite different from those used here.

Next, we provide more information on the sets  $\mathcal{R}$ ,  $\mathcal{R}'$ ,  $Bl_1$  and  $Bl_2$ , and obtain a full description of  $\mathcal{I}_e$  for a set of parameter values including (48) and (50)–(51). This is the content of Theorem 2.3 below. Its statement will be completed in Section 2.4.5 by the explicit description of the sets  $Bl_1$  and  $Bl_2$  referred to in the following statements. Recall (23). We define the reduced parameters

$$r := \frac{q}{p}, \quad d := \frac{\gamma_0}{\gamma_0 + \gamma_1} \text{ if } \gamma_0 + \gamma_1 \neq 0 \quad (52)$$

and set

$$r_0 := \frac{1 - 2\sqrt{-7 + \sqrt{52}}}{1 + 2\sqrt{-7 + \sqrt{52}}} = 0,042 \dots \quad (53)$$

Due to (23), we have  $(d, r) \in [0, 1] \times [0, 1]$ .

**Theorem 2.3.** (o) *If  $\gamma_0 > 0$  and  $\gamma_1 > 0$ , elements of  $Bl_2$  are supported on the set  $\mathcal{X}_2$ .*

(i) *Assume (50). Then  $\mathcal{R} = \mathcal{R}' = Bl_2 = \emptyset$ , hence*

$$\mathcal{I}_e = \{\nu_\rho : \rho \in [0, 2]\} \quad (54)$$

- Assume  $q > 0$ . Then:

(ii) Assume either: (a)  $d_0/l_0 = d_1/l_1 > 1$ ; or (b)  $l_0 = l_1 = 0$  and  $d_0, d_1 > 0$ . Then  $Bl_2$  is nonempty and given by (58).

(iii) There exists an open subset  $\mathcal{Z}$  of  $[0, 1] \times [0, 1]$ , containing  $\{1/2\} \times (0, r_0)$ , such that  $\mathcal{R} = \mathcal{R}' = \emptyset$  for every  $(d, r) \in \mathcal{Z}$ . In particular, if  $r \in (0, r_0)$ ,  $d_1 = \lambda d_0$  and  $l_1 = \lambda l_0$  with  $\lambda$  close enough to 1, then (46) holds with  $Bl_2$  as in (ii); this yields a complete description of  $\mathcal{I}_e$ .

- Assume now  $q = 0 < p$ . Then a complete description of  $\mathcal{I}_e$  can be obtained whenever  $\gamma_0 \neq \gamma_1$ . More precisely:

(iv) (a). If  $\gamma_0 > 0$  and  $\gamma_1 > 0$ , then  $\mathcal{R}' = \{(0, 1); (1, 2)\}$ ;  $\mathcal{R}$  is empty if  $\gamma_0 \neq \gamma_1$ , or contained in  $\{(3/2, 1/2)\}$  if  $\gamma_0 = \gamma_1$ . The set  $Bl_1$  is given by (59). The set  $Bl_2$  is empty unless  $l_0 = l_1 = 0$ . (b) If  $l_0 = l_1 = 0$ ,  $Bl_2$  is given by (61).

(v) If  $\gamma_1 < 0 < \gamma_0$ , then  $\mathcal{R}' = \{(1, 0), (1, 2)\}$ ,  $\mathcal{R} = Bl_2 = \emptyset$ . The set  $Bl_1$  is given by (62).

(vi) If  $\gamma_0 = 0 < \gamma_1$ , then  $\mathcal{R}' = \{(0, 1)\}$ ,  $\mathcal{R} = Bl_2 = \emptyset$ . The set  $Bl_1$  is given by (63).

**Remark 2.2.** In case (i), when  $p = q$ , the kernel defined by (19) is symmetric. The result is then a particular case of the general picture ([10, 9]) for symmetric exclusion processes, although our method of proof is different. However when in case (i) we have  $p \neq q$ , the two-lane SEP is not a symmetric exclusion process, and our result is new.

### 2.4.5 Explicit blocking measures in Theorem 2.3

We here complete the statement of Theorem 2.3 by giving the explicit description of  $Bl_1$  and  $Bl_2$  in each case.

Case (ii), (a). Let  $\theta = d_0/l_0 = d_1/l_1$ . Then the reversibility equation (24) with the kernel (19) has the following  $(0, 1)$ -valued solutions similar to (27):

$$\rho_{z,i}^c := \frac{c\theta^z \left(\frac{p}{q}\right)^i}{1 + c\theta^z \left(\frac{p}{q}\right)^i}, \quad (z, i) \in \mathbb{Z} \times W, c > 0 \quad (55)$$

The product measure  $\mu^{\rho^c}$ , cf. (25), is thus reversible for the two-lane SEP. It is supported on the set  $\mathcal{X}_2$  defined by (40). As in (32)–(33), we fix  $c > 0$  and define the conditioned measures (independent of the choice of  $c > 0$ , see Appendix A.1,

where the equalities below are also proved).

$$\begin{aligned}\check{\nu}_n &:= \mu^{\rho^c}(\cdot | H_2(\eta) = 2n) = \tau_n \check{\nu}_0, \quad n \in \mathbb{Z} \\ \hat{\nu}_n &:= \mu^{\rho^c}(\cdot | H_2(\eta) = 2n + 1) = \tau_n \hat{\nu}_0, \quad n \in \mathbb{Z}\end{aligned}\tag{56}$$

where now

$$H_2(\eta) := \sum_{x \in V: x(0) \leq 0} \eta(x) - \sum_{x \in V: x(0) > 0} [1 - \eta(x)]\tag{57}$$

Case (ii), (b). Let, for  $x \in V$ ,

$$\begin{aligned}\check{\eta}(x) &= 1_{\{x(0) > 0\}} \\ \hat{\eta}^0(x) &= 1_{\{x(0) > 0\}} + 1_{\{x=(0,0)\}} \\ \hat{\eta}^1(x) &= 1_{\{x(0) > 0\}} + 1_{\{x=(0,1)\}}.\end{aligned}$$

We define the measures  $\check{\nu}_0$  and  $\hat{\nu}_0$  through

$$\begin{aligned}\check{\nu}_0 &= \delta_{\check{\eta}} \\ \hat{\nu}_0 &= \frac{q}{p+q} \delta_{\hat{\eta}^0} + \frac{p}{p+q} \delta_{\hat{\eta}^1}\end{aligned}$$

We define also  $\check{\nu}_z = \tau_{-z} \check{\nu}_0$  and  $\hat{\nu}_z = \tau_{-z} \hat{\nu}_0$  for every  $z \in \mathbb{Z}$ .

In cases (ii), (a)–(b) above, we set

$$Bl_2 := \{\check{\nu}_z : z \in \mathbb{Z}\} \cup \{\hat{\nu}_z : z \in \mathbb{Z}\}\tag{58}$$

In cases (iv)–(vi) below, for  $n \in \mathbb{Z}$ , we denote by  $\nu^{\perp, +\infty, n}$  and  $\nu^{\perp, n, -\infty}$  the probability measures on  $\mathcal{X}$  defined as follows. Under  $\nu^{\perp, +\infty, n}$ ,  $\eta^0 = \eta_{+\infty}^*$  (i.e. lane 0 is empty, cf. (34)) and  $\eta^1 \sim \hat{\mu}_n$ , where  $\hat{\mu}_n$  is given by (32) with  $l = l_1$  and  $d = d_1$  if  $l_1 > 0$ , or by (34) if  $l_1 = 0$ . Under  $\nu^{\perp, n, -\infty}$ ,  $\eta^1 = \eta_{-\infty}^*$  (i.e. lane 1 is full) and  $\eta^0 \sim \hat{\mu}_n$ , where  $\hat{\mu}_n$  is given by (32) with  $l = l_0$  and  $d = d_0$  if  $l_0 > 0$ , or by (34) if  $l_0 = 0$ .

Case (iv), (a). We set

$$Bl_1 := \{\nu^{\perp, +\infty, n} : n \in \mathbb{Z}\} \cup \{\nu^{\perp, n, -\infty} : n \in \mathbb{Z}\}\tag{59}$$

Case (iv), (b). Let  $\mathbb{B}$  denote the set of  $(i, j) \in \mathbb{Z}^2$  such that  $i \geq j$ , and set  $\overline{\mathbb{B}} := \mathbb{B} \cup \{(+\infty, n), (n, -\infty) : n \in \mathbb{Z}\}$ . For  $(i, j) \in \overline{\mathbb{B}}$ , let  $\nu^{\perp, i, j}$  denote the Dirac measure supported on the configuration  $\eta^{\perp, i, j}$  defined by (recalling (34))

$$\eta^{\perp, i, j}(z, 0) = \eta_i^*(z), \quad \eta^{\perp, i, j}(z, 1) = \eta_j^*(z)\tag{60}$$

for every  $z \in \mathbb{Z}$ . The set  $Bl_2$  is given by

$$Bl_2 := \{\nu^{\perp, i, j} : (i, j) \in \mathbb{B}\}\tag{61}$$



Case (v). For  $n \in \mathbb{Z}$ , we denote by  $\nu^{\perp, +\infty, n\leftarrow}$  the probability measure on  $\mathcal{X}$  defined as follows. Recall the lane symmetry operator  $\sigma$  defined by (22). Under  $\nu^{\perp, +\infty, n\leftarrow}$ ,  $\eta^0 = \eta_{+\infty}^*$  and  $\sigma\eta^1 \sim \widehat{\mu}_n$ , where  $\widehat{\mu}_n$  is given by (32) with  $l = l_1$  and  $d = d_1$  if  $l_1 > 0$ , or by (34) if  $l_1 = 0$ . The set  $Bl_1$  is then given by

$$Bl_1 := \{\nu^{\perp, +\infty, n\leftarrow} : n \in \mathbb{Z}\} \cup \{\nu^{\perp, n, -\infty} : n \in \mathbb{Z}\} \quad (62)$$

Case (vi). The set  $Bl_1$  is given by

$$Bl_1 := \{\nu^{\perp, +\infty, n} : n \in \mathbb{Z}\} \quad (63)$$

## 2.5 Multilane SEP and rotational invariance

In this section, we consider the general model defined by (3) with (6), (10) and (11). Without loss of generality, we may consider  $W = \{0, \dots, n-1\}$ . We are interested in a generalization of the two-lane model with  $p = q$  (cf. (18)). To this end, we introduce the following assumption.

**Assumption 2.1.**  $W = \mathbb{T}_n$  is a torus, and  $q(\cdot, \cdot)$  is an irreducible translation-invariant kernel, that is  $q(i, j) = Q(j-i)$  for some function  $Q : \mathbb{T}_n \rightarrow [0, +\infty)$ .

For  $\rho \in [0, n]$ , we denote by  $\nu_\rho$  the product measure on  $\mathcal{X}$  such that

$$\nu_\rho \{\eta(z, i) = 1\} = \frac{\rho}{n} \quad (64)$$

for every  $(z, i) \in \mathbb{Z} \times W$ . For Theorems 2.1, 2.2 and 2.3, the scheme of proof laid out in Sections 3 to 5 carries over to the multi-lane model. Here, since  $W = \mathbb{T}_n$ , in addition to the shift operator  $\tau$  along  $\mathbb{Z}$  already considered, we can consider the translation operator  $\tau'$  along  $W$ . Following [5, page 2309], we shall call a probability measure on  $\mathcal{X}$  *rotationally invariant* if it is invariant by  $\tau'$ . The *open question 1* for the ladder process raised in [5] is whether, when  $d_i$  and  $l_i$  are independent of  $i$  (i.e. the horizontal dynamics is the same on each lane), all invariant measures are rotationally invariant. We give a positive answer to this question in item (3) of the following theorem.

**Theorem 2.4.** *Under Assumption 2.1, the following hold:*

(0) *We have  $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho, \rho \in [0, n]\}$ .*

(1) *For  $k = 1, \dots, n$ , let  $(\rho_k^-, \rho_k^+) = (\frac{n-k}{2}, \frac{n+k}{2} = n - \rho_k^-)$ . Then: (a)*

$$\mathcal{I}_e = \{\nu_\rho : \rho \in [0, n]\} \cup \bigcup_{k=1}^n \mathcal{I}_k \quad (65)$$

where  $\mathcal{I}_k$  is a (possibly empty) set of  $(\rho_k^-, \rho_k^+)$ -shock measures of amplitude  $k$ , which contains at most (up to horizontal translations)  $k$  measures. (b) *If  $\gamma_i > 0$  for all  $i$ ,  $\mathcal{I}_n$  is supported on the set  $\mathcal{X}_n$  defined by the right-hand side of (40).*

(c) If  $d_i/l_i$  does not depend on  $i$ ,  $\mathcal{I}_n$  consists (up to horizontal translations) of  $n$  explicit blocking measures  $\nu_i$  defined below for  $i = 0, \dots, n-1$ .

(2) If  $\gamma_i := d_i - l_i = 0$  for all  $i \in W$ , then  $\mathcal{I}_e = \{\nu_\rho : \rho \in [0, n]\}$ .

(3) If  $d_i$  and  $l_i$  do not depend on  $i$ , any invariant measure is rotationally invariant.

The blocking measures in (1), (c) are defined as in cases (ii), (a) and (ii), (b) of Theorem 2.3:

*First case.* If  $l_i > 0$  for all  $i$ , we define  $\rho_{z,i}^c$  as in (55), with  $\theta = d_i/l_i$  and  $p/q$  replaced by 1. For  $i = 0, \dots, n-1$ , we define the conditioned measures (independent of the choice of  $c > 0$  as in (56))

$$\nu_i := \mu^{\rho^c}(\cdot | H_n(\eta) = i) \quad (66)$$

where  $H_n$  is defined as the right-hand side of (57).

*Second case.* If  $l_i = 0 < d_i$  for all  $i$ , we define the configurations

$$\eta_A := \mathbf{1}_{\{x(0) \geq 0\}} + \mathbf{1}_{\{x(0) = -1, x(1) \in A\}}, \quad A \subset \{0, \dots, n-1\}$$

Then  $\nu_i$  is the law of a random configuration  $\eta_A$ , where  $A$  is a uniformly chosen random subset of  $\{0, \dots, n-1\}$  such that  $|A| = i$ .

### 3 Proof of Theorem 2.1

The proof of Theorem 2.1 mainly adapts the scheme of [8, Theorem 1.1] to our model. However when  $q = l_0 = l_1 = 0$ , additional arguments are required because the kernel (19) does not satisfy usual irreducibility assumptions.

First, in Subsection 3.1, we show how to parametrize the set  $\mathcal{F}$  in (36) by the global density over the two lanes and establish invariance of the associated product measures given in (37). Next, we introduce coupling prerequisites in Subsection 3.2, and complete the proof of characterization in Subsection 3.4.

#### 3.1 Parametrization and proof of invariance

The following lemma will lead to the parametrization (38).

**Lemma 3.1.** (i) *The mapping  $\psi : \mathcal{F} \rightarrow [0, 2]$ ;  $(\rho_0, \rho_1) \mapsto \psi(\rho_0, \rho_1) := \rho_0 + \rho_1$ , is a bijection.*

(ii) *Its inverse is of the form  $\psi^{-1}(\rho) = (\tilde{\rho}_0(\rho), \tilde{\rho}_1(\rho))$ , where  $\tilde{\rho}_1(\rho) := \rho - \tilde{\rho}_0(\rho)$ , and the function  $\rho \mapsto \tilde{\rho}_0(\rho)$  is given by the following formulae:*

Case 1a.  $pq \neq 0$ ,  $p \neq q$ . Then, for  $r = q/p$ ,

$$\tilde{\rho}_0(\rho) := \frac{\rho}{2} + \frac{r+1 - \sqrt{(r+1)^2 + \rho(r-1)^2(\rho-2)}}{2(r-1)}, \quad (67)$$

Case 1b.  $p = q \neq 0$ . Then

$$\tilde{\rho}_0(\rho) := \frac{\rho}{2} \quad (68)$$

Case 2.  $p = 0 < q$ . Then

$$\tilde{\rho}_0(\rho) := \min(\rho, 1) \quad (69)$$

Case 3.  $q = 0 < p$ . Then

$$\tilde{\rho}_0(\rho) := \max(\rho - 1, 0) \quad (70)$$

**Remark 3.1.** In Lemma 3.1, the formulae in (ii) imply that for  $pq > 0$ ,  $\tilde{\rho}_i(\rho)$  strictly increases from 0 to 1 as  $\rho$  increases from 0 to 2, and  $\tilde{\rho}_i \in C^1([0, 2])$ .

Next we define

$$\nu_\rho := \nu^{\tilde{\rho}_0(\rho), \tilde{\rho}_1(\rho)} \quad (71)$$

By (71) and (35), we have (recalling definition (8)), for every  $i \in \{0, 1\}$ ,

$$\mathbb{E}_{\nu_\rho}[\eta^i(0)] = \tilde{\rho}_i(\rho) \quad (72)$$

which implies (39).

**Remark 3.2.** It follows from (ii) of Lemma 3.1 that the measure  $\nu_\rho$  is weakly continuous and stochastically nondecreasing with respect to  $\rho$ . Namely, if  $\rho < \rho'$  then  $\nu_\rho \leq \nu_{\rho'}$ .

*Proof of Lemma 3.1.* We have to prove that, for  $\rho \in [0, 2]$ , the equation  $\rho_0 + \rho_1 = \rho$  has a unique solution  $(\rho_0, \rho_1) \in \mathcal{F}$ ; by definition  $\rho_i = \tilde{\rho}_i(\rho)$  for  $i \in \{0, 1\}$ . For  $r > 0$ , we define a mapping  $\phi_r$  from  $[0, 1]$  to  $[0, 1]$  by

$$\phi_r(\rho_0) := \frac{r\rho_0}{1 - \rho_0 + r\rho_0}, \quad \forall \rho_0 \in [0, 1] \quad (73)$$

One can then distinguish the following cases for  $\mathcal{F}$ :

Case 1.  $pq \neq 0$ . Then

$$\mathcal{F} := \{(\rho_0, \rho_1) \in [0, 1]^2 : \rho_1 = \phi_s(\rho_0)\}, \quad s = \frac{q}{p} \quad (74)$$

Case 2.  $p = 0 < q$ . Then

$$\mathcal{F} := ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1]) \quad (75)$$

Case 3.  $q = 0 < p$ . Then

$$\mathcal{F} := ([0, 1] \times \{1\}) \cup (\{0\} \times [0, 1]) \quad (76)$$

Equalities (69)–(70) follow from (75)–(76). For (67)–(68), using (74), we equivalently show that, for  $\rho \in [0, 2]$ , the equation

$$\rho_0 + \phi_s(\rho_0) = \rho \quad (77)$$

has a unique solution  $\rho_0 =: \tilde{\rho}_0(\rho) \in [0, 1]$  and deduce  $\tilde{\rho}_1(\rho)$ . If  $p = q > 0$ , (74) with  $s = 1$  yields  $\phi_1(\rho_0) = \rho_0$ , whence (68). If  $p \neq q$ ,  $p > 0$  and  $q > 0$ , (74) and (77) yield a quadratic equation for  $\rho_0$ , and (67) is its unique solution in  $[0, 1]$ .  $\square$

The following lemma shows that the measures in (37) of Theorem 2.1 are indeed extremal translation invariant and invariant probability measures. Stationarity can be derived from [5, Theorem 1], but we give an independent proof based on prior knowledge of invariance along horizontal and vertical layers.

**Lemma 3.2.** *Let  $(\rho_0, \rho_1) \in \mathcal{F}$ . Then  $\nu^{\rho_0, \rho_1} \in (\mathcal{I} \cap \mathcal{S})_e$ .*

*Proof of Lemma 3.2.* Let  $f$  be a cylinder function on  $\mathcal{X}$ . Note that the generator (3) has the following structure,

$$L = \sum_{i \in W} L_h^i + \sum_{z \in \mathbb{Z}} L_v^z \quad (78)$$

where, for  $i \in W$  and  $z \in \mathbb{Z}$ ,

$$L_h^i f(\eta) = \sum_{z \in \mathbb{Z}} p((z, i), (z+1, i)) \eta((z, i)) (1 - \eta((z+1, i))) \left( f\left(\eta^{(z, i), (z+1, i)}\right) - f(\eta) \right)$$

$$L_v^z f(\eta) = \sum_{i, j \in W} p((z, i), (z, j)) \eta((z, i)) (1 - \eta((z, j))) \left( f\left(\eta^{(z, i), (z, j)}\right) - f(\eta) \right)$$

In other words,  $L_h^i$ , acting only on  $\eta^i$ , and being translation invariant along the  $\mathbb{Z}$  direction, describes the evolution of the process on  $\mathbb{L}_i$ , which is the one of a (single-lane) SEP; while  $L_v^z$ , acting only on  $\{z\} \times W$ , describes the motion of particles along  $\{z\} \times W$ , that is, the displacements from one lane to another at a fixed spatial location  $z$ .

The statement  $\nu^{\rho_0, \rho_1} \in \mathcal{S}$  holds because  $\nu^{\rho_0, \rho_1}$  is a product Bernoulli measure whose parameters are uniform in the  $\mathbb{Z}$ -direction. Considering (78), to prove that  $\nu^{\rho_0, \rho_1}$  belongs to  $\mathcal{I}$ , it is enough to show that, for  $i \in W$  and  $z \in \mathbb{Z}$ ,

$$\int L_h^i f(\eta) \nu^{\rho_0, \rho_1}(d\eta) = 0 \quad (79)$$

and

$$\int L_v^z f(\eta) \nu^{\rho_0, \rho_1}(d\eta) = 0. \quad (80)$$

Let us write, for a fixed  $i \in W$ ,

$$\eta = (\eta^i, \eta'), \quad \nu^{\rho_0, \rho_1}(d\eta) = \nu^i(d\eta^i) \otimes \nu'(d\eta') \quad (81)$$

where  $\eta'$  denotes the restriction of  $\eta$  to lanes other than  $i$ . Note that  $\nu^i$  is invariant for  $L_h^i$  because  $L_h^i$  is the generator of a single-lane SEP on  $\mathbb{L}_i$  and  $\nu^i$  is a homogeneous product Bernoulli measure. Since  $L_h^i$  acts only on  $\eta^i$ , we have

$$\int L_h^i f(\eta) \nu^{\rho_0, \rho_1}(d\eta) = \int \left( \int L_h^i [f(\cdot, \eta')] (\eta^i, \eta') d\nu^i(\eta^i) \right) d\nu'(\eta') = 0$$

This establishes (79). We can similarly write, for a fixed  $z \in \mathbb{Z}$ ,

$$\eta = ({}^z\eta, \eta''), \quad \nu_\rho(d\eta) = {}^z\nu(d{}^z\eta) \otimes \nu''(d\eta'') \quad (82)$$

where  ${}^z\eta$  is the restriction of  $\eta$  to  $\{z\} \times W$ , and  $\eta''$  its restriction to the complement of  $\{z\} \times W$ . So, to prove (80), it is enough to prove that  ${}^z\nu$  is invariant for  $L_v^z$ . The latter is the generator of a simple exclusion process on  $\{z\} \times W$ . The invariance of  ${}^z\nu$  follows from (24)–(25) applied to  $S = \{z\} \times W$ ,  $\pi((z, 0), (z, 1)) = p$ ,  $\pi((z, 1), (z, 0)) = q$ , and definition (36) of  $\mathcal{F}$ . Finally, since  $\nu^{\rho_0, \rho_1} \in \mathcal{S}$  is a homogeneous product measure, it is spatially ergodic, that is extremal in  $\mathcal{S}$ , and thus also in  $\mathcal{I} \cap \mathcal{S}$ .  $\square$

### 3.2 Coupling, attractiveness and discrepancies

Let us first recall these properties for a general SEP; we refer to [9, chapter VIII, Section 2] for details.

*Coupling.* We recall the so-called *Harris graphical representation* ([7]). Suppose  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space that supports a family  $\mathcal{N} = \{\mathcal{N}_{(x,y)} : (x,y) \in V \times V\}$  of independent Poisson processes  $\mathcal{N}_{(x,y)}$  with respective intensities  $p(x,y)$ . For a given  $\omega \in \Omega$ , we let the process evolve according to the following rule: if there is a particle at site  $x \in V$  at time  $t^-$  where  $t \in \mathcal{N}_{(x,y)}$ , it shall attempt to jump to site  $y$ . The attempt is suppressed if at time  $t^-$  site  $y$  is occupied.

The graphical construction allows to couple the evolutions from different initial configurations through *basic coupling*, that is, by using the same Poisson processes for them. In particular, if  $(\eta_t)_{t \geq 0}$  and  $(\xi_t)_{t \geq 0}$  are two processes coupled in this way,  $(\eta_t, \xi_t)_{t \geq 0}$  is a Markov process on  $\mathcal{X} \times \mathcal{X}$  whose generator  $\tilde{L}$  is given by

$$\begin{aligned} \tilde{L}f(\eta, \xi) &:= \sum_{x,y \in V} p(x,y) [\eta(x)(1-\eta(y)) \wedge \xi(x)(1-\xi(y))] [f(\eta^{x,y}, \xi^{x,y}) - f(\eta, \xi)] \\ &+ \sum_{x,y \in V} p(x,y) [\eta(x)(1-\eta(y)) - \xi(x)(1-\xi(y))]^+ [f(\eta^{x,y}, \xi) - f(\eta, \xi)] \\ &+ \sum_{x,y \in V} p(x,y) [\eta(x)(1-\eta(y)) - \xi(x)(1-\xi(y))]^- [f(\eta, \xi^{x,y}) - f(\eta, \xi)] \end{aligned} \quad (83)$$

We shall denote by  $(\tilde{S}_t)_{t \geq 0}$  the semigroup generated by  $\tilde{L}$ , by  $\tilde{\mathcal{I}}$  the set of invariant probability measures for  $\tilde{L}$ , by  $\tilde{\mathcal{S}}$  the set of probability measures on  $\mathcal{X} \times \mathcal{X}$

that are invariant with respect to translations along  $\mathbb{Z}$ , and by  $\tilde{\mathbb{E}}_{\tilde{\mu}}$  the expectation for the coupled process with initial distribution a probability measure  $\tilde{\mu}$  on  $\mathcal{X} \times \mathcal{X}$ .

*Attractiveness.* There is a natural partial order on  $\mathcal{X}$ , namely, for  $\eta, \xi \in \mathcal{X}$ ,

$$\eta \leq \xi \quad \text{if and only if} \quad \forall x \in V, \eta(x) \leq \xi(x) \quad (84)$$

We shall write  $\eta < \xi$  if  $\eta \leq \xi$  and  $\eta \neq \xi$ .

If  $\eta \leq \xi$  or  $\eta \geq \xi$ , we say that  $\eta$  and  $\xi$  are *ordered* configurations.

The order (84) endows an order on the set  $\mathcal{M}_1$  in the following way. A function  $f$  on  $\mathcal{X}$  is said to be *increasing* if and only if  $\eta \leq \xi$  implies  $f(\eta) \leq f(\xi)$ . For two probability measures  $\mu_0, \mu_1$  on  $\mathcal{X}$ , we write  $\mu_0 \leq \mu_1$  if and only if for every increasing function  $f$  on  $\mathcal{X}$  we have  $\int f d\mu_0(\eta) \leq \int f d\mu_1(\eta)$ . We shall write  $\mu_1 < \mu_2$  if  $\mu_1 \leq \mu_2$  and  $\mu_1 \neq \mu_2$ . We say  $\mu_1$  and  $\mu_2$  are *ordered* if  $\mu_1 \leq \mu_2$  or  $\mu_2 \leq \mu_1$ . In particular,  $\mu_1 \leq \mu_2$  if there exists a measure  $\tilde{\mu}(d\eta, d\xi)$  with marginals  $\mu_1(d\eta)$  and  $\mu_2(d\xi)$  (that is a *coupling* of  $\mu_1$  and  $\mu_2$ ) supported on  $\{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta \leq \xi\}$ ; such a coupling is called an *ordered coupling*.

The basic coupling shows that the simple exclusion process is *attractive*, that is, the partial order (84) is conserved by the dynamics. In other words,

$$\forall \eta_0, \xi_0 \in \mathcal{X}, \eta_0 \leq \xi_0 \Rightarrow \forall t \geq 0, \eta_t \leq \xi_t \quad \text{a.s.} \quad (85)$$

This implies, for two probability measures  $\mu, \nu$  on  $\mathcal{X}$ ,

$$\mu \leq \nu \Rightarrow \mu S_t \leq \nu S_t \quad (86)$$

*Discrepancies.* If  $(\eta, \xi) \in \mathcal{X} \times \mathcal{X}$ , we say that at  $x \in V$  there is an  $\eta$  *discrepancy* if  $\eta(x) > \xi(x)$ , a  $\xi$  *discrepancy* if  $\eta(x) < \xi(x)$ , a coupled particle if  $\eta(x) = \xi(x) = 1$ , a hole if  $\eta(x) = \xi(x) = 0$ . An  $\eta$  and a  $\xi$  discrepancy are called *opposite discrepancies*, or *discrepancies of opposite type*. The evolution of the coupled process can be formulated as follows. At a time  $t \in \mathcal{N}_{(x,y)}$ , a discrepancy or a coupled particle at  $x$  exchanges with a hole at  $y$ ; a coupled particle at  $x$  exchanges with a discrepancy at  $y$ ; if there is a pair of opposite discrepancies at  $x$  and  $y$ , they are replaced by a hole at  $x$  and a coupled particle at  $y$ . We call this a *coalescence*. This shows that no new discrepancy can ever be created.

Given an initial tagged discrepancy, we may follow its motion over time. We state in this context a classical *finite propagation* property for discrepancies. Single-lane versions of this statement can be found e.g. in [4, Lemma 3.1] or [2, Lemma 3.1, Lemma 3.2]. Proofs are similar for the two-lane model.

**Proposition 3.1.** *There exist constants  $\sigma, C, C' > 0$  such that the following holds. Assume  $(\eta_t)_{t \geq 0}$  and  $(\xi_t)_{t \geq 0}$  are two coupled two-lane SEP's with at least one discrepancy at time 0. Let  $X_t = (X_t(0), X_t(1)) \in \mathbb{Z} \times W$  denote the position*

of a tagged discrepancy at time  $t$ . Then:

(i) outside probability  $e^{-Ct}$ , it holds that  $|X_t(0) - X_0(0)| \leq (1 + \sigma)t$ .

(ii) Similarly, if we assume  $\eta_0(z, i) = \xi_0(z, i)$  for all  $z \in [a, b]$  and  $i \in \{0, 1\}$ , where  $a, b \in \mathbb{Z}$  and  $a < b$ , then outside probability  $e^{-C't}$ ,  $\eta_t(z, i) = \xi_t(z, i)$  for all  $z \in [a + \sigma t, b - \sigma t]$  and  $i \in \{0, 1\}$ .

### 3.3 Irreducibility and discrepancies

As for general SEP, a crucial tool to prove Theorem 2.1 is an irreducibility property. We thus begin with the following definitions and properties.

For  $x, y \in V$  such that  $x \neq y$ , and  $n \in \mathbb{N}$ , we write  $x \xrightarrow{n}_p y$  if there exists a path  $(x = x_0, \dots, x_{n-1} = y)$  of length  $n$  such that  $p(x_k, x_{k+1}) > 0$  for  $k = 0, \dots, n-1$ .

We write  $x \rightarrow_p y$  if there exists  $n \in \mathbb{N}$  such that  $x \xrightarrow{n}_p y$ . We omit mention of  $p$  whenever there is no ambiguity on the kernel. We say  $x$  and  $y$  are  $p$ -connected if  $x \rightarrow_p y$  or  $y \rightarrow_p x$ . We say two configurations  $\eta, \xi$  in  $\mathcal{X}$  are  $p$ -ordered if there exists no  $(x, y) \in V \times V$  such that  $x$  and  $y$  are  $p$ -connected and  $(\eta, \xi)$  has discrepancies of opposite types at  $x$  and  $y$ .

**Definition 3.1.** We say the kernel  $p(\cdot, \cdot)$  is weakly irreducible if, for every  $(x, y) \in V \times V$  such that  $x \neq y$ ,  $x$  and  $y$  are  $p$ -connected.

The above notion is weaker than the *usual* irreducibility property, for which a stronger notion of  $p$ -connection is required, namely  $x \rightarrow_p y$  and  $y \rightarrow_p x$ . For instance, the kernel (4) is irreducible if and only if  $dl > 0$ ; if  $dl = 0$ , it is weakly irreducible but not irreducible. For our two-lane and multi-lane models, we need the following lemma.

**Lemma 3.3.** (i) The two-lane kernel  $p(\cdot, \cdot)$  given by (19) is weakly irreducible except when  $q = 0$  and both lanes are totally asymmetric in the same direction, that is  $d_0l_0 + d_1l_1 = 0 < d_0d_1 + l_0l_1$ .

(ii) The multi-lane kernel  $p(\cdot, \cdot)$  given by (10) is weakly irreducible under assumption 2.1.

*Proof of Lemma 3.3.*

*Proof of (i).* Let  $x, y \in \mathbb{Z}$  such that  $x \neq y$ . We need to go either from  $(x, 0)$  to  $(y, 1)$ , or from  $(y, 1)$  to  $(x, 0)$ , with the kernel  $p(\cdot, \cdot)$ .

(a) Assume first  $q > 0$ . Since the kernel (4) is weakly irreducible, the horizontal kernel on lane 0 can either go from  $(x, 0)$  to  $(y, 0)$  or from  $(y, 0)$  to  $(x, 0)$ . In the former case, since  $p > 0$ , we go from  $(y, 0)$  to  $(y, 1)$  with the vertical kernel. In the latter, since  $q > 0$ , we can go from  $(y, 1)$  to  $(y, 0)$  vertically and then from  $(y, 0)$  to  $(x, 0)$  horizontally.

(b) Assume now  $q = 0$ . If the two lanes are totally asymmetric in the same direction, say e.g.  $l_0 = l_1 = 0 < d_0d_1$ , and  $x > y$ , we can neither go from  $(x, 0)$  to  $(y, 1)$  (because  $l_0 + l_1 = 0$ ), nor from  $(y, 1)$  to  $(x, 0)$  (because  $q = 0$ ); otherwise, we have either  $d_0l_1 > 0$  or  $d_1l_0 > 0$ , say for instance the former. Then we can go from  $(x, 0)$  to  $(y, 1)$  via  $(y, 0)$  if  $x < y$ , or via  $(x, 1)$  if  $x > y$ .

*Proof of (ii).* Let  $x, y \in \mathbb{Z}$  such that  $x \neq y$  and  $i, j \in W$  such that  $i \neq j$ . Since the vertical kernel  $q(\cdot, \cdot)$  is irreducible, the same argument as in case (a) of (i) shows that we can either go from  $(x, i)$  to  $(y, j)$  or from  $(y, j)$  to  $(x, i)$ .  $\square$

The next lemma gives a characterization of  $p$ -ordered configurations. This requires the following definition.

**Definition 3.2.** For  $(\eta, \xi) \in \mathcal{X} \times \mathcal{X}$ , we write  $\eta \succ \xi$  if and only if there exist  $x, y \in \mathbb{Z}$  such that  $x < y$  and the following hold: (a) there are discrepancies of opposite type at  $(x, 1)$  and  $(y, 0)$ ; (b)  $\eta^0 \leq \xi^0$  and  $\eta^1 \geq \xi^1$  if the discrepancy at  $(x, 1)$  is an  $\eta$  discrepancy; or  $\eta^0 \geq \xi^0$  and  $\eta^1 \leq \xi^1$  if the discrepancy at  $(x, 1)$  is a  $\xi$  discrepancy; (c) There is no discrepancy at  $(z, 1)$  if  $z > x$ , nor any discrepancy at  $(z, 0)$  if  $z < y$ .

We define

$$E_{\succ} := \{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta \succ \xi\} \quad (87)$$

**Lemma 3.4.** For the kernel  $p(\cdot, \cdot)$  in (19), under (23), we have the following:

- (i) Unless  $q = 0$  and  $l_0 = l_1 = 0$ , two configurations  $\eta$  and  $\xi$  are  $p$ -ordered if and only if they are ordered, i.e.  $\eta \leq \xi$  or  $\xi \leq \eta$ .
- (ii) If  $q = 0$  and  $l_0 = l_1 = 0$ , two configurations  $\eta$  and  $\xi$  are  $p$ -ordered if and only if either they are ordered, or  $\eta \succ \xi$ .

*Proof of Lemma 3.4.* Let  $\eta$  and  $\xi$  be two  $p$ -ordered configurations. Note that two configurations are ordered (see (84)) if and only if they have no pair of opposite discrepancies. If  $pq \neq 0$  or  $l_0 + l_1 > 0$ , because of (23), any two distinct points of  $V$  are  $p$ -connected, hence  $\eta$  and  $\xi$  are ordered. Assume  $q = 0 < p$ . First we note that for all  $x, y \in \mathbb{Z}$ ,  $(x, 0)$  and  $(y, 0)$  are  $p$ -connected, and so are  $(x, 1)$  and  $(y, 1)$ . Thus  $\eta^i$  and  $\xi^i$  are ordered. If the ordering is the same, then  $\eta$  and  $\xi$  are ordered. Otherwise, there exists a pair of opposite discrepancies, one at  $(\mathbf{x}, 1)$  and one at  $(\mathbf{y}, 0)$  for  $\mathbf{x}, \mathbf{y} \in \mathbb{Z}$ . We must have  $\mathbf{x} < \mathbf{y}$ , otherwise  $(\mathbf{x}, 1)$  and  $(\mathbf{y}, 0)$  are  $p$ -connected. The ordering on each lane is imposed by the nature of the discrepancies at  $(\mathbf{x}, 1)$  and  $(\mathbf{y}, 0)$ . Assume for instance that there is an  $\eta$  discrepancy at  $(\mathbf{x}, 1)$  and a  $\xi$  discrepancy at  $(\mathbf{y}, 0)$ . Then  $\eta^0 \leq \xi^0$  and  $\eta^1 \geq \xi^1$ . For every  $z > \mathbf{y}$ , since  $\eta^1 \geq \xi^1$ , we have  $\eta^1(z) = \xi^1(z)$  or an  $\eta$  discrepancy at  $(z, 1)$ . The latter is ruled out because  $(\mathbf{y}, 0)$  and  $(z, 1)$  are  $p$ -connected. Similarly there can be no discrepancy at  $(z, 0)$  if  $z < \mathbf{x}$ . We can then redefine  $x$  as the location of the rightmost  $\eta$  discrepancy on lane 1, and  $y$  denote the location of the leftmost  $\xi$  discrepancy on lane 0.  $\square$

### 3.4 Proof of characterization

The next two results will enable us to deal with discrepancies in the proof of Theorem 2.1.

**Lemma 3.5.** Let  $\tilde{\nu} \in (\tilde{\mathcal{I}} \cap \tilde{\mathcal{S}})$ . If  $l_0 = l_1 = q = 0$ , then  $\tilde{\nu}(E_{\succ}) = 0$ .



*Proof of Lemma 3.5.* We define the following random variables taking values in  $\mathbb{Z} \cup \{\pm\infty\}$ :

$$X = X(\eta, \xi) := \sup\{x \in \mathbb{Z} : \eta^1(x) \neq \xi^1(x)\} \quad (88)$$

$$Y = Y(\eta, \xi) := \inf\{x \in \mathbb{Z} : \eta^0(x) \neq \xi^0(x)\} \quad (89)$$

with the convention  $\sup \emptyset = -\infty = -\inf \emptyset$ . That is,  $X$  is the location (if it exists) of the rightmost discrepancy on lane 1. Indeed on  $E_{><}$ , we have  $X(\eta, \xi) \in \mathbb{Z}$  by Lemma 3.4. Hence

$$\tilde{\nu}(E_{><}) \leq \sum_{k \in \mathbb{Z}} \tilde{\nu}(X = k) \leq 1 \quad (90)$$

Since  $\tilde{\nu} \in \tilde{\mathcal{S}}$ ,  $\tilde{\nu}(X = k)$  does not depend on  $k \in \mathbb{Z}$ . This quantity must vanish by the second inequality in (90), hence the result follows from the first one.  $\square$

For  $m, n \in \mathbb{Z} \cup \{\pm\infty\}$ , where  $m \leq n$ , let

$$D_{m,n}(\eta, \xi) := \sum_{x \in V: m \leq x(0) \leq n} |\eta(x) - \xi(x)| \quad (91)$$

denote the number of discrepancies in the space interval  $[m, n] \cap \mathbb{Z}$ . We simply write  $D(\eta, \xi)$  when  $(m, n) = (-\infty, +\infty)$ .

**Proposition 3.2.** *Let  $\tilde{\lambda} \in \tilde{\mathcal{I}}$ . Assume either  $\tilde{\lambda} \in \tilde{\mathcal{S}}$ , or*

$$\int_{\mathcal{X} \times \mathcal{X}} D(\eta, \xi) \tilde{\lambda}(d\eta, d\xi) < +\infty \quad (92)$$

*Then, for every  $(x, y) \in V \times V$  such that  $x$  and  $y$  are  $p$ -connected,*

$$\tilde{\lambda}(E_{x,y}) = 0 \quad (93)$$

*where*

$$E_{x,y} := \{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \text{there are discrepancies of opposite types at } x \text{ and } y\} \quad (94)$$

An equivalent formulation of Proposition 3.2 is

$$\tilde{\lambda}\{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta \text{ and } \xi \text{ are } p\text{-ordered}\} = 1 \quad (95)$$

In [8, Theorem 1.1] it is proved that if  $\tilde{\lambda}$  is a translation invariant and invariant probability measure for a one-dimensional translation invariant SEP (coupled via basic coupling), then (93) holds whenever  $x$  and  $y$  are  $p$ -connected. The argument carries over to our setting by using only translation invariance in the  $\mathbb{Z}$  direction. For the sake of completeness, details of the proof of Proposition 3.2 are given in Appendix A.2.

We are now in a position to complete the proof of Theorem 2.1.

*Proof of Theorem 2.1.* Let  $\mu \in (\mathcal{I} \cap \mathcal{S})_e$  and  $\rho \in [0, 2]$ . Since  $\nu_\rho \in (\mathcal{I} \cap \mathcal{S})_e$  (cf. Lemma 3.2), using [9, Proposition 2.14 in Chapter VIII], we obtain a measure  $\tilde{\lambda}$  on  $\mathcal{X} \times \mathcal{X}$ , which belongs to  $(\tilde{\mathcal{I}} \cap \tilde{\mathcal{S}})_e$  and whose marginals are  $\mu$  and  $\nu_\rho$ . The events

$$E_- := \{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta \leq \xi\} \quad \text{and} \quad (96)$$

$$E_+ := \{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta \geq \xi\} \quad (97)$$

are invariant with respect to spatial translations, and (by attractiveness) they are conserved by the coupled dynamics. Since  $\tilde{\lambda} \in (\tilde{\mathcal{I}} \cap \tilde{\mathcal{S}})_e$ ,  $E_+$  and  $E_-$  both have  $\tilde{\lambda}$ -probability 0 or 1. The main step is to prove that

$$\tilde{\lambda}(E_+ \cup E_-) = \tilde{\lambda}\{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta \leq \xi \text{ or } \xi \leq \eta\} = 1 \quad (98)$$

implying that one of the events  $E_+$  and  $E_-$  has probability 1. It follows that for every  $0 < \rho < 2$  we either have  $\mu \leq \nu_\rho$  or  $\mu \geq \nu_\rho$ . By Remark 3.2 we conclude that there exists some  $r \in [0, 2]$  such that  $\mu = \nu_r$ .

We now turn to the proof of (98). Outside the case  $q = 0 = l_0 = l_1 < p$ , the kernel  $p(\cdot, \cdot)$  in (19) is weakly irreducible; thus (98) follows from (95) and (i) of Lemma 3.4. Now assume  $q = 0 = l_0 = l_1 < p$ . By (ii) of Lemma 3.4, we obtain

$$\tilde{\lambda}(E_- \cup E_+ \cup E_{><}) = 1 \quad (99)$$

and the conclusion follows from Lemma 3.5.  $\square$

## 4 Proofs of Theorems 2.2, 2.3 and 2.4

The proofs of Theorems 2.2 and 2.3 are developed respectively in Subsections 4.1 and 4.3. They rely on a series of intermediate results, all established in Section 5, except Proposition 4.6, established in Subsection 4.2. Indeed this proposition is necessary for Theorem 2.2, but its proof introduces material (namely current and flux function) also used for Theorem 2.3. Finally, Theorem 2.4 is proved in Subsection 4.4.

### 4.1 Proof of Theorem 2.2

We have to distinguish the case (48), where the kernel  $p(\cdot, \cdot)$  in (19) is not weakly irreducible, cf. Lemma 3.3. In this case, we introduce the following definition.

**Definition 4.1.** For  $(\eta, \xi) \in \mathcal{X} \times \mathcal{X}$ , we write  $\eta \bowtie \xi$  if and only if the following hold: (i)  $\eta >< \xi$  (cf. Definition 3.2); (ii) the number of locations  $z \in \mathbb{Z}^+$  on lane 1 that are not occupied by a coupled particle is finite; (iii) the number of locations  $z \in \mathbb{Z}^-$  on lane 0 that are not occupied by a hole is finite.

We define

$$E_{\bowtie} := \{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta \bowtie \xi\} \quad (100)$$

The main steps for the proof of Theorem 2.2 are Propositions 4.1–4.6 and Corollary 4.1, stated below and proved in Subsection 5.1.

*Step one.* Let  $\mu \in \mathcal{I}_e$ . We prove the following proposition.

**Proposition 4.1.** (i) *There exists a measure  $\tilde{\lambda}(d\eta, d\xi)$  on  $\mathcal{X} \times \mathcal{X}$  with marginals  $\mu(d\eta)$  and  $\tau_1\mu(d\xi)$ , satisfying one of (101)–(103) below (if  $q > 0$ ), or one of (101)–(104) below (if  $l_0 = l_1 = q = 0 < p$ ):*

$$\tilde{\lambda}(E_1) = 1 \quad \text{where} \quad E_1 := ((\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta < \xi) \quad (101)$$

$$\tilde{\lambda}(E_2) = 1 \quad \text{where} \quad E_2 := ((\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \xi < \eta) \quad (102)$$

$$\tilde{\lambda}(E_3) = 1 \quad \text{where} \quad E_3 := ((\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta = \xi) \quad (103)$$

$$\tilde{\lambda}(E_{\boxtimes}) = 1 \quad (104)$$

(ii) *For any measure  $\tilde{\lambda}(d\eta, d\xi)$  with marginals  $\mu(d\eta)$  and  $\tau_1\mu(d\xi)$  satisfying (101) or (102), there exists  $k \in \{1, 2\}$  such that (cf. definition of  $D(\eta, \xi)$  below (91))*

$$\tilde{\lambda}((\eta, \xi) \in \mathcal{X} \times \mathcal{X} : D(\eta, \xi) = k) = 1 \quad (105)$$

This has the following consequences.

**Corollary 4.1.** (i) *In cases (101)–(103), the family  $(\tau_n\mu)_{n \in \mathbb{Z}}$  is stochastically monotone.*

(ii) *If a probability measure  $\hat{\mu}$  on  $\mathcal{X}$  is such that  $\hat{\mu} \in \mathcal{I}$  and  $(\tau_n\hat{\mu})_{n \in \mathbb{Z}}$  is stochastically monotone, then there exist probability measures  $\gamma^-(d\rho)$  and  $\gamma^+(d\rho)$  on  $[0, 2]$  such that the limits*

$$\hat{\mu}_{\pm} := \lim_{n \rightarrow \pm\infty} \tau_n\hat{\mu} = \int_{[0,2]} \nu_{\rho}\gamma^{\pm}(d\rho) \quad (106)$$

*hold in the sense of weak convergence.*

*Step two.* We conclude in case (104) of Proposition 4.1.

**Proposition 4.2.** *In case (104), we have  $\mu \in Bl$ .*

*Step three.* We show that the measures  $\gamma^{\pm}$  of Corollary 4.1 are Dirac measures.

**Proposition 4.3.** *In cases (101)–(102), there exists  $(\rho^-, \rho^+) \in [0, 2]^2 \setminus \mathcal{D}$  such that (i)  $\gamma^{\pm} = \delta_{\rho^{\pm}}$ , thus  $\mu$  is a  $(\rho^-, \rho^+)$ -shock measure, cf. (42); (ii)  $|\rho^+ - \rho^-| = k$ , where  $k$  is defined in (ii) of Proposition 4.1.*

*Step four.* In Proposition 4.5 below, we study the relation between extremal invariant measures that are  $(\rho^-, \rho^+)$ -shock measures for a common pair  $(\rho^-, \rho^+)$ . The proof of Proposition 4.5 requires the following variant of Proposition 4.1.

**Proposition 4.4.** *Let  $(\rho^-, \rho^+) \in [0, 2]^2 \setminus \mathcal{D}$ , and assume  $\nu, \nu' \in \mathcal{I}_e$  are two  $(\rho^-, \rho^+)$ -shock measures. Then: (i) there exists a coupling of  $\nu$  and  $\nu'$  that satisfies one of the properties (101)–(104), property (104) being possible only under assumption (48); (ii) in case (104),  $\nu$  and  $\nu'$  lie in  $Bl$ ; (iii) in cases (101)–(102), (105) holds for some  $k \in (\mathbb{N} \setminus \{0\}) \cup \{+\infty\}$ .*

**Proposition 4.5.** *Let  $\nu, \nu' \in \mathcal{I}_e$  be two  $(\rho^-, \rho^+)$ -shock measures. (i) Assume  $|\rho^+ - \rho^-| = 1$ , and we do not simultaneously have (48) and  $\nu \in Bl_1$ . Then  $\nu'$  is a translate of  $\nu$ , i.e. there exists  $n \in \mathbb{Z}$  such that  $\nu' = \tau_n \nu$ . (ii) Assume  $|\rho^+ - \rho^-| = 2$ ,  $\nu'$  is not a translate of  $\nu$ , and we do not have (48). Then every  $(\rho^-, \rho^+)$ -shock measure is either a translate of  $\nu$ , or a translate of  $\nu'$ .*

*Final step.* We assemble the previous steps to conclude the proof of Theorem 2.2. We first introduce the sets  $\mathcal{R}$  and  $\mathcal{R}'$  involved in Theorem 2.2.

**Definition 4.2.** *We denote by  $\mathcal{R}$  the set of  $(\rho^-, \rho^+) \in [0, 2]^2 \setminus (\mathcal{D} \cup \mathcal{B})$  such that  $\mathcal{I}_e$  contains at least one  $(\rho^-, \rho^+)$ -shock measure, and by  $\mathcal{R}'$  the set of  $(\rho^-, \rho^+) \in \mathcal{B}_1$  such that  $\mathcal{I}_e$  contains at least one  $(\rho^-, \rho^+)$ -shock measure.*

In Subsection 4.2 below, we prove the following proposition, after introducing the *macroscopic flux function* of our model. We can then end this section with the proof of Theorem 2.2.

**Proposition 4.6.** *Outside (49)–(50), the following holds: (i) in cases (101)–(102) with  $k = 2$ ,  $\mu$  is a  $(0, 2)$ -shock measure; (ii) Statements (ii), (b) and (c) of Theorem 2.2 hold. (iii) Statement (i) of Proposition 4.5 still holds if we have (48) and  $\nu \in Bl_1$ .*

*Proof of Theorem 2.2.* First, for  $\mu \in \mathcal{I}_e$ , we consider the different possibilities in Proposition 4.1. In case (103), we have  $\mu \in (\mathcal{I} \cap \mathcal{S})_e$ ; by Theorem 2.1,  $\mu = \nu_\rho$  for some  $\rho \in [0, 2]$ . In case (104) (which may only occur under (48)), Proposition 4.2 implies  $\mu \in Bl_1 \cup Bl_2$ , with  $Bl_1$  and  $Bl_2$  given by (59)–(61). In cases (101)–(102) with  $k = 2$  in (105), Proposition 4.3 and (i) of Proposition 4.6 lead to  $\mu \in Bl_2$ . In cases (101)–(102) with  $k = 1$  in (105), by Proposition 4.3,  $\mu$  is a shock measure of amplitude 1.

Next, to obtain (46), we consider the structure modulo translations of shock measures. Cardinality bounds for  $\mathcal{R}$  and  $\mathcal{R} \cup \mathcal{R}'$  are given by Proposition 4.6. By (i) of Proposition 4.5 and (iii) of Proposition 4.6, for every  $(\rho^-, \rho^+) \in \mathcal{R} \cup \mathcal{R}'$ , the set of  $(\rho^-, \rho^+)$ -shock measures in  $\mathcal{I}_e$  consists of translates of a single measure. The set  $Bl_2$  is stable by translation because the generator (3) with transition kernel (19) is translation invariant. By (ii) of Proposition 4.5, outside (48),  $Bl_2$  consists of at most (up to translations) two measures. This concludes the proof of (i) and (ii), (a). Finally, (ii), (b) and (ii), (c) are contained in statement (ii) of Proposition 4.6.  $\square$

## 4.2 Proof of Proposition 4.6

We begin by defining the flux function, which will also play an important role in the proof of Theorem 2.3, and state some of its properties.

### 4.2.1 Microscopic current and macroscopic flux

We first define the microscopic current by

$$j(\eta) := \sum_{x(0) \leq 0, y(0) > 0} p(x, y) \eta(x) (1 - \eta(y)) - \sum_{x(0) \leq 0, y(0) > 0} p(y, x) \eta(y) (1 - \eta(x)) \quad (107)$$

for  $\eta \in \mathcal{X}$ . With the kernel defined by (19), this yields

$$\begin{aligned} j(\eta) &= \sum_{i=0}^1 \{d_i \eta^i(0) [1 - \eta^i(1)] - l_i \eta^i(1) [1 - \eta^i(0)]\} \\ &= \sum_{i=0}^1 \{\gamma_i \eta^i(0) [1 - \eta^i(1)] + l_i [\eta^i(0) - \eta^i(1)]\} \end{aligned} \quad (108)$$

The macroscopic flux is then given by, for  $\rho \in [0, 2]$ ,

$$G(\rho) := \int j(\eta) \nu_\rho(d\eta). \quad (109)$$

Using (71) and (35), this yields

$$G(\rho) = \gamma_0 G_0[\tilde{\rho}_0(\rho)] + \gamma_1 G_0[\tilde{\rho}_1(\rho)] \quad (110)$$

where  $G_0$  is the flux function of the single-lane TASEP, given by

$$G_0(\alpha) = \alpha(1 - \alpha) \quad \forall \alpha \in [0, 1]. \quad (111)$$

In the following two special cases, the function  $G$  has a simple expression.

**Example 4.1.** Assume  $q = 0 < p$ . Then, by (110) and (70),

$$G(\rho) = \begin{cases} \gamma_1 \rho(1 - \rho) & \text{if } \rho \in [0, 1] \\ \gamma_0(\rho - 1)(2 - \rho) & \text{if } \rho \in (1, 2] \end{cases} \quad (112)$$

In particular, when  $\gamma_0 = \gamma_1$ , the flux is a function of period 1 whose restriction to  $[0, 1]$  is the TASEP flux. It exhibits a change of convexity at  $\rho = 1$ , where it is also non differentiable. Note that the latter property is not seen in usual single-lane models with product invariant measures.

**Example 4.2.** Assume  $p = q > 0$ . Then, by (110) and (68),

$$G(\rho) = \frac{\gamma_0 + \gamma_1}{4} \rho(2 - \rho) \quad (113)$$

Here, the flux has the same shape as the single-lane TASEP flux (from which it is obtained by a scale change in the horizontal and vertical directions). It is in particular strictly concave.

Useful properties of  $G$  are gathered in the following proposition.

**Proposition 4.7.**

(o)  $G(0) = G(2) = 0$ .

(i) Outside cases (49), (50) and (51),  $G$  has at least one and at most three local extrema.

(ii) (a)  $G(1) = 0$  if and only if  $q = 0$  or  $\gamma_0 + \gamma_1 = 0$ ; (b) if  $q > 0$ ,  $G$  is continuously differentiable on  $[0, 2]$ , and  $G'(1) = 0$  if and only if  $\gamma_0 = \gamma_1$  or  $p = q$ .

(iii) Under (23),  $G'(2) \leq 0$ . Besides,  $G'(2) < 0$  holds unless we have (49), or (50), or  $q = \gamma_0 = 0$ .

(iv) The function  $G$  depends only on the parameters  $\gamma_0, \gamma_1$  and  $r$  defined in (52). Denoting  $G = G_{\gamma_0, \gamma_1, r}$ , it holds that

$$G_{\gamma_0, \gamma_1, r}(2 - \rho) = G_{\gamma_1, \gamma_0, r}(\rho) = G_{\gamma_0, \gamma_1, r^{-1}}(\rho) \quad (114)$$

where the last equality holds when  $r > 0$ . If  $\gamma_0 + \gamma_1 \neq 0$ , for  $d \in \mathbb{R}$ , it holds that

$$G_{\gamma_0, \gamma_1, r} = (\gamma_0 + \gamma_1)G_{d, 1-d, r}, \quad \text{with } d \text{ defined in (52)}. \quad (115)$$

(v) Assume  $\gamma_0 = \gamma_1 \neq 0$ , that is  $d = 1/2$ . Then: (a)  $G'(1/2) > 0$ ; (b) for  $r \in (0, r_0)$ , with  $r_0$  given by (53), we have  $G(1/2) > G(1)$ .

(vi) If  $q \neq 0$  and  $\gamma_0 + \gamma_1 \neq 0$ , the equation  $G(\rho + 1) - G(\rho) = 0$  has a unique solution in  $[0, 1]$ . If  $q \neq 0$ ,  $p \neq q$  and  $\gamma_0 + \gamma_1 = 0 \neq \gamma_0\gamma_1$ , the solutions of this equation in  $[0, 1]$  are  $\rho = 0$  and  $\rho = 1$ .

**4.2.2 Proof of Proposition 4.6**

The scheme of proof of Proposition 4.6 is the following. We introduce in Definition 4.3 a set denoted by  $\mathcal{R}_0$ , which depends only on the flux function. Lemma 4.1 (which will be proved using Proposition 4.7) says that  $\mathcal{R}_0$  contains at most three elements, in most cases no more than one, and sometimes none. Proposition 4.8 provides information on possible stationary shock measures, implying that  $\mathcal{R}$  is contained in  $\mathcal{R}_0$ ; part of this proposition will be useful for the proof of Theorem 2.3. The proof of Proposition 4.6 is concluded using Lemma 4.1 and Proposition 4.8; these are proved in Subsection 5.2.

**Definition 4.3.** Let  $\mathcal{R}_0$  denote the set of pairs  $(\rho^-, \rho^+) \in [0, 2]^2 \setminus \mathcal{D}$  satisfying the following conditions: (i)  $|\rho^+ - \rho^-| = 1$ ; (ii)  $G(\rho^+) = G(\rho^-) = \min_{\rho \in [\rho^-, \rho^+]} G$  if  $\rho^- < \rho^+$ ; or  $G(\rho^+) = G(\rho^-) = \max_{\rho \in [\rho^+, \rho^-]} G$  if  $\rho^- > \rho^+$ , where  $G$  is defined by (109)–(110).

**Remark 4.1.** Condition (ii) in Definition 4.3 means that  $(\rho^-, \rho^+)$  is an entropy shock for the scalar conservation law with flux function  $G$ , that is the expected

hydrodynamic equation for our model. Thus  $\mathcal{R}_0$  is exactly the set of entropy shocks of amplitude 1.

**Lemma 4.1.** *Outside (49)–(51), the set  $\mathcal{R}_0$  contains at most three elements. More precisely:*

- (i) *If  $q > 0$  and  $\gamma_0 + \gamma_1 \neq 0$ ,  $\mathcal{R}_0$  contains one element, and  $\mathcal{B}_1 \cap \mathcal{R}_0 = \emptyset$ .*
- (ii) *If  $q > 0$ ,  $p \neq q$  and  $\gamma_0 + \gamma_1 = 0 \neq \gamma_0\gamma_1$ , or if  $q = 0$  and  $\gamma_0 \neq \gamma_1$ ,  $\mathcal{R}_0$  contains two elements, and  $\mathcal{R}_0 \subset \mathcal{B}_1$ .*
- (iii) *Assume  $d = 1/2$ , and recall  $r_0$  defined by (53). Then  $\mathcal{R}_0 = \{(1/2, 3/2)\}$  if and only if  $r \in [r_0, 1]$ ,  $\mathcal{R}_0 = \emptyset$  if and only if*

$$r \in (0, r_0), \tag{116}$$

and  $\mathcal{R}_0 = \{(3/2, 1/2); (0, 1); (1, 2)\}$  if and only if  $r = 0$ .

- (iv) *There exists an open subset  $\mathcal{Z}$  of  $[0, 1]^2$ , containing  $\{1/2\} \times (0, r_0)$ , such that  $\mathcal{R}_0 = \emptyset$  for  $(d, r) \in \mathcal{Z}$ .*

**Proposition 4.8.** (i) *Assume that a measure  $\nu \in \mathcal{I}$  is a  $(\rho^-, \rho^+)$ -shock measure. Then  $(\rho^-, \rho^+)$  satisfies condition (ii) of Definition 4.3.*

(ii) *Assume that in Proposition 4.1 we have (101) or (102), and  $k = 1$ . Then the pair  $(\rho^-, \rho^+)$  in Proposition 4.3 satisfies  $(\rho^-, \rho^+) \in \mathcal{R}_0$ .*

(iii) *Under the assumptions of (ii), suppose in addition that  $(\rho^-, \rho^+) \in \mathcal{B}_1$ ; then either  $\gamma_0 + \gamma_1 = 0$ , or  $q = 0$ . If the latter holds, we are in one of the cases (iv), resp. (v), (vi) of Theorem 2.3, and  $\nu$  lies in the set given by (59), resp. (62), (63).*

*Proof of Proposition 4.6.*

*Proof of (i).* By (i) of Proposition 4.3,  $\mu$  is a shock measure of amplitude 2, that is either a  $(0, 2)$  or a  $(2, 0)$ -shock measure. The second possibility and (i) of Proposition 4.8 would imply that  $(2, 0)$  satisfies condition (ii) of Definition 4.3, thus that the maximum of  $G$  is 0; whereas (iii) of Proposition 4.7 (when  $q > 0$ ) and (112) (when  $q = 0$ ) imply that this maximum is positive under (23).

*Proof of (ii).* By Definition 4.2 and (ii) of Proposition 4.3,  $\mathcal{R}$  and  $\mathcal{R}'$  contain only shocks of amplitude 1 associated with stationary shock measures. By Proposition 4.8, (i), any shock associated with a stationary shock measure is an entropy shock (cf. Remark 4.1). Thus by Definition 4.2 and Remark 4.1, we have  $\mathcal{R} \cup \mathcal{R}' \subset \mathcal{R}_0$ ,  $\mathcal{R} \subset \mathcal{R}_0 \setminus \mathcal{B}_1$  and  $\mathcal{R}' \subset \mathcal{R}_0 \cap \mathcal{B}_1$ . The results then follow from (i) and (ii) of Lemma 4.1 if  $q > 0$ . If  $q = 0$ ,  $\gamma_0\gamma_1 \neq 0$  and  $\gamma_0 \neq \gamma_1$ , (112) and Definition 4.3 show that  $\mathcal{R}_0$  contains at most two points and  $\mathcal{R}_0 \subset \mathcal{B}_1$ , thus  $\mathcal{R} = \emptyset$  and  $\mathcal{R}' \subset \mathcal{R}_0$ .

*Proof of (iii).* In this case, by (iii) of Proposition 4.8,  $\mathcal{B}l_1$  is contained in the right-hand side of (59). Each of the two sets on this right-hand side consists of translates of a single measure; the first set contains only  $(1, 2)$ -shock measures and the second one only  $(0, 1)$ -shock measures.  $\square$

### 4.3 Proof of Theorem 2.3

We will need the following lemma, proved in Section 5.2.

**Lemma 4.2.** *Assume  $\nu^1, \nu^2 \in \mathcal{I}$  are supported on  $\mathcal{X}_2$ , and  $H_2(\eta)$  defined by (57) has the same constant value under  $\nu^1$  and  $\nu^2$ . Then, unless  $l_0 = l_1 = q = 0 < p$ , it holds that  $\nu^1 = \nu^2$ .*

*Proof of Theorem 2.3.*

*Proof of (o).* Since  $\mu \in \mathcal{I}$ , we have (recalling (9))

$$\int L \sum_{x \in \mathbb{Z}: m \leq x \leq n} \bar{\eta}(x) d\mu(\eta) = 0$$

where

$$L\bar{\eta}(x) = \tau_{x-1}j(\eta) - \tau_x j(\eta) \quad (117)$$

with  $j$  defined by (108). Hence, for arbitrary  $n, m \in \mathbb{Z}$ , we conclude that the quantity  $\mu[\tau_x j(\eta)]$  is independent of  $x$ . Since  $\mu$  is a  $(0, 2)$ -shock measure (see (42)), we have

$$\lim_{n \rightarrow +\infty} \mu[\eta(n, i)] = 1, \quad \lim_{n \rightarrow -\infty} \mu[\eta(n, i)] = 0$$

Since  $0 \leq \eta(x, i)(1 - \eta(y, i)) \leq \min[\eta(x, i), 1 - \eta(y, i)]$  for  $x, y \in \mathbb{Z}$ , this implies

$$\lim_{n \rightarrow +\infty} \mu[\eta(x, i)(1 - \eta(x+1, i))] = \lim_{n \rightarrow +\infty} \mu[\eta(x+1, i)(1 - \eta(x, i))] = 0 \quad (118)$$

for  $i \in \{0, 1\}$ . Thus  $\mu[\tau_x j(\eta)] = 0$ , which can be written

$$\mu \left\{ \sum_{i=0}^1 \gamma_i \eta^i(x)[1 - \eta^i(x+1)] \right\} = \mu \left\{ \sum_{i=0}^1 l_i [\eta^i(x) - \eta^i(x+1)] \right\} \quad (119)$$

Summing (119) over  $x \in \mathbb{Z}$  and using (118), we obtain

$$\sum_{i=0}^1 \gamma_i \mu \left\{ \sum_{x \in \mathbb{Z}} \eta^i(x)[1 - \eta^i(x+1)] \right\} < +\infty \quad (120)$$

For each  $i \in \{0, 1\}$ ,  $\gamma_i > 0$ , hence the series inside braces in (120) converges  $\mu$ -almost surely. Thus,  $\mu$ -almost surely,  $\eta^i(x)[1 - \eta^i(x+1)] \rightarrow 0$  as  $x \rightarrow \pm\infty$  implying  $\eta^i(x)[1 - \eta^i(x+1)] = 0$  for  $|x|$  large enough, and  $\eta \in \mathcal{X}_2$ .

*Proof of (i).* Let  $\varphi \in C_K^0(\mathbb{R})$ , that is, a continuous function with compact support. We consider the function  $F_N : \mathcal{X} \rightarrow \mathbb{R}$  defined by

$$F_N(\eta) := N \sum_{x \in \mathbb{Z}} \varphi\left(\frac{x}{N}\right) \bar{\eta}(x)$$



Since  $\gamma_0 = \gamma_1 = 0$ , i.e.  $l_0 = d_0$  and  $l_1 = d_1$ , the microscopic current (108) writes

$$j(\eta) = \sum_{i=0}^1 d_i [\eta^i(0) - \eta^i(1)] \quad (121)$$

Using (117), (121) and two summations by parts, we obtain

$$LF_N(\eta) = N^{-1} \sum_{i=0}^1 d_i \sum_{x \in \mathbb{Z}} \varphi''\left(\frac{x}{N}\right) \eta^i(x) + o_N(1) \quad (122)$$

where  $o_N(1)$  is a quantity bounded in modulus by a deterministic sequence vanishing as  $N \rightarrow +\infty$ . By Theorem 2.2,  $\mu$  satisfies (42), where either  $\rho^+ = \rho^-$  and  $\mu$  is a product measure given by Theorem 2.1, or  $\rho^+ \neq \rho^-$  and  $\mu$  is a  $(\rho^-, \rho^+)$ -shock measure. We show that we are in the first situation. Indeed, (42) implies

$$\lim_{x \rightarrow \pm\infty} \int_{\mathcal{X}} \eta^i(x) d\mu(\eta) = \tilde{\rho}_i(\rho^\pm) \quad (123)$$

Thus, taking the expectation of (122) and using stationarity of  $\mu$ , we have

$$0 = \int_{\mathcal{X}} LF_N(\eta) d\mu(\eta) = \varphi''(0) \sum_{i=0}^1 d_i [\tilde{\rho}_i(\rho^-) - \tilde{\rho}_i(\rho^+)] + \varepsilon_N \quad (124)$$

where  $\varepsilon_N \rightarrow 0$  as  $N \rightarrow +\infty$ . Since  $\tilde{\rho}_i$  is increasing and  $\varphi$  arbitrary, it follows that  $\rho^+ = \rho^-$ .

*Proof of (ii), (a).* The product measure  $\mu^\rho$  defined by (25) for  $\rho$  given by (55) is reversible because  $\rho$  is a solution of (24). The result follows since the measures in (56) are defined by conditioning the reversible measure  $\mu^\rho$  on the conserved quantity  $H_2$ . Assume now  $\check{\nu}_n = (1 - \alpha)\nu^1 + \alpha\nu^2$  with  $\nu^1, \nu^2 \in \mathcal{I}$ . Since  $\check{\nu}$  is supported on  $\mathcal{X}_2$  and  $H_2$  has constant value  $2n$  under  $\check{\nu}_n$ , the same holds for  $\nu^1$  and  $\nu^2$ . Thus  $\nu^1 = \nu^2$  by Lemma 4.2, implying that  $\check{\nu}_n$  is extremal. The same argument applies to  $\hat{\nu}_n$ .

*Proof of (ii), (b).* The measure  $\check{\nu}_0$  is a product measure of the form (25), with  $\rho$  given by

$$\rho_{x,i} = \check{\rho}_{x,i} := \mathbf{1}_{\{x>0\}}, \quad (x, i) \in \mathbb{Z} \times W, \quad (125)$$

On the other hand, let  $\hat{\mu}_0$  denote the product measure (25) where

$$\begin{aligned} \rho_{x,i} &= \hat{\rho}_{x,i} := \mathbf{1}_{\{x>0\}} \\ &+ \rho_0 \mathbf{1}_{\{(0,0)\}}(x, i) + \rho_1 \mathbf{1}_{\{(0,1)\}}(x, i), \quad (x, i) \in \mathbb{Z} \times W \end{aligned} \quad (126)$$

with

$$\rho_0 := \frac{c}{1+c}, \quad \rho_1 := \frac{c \frac{p}{q}}{1+c \frac{p}{q}}; \quad c > 0 \quad (127)$$

The functions defined by (125) and (126)–(127) are solutions of (24). Thus,  $\widehat{\mu}_0$  and  $\check{\nu}_0$  are reversible. Under  $\widehat{\mu}_0$ , we have a.s. that

$$\eta(x, i) = 1 \text{ for } x > 0, \quad \eta(x, i) = 0 \text{ for } x < 0; \quad i \in \{0, 1\} \quad (128)$$

which does not evolve in time. Hence under  $\widehat{\mu}_0$ ,  $\eta(-1, 0) + \eta(-1, 1)$  is conserved by the evolution, and conditioning  $\widehat{\mu}_0$  on  $\{\eta(-1, 0) + \eta(-1, 1) = 1\}$  yields a reversible measure satisfying (128), under which the vertical layer  $\{0\} \times \{0, 1\}$  contains a single particle located at  $i \in \{0, 1\}$  with probability  $p_i$  given by

$$\begin{aligned} p_0 &= \frac{\rho_0(1 - \rho_1)}{\rho_0(1 - \rho_1) + \rho_1(1 - \rho_0)} = \frac{q}{p + q} \\ p_1 &= \frac{\rho_1(1 - \rho_0)}{\rho_0(1 - \rho_1) + \rho_1(1 - \rho_0)} = \frac{p}{p + q} \end{aligned}$$

This measure is exactly  $\widehat{\nu}_0$ . Note that the process starting with (128) and a single particle on  $\{0\} \times \{0, 1\}$  reduces to the two state Markov-process followed by this single particle jumping between lanes 0 and 1, and  $\widehat{\nu}_0$  reduces to the unique invariant measure of this process (which is reversible).

For the measures  $\check{\nu}_0$  and  $\widehat{\nu}_0$ , the proof of extremality in (ii), (a) also applies here. Finally, by Theorem 2.2, (ii), (a), the above measures are (modulo horizontal translations) the only elements of  $Bl_2$ .

*Proof of (iii).* That  $\mathcal{R} = \mathcal{R}' = \emptyset$  follows from Definition 4.2, (ii) of Proposition 4.8 and (ii)–(iv) of Lemma 4.1. When  $d_1 = \lambda d_0$  and  $l_1 = \lambda l_0$  with  $\lambda$  close to 1, then  $d$  is close to  $1/2$ , and (54) follows from (46) and (ii) of Theorem 2.3 proven above.

*Proof of (iv)–(vi).* We first treat  $Bl_1$  with an argument common to the three situations. Indeed in (iv), resp. (v), (vi), by statement (iii) of Proposition 4.8, any element of  $Bl_1$  must belong to the set (59), resp. (62), (63). Conversely, elements of these sets are extremal invariant probability measures in each case. We detail the argument for  $\nu^{\perp, +\infty, n}$  in case (iv), all others are similar. Assume  $\nu^{\perp, +\infty, n} = (1 - \alpha)\nu^1 + \alpha\nu^2$ , with  $\nu^1, \nu^2 \in \mathcal{I}$  and  $\alpha \in (0, 1)$ . Since lane 0 is empty under  $\nu^{\perp, +\infty, n}$ , the same holds under  $\nu^1$  and  $\nu^2$ . Thus under the three measures, lane 1 evolves as an autonomous SEP with jump rate  $d_1$  to the right and  $l_1$  to the left, i.e., (4) with transition kernel (3) with  $(d, l) = (d_1, l_1)$ . The marginal of each measure on lane 1 is then an invariant measure for this SEP. Since the marginal of  $\nu^{\perp, +\infty, n}$  is an extremal (blocking) invariant measure for the SEP on lane 1, we have  $\nu^1 = \nu^2 = \nu^{\perp, +\infty, n}$ . Since lane 0 remains empty under the evolution,  $\nu^{\perp, +\infty, n}$  is indeed an invariant measure for the two-lane SEP.

We next treat  $\mathcal{R}$  and  $Bl_2$  with arguments specific to each situation.

*Proof of (iv), (a).* This corresponds to (112), that is, example 4.1. Note first that, since  $\gamma_0 > 0$  and  $\gamma_1 > 0$ ,

$$(\rho^-, \rho^+) = (0, 1); \quad (\rho^-, \rho^+) = (1, 2) \quad (129)$$

satisfy the conditions of Definition 4.3. These two shocks belong to  $\mathcal{B}_1$ .

If  $\gamma_0 = \gamma_1$ , we have  $G(\rho+1) = G(\rho)$  for every  $\rho \in [0, 1]$ ; thus  $(\rho^-, \rho^+) = (\rho, \rho+1)$  and  $(\rho^-, \rho^+) = (\rho+1, \rho)$  satisfy  $G(\rho^+) = G(\rho^-)$  for every  $\rho \in [0, 1]$ . Among such shocks different from the ones in (129), only  $(\rho^-, \rho^+) = (3/2, 1/2)$ , satisfies the variational equality in condition (ii) of Definition 4.3. Thus  $\mathcal{R}_0 \subset \{(3/2, 1/2); (0, 1); (1, 2); (0, 2)\}$ , and  $\mathcal{R} \subset \{(3/2, 1/2)\}$ .

If  $\gamma_0 \neq \gamma_1$ , then for every  $\rho \in (0, 1)$ , we have  $G(\rho+1) \neq G(\rho)$ . Thus  $\mathcal{R}_0$  contains no other shock than those in (129). Hence  $\mathcal{R} = \emptyset$ .

We finally prove that  $Bl_2$  is empty unless  $l_0 = l_1 = 0$ . Let  $\mu \in Bl_2$ , and

$$H_1(\eta) := \sum_{z \in \mathbb{Z}: z \leq 0} \eta(z, 1) - \sum_{z \in \mathbb{Z}: z > 0} [1 - \eta(z, 1)]$$

(which is well-defined since  $\mu$  is supported on  $\mathcal{X}_2$ ). The function  $H_1$  is constant along horizontal jumps and is increased by vertical jumps from lane 0 to lane 1. We claim and prove below that if  $l_0 > 0$ , there is a positive probability that by time 1, the leftmost particle on lane 0 has jumped to lane 1. This implies

$$\mathbb{E}_\mu [H_1(\eta_1) - H_1(\eta_0)] > 0 \quad (130)$$

which contradicts stationarity. Similarly, if  $l_1 > 0$ , there is a positive probability that by time 1, the leftmost particle on lane 1 has jumped to lane 0, which implies the reverse strict inequality in (130).

We now prove the claim when  $l_0 > 0$  (the proof in the case  $l_1 > 0$  is similar). In the sequel, on each lane, we call *active* those particles initially on the left of the rightmost hole and the next particle to the right of this hole (we also call active those sites where active particles are initially sitting). For  $x, y \in V$ , we say a Poisson process  $\mathcal{N}_{(x,y)}$  of the Harris construction is *attached* to some site  $z \in V$  if  $z \in \{x, y\}$ . We condition  $\mu$  on the number and positions of active particles on each lane. Denote respectively by  $x_0, y_0, x_1$  the initial positions of the leftmost particle on lane 0, the next particle on its right, and the leftmost particle on lane 1. We couple our two-lane SEP with a random walk on lane 0 starting from  $(x_0, 0)$ , that jumps to the right and left with respective rates  $d_0, l_0$  and is reflected at  $(y_0, 0)$ . The random walk is defined from the Harris system as follows: if its current position is  $(x, 0) \in V$ , at the first point of a Poisson process  $\mathcal{N}_{(x,0),(x+\varepsilon,0)}$  where  $\varepsilon \in \{-1, 1\}$ , it jumps to  $x + \varepsilon$ , except if  $x = y_0 - 1$  and  $\varepsilon = 1$ . Let  $x'_0 := \min(x_0, x_1 - 1, y_0 - 2)$ , and  $E_0$  denote the event that the random walk hits  $(x'_0, 0)$  for the first time before time 1/2 and stays there

at least until time 1 (if  $x'_0 = x_0$ ,  $E_0$  corresponds to the return time to  $(x_0, 0)$ ). This event has positive probability and depends only on the Poisson processes  $\mathcal{N}_{(x,x+1)}$  and  $\mathcal{N}_{(x+1,x)}$  for  $x'_0 - 1 \leq x \leq y_0 - 2$ . Let  $T_0$  denote the first time among all the following Poisson processes:

- (a)  $\mathcal{N}_{(y_0,0),(y_0-1,0)}$ ;
- (b)  $\mathcal{N}_{(x,0),(x,1)}$  for  $x'_0 < x \leq y_0 - 1$ ; and
- (c) the Poisson processes attached to active sites on lane 1;  $T_0$  is an exponential random variable. Consider the event

$$E'_0 := E_0 \cap \{T_0 > 1\} \\ \cap \{\mathcal{N}_{(x'_0,0),(x'_0,1)} \text{ has at least one point in the time interval } [1/2, 1]\}$$

On  $E_0$ , in the two-lane SEP starting from the conditioned measure, the particle initially at  $(x_0, 0)$  coincides with the random walk until it reaches  $(x'_0, 0)$ ; then its next motion is a jump from there to  $(x'_0, 1)$  before time 1; all this occurs before any particle initially on lane 1 has moved and before the particle initially at  $(y_0, 0)$  has moved. The three events of which  $E'_0$  is the intersection are independent, because they depend on disjoint sets of Poisson processes. It follows that  $E'_0$  has positive probability when starting from the conditioned measure, irrespective of the conditioning.

*Proof of (iv), (b).* Assume  $l_0 = l_1 = 0$ . The measure  $\nu^{\perp,i,j}$  is shown to be reversible as in (ii) above, since it is a product measure of the form (25) with

$$\rho_{x,0} = \mathbf{1}_{\{x>i\}}, \quad \rho_{x,1} = \mathbf{1}_{\{x>j\}}$$

for  $x \in \mathbb{Z}$ , and the above function  $\rho$  satisfies the reversibility conditions (24). The measure  $\nu^{\perp,i,j}$  is a Dirac measure, hence it is extremal in the set of probability measures on  $\mathcal{X}$ , thus *a fortiori* extremal in  $\mathcal{I}$ .

Now we prove that any element  $\mu$  of  $B\ell_2$  is of the form  $\nu^{\perp,i,j}$  for  $(i,j) \in \mathbb{B}$ . Indeed, since  $\mu$  is supported on  $\mathcal{X}_2$ , the random variable

$$n_0 := \inf\{z \in \mathbb{Z} : \eta(z, i) = 1, \quad \forall z \geq n_0, i \in \{0, 1\}\}$$

is  $\mu$ -a.s. finite, as well as the number of particles to the left of  $n_0$ . Since jumps are totally asymmetric both horizontally and vertically, conditioned on this number and on  $n_0$ , the process lives on a finite space, and its irreducible classes are singletons containing states  $\{\eta^{\perp,i,j}\}$  that can be reached from the initial state (indeed, states  $\eta^{\perp,i,j}$  are the only ones from which no transition is possible, and no return is possible from a state not belonging to this class). This implies that  $\mu$  is a mixture of the invariant measures  $\nu^{\perp,i,j}$ , and by extremality, it must be one of them.

*Proof of (v).* The flux function  $G$  has the form (112), that is, example 4.1. Since  $\gamma_1 < 0 < \gamma_0$ , the only pairs  $(\rho^-, \rho^+)$  satisfying the requirements of Definition 4.3 are  $(\rho^-, \rho^+) = (1, 0)$  and  $(\rho^-, \rho^+) = (1, 2)$ . These shocks belong

to  $\mathcal{B}_1$ , hence  $\mathcal{R} = \emptyset$ . We next prove that  $Bl_2$  is empty. Indeed by statement (i) of Proposition 4.8, a  $(\rho^-, \rho^+)$ -shock measure must satisfy condition (ii) of Definition 4.3. This is not the case for  $(\rho^-, \rho^+) = (0, 2)$  or  $(\rho^-, \rho^+) = (2, 0)$ , because by (112),  $G(0) = G(2)$  and  $\gamma_1 < 0 < \gamma_0$  implies that  $G$  is negative on  $(0, 1)$  and positive on  $(1, 2)$ .

*Proof of (vi).* The flux function  $G$  has the form (112), that is, example 4.1. The proof that  $\mathcal{R} = \emptyset$  is similar to the case  $\gamma_0 \neq \gamma_1$  in (iv), (a) (the fact that  $\gamma_0 = 0$  being irrelevant there). The proof that  $Bl_2$  is empty is similar to case (iv), (a) above, since  $\gamma_0 = 0$  implies  $l_0 > 0$ .  $\square$

#### 4.4 Proof of Theorem 2.4

*Proof of (0).* The proof of Theorem 2.1 requires only minor changes. First we can repeat the proof of Lemma 3.2. The only difference is that on a vertical layer  $\{z\} \times W$ , we now use the fact that  $L_v^z$  is the generator of a translation-invariant SEP on a torus, and  $\nu_\rho$  is a homogeneous product Bernoulli measure, which is thus invariant for  $L_v^z$ . The rest of the proof is exactly similar to that of Theorem 2.1. Note that here by (ii) of Lemma 3.3, a  $p$ -ordered pair of configurations is ordered, so we do not need analogues of Lemmas 3.4 and 3.5 in this context.

*Proof of (1).* (a). We can repeat the following steps of the proof of Theorem 2.2: Proposition 4.1 (leading to (101)–(103) with  $k \in \{1, \dots, n\}$  instead of  $k \in \{1, 2\}$  in (105), (104) is irrelevant here because  $p(\cdot, \cdot)$  is weakly irreducible), Corollary 4.1, Proposition 4.3 and Proposition 4.5. This yields that an extremal invariant measure that is not invariant by horizontal translations is a shock measure whose amplitude lies in  $[1, n] \cap \mathbb{Z}$ . Similarly to Proposition 4.5, we can prove that there are at most (up to translations)  $k$  shock measures of amplitude  $k$ . To further characterize possible shocks  $(\rho^-, \rho^+)$ , we consider the macroscopic flux function  $G$  defined by (107) and (109) in this setup. By definition (64) of  $\nu_\rho$ ,

$$G(\rho) = \left( \sum_{i=0}^{n-1} \gamma_i \right) \frac{\rho}{n} \left( 1 - \frac{\rho}{n} \right)$$

We can then repeat the proof of statement (i) of Proposition 4.8. Since the above function  $G$  is strictly concave and symmetric around  $\rho = n/2$ , shocks satisfying condition (ii) of Definition 4.3 are those specified in the theorem.

(b). The proof is similar to Theorem 2.3, (o).

(c) Stationarity of the product measure  $\nu^{\rho^c}$  is proved as in Lemma 3.2, observing that on vertical layers we have a periodic SEP for which a homogeneous product measure is invariant. Stationarity and extremality of the conditioned measure are proved as in Theorem 2.3, (ii).

*Proof of (2).* We can repeat with minor modifications the proof of statement (i) of Theorem 2.3. The microscopic current is as in (121). Summation there and in (123)–(124) is now over  $i \in W := \{0, \dots, n-1\}$ . In the latter two displays,

$\tilde{\rho}_i(\rho)$  is replaced by  $\rho/n$ .

*Proof of (3).* Let  $\mu \in \mathcal{I}_e$ . If  $\mu \in \mathcal{S}$ , the conclusion follows from (o). Otherwise, since the generator (3) is invariant by  $\tau'$ ,  $\mu' := \tau'\mu \in \mathcal{I}_e$ . By (1),  $\mu$  and  $\mu'$  are shock measures, and  $\mu' = \tau'\mu$  implies that they are  $(\rho^-, \rho^+)$ -shock measures for the same pair  $(\rho^-, \rho^+)$ . The proof of Proposition 4.4 carries over to the multilane model (notice indeed that under condition (iii'), the global kernel  $p(\cdot, \cdot)$  is weakly irreducible; thus when repeating the part of the proof of Proposition 4.1 that is used to derive Proposition 4.4, we always obtain (139), and do not need an analogue of step three). Hence, we have either  $\mu \leq \mu'$  or  $\mu' \leq \mu$ . Since  $\tau'$  is a periodic shift, this implies  $\mu' = \mu$ .

## 5 Proofs of intermediate results from Subsections 4.1–4.3

### 5.1 Proofs of intermediate results from Subsection 4.1

*Proof of Proposition 4.1.*

*Proof of (i).* Cases (101)–(103) are an adaptation of [4, Proposition 3.2], the main ingredients of which we recall in steps one and two below, whereas an additional argument (step three below) is required for (104). Let us fix  $T > 0$ .

*Step one.* Let  $\tilde{\lambda}_0$  denote the distribution on  $\mathcal{X} \times \mathcal{X}$  of the coupled configuration  $(\eta_0, \xi_0)$ , where  $\eta_0 \sim \mu$  and  $\xi_0 = \tau\eta_0$ . We denote by  $(\eta_t, \xi_t)_{t \geq 0}$  the coupled process starting from  $(\eta_0, \xi_0)$ . Define

$$\mathcal{R}_T = \{x \in V : -T \leq x(0) \leq T\} \quad (131)$$

Let  $\mathcal{N}_T$  be the number of discrepancies of  $(\eta_t, \xi_t)_{t \geq 0}$  that visit  $\mathcal{R}_T$  at any time in  $[\sqrt{T}, T]$ ,  $\mathcal{N}_T^{in}$  the number of these starting from  $[-(1+\sigma)T, (1+\sigma)T]$  (where  $\sigma$  is the constant in Proposition 3.1), and  $\mathcal{N}_T^{out}$  the number of these starting outside this interval. Adapting the proof in [4, Proposition 2.5] to our model yields

$$\mathbb{E}^{\tilde{\lambda}_0}(\mathcal{N}_T) = o(T) \quad \text{when } T \rightarrow \infty. \quad (132)$$

The proof of [4, Proposition 2.5] used only the following properties of single-lane SEP, which hold also for our two-lane model.

(a) The finite propagation property (Proposition 3.1) is used to show

$$\mathbb{E}^{\tilde{\lambda}_0}(\mathcal{N}_T^{out}) = o(T) \quad (133)$$

(b) the invariance of the generator with respect to horizontal translations, and (c) the characterization theorem (here Theorem 2.1) for stationary measures invariant with respect to such translations: these are used to show

$$\mathbb{E}^{\tilde{\lambda}_0}(\mathcal{N}_T^{in}) = o(T) \quad (134)$$

*Step two.* For  $x, y \in \mathbb{Z}$ , let  $\mathcal{N}_T^{x,y}$  denote the number of discrepancies that visit either  $x$  or  $y$  and disappear during the time interval  $[\sqrt{T}, T]$ . Recall the definition (94) of  $E_{x,y}$ , and define

$$e_{x,y} := \inf_{(\eta,\xi) \in E_{x,y}} \mathbb{P}_{(\eta,\xi)} \text{ (one of the discrepancies at } x \text{ and } y \text{ has coalesced by time } 1)$$

where  $\mathbb{P}_{(\eta,\xi)}$  denotes the law of the coupled process starting from  $(\eta, \xi)$ . The same argument as in [4, Lemma 3.1] shows that

$$e_{x,y} > 0 \text{ if } x \rightarrow_p y \text{ or } y \rightarrow_p x \quad (135)$$

Let

$$\tilde{\lambda}^T = \frac{1}{T - \sqrt{T}} \int_{\sqrt{T}}^T \tilde{\lambda}_0 \tilde{S}_t dt \quad (136)$$

and let  $\tilde{\lambda} = \lim_{i \rightarrow \infty} \tilde{\lambda}^{T_i}$  be a subsequential weak limit. Then

$$\tilde{\lambda} \in \tilde{\mathcal{I}} \quad (137)$$

Since  $\mu \in \mathcal{I}_e$  and the two-lane SEP is translation-invariant in the  $\mathbb{Z}$ -direction, we have  $\tau\mu \in \mathcal{I}_e$ . Since  $\tilde{\lambda}_0$  has marginals  $\mu \in \mathcal{I}$  and  $\tau\mu \in \mathcal{I}$ ,  $\tilde{\lambda}$  has marginals  $\mu$  and  $\tau\mu$ . As in [4, Proposition 3.2], (132) and the strong Markov property yield respectively the following equality and inequality:

$$0 = \liminf_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E}_{\tilde{\lambda}_0} (\mathcal{N}_T^{x,y}) \geq e_{x,y} \tilde{\lambda}(E_{x,y}) \quad (138)$$

Combining (135) and (138), we obtain

$$\tilde{\lambda} \{(\eta, \xi) \text{ is } p\text{-ordered}\} = 1. \quad (139)$$

that is, (95). In the case  $q > 0$ , by (i) of Lemma 3.4, (139) implies (98). When  $q = 0$ , we only arrive at (99).

*Step three.* Assuming  $q = 0$ , we prove below that

$$\tilde{\lambda}(E_{><} \setminus E_{\bowtie}) = 0 \quad (140)$$

This together with (99) implies

$$\tilde{\lambda}(E_0 \cup E_1 \cup E_2 \cup E_{\bowtie}) = 1 \quad (141)$$

Moreover, each of the events in (141) is invariant under the coupled dynamics. Then using the fact that  $\mu$  and  $\tau\mu$  lie in  $\mathcal{I}_e$ , we can conclude as in [4, Proposition 3.2] that  $\tilde{\lambda}$  actually satisfies one of the conditions (101)–(104).

We now prove the claim (140). Recall the random variables  $X, Y$  defined by (88)–(89). Then, by conditions (ii)–(iii) of Definition 4.1,

$$E_{><} \setminus E_{\bowtie} \subset \bigcup_{x,y \in \mathbb{Z}: x < y} E'_{\bowtie,x,y} \cup \bigcup_{x,y \in \mathbb{Z}: x < y} F'_{\bowtie,x,y} \quad (142)$$

where, for  $x < y$ ,

$$\begin{aligned} E'_{\triangleright, x, y} &:= E_{><} \cap \{X = x, Y = y\} \\ &\quad \cap \{\text{There are at least } y - x \text{ holes on lane 1 to the right of } x\} \\ F'_{\triangleright, x, y} &:= E_{><} \cap \{X = x, Y = y\} \\ &\quad \cap \{\text{There are at least } y - x \text{ coupled particles on lane 0 to the} \\ &\quad \text{left of } y\} \end{aligned}$$

We claim that  $\tilde{\lambda}(E'_{\triangleright, x, y}) = \tilde{\lambda}(F'_{\triangleright, x, y}) = 0$  which, in view of (142), implies (140). On  $E'_{\triangleright, x, y}$ , there is a possible sequence of moves with positive probability that brings the discrepancy from  $(x, 1)$  to  $(y, 1)$  that is  $p$ -connected to  $(y, 0)$ . Indeed one can construct an event on the Harris system prescribing that on the time interval  $[0, 1]$ , the corresponding Harris clocks will ring in the desired order while no other clock rings. Hence, by stationarity,  $\tilde{\lambda}(E'_{\triangleright, x, y}) > 0$  implies  $\tilde{\lambda}(E_{(y, 0), (y, 1)}) > 0$ , in contradiction with (139). Similarly on  $F'_{\triangleright, x, y}$ , there is a possible sequence of moves with positive probability that brings the discrepancy from  $(y, 0)$  to  $(x, 0)$  that is  $p$ -connected to  $(x, 1)$ .

*Proof of (ii).* Since the coupled configurations  $\eta$  and  $\xi$  are a.s. ordered under  $\tilde{\lambda}$ , all discrepancies (if any) are of the same type (that is  $\eta$  or  $\xi$  discrepancies), so no coalescence occurs. Thus, recalling the definition of  $D(\eta, \xi)$  from (91), the sets  $A_k := \{D(\eta, \xi) = k\}$ , with  $k \in \mathbb{N} \cup \{+\infty\}$ , are invariant under the dynamics. Hence,

$$\tilde{\lambda} = \sum_{k \in \mathbb{N} \cup \{+\infty\}: \tilde{\lambda}(A_k) > 0} \tilde{\lambda}(A_k) \tilde{\lambda}_k \quad (143)$$

where  $\tilde{\lambda}_k := \tilde{\lambda}(\cdot | A_k) \in \tilde{\mathcal{I}}$ . Since  $\mu$  and  $\tau\mu$  are extremal elements of  $\mathcal{I}$ , for each  $k$  such that  $\tilde{\lambda}(A_k) > 0$ ,  $\tilde{\lambda}_k$  has marginals  $\mu$  and  $\tau\mu$ . Assume for instance that  $\tilde{\lambda}$  (and thus  $\tilde{\lambda}_k$ ) satisfies (101). Then

$$\begin{aligned} \tilde{\lambda}_k[D_{m, n}(\eta, \xi)] &= \tilde{\lambda}_k \left\{ \sum_{x \in V: m \leq x(0) \leq n} [\xi(x) - \eta(x)] \right\} \\ &= \mu[\bar{\eta}(n+1)] - \mu[\bar{\eta}(m)] \in [0, 2] \end{aligned} \quad (144)$$

Letting  $m \rightarrow -\infty$  and  $n \rightarrow +\infty$ , by monotone convergence, and because  $\tilde{\lambda}_k$  is supported on  $A_k$ , we obtain

$$k = \tilde{\lambda}_k[D(\eta, \xi)] \in \{0, 1, 2\} \quad (145)$$

Notice that the right-hand side of (144), and thus also its limit, depends only on  $\mu$ . Hence  $k$  depends only on  $\mu$ . This shows that  $\tilde{\lambda} = \tilde{\lambda}_k$  for a unique  $k \in \{0, 1, 2\}$ . Since we are in case (101),  $k = 0$  would yield a contradiction. Thus  $k \in \{1, 2\}$ . Dealing with the case (102) is similar.  $\square$



*Proof of Corollary 4.1.* (i) The marginals of  $\tilde{\lambda}$  are  $\mu$  and  $\tau\mu$ , thus  $\mu \leq \tau\mu$  in case (101), or  $\tau\mu \leq \mu$  in case (102), or  $\tau\mu = \mu$  in case (103).

(ii) Given the assumptions, the limits (106) exist and satisfy  $\tau\hat{\mu}_\pm = \hat{\mu}_\pm$ , that is  $\hat{\mu}_\pm \in \mathcal{S}$ . Besides, we have  $\hat{\mu}_\pm \in \mathcal{I}$ . Indeed if  $f$  is a local function on  $\mathcal{X}$ ,

$$\begin{aligned} \int_{\mathcal{X}} Lf(\eta) d\hat{\mu}_\pm(\eta) &= \lim_{n \rightarrow \pm\infty} \int_{\mathcal{X}} Lf(\eta) d(\tau_n \hat{\mu})(\eta) \\ &= \lim_{n \rightarrow \pm\infty} \int_{\mathcal{X}} L[\tau_n f](\eta) d\hat{\mu}(\eta) = 0 \end{aligned}$$

where we used that  $L$  commutes with the shift and  $\hat{\mu} \in \mathcal{I}$ . The last equality in (106) follows from Theorem 2.1 and (38).  $\square$

*Proof of Proposition 4.2.* Recall from Proposition 4.1 that (104) may only occur when  $q = l_0 = l_1 = 0$ . Hence the dynamics of horizontal jumps on each lane is a TASEP, and these TASEP's interact through vertical jumps from lane 0 to lane 1. Let  $(\eta_0, \xi_0) = (\eta, \xi) \sim \tilde{\lambda}$ , where  $\tilde{\lambda}$  is the measure in Proposition 4.1. We couple the process  $\eta$  through basic coupling with a process  $\zeta$  such that for every  $i \in \{0, 1\}$ ,  $\zeta_t^i$  is a TASEP on lane  $i$  starting from configuration  $\eta_0^i = \eta^i$  (with jumping rates  $d_i, l_i$ ). Then one has, for every  $t \geq 0$ ,

$$\eta_t^1 \geq \zeta_t^1, \quad \eta_t^0 \leq \zeta_t^0 \quad (146)$$

Indeed, to derive the first inequality in (146), note that at certain random times belonging to one of the Poisson processes  $\mathcal{N}_{(z,0),(z,1)}$ , a new particle may appear (following a jump of a particle from lane 0) at site  $z \in \mathbb{Z}$  in  $\eta_t^1$  that does not appear in  $\zeta_t^1$ . On the other hand, between such times, both processes evolve as coupled TASEP's on lane 1, whose order is preserved by attractiveness property (86). A similar argument holds for the second inequality in (146).

For  $t > 0$ , define the empirical measures

$$M_t^i := \frac{1}{t} \int_0^t \mu_s^i ds, \quad N_t^i := \frac{1}{t} \int_0^t \nu_s^i ds \quad (147)$$

where  $\mu_t^i$  denotes the law of  $\eta_t^i$  and  $\nu_t^i$  that of  $\zeta_t^i$ . Since  $\mu \in \mathcal{I}$ ,  $\mu_t = \mu$  does not depend on  $t$ , hence  $M_t^i =: \mu^i$  does not depend on  $t$  and is the marginal of  $\mu$  on lane  $i$ .

Let  $t_n \uparrow +\infty$  be a subsequence along which  $N_{t_n}^i \rightarrow \nu_\infty^i$ , where  $\nu_\infty^i$  is an invariant measure for TASEP. Since  $\eta \bowtie \xi$ , there is a random variable  $N \in \mathbb{Z} \cup \{-\infty\}$  such that  $\zeta_0^i(x) = \eta_0^i(x) = 1$  for  $i \in \{0, 1\}$  and  $x \in \mathbb{Z}$  with  $x \geq N$ . By TASEP dynamics, this remains true at time  $t$  for  $\zeta_t^i$  with the same  $N$ . Thus if  $\zeta$  is a random configuration with distribution  $\nu_\infty^i$ , we a.s. have

$$\zeta(x) = 1, \quad \forall x \geq N' \quad (148)$$

where  $N'$  has the same law as  $N$ . As  $\nu_\infty^i$  is invariant for TASEP by [8, Theorem 1.4], it is a mixture of Bernoulli and blocking measures. But by (148), the only possible Bernoulli measure is the one with density 1. Thus there exists a random variable  $N_i \in \mathbb{Z} \cup \{-\infty\}$  such that the random configuration  $\zeta_\infty^i := \eta_{N_i}^*$  has distribution  $\nu_\infty^i$ .

By (146),  $\mu^1 \geq \nu_\infty^1$  and  $\mu^0 \leq \nu_\infty^0$ . It follows from the above that there exist random variables  $M_0$  and  $M_1$  with values in  $\mathbb{Z} \cup \{-\infty\}$  such that  $\eta_0^1 \geq \eta_{M_1}^*$  and  $\eta_0^0 \leq \eta_{M_0}^*$  a.s. Since  $\eta \bowtie \xi$ , the dynamics of  $(\eta_t^1)_{t \geq 0}$  can only possibly create a finite number of particles (from lane 0) to the left of  $M_1$  and move these particles to the right until they pile up and get blocked. The same argument applies to holes in  $\eta^0$ , since the dynamics of holes is a two-lane TASEP with jumps to the left and from lane 1 to lane 0. Thus there exist random variables  $-\infty \leq M'_1 \leq M_1 < +\infty$  and  $-\infty < M_0 \leq M'_0 \leq +\infty$  such that

$$\lim_{t \rightarrow +\infty} \eta_t^1 = \eta_{M'_1}^*, \quad \lim_{t \rightarrow +\infty} \eta_t^0 = \eta_{M'_0}^* \quad (149)$$

Since particles can jump from lane 0 to lane 1 but not the other way, the dynamics imposes

$$M'_0 \geq M'_1 \quad (150)$$

The limits in (149) imply that  $\eta_t$  converges in law to the distribution of the random configuration  $\eta_\infty$  defined by  $\eta_\infty^i = \eta_{M'_i}^*$  for  $i \in \{0, 1\}$ , that is (cf. (60))  $\eta_\infty = \eta^{\perp, M'_0, M'_1}$ . By stationarity,  $\mu$  is the distribution of this configuration; hence, recalling the definition of  $\mathbb{B}$  above (60),

$$\mu = \int_{\mathbb{B}} \nu^{\perp, i, j} dm(i, j) \quad (151)$$

where  $m(di, dj)$  denotes the law of  $(M'_0, M'_1)$ . This with (59)–(61) implies that  $\mu$  is a mixture of the measures in  $Bl$ .  $\square$

*Proof of Proposition 4.3.*

*Proof of (i).* Without loss of generality, we may assume  $\mu \leq \tau\mu$ . Since  $\mu \leq \mu_+$  (where  $\mu_+$  is defined as in Corollary 4.1) and  $\mu \neq \mu_+$  (because we are not in case (103)), by [9, Proposition 2.14 in Chapter VIII] there exists a coupling measure  $\tilde{\mu}(d\eta, d\xi)$  with marginals  $\mu(d\eta)$  and  $\mu_+(d\xi)$ , such that

$$\tilde{\mu}((\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta < \xi) = 1 \quad (152)$$

and which is invariant for the coupled process.

For  $n, m \in \mathbb{Z}$  such that  $m \leq n$ , and  $\xi \in \mathcal{X}$ , we set

$$M_{m,n}(\xi) := \frac{1}{n - m + 1} \sum_{x \in V: m \leq x(0) \leq n} \xi(x)$$

and simply write  $M_n(\xi)$  when  $m = 1$ . Because  $\mu_+$  is a mixture of Bernoulli measures, by the ergodic theorem, the limit

$$M(\xi) := \lim_{n \rightarrow +\infty} M_n(\xi) = \lim_{n \rightarrow +\infty} M_{-n,n}(\xi) \quad (153)$$

exists  $\tilde{\mu}$ -a.s. The distribution of  $M(\xi)$  is exactly  $\gamma^+$ . Besides,  $M(\xi)$  is a conserved quantity for the dynamics of the stationary coupled process  $(\eta_t, \xi_t)_{t \geq 0}$  starting from  $\tilde{\mu}(d\eta_0, d\xi_0)$ . Indeed, by the finite propagation property (Proposition 3.1),

$$\begin{aligned} \frac{2n+1-4\lfloor\sigma t\rfloor}{2n+1} M_{-n+2\lfloor\sigma t\rfloor, n-2\lfloor\sigma t\rfloor}(\xi_0) &\leq M_{-n,n}(\xi_t) \\ &\leq \frac{2n+1+4\lfloor\sigma t\rfloor}{2n+1} M_{-n-2\lfloor\sigma t\rfloor, n+2\lfloor\sigma t\rfloor}(\xi_0) \end{aligned}$$

with probability greater than  $1 - e^{-Cn}$ . Letting  $n \rightarrow +\infty$  yields

$$M(\xi_t) = M(\xi_0) \quad (154)$$

It follows that for every  $\rho$  in the support of  $\gamma^+$ , the conditioned measure

$$\tilde{\mu}_\rho(d\eta, d\xi) := \tilde{\mu}((d\eta, d\xi) | M(\xi) = \rho) \quad (155)$$

is invariant for the coupled process. Indeed, for every bounded function  $f$  on  $\mathcal{X} \times \mathcal{X}$  and every bounded measurable function  $g$  on  $[0, 2]$ ,

$$\begin{aligned} \int_{[0,2]} \langle \tilde{\mu}_\rho, \tilde{S}_t f \rangle g(\rho) \gamma^+(d\rho) &= \tilde{\mathbb{E}}_{\tilde{\mu}} [f(\eta_t, \xi_t) g(M(\xi_0))] \\ &= \tilde{\mathbb{E}}_{\tilde{\mu}} [f(\eta_t, \xi_t) g(M(\xi_t))] \\ &= \tilde{\mathbb{E}}_{\tilde{\mu}} [f(\eta_0, \xi_0) g(M(\xi_0))] \\ &= \int_{[0,2]} \langle \tilde{\mu}_\rho, f \rangle g(\rho) \gamma^+(d\rho) \end{aligned}$$

In the above display, the first and last equality follow from definition (155), the second one from (154), and the third one from stationarity.

Hence, the  $\eta$ -marginal of  $\tilde{\mu}_\rho$ , that is  $\mu_\rho(d\eta) := \tilde{\mu}(d\eta | M(\xi) = \rho)$  is invariant for  $L$ . Since

$$\mu = \int_{[0,2]} \mu_\rho \gamma^+(d\rho), \quad (156)$$

by extremality of  $\mu$ , we must have  $\mu_\rho = \mu$  for  $\gamma^+$ -a.e.  $\rho \in [0, 2]$ . This means that that under  $\tilde{\mu}(d\eta, d\xi)$ ,  $\eta$  is independent of  $M(\xi)$ .

Now we consider  $A, B, A', B' \in \mathbb{R}$  such that  $A$  lies in the support of  $\gamma^+$  and

$B < B' < A' < A$ . Let  $f, g$  be nondecreasing continuous functions on  $\mathcal{X}$  supported respectively on  $[A', +\infty)$  and  $(-\infty, B']$ , taking constant value 1 respectively on  $[A, +\infty)$  and  $(-\infty, B]$ . By (152), (153), and independence of  $M_n(\eta)$  and  $M(\xi)$ , the following holds under  $\tilde{\mu}$ :

$$\begin{aligned} 0 = \tilde{\mathbb{E}}_{\tilde{\mu}} [f(M_n(\eta))g(M_n(\xi))] &= \tilde{\mathbb{E}}_{\tilde{\mu}} [f(M_n(\eta))g(M(\xi))] + \varepsilon_n \\ &= \tilde{\mathbb{E}}_{\tilde{\mu}} [f(M_n(\eta))] \tilde{\mathbb{E}}_{\tilde{\mu}} [g(M(\xi))] + \varepsilon_n \end{aligned}$$

for some sequence  $\varepsilon_n \rightarrow 0$ . It follows that

$$\lim_{n \rightarrow +\infty} \tilde{\mu}(M_n(\eta) > A) \tilde{\mu}(M(\xi) < B) = 0$$

Choosing  $B$  strictly larger than the infimum of the support of  $\gamma^+$  yields

$$\lim_{n \rightarrow +\infty} \tilde{\mu}(M_n(\eta) > A) = 0$$

It follows that

$$\limsup_{n \rightarrow +\infty} \tilde{\mathbb{E}}_{\tilde{\mu}} [M_n(\eta)] \leq A \tag{157}$$

Set

$$\bar{\mu}_n := \frac{1}{n} \sum_{x=1}^n \tau_x \mu$$

so that (157) also writes

$$\limsup_{n \rightarrow +\infty} \int_{\mathcal{X}} \eta(0) d\bar{\mu}_n(\eta) \leq A \tag{158}$$

On the other hand, by Proposition 4.1,  $\bar{\mu}_n \rightarrow \mu_+$ , thus

$$\int_{\mathcal{X}} \eta(0) d\mu_+(\eta) = \int_{[0,2]} \rho d\gamma^+(\rho) \leq A$$

for every  $A$  in the support of  $\gamma^+$ . Hence  $\gamma^+ = \delta_{\rho^+}$  for some  $\rho^+ \in [0, 2]$ .

*Proof of (ii).* Assume for instance  $\rho^- < \rho^+$ , the other case being similar. The equality (144) yields (recall that  $\tilde{\lambda} = \tilde{\lambda}_k$  for  $k \in \{0, 1, 2\}$ , cf. (145))

$$\tilde{\lambda}[D(\eta, \xi)] = k = \lim_{n \rightarrow +\infty} \mu[\eta(n)] - \lim_{m \rightarrow -\infty} \mu[\eta(m)] = \rho^+ - \rho^-$$

□

*Proof of Proposition 4.4.* The proof of (ii) is similar to that of Proposition 4.2. We prove (i) and (iii) below.

*Proof of (i), step one.* We show that if  $\tilde{\lambda}_0 \in \tilde{\mathcal{I}}_e$  is a coupling of  $\nu$  and  $\nu'$  (that exists by [9, Proposition 2.14 in Chapter VIII]), then

$$\int_{\mathcal{X} \times \mathcal{X}} \left( \sum_{x \in \mathbb{Z}, i \in W, |x| \leq T} |\eta^i(x) - \xi^i(x)| \right) d\tilde{\lambda}_0(\eta, \xi) = o(T), \quad \text{as } T \rightarrow +\infty \quad (159)$$

Let (recall definition (91))

$$\tilde{\lambda}_T^\pm := \frac{1}{|[-T, T] \cap \mathbb{Z}^\pm|} \sum_{x \in \mathbb{Z}^\pm: |x| \leq T} \tau_x \tilde{\lambda}_0,$$

$$A_l(\eta, \xi) := \frac{1}{2(2l+1)} \sum_{y \in \mathbb{Z}, i \in W: |y| \leq l} |\eta^i(y) - \xi^i(y)| = \frac{1}{2(2l+1)} D_{-l, l}(\eta, \xi)$$

$$B_l(\eta, \xi) := \left| \frac{1}{2(2l+1)} \sum_{y \in \mathbb{Z}: i \in W: |y| \leq l} \eta^i(y) - \frac{1}{2(2l+1)} \sum_{y \in \mathbb{Z}: i \in W: |y| \leq l} \xi^i(y) \right|$$

Every subsequential weak limit  $\tilde{\lambda}_\infty^\pm$  of the family  $(\tilde{\lambda}_T^\pm)_{T \geq 0}$  (which is tight as it lives on a compact space) lies in  $\tilde{\mathcal{I}} \cap \tilde{\mathcal{S}}$ . Thus by Proposition 3.2 and Lemma 3.5, it is supported on  $E_- \cup E_+$  (see (96)–(97)), where  $A_l = B_l$ . The desired conclusion (159) is equivalent to having, for any subsequential limit  $\tilde{\lambda}_\infty$ ,

$$\begin{aligned} 0 &= \lim_{l \rightarrow +\infty} \lim_{T \rightarrow +\infty} \int_{\mathcal{X} \times \mathcal{X}} \left( \frac{1}{T} \sum_{x \in \mathbb{Z}^\pm, |x| \leq T} \tau_x A_l(\eta, \xi) \right) d\tilde{\lambda}_0(\eta, \xi) \\ &= \lim_{l \rightarrow +\infty} \int_{\mathcal{X} \times \mathcal{X}} B_l(\eta, \xi) d\tilde{\lambda}_\infty^\pm(\eta, \xi) \end{aligned} \quad (160)$$

By definition (42) of shock measures,  $\tilde{\lambda}_\infty^\pm$  has marginals  $\nu_{\rho^\pm}$ . It follows that under  $\tilde{\lambda}_\infty^\pm$ , the spatial averages in  $B_l(\eta, \xi)$  both converge in probability and (being bounded by 2) in  $L^1$  to  $\rho^\pm$ , thus implying the limits in (160).

*Proof of (i), step two.* We now adapt the proof of Proposition 4.1, defining  $\mathcal{N}_T$ ,  $\mathcal{N}_T^{in}$  and  $\mathcal{N}_T^{out}$  as we did there, and replacing the initial distribution  $\tilde{\lambda}_0$  defined there by the one considered in the first step of the current proof. In the first step of the proof of Proposition 4.1, we similarly derive (133) from Proposition 3.1, whereas we can now obtain (134) as a consequence of (159). Steps two and three are unchanged and yield (139), where the measure  $\tilde{\lambda}$  now coincides with  $\tilde{\lambda}_0$ , because the latter is invariant. Hence, we obtain (98) if  $q > 0$ , or (141) if  $q = 0$ . By extremality, this implies that  $\tilde{\lambda}_0$  satisfies one of (101)–(104).

*Proof of (iii).* The proof is similar to that of Proposition 4.1, statement (ii).

The only differences lie in the following points, assuming for instance that the conclusion of (i) is (101). First, the second line of (144) is now

$$\sum_{x \in V: m \leq x(0) \leq n} [\nu'(\xi(x)) - \nu(\xi(x))] \in [0, +\infty] \quad (161)$$

which depends only on  $\nu, \nu'$ . Next, in (145),  $k$  can be a priori any value in  $\mathbb{N} \cup \{+\infty\}$  instead of only 0, 1, 2.  $\square$

*Proof of Proposition 4.5.* For two ordered probability measures  $\gamma, \gamma'$  on  $\mathcal{X}$ , let

$$\Delta(\gamma, \gamma') := \sum_{x \in V} |\gamma(\eta(x)) - \gamma'(\eta(x))| \in [0, +\infty] \quad (162)$$

Note that  $\Delta(\gamma, \gamma')$  satisfies the three following properties:

$$\Delta(\gamma, \gamma') = 0 \text{ if and only if } \gamma = \gamma' \quad (163)$$

If  $\tilde{\gamma}$  is an ordered coupling of  $\gamma$  and  $\gamma'$ , we have

$$\Delta(\gamma, \gamma') = \int_{\mathcal{X} \times \mathcal{X}} D(\eta, \xi) d\tilde{\gamma}(\eta, \xi) \quad (164)$$

If a probability measure  $\gamma''$  on  $\mathcal{X}$  is such that  $\gamma \leq \gamma' \leq \gamma''$  or  $\gamma'' \leq \gamma' \leq \gamma$ , then

$$\Delta(\gamma, \gamma'') = \Delta(\gamma, \gamma') + \Delta(\gamma', \gamma'') \quad (165)$$

*Proof of (i).* Without loss of generality, we assume  $\rho^- < \rho^+$ . For  $n \in \mathbb{Z}$ , let us denote  $\nu_n := \tau_n \nu$ . We can apply Proposition 4.1 to  $\nu$  and rule out the case (104) by assumption and Proposition 4.2. Thus by (i) of Corollary 4.1,  $\nu_n \leq \nu_{n+1}$  for all  $n \in \mathbb{Z}$ . We can also exclude the case  $k = 2$  by (ii) of Proposition 4.3 because  $|\rho^+ - \rho^-| = 1$ ; and the case (103) because  $\rho^- \neq \rho^+$ . Thus by (164), (ii) of Proposition 4.1 and (ii) of Proposition 4.3,

$$\Delta(\nu_{n-1}, \nu_n) = 1, \quad \forall n \in \mathbb{Z} \quad (166)$$

By (i) of Proposition 4.4, for  $n \in \mathbb{Z}$ , there exists a coupling  $\tilde{\nu}_n(d\eta, d\xi)$  of  $\nu_n(d\eta)$  and  $\nu'(d\xi)$  that satisfies one of the properties (101)–(104) of Proposition 4.1. By assumption and (ii) of Proposition 4.4, we can rule out (104). Thus  $\nu_n$  and  $\nu'$  are ordered. Besides, (iii) of Proposition 4.4 and (164) imply

$$\Delta(\nu_n, \nu') \in \mathbb{N}, \quad \forall n \in \mathbb{Z} \quad (167)$$

Let  $S := \{n \in \mathbb{Z} : \nu' \leq \nu_n\}$ . We claim that  $S$  is non-empty and bounded from below. Indeed if  $S$  were empty, since  $\nu$  a  $(\rho^-, \rho^+)$ -shock measure (cf. definition (42)),  $\nu' \geq \nu_n$  and  $n \rightarrow +\infty$  would imply  $\nu' \geq \nu_{\rho^+}$ ; if  $S$  were not bounded from below,  $n \rightarrow -\infty$  along a subsequence where  $\nu' \leq \nu_n$  would imply  $\nu' \leq \nu_{\rho^-}$ . Both conclusions would contradict  $\nu'$  being a  $(\rho^-, \rho^+)$ -shock measure. We set  $n_0 := \min(S)$ , thus

$$\nu_{n_0-1} < \nu' \leq \nu_{n_0} \quad (168)$$

By (168) and (165),

$$\Delta(\nu_{n_0-1}, \nu_{n_0}) = \Delta(\nu_{n_0-1}, \nu') + \Delta(\nu', \nu_{n_0}) \quad (169)$$

By (168), (163) and (167), the first term on the right-hand side of (169) is a nonzero integer; thus by (166) for  $n = n_0$ , the second term is zero, and the conclusion follows from (163).

*Proof of (ii).* We can consider  $n_0$  and the couplings of  $\nu_{n_0}$  with  $\nu'$  and  $\nu_{n_0-1}$  with  $\nu_{n_0}$  as in (i). Let  $\nu''$  be a  $(\rho^-, \rho^+)$ -shock measure. For  $n \in \mathbb{Z}$ , we can also apply (i) of Proposition 4.4 to  $\nu''_n := \tau_n \nu''$  and  $\nu$ , and rule out case (104), since by assumption we exclude (48); thus these measures are ordered. The same holds for  $\nu''_n$  and  $\nu'$ . Similarly to  $n_0$ , we can then define  $n_1 \in \mathbb{Z}$  such that

$$\nu''_{n_1-1} < \nu_{n_0-1} \leq \nu''_{n_1} \quad (170)$$

Property (167) holds, but instead of (166), (iii) of Proposition 4.4 now implies

$$\Delta(\nu_{n-1}, \nu_n) = 2 = \Delta(\nu''_{n-1}, \nu''_n), \quad \forall n \in \mathbb{Z} \quad (171)$$

Since  $\nu'$  is not a translate of  $\nu$ , both terms on the right-hand side of (169) are now nonzero integers. The first equality in (171) for  $n = n_0$ , combined with (169), then yields

$$\Delta(\nu_{n_0-1}, \nu') = \Delta(\nu', \nu_{n_0}) = 1 \quad (172)$$

We now distinguish the following cases.

(1) If  $\nu''_{n_1} \geq \nu_{n_0}$ , by (170) we have  $\nu''_{n_1-1} \leq \nu_{n_0-1} \leq \nu_{n_0} \leq \nu''_{n_1}$ ; by (165),

$$\Delta(\nu''_{n_1-1}, \nu''_{n_1}) = \Delta(\nu_{n_1'-1}, \nu_{n_0-1}) + \Delta(\nu_{n_0-1}, \nu_{n_0}) + \Delta(\nu_{n_0}, \nu''_{n_1})$$

From (170), (171) with  $n = n_1$ , and (163), we obtain  $\nu''_{n_1} = \nu_{n_0}$ .

(2) If  $\nu_{n_0-1} \leq \nu''_{n_1} \leq \nu_{n_0}$ , we distinguish whether (a)  $\nu_{n_0-1} \leq \nu''_{n_1} \leq \nu'$  or (b)  $\nu' \leq \nu''_{n_1} \leq \nu_{n_0}$ . In the former case, (165) and (172) yield

$$1 = \Delta(\nu_{n_0-1}, \nu') = \Delta(\nu_{n_0-1}, \nu''_{n_1}) + \Delta(\nu''_{n_1}, \nu')$$

and one of the terms on the r.h.s. must be 0. Case (b) is similar.  $\square$

## 5.2 Proofs of intermediate results from Subsections 4.2–4.3

*Proof of Proposition 4.7.*

*Proof of (o).* This follows from (110), (111), and Lemma 3.1.

*Proof of (i).* For the following, we rely on the expression for  $G$  given in (110).

Therefore  $G \geq 0$ . In cases (49) and (50),  $G$  is identically 0 (the former follows from example 4.2). We henceforth exclude these cases. If  $q = 0$ , the conclusion follows from example 4.1. If  $q > 0$ ,  $G$  is continuously differentiable. First,  $G'$  vanishes at least once because  $G(0) = G(2) = 0$ , cf. (o). Next,

$$\begin{aligned} G(\rho) &= (\gamma_0 + \gamma_1) \frac{\rho}{2} \left(1 - \frac{\rho}{2}\right) + (\gamma_0 - \gamma_1)(1 - \rho)\varphi(\rho) \\ &\quad - (\gamma_0 + \gamma_1)\varphi(\rho)^2, \quad \text{with} \end{aligned} \quad (173)$$

$$\begin{aligned} \varphi(\rho) &= \frac{1}{2} \left(\frac{r+1}{r-1}\right) \left(1 - \sqrt{\psi(\rho)}\right) \text{ if } r \neq 1 \\ &= 0 \text{ if } r = 1 \end{aligned} \quad (174)$$

$$\psi(\rho) = 1 + \left(\frac{r-1}{r+1}\right)^2 \rho(\rho-2) \quad (175)$$

Note that  $\psi(\rho) \leq 1$ . We then compute

$$\psi'(\rho) = \left(\frac{r-1}{r+1}\right)^2 2(\rho-1) \quad (176)$$

$$\varphi'(\rho) = -\frac{1}{2} \left(\frac{r-1}{r+1}\right) \frac{(\rho-1)}{\sqrt{\psi(\rho)}} \quad (177)$$

$$\varphi''(\rho) = -2r \left(\frac{r-1}{r+1}\right)^3 \psi(\rho)^{-3/2} \quad (178)$$

$$\varphi^{(3)}(\rho) = 6r(\rho-1) \left(\frac{r-1}{r+1}\right)^5 \psi(\rho)^{-5/2} \quad (179)$$

$$\begin{aligned} G'(\rho) &= (\gamma_0 + \gamma_1) \frac{1}{2}(1 - \rho) + (\gamma_0 - \gamma_1) [-\varphi(\rho) + (1 - \rho)\varphi'(\rho)] \\ &\quad - 2(\gamma_0 + \gamma_1)\varphi(\rho)\varphi'(\rho) \end{aligned} \quad (180)$$

$$\begin{aligned} G''(\rho) &= -\frac{1}{2}(\gamma_0 + \gamma_1) + (\gamma_0 - \gamma_1) [-2\varphi'(\rho) + (1 - \rho)\varphi''(\rho)] \\ &\quad - 2(\gamma_0 + \gamma_1)[\varphi'(\rho)^2 + \varphi(\rho)\varphi''(\rho)] \end{aligned} \quad (181)$$

$$\begin{aligned} G^{(3)}(\rho) &= (\gamma_0 - \gamma_1) \left[-3\varphi''(\rho) + (1 - \rho)\varphi^{(3)}(\rho)\right] \\ &\quad - (\gamma_0 + \gamma_1)[6\varphi'(\rho)\varphi''(\rho) + 2\varphi(\rho)\varphi^{(3)}(\rho)] \end{aligned} \quad (182)$$

$$\begin{aligned} &= 6r \frac{(r-1)^2}{(r+1)^4} \psi(\rho)^{-5/2} \times \\ &\quad \left[ (\gamma_0 - \gamma_1) \frac{4r}{(r-1)(r+1)} + (\gamma_0 + \gamma_1)(1 - \rho) \right] \end{aligned} \quad (183)$$

Hence, if  $\gamma_0 + \gamma_1 = 0$ ,  $G^{(3)}$  has a constant sign. Whereas if  $\gamma_0 + \gamma_1 \neq 0$ , we have that  $G^{(3)}(\rho)$  changes sign exactly once, for the value

$$\tilde{\rho}_0 = \tilde{\rho}_0(r, d) = 1 + \frac{\gamma_0 - \gamma_1}{\gamma_0 + \gamma_1} \frac{4r}{(r-1)(r+1)} \quad (184)$$



Therefore  $G''$  is increasing before  $\tilde{\rho}_0$  and decreasing afterwards. Hence  $G''$  changes sign at most twice and  $G'$  changes sign at most three times.

*Proof of (ii).* If  $q = 0$ , then  $G(1) = 0$  by (112). If  $q \neq 0$ , the functions  $\tilde{\rho}_i$  in Lemma 3.1 are continuously differentiable on  $[0, 2]$ , thus the same holds for  $G$ . By (173), (174)–(175), (177) and (180),

$$G(1) = \frac{\gamma_0 + \gamma_1}{4}, \quad G'(1) = \frac{\gamma_1 - \gamma_0}{2} \frac{\sqrt{r} - 1}{\sqrt{r} + 1},$$

whence the desired conclusions.

*Proof of (iii).* Here we obtain

$$G'(2) = -\frac{\gamma_0 + r\gamma_1}{r + 1}$$

Under (23), we have  $\gamma_0 + r\gamma_1 \geq r(\gamma_0 + \gamma_1) \geq 0$ . The lower bound is positive if  $r > 0$  and  $\gamma_0 + \gamma_1 > 0$ . On the other hand,  $\gamma_0 + r\gamma_1 = (1-r)\gamma_0 > 0$  if  $\gamma_0 + \gamma_1 = 0$  and  $\gamma_0 \neq 0$ ; and  $\gamma_0 + r\gamma_1 > 0$  if  $r = 0$  and  $\gamma_0 > 0$ .

*Proof of (iv).* This follows from (173), (174) and (175).

*Proof of (v).* Without loss of generality, we assume  $\gamma_0 = \gamma_1 = 1$ . Then (173) becomes

$$G(\rho) = \frac{\rho}{2} \left(1 - \frac{\rho}{2}\right) - \varphi(\rho)^2 \quad (185)$$

and (180) becomes

$$G'(\rho) = \frac{1}{2}(1 - \rho) - 2\varphi(\rho)\varphi'(\rho) = (1 - \rho) \left(1 - \frac{1}{2\sqrt{\psi(\rho)}}\right) \quad (186)$$

We have that

$$\begin{aligned} G'(1/2) &= \frac{1}{2} \left(1 - \frac{1}{2\sqrt{\psi(1/2)}}\right) \\ G'(1/2) &> 0 \Leftrightarrow \psi(1/2) > \frac{1}{4} \Leftrightarrow 1 > \left(\frac{r-1}{r+1}\right)^2 \end{aligned}$$

which is true. Then after some computations, one can see that

$$\begin{aligned} G(1/2) > G(1) &\Leftrightarrow 4\psi\left(\frac{1}{2}\right)\psi(1) < \left[-1 + \frac{7}{4} \left(\frac{r-1}{r+1}\right)^2\right]^2 \\ &\Leftrightarrow 3 - \frac{7}{2} \left(\frac{r-1}{r+1}\right)^2 - \frac{1}{16} \left(\frac{r-1}{r+1}\right)^4 < 0 \end{aligned} \quad (187)$$

Solving this inequation with respect to  $r$  gives the condition in (b).

*Proof of (vi).* In view of (114), we may consider  $\gamma_0 \geq \gamma_1$  and  $r \geq 1$ . Let  $F(\rho) := G(\rho + 1) - G(\rho)$ . Note that

$$F(\rho) = \mathcal{F}(\rho + 1) - \mathcal{F}(\rho) \quad \text{with} \quad (188)$$

$$\mathcal{F}(\rho) = (\gamma_0 + \gamma_1) \left[ -\frac{1}{4}(\rho - 1)^2 - \varphi(\rho)^2 \right] - (\gamma_0 - \gamma_1)(\rho - 1)\varphi(\rho) \quad (189)$$

*First case.* We assume  $\gamma_0 + \gamma_1 \neq 0$ . By (115), without loss of generality, we may consider  $\gamma_0 = d$  and  $\gamma_1 = 1 - d$  with  $d \geq 1/2$ . We have

$$\begin{aligned} F'(\rho) &= -1 + \frac{1}{2} \left[ \frac{\rho}{\sqrt{\psi(\rho + 1)}} - \frac{(\rho - 1)}{\sqrt{\psi(\rho)}} \right] \\ &\quad + \frac{(2d - 1)}{2} \left( \frac{r - 1}{r + 1} \right) \left[ \frac{\rho^2}{\sqrt{\psi(\rho + 1)}} - \frac{(\rho - 1)^2}{\sqrt{\psi(\rho)}} \right] \\ &\quad + \frac{(2d - 1)}{2} \left( \frac{r + 1}{r - 1} \right) \left[ \sqrt{\psi(\rho + 1)} - \sqrt{\psi(\rho)} \right] \end{aligned}$$

then

$$F''(\rho) = -\frac{1}{2} \left( \frac{r - 1}{r + 1} \right)^2 \left[ \frac{\rho^2}{\psi(\rho + 1)} - \frac{(\rho - 1)^2}{\psi(\rho)} \right] \quad (190)$$

$$+ \frac{2r}{(r + 1)^2} \left[ \frac{1}{\psi(\rho + 1)^{3/2}} - \frac{1}{\psi(\rho)^{3/2}} - \frac{1}{\psi(\rho + 1)} + \frac{1}{\psi(\rho)} \right] \quad (191)$$

$$+ (2d - 1) \left( \frac{r - 1}{r + 1} \right) \left[ \left( \frac{\rho}{\sqrt{\psi(\rho + 1)}} + \frac{2r}{(r + 1)} \frac{\rho}{\psi(\rho)^{3/2}} \right) \right] \quad (192)$$

$$- \left( \frac{(\rho - 1)}{\sqrt{\psi(\rho)}} + \frac{2r}{(r + 1)^2} \frac{(\rho - 1)}{\psi(\rho)^{3/2}} \right) \quad (193)$$

We check the sign of each term.

$$f(\rho) = \frac{(\rho-1)}{\sqrt{\psi(\rho)}} \quad (194)$$

$$f'(\rho) = \frac{1}{\psi(\rho)^{3/2}} \frac{4r}{(r+1)^2} > 0 \quad (195)$$

$$\bar{f}(\rho) = \frac{1}{\psi(\rho)^{3/2}} - \frac{1}{\psi(\rho)} \quad (196)$$

$$\bar{f}'(\rho) = \frac{\psi'(\rho)}{2\psi(\rho)^{5/2}[2\sqrt{\psi(\rho)}+3]} \left[ -5 + 4 \left( \frac{r-1}{r+1} \right)^2 \rho(\rho-2) \right] \geq 0 \quad (197)$$

$$g(\rho) = f(\rho)^2 \quad (198)$$

$$g'(\rho) = 2f(\rho)f'(\rho) < 0 \quad \text{for } \rho \in [0, 1] \quad (199)$$

$$h(\rho) = \frac{(\rho-1)}{\psi(\rho)^{3/2}} \left[ \psi(\rho) + \frac{2r}{(r+1)^2} \right] \quad (200)$$

$$h'(\rho) = \frac{1}{\psi(\rho)^{5/2}} \frac{2r}{(r+1)^2} \frac{12r}{(r+1)^2} > 0 \quad (201)$$

(note that  $\bar{f}'(\rho) = 0$  if  $r = 1$ , and  $\bar{f}'(\rho) > 0$  if  $r \neq 1$ ). Hence  $F''(\rho) > 0$  for  $\rho \in [0, 1)$ . Then

$$F'(0) = -\frac{1}{2} - \frac{2d-1}{2} \left[ \left( \frac{r-1}{r+1} \right) + \left( \frac{\sqrt{r}-1}{\sqrt{r}+1} \right) \right] < 0 \quad (202)$$

$$F'(1) = -\frac{1}{2} + \frac{2d-1}{2} \left[ \left( \frac{r-1}{r+1} \right) + \left( \frac{\sqrt{r}-1}{\sqrt{r}+1} \right) \right] < 0 \quad (\text{see below}) \quad (203)$$

$$F(0) = G(1) = \frac{\sqrt{r}}{(\sqrt{r}+1)^2} > 0 \quad (204)$$

$$F(1) = -G(1) < 0 \quad (205)$$

We now show that  $F'(1) < 0$ . We write  $X = \sqrt{r}$ , and we consider  $X \geq 1$ .

$$\begin{aligned} \mathfrak{f}(X) &:= 2(r+1)(\sqrt{r}+1)F'(1) \\ &= (4d-3)X^3 - X^2 - X - (4d-1) \end{aligned} \quad (206)$$

$$\mathfrak{f}(1) = -4 < 0 \quad (207)$$

$$\mathfrak{f}'(X) = 3(4d-3)X^2 - 2X - 1 \quad (208)$$

If if  $d = 3/4$ ,  $\mathfrak{f}'(X) < 0$ . Otherwise we solve  $\mathfrak{f}'(X) = 0$ .

$$\delta = 4(3d-2) > 0 \quad \text{for } d > 2/3 \quad (209)$$

$$X_{\pm} = \frac{1 \pm \sqrt{\delta}}{3(4d-3)} \quad \text{for } \delta \geq 0 \quad (210)$$

Then

- if  $d < 2/3$ ,  $\delta < 0$ ,  $\mathfrak{f}'(X) < 0$ ,  $\mathfrak{f}$  is decreasing hence  $F'(1) < 0$ .

- if  $2/3 \leq d < 3/4$ ,  $f'(X) > 0$  for  $X \in (X_-, X_+)$ ; but  $X_{\pm} < 0$ , hence  $f'(X) < 0$ ,  $f$  is decreasing and  $F'(1) < 0$ .
- if  $d = 3/4$ ,  $f'(X) < 0$ , hence  $F'(1) < 0$ .
- if  $d > 5/6$ ,  $X_- < 0 < X_+$  and  $X_+ > 1$  because

$$X_+ < 1 \Leftrightarrow 9(d-1)(4d-3) > 0 \Leftrightarrow d \notin (3/4, 1)$$

thus  $f'(X) < 0$ ,  $f$  is decreasing hence  $F'(1) < 0$ .

- if  $3/4 < d < 5/6$  we also have  $X_+ > 1$ , thus  $F'(1) < 0$ .
- if  $d = 5/6$ , then  $X_+ = 1 + \frac{2}{\sqrt{2}} > 1$  hence  $f$  is decreasing and  $F'(1) < 0$ .

*Second case.* We assume  $\gamma_0 + \gamma_1 = 0$  and  $p \neq q$ . Without loss of generality, we can consider  $\gamma_0 = 1$ . This amounts to repeating the computations of the first case keeping only in  $F'(\rho)$  and  $F''(\rho)$  those terms with the factor  $(2d-1)/2$ , which we replace by 1. This leads similarly to  $F''(\rho) < 0$  for  $\rho \in [0, 1]$ . However, we now have  $F'(0) < 0$  and  $F'(1) > 0$ . Thus there exists  $\rho^* \in (0, 1)$  such that  $F$  is decreasing on  $[0, \rho^*]$  and increasing on  $[\rho^*, 1]$ . Besides, (204)–(205) are now replaced by  $F(0) = F(1) = 0$ , cf. (o) and (ii) of Proposition 4.7. This implies that 0 and 1 are the only solutions of the equation  $G(\rho+1) - G(\rho) = 0$ .  $\square$

*Proof of Lemma 4.1.* In cases (i)–(ii) below, we always have  $|\mathcal{R}_0| \leq 3$ . The only case not covered below is  $q = 0 < p$  and  $\gamma_0 \neq \gamma_1$ . Then (112) and Definition 4.3 show that  $\mathcal{R}_0$  is reduced to two elements of  $\mathcal{B}_1$ .

*Proof of (i).* By Definition 4.3, for any  $(\rho^-, \rho^+)$  in  $\mathcal{R}_0$ ,  $\rho = \min(\rho^-, \rho^+)$  must be a solution of the equation  $G(\rho+1) - G(\rho) = 0$ . By (vi) of Proposition 4.7, this equation has exactly one solution  $\rho$  in  $[0, 1]$ . This implies  $\mathcal{R}_0 \subset \{(\rho, \rho+1); (\rho+1, \rho)\}$ . But condition (ii) of Definition 4.3 implies that  $(\rho, \rho+1)$  and  $(\rho+1, \rho)$  cannot both lie in  $\mathcal{R}_0$ . Indeed,  $G$  would then be constant on  $[\rho, \rho+1]$ , and the only situations where  $G$  can be constant on a nontrivial interval are (49), (50) and (51), which are excluded here.

Since  $G(0) = G(2) = 0$  by (o) of Proposition 4.7, in order to have  $\mathcal{B}_1 \cap \mathcal{R}_0 \neq \emptyset$ , it is necessary to have  $G(1) = 0$ . By (ii) of Proposition 4.7, this only occurs if  $q = 0$  or  $\gamma_0 + \gamma_1 = 0$ .

*Proof of (ii).* *First case:*  $q > 0$ ,  $p \neq q$  and  $\gamma_0 + \gamma_1 = 0 \neq \gamma_0\gamma_1$ . Similarly to (i), using (vi) of Proposition 4.7, we see that  $\mathcal{R}_0 \subset \mathcal{B}_1$ . By (o) and (ii) of Proposition 4.7,  $G$  only vanishes for  $\rho \in \{0, 1, 2\}$ ; thus by Definition 4.3, one of the points  $(0, 1)$  or  $(1, 0)$ , and one of the points  $(1, 2)$  or  $(2, 1)$ , lie in  $\mathcal{R}_0$ . And since  $(\rho, \rho+1)$  and  $(\rho+1, \rho)$  cannot both lie in  $\mathcal{R}_0$ ,  $\mathcal{R}_0$  contains two elements. *Second case:*  $q = 0$  and  $\gamma_0 \neq \gamma_1$ . Then (112) and Definition 4.3 shows that  $\mathcal{R}_0$  is reduced to two elements of  $\mathcal{B}_1$ .

*Proof of (iii).* Assume first  $r > 0$ . By (114), since  $\gamma_0 = \gamma_1$ , we have  $G(2 - \rho) = G(\rho)$  for all  $\rho \in [0, 2]$ . Thus  $G(1/2) = G(3/2)$  and  $G'(1) = 0$ . Recalling (i), there can be no shock of amplitude 1 other than  $(1/2, 3/2)$  or  $(3/2, 1/2)$ ; and at most one of these lies in  $\mathcal{R}_0$ . If  $G$  has a single extremum (which must be at 1), by (iii) of Proposition 4.7,  $G$  is bell-shaped and this extremum is a maximum. Thus  $\mathcal{R}_0 = \{(1/2, 3/2)\}$ . If  $G$  has more than one extremum, by symmetry it must have three. Still by (iii) of Proposition 4.7, the extremum at 1 is then a local minimum and the other two are local maxima symmetric with respect to 1. Since  $G'(1/2) > 0$  by (v) of Proposition 4.7, condition (ii) of Definition 4.3 cannot hold with  $(\rho^-, \rho^+) = (3/2, 1/2)$ . On the other hand, this condition holds with  $(\rho^-, \rho^+) = (1/2, 3/2)$  if and only if  $G(1/2) \leq G(1)$ . The conclusion then follows from (v) of Proposition 4.7. Finally, for  $r = 0$ ,  $\mathcal{R}_0$  follows from (112) and Definition 4.3 (recall (23), implying here that  $\gamma_0 > 0$  and  $\gamma_1 > 0$ ).

*Proof of (iv).* For  $(d, r) \in [1/2, 1] \times [1, +\infty)$ , let us denote by  $\rho(d, r)$  the unique solution given by (vi) of Proposition 4.7 of  $F_{d,1-d,r}(\rho) = 0$ , where  $F_{d,1-d,r}(\rho) := G_{d,1-d,r}(\rho + 1) - G_{d,1-d,r}(\rho)$ . The proof of Proposition 4.7, (vi) showed that  $F'_{d,1-d,r}(\rho) < 0$  for every  $\rho \in [0, 1]$ . Besides, by (173), (174) and (175),  $F_{d,1-d,r}$  is continuously differentiable with respect to  $(d, r)$ . Thus the implicit function theorem implies that  $(d, r) \mapsto \rho(d, r)$  is continuously differentiable. Let

$$I(d, r) := \inf_{\rho \in [\rho(d, r), 1 + \rho(d, r)]} G(\rho), \quad S(d, r) := \sup_{\rho \in [\rho(d, r), 1 + \rho(d, r)]} G(\rho)$$

We define

$$\mathcal{Z} := \{(d, r) \in [0, 1] \times [0, 1] : I(d, r) < G[\rho(d, r)] < S(d, r)\} \quad (211)$$

The set  $\mathcal{Z}$  is an open subset of  $[0, 1]^2$  because  $(d, r) \mapsto \rho(d, r)$  is continuous. By (iii), it contains  $\{1/2\} \times (0, r_0)$ . Finally, by (ii) of Definition 4.3, for  $(d, r) \in \mathcal{Z}$ , neither  $(\rho(d, r), 1 + \rho(d, r))$  nor  $(1 + \rho(d, r), \rho(d, r))$  lies in  $\mathcal{R}_0$ , thus  $\mathcal{R}_0 = \emptyset$ .  $\square$

*Proof of Proposition 4.8.*

*Proof of (i).* Assume for instance  $\rho^- < \rho^+$ , the case  $\rho^- > \rho^+$  being similar. Let  $r \in [\rho^-, \rho^+]$ . Let  $\tilde{\nu}(d\eta, d\xi)$  be a coupling of  $\nu(d\eta)$  and  $\nu_r(d\xi)$  that is invariant for the coupled generator (83) (it exists by [9, Proposition 2.14 in Chapter VIII]). Since  $\tilde{\nu}$  is supported on a compact space, there exists an increasing  $\mathbb{N}$ -valued sequence  $x_n \rightarrow +\infty$  such that  $\tau_{-x_n-1}\tilde{\nu}$  and  $\tau_{x_n}\tilde{\nu}$  have weak limits denoted respectively by  $\tilde{\nu}_{-\infty}$  and  $\tilde{\nu}_{+\infty}$ . By (42) and translation invariance of  $\nu_r$ ,  $\tilde{\nu}_{\pm\infty}$  is a coupling of  $\nu_{\rho^\pm}$  and  $\nu_r$ . Since the coupled generator  $\tilde{L}$  given by (83) for the transition kernel (19) is translation invariant in the  $\mathbb{Z}$ -direction, we have  $\tilde{\nu} \in \tilde{\mathcal{I}} \cap \tilde{\mathcal{S}}$ . Hence, by (98) in the proof of Theorem 2.1,  $\tilde{\nu}$  is supported on ordered pairs  $(\eta, \xi)$ . On the other hand, under  $\tilde{\nu}_{\pm\infty}$ , empirical averages (cf. (153)) exist by the law of large numbers and are given by  $M(\eta) = \rho^\pm$  and  $M(\xi) = r$ . These averages must be ordered like  $\eta$  and  $\xi$ , hence  $\tilde{\nu}_{-\infty}$  and  $\tilde{\nu}_{+\infty}$  are supported respectively on  $E_-$  and  $E_+$ .

Let  $N \in \mathbb{N}$ ,  $R_N := (\mathbb{Z} \cap [-N, N]) \times W$ , and

$$\tilde{F}_N(\eta, \xi) := D_{-N, N}(\eta, \xi) = \sum_{i \in W} \sum_{z \in \mathbb{Z} \cap [-N, N]} |\eta(z, i) - \xi(z, i)| \quad (212)$$

Since  $\tilde{\nu} \in \tilde{\mathcal{I}}$ , we have

$$\int_{\mathcal{X} \times \mathcal{X}} \tilde{L}\tilde{F}_N(\eta, \xi) d\tilde{\nu}(\eta, \xi) = 0 \quad (213)$$

By [8, Lemma 2.4], we have

$$\tilde{L}\tilde{F}_N(\eta, \xi) = \sum_{x \notin R_N, y \in R_N} p(x, y) J_{x, y}(\eta, \xi) \quad (214)$$

$$- \sum_{x \in R_N, y \notin R_N} p(x, y) J_{x, y}(\eta, \xi) \quad (215)$$

$$- \sum_{x \in R_N, y \in R_N, x \neq y} [p(x, y) + p(y, x)] D_{x, y}(\eta, \xi) \quad (216)$$

where

$$J_{x, y}(\eta, \xi) := [\eta(x)(1 - \eta(y)) - \xi(x)(1 - \xi(y))] \{ \mathbf{1}_{\{\eta(x) \geq \xi(x), \eta(y) \geq \xi(y)\}} - \mathbf{1}_{\{\eta(x) \leq \xi(x), \eta(y) \leq \xi(y)\}} \} \quad (217)$$

and

$$D_{x, y}(\eta, \xi) := \mathbf{1}_{E_{x, y}}(\eta, \xi) = \mathbf{1}_{\{\eta(x) > \xi(x), \eta(y) < \xi(y)\}} + \mathbf{1}_{\{\eta(x) < \xi(x), \eta(y) > \xi(y)\}} \quad (218)$$

Let

$$\tilde{j}(\eta, \xi) := \sum_{x(0) \leq 0, y(0) > 0} p(x, y) J_{x, y}(\eta, \xi) - \sum_{x(0) \leq 0, y(0) > 0} p(y, x) J_{y, x}(\eta, \xi)$$

where  $J_{x, y}$  is defined by (217). Then (214)–(215) can be written as  $\tau_{-N-1}\tilde{j}(\eta, \xi) - \tau_N\tilde{j}(\eta, \xi)$ . By (217) and (107)

$$\tilde{j}(\eta, \xi) = j(\eta) - j(\xi) \text{ if } \eta \leq \xi, \quad \tilde{j}(\eta, \xi) = j(\xi) - j(\eta) \text{ if } \xi \leq \eta \quad (219)$$

Since  $D_{x, y}(\eta, \xi)$  defined by (218) is nonnegative, the stationarity relation (213) combined with (214)–(216) yields

$$\tilde{\nu}(\tau_{-N-1}\tilde{j}) - \tilde{\nu}(\tau_N\tilde{j}) \geq 0 \quad (220)$$

Taking  $N = x_n$  and letting  $n \rightarrow +\infty$  yields

$$\tilde{\nu}_{-\infty}(\tilde{j}) - \tilde{\nu}_{+\infty}(\tilde{j}) \geq 0 \quad (221)$$

Under  $\tilde{\nu}_{\pm\infty}$ , we can use (219) for ordered configurations. The marginals of  $\tilde{\nu}_{\pm\infty}$  then yield

$$G(r) - G(\rho^-) \geq G(\rho^+) - G(r) \quad (222)$$

Since  $r \in [\rho^-, \rho^+]$  is arbitrary, we first obtain  $G(\rho^+) = G(\rho^-)$  by letting  $r = \rho^\pm$ , and then  $G(\rho^+) = G(\rho^-) = \min_{r \in [\rho^-, \rho^+]} G(r)$ .

*Proof of (ii).* This follows from (i) above, and (ii) of Proposition 4.3.

*Proof of (iii).* By Lemma 4.1,  $\gamma_0 + \gamma_1 = 0$  or  $q = 0$ . Assume from now on that the latter holds.

(a) We assume first  $\gamma_1 \geq 0$ . Then by (112) and Definition 4.3, if  $\gamma_0$  and  $\gamma_1$  are not both 0, we have  $\mathcal{R}_0 \cap \mathcal{B}_1 = \{(0, 1); (1, 2)\}$ .

We consider first  $(\rho^-, \rho^+) = (0, 1)$ . We show that this case is impossible if  $\gamma_1 = 0$ , whereas if  $\gamma_1 > 0$ ,  $\mu$  is one of the measures  $\nu^{\pm, +\infty, j}$  in (59). To this end, observe first that since  $q = 0 < p$ ,  $\nu^0$  is the probability measure supported on the empty configuration and  $\nu^1$  is supported on the configuration that is empty on lane 0 and full on lane 1. Since  $\mu$  is a  $(0, 1)$ -shock measure, we have

$$\lim_{x \rightarrow -\infty} \tau_x \eta_0^0 = \mu_0, \quad \lim_{x \rightarrow +\infty} \tau_x \eta_0^0 = \mu_0, \quad (223)$$

$$\lim_{x \rightarrow -\infty} \tau_x \eta_0^1 = \mu_0, \quad \lim_{x \rightarrow +\infty} \tau_x \eta_0^1 = \mu_1 \quad (224)$$

where  $\mu_\rho$ , cf. Section 2.4.2, denotes the product Bernoulli measure on  $\{0, 1\}^{\mathbb{Z}}$  with parameter  $\rho$ . As in the proof of Proposition 4.2, we couple  $\eta$  with an ASEP  $\zeta^0$  on lane 0 starting from  $\zeta_0^0 := \eta_0^0$ , with jump rate  $d_0$  to the right and  $l_0$  to the left, that is (4)–(3) with  $(l, d) = (l_0, d_0)$ . The limit (223) implies

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=1}^n \zeta_0^0(x) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=-n}^1 \zeta_0^0(x) = 0 \quad (225)$$

in probability. Since the initial configuration satisfies (225),  $\zeta_t^0$  converges in law as  $t \rightarrow +\infty$  to the Bernoulli invariant measure with zero density, that is the empty configuration; this follows from [3, Theorem 1] when  $\gamma_0 > 0$ , or [9, Chapter VIII] when  $\gamma_0 = 0$ . Since  $\eta_t^0 \leq \zeta_t^0$ , the same limit holds for  $\eta_t^0$ . By stationarity of  $\mu$ , this implies that under  $\mu$ , lane 0 is almost surely empty. It follows that  $(\eta_t^1)_{t \geq 0}$  is itself an autonomous SEP. Thus the marginal of  $\mu$  on lane 1 is an invariant measure for SEP. By [8, Theorem 1.4], it is a mixture of Bernoulli and blocking measures. Because of (224), only blocking measures are present in the mixture. Note that this is only possible if  $\gamma_1 > 0$ . In this case,  $\mu$  is a mixture of the invariant measures  $\nu^{\pm, +\infty, j}$  in (59) for  $j \in \mathbb{Z}$ . Since  $\mu$  is extremal, it is one of them.

Next, we consider  $(\rho^-, \rho^+) = (1, 2)$ . This can be reduced to the previous case

by Lemma 2.1, considering the image of  $\eta_t$  by  $\sigma\sigma'\sigma''$ . The resulting process has drift  $\gamma'_0 = \gamma_1$  on lane 0, and  $\gamma'_1 = \gamma_0$  on lane 1. The image  $\mu''$  of  $\mu$  is a  $(0, 1)$ -shock measure invariant for the transformed process. It follows from the above that:

- If  $\gamma_0 > 0$ ,  $\mu'' = \nu^{\perp, +\infty, j}$ , thus  $\mu = \nu^{\perp, j, -\infty}$ , for some  $j \in \mathbb{Z}$ .

- If  $\gamma_0 = 0$ , that is  $\gamma'_1 = 0$ , from the above discussion, it is impossible for  $\mu''$  to be a  $(0, 1)$ -shock measure, and thus for  $\mu$  to be a  $(1, 2)$ -shock measure.

Putting together the cases  $(\rho^-, \rho^+) = (0, 1)$  and  $(\rho^-, \rho^+) = (1, 2)$ , we conclude that in case (iv) of Theorem 2.3, a  $(\rho^-, \rho^+)$ -shock measure with  $(\rho^-, \rho^+) \in \mathcal{B}_1$  lies in the set (59); whereas in case (vi) it lies in the set (63). In the former case  $\mathcal{R}' = \mathcal{R}_0 \cap \mathcal{B}_1 = \{(0, 1); (1, 2)\}$ , whereas in the latter case  $\mathcal{R}' = \{(0, 1)\} \neq \mathcal{R}_0 \cap \mathcal{B}_1 = \{(0, 1); (1, 2)\}$ .

(b) We consider now  $\gamma_1 < 0 < \gamma_0$ . Here, by (112) and Definition 4.3, we have  $\mathcal{R}_0 \cap \mathcal{B}_1 = \{(1, 0); (2, 1)\}$ . The case  $(\rho^-, \rho^+) = (1, 0)$  is treated like  $(\rho^-, \rho^+) = (0, 1)$  in (a) above; except that on lane 1 we have a  $(1, 0)$ -shock with a negative drift. The case  $(\rho^-, \rho^+) = (1, 2)$  is deduced by Lemma 2.1 and particle-hole symmetry (recall (22)).  $\square$

*Proof of Lemma 4.2.* Let  $\tilde{\nu}$  denote a coupling of  $\nu^1$  and  $\nu^2$  such that  $\tilde{\nu} \in \tilde{\mathcal{L}}$ . Since  $\nu^1$  and  $\nu^2$  are supported on  $\mathcal{X}_2$ ,  $\tilde{\nu}$  satisfies assumption (92) of Proposition 3.2. Since we excluded the case  $l_0 = l_1 = q = 0 < p$ , by Lemma 3.3,  $p(\cdot, \cdot)$  is weakly irreducible. Thus, by (95) and the proof of Theorem 2.1,  $\tilde{\nu}$  is supported on ordered pairs of configurations. Since  $H_2$  is a nondecreasing function on  $\mathcal{X}_2$  and has the same value under both marginals of  $\tilde{\nu}$ , it follows that  $\tilde{\nu}$  is supported on  $E_3$ , whence the conclusion.  $\square$

## A Additional proofs

### A.1 Proof of (56)

We prove the first equality in (56), the proof of the second one being similar. Let  $\mathcal{X}_n := \{\eta \in \mathcal{X} : H_2(\eta) = 2n\}$ , and  $\xi^n$  denote the element of  $\mathcal{X}_n$  defined by

$$\xi^n(z, i) = \mathbf{1}_{\{z > -n\}}; \quad z \in \mathbb{Z}, i \in W$$

so that  $\xi^n = \tau_n \xi^0$ , with  $\xi^0 \in \mathcal{X}_0$ . For  $\eta, \xi \in \mathcal{X}_n$ , let  $\mathcal{A}$ , resp.  $\mathcal{B}$ , denote the set of  $x \in V$  for which  $\eta(x) = 1 - \xi(x) = 0$ , resp.  $\eta(x) = 1 - \xi(x) = 1$ . Then



$|\mathcal{A}| = |\mathcal{B}| < +\infty$ , and

$$\begin{aligned} \frac{\nu^{\rho^c}(\eta)}{\nu^{\rho^c}(\xi)} &= \prod_{x \in \mathcal{B}} \frac{\rho_x}{1 - \rho_x} \prod_{x \in \mathcal{A}} \frac{1 - \rho_x}{\rho_x} \\ &= \left(\frac{p}{q}\right)^{\sum_{z \in \mathbb{Z}, i \in W} i[\eta(z) - \xi(z)]} \theta^{\sum_{z \in \mathbb{Z}, i \in W} z[\eta(z) - \xi(z)]} =: r(\eta, \xi) \end{aligned}$$

The second equality above follows from  $\rho_{z,i}/(1 - \rho_{z,i}) = c(p/q)^i \theta^z$  for  $(z, i) \in V$ . We apply this to  $\eta \in \mathcal{X}_n$  and  $\xi^n$ :

$$\check{\nu}_n(\eta) = \frac{\nu^{\rho^c}(\eta)}{\nu^{\rho^c}(\mathcal{X}_n)} = \frac{r(\eta, \xi^n)}{Z^n}$$

where

$$Z^n := \sum_{\xi \in \mathcal{X}_n} r(\xi, \xi^n)$$

Thus  $\check{\nu}_n$  does not depend on  $c$ . Note that if  $\eta \in \mathcal{X}_0$ , we have  $\tau_n \eta \in \mathcal{X}_n$ , and  $r(\tau_n \eta, \xi^n) = r(\eta, \xi^0)$ . This implies that  $Z^n$  does not depend on  $n$  and that

$$\check{\nu}_n(\tau_n \eta) = \check{\nu}_0(\eta)$$

## A.2 Proof of Proposition 3.2

Let us rewrite the coupled generator (83) as

$$\bar{L}f(\eta, \xi) = \sum_{(\eta', \xi') \in \mathcal{X} \times \mathcal{X}} a[(\eta, \xi); (\eta', \xi')] [f(\eta', \xi') - f(\eta, \xi)] \quad (226)$$

where the rates  $a[(\eta, \xi); (\eta', \xi')]$  are defined as follows. First, for any  $(x, y) \in V$  such that  $x \neq y$ ,  $a[(\eta, \xi); (\eta', \xi')]$  is given by

$$\begin{cases} p(x, y)[\eta(x)(1 - \eta(y))] \vee [\xi(x)(1 - \xi(y))] & \text{if } (\eta', \xi') = (\eta^{x,y}, \xi^{x,y}) \\ p(x, y)[\eta(x)(1 - \eta(y)) - \xi(x)(1 - \xi(y))]^+ & \text{if } (\eta', \xi') = (\eta^{x,y}, \xi) \\ p(x, y)[\eta(x)(1 - \eta(y)) - \xi(x)(1 - \xi(y))]^- & \text{if } (\eta', \xi') = (\eta, \xi^{x,y}) \end{cases} \quad (227)$$

with the kernel  $p(\cdot, \cdot)$  given by (19). Next,  $a[(\eta, \xi); (\eta', \xi')] = 0$  if there exists no  $(x, y) \in V^2$  such that  $x \neq y$  and  $(\eta', \xi') \in \{(\eta^{x,y}, \xi^{x,y}), (\eta^{x,y}, \xi), (\eta, \xi^{x,y})\}$ .

If  $a[(\eta, \xi); (\eta', \xi')] \neq 0$ , we say there is a transition from  $(\eta, \xi)$  to  $(\eta', \xi')$ . Recalling the notation  $x \xrightarrow{k} y$  introduced before Definition 3.1, we shall prove the following.

**Lemma A.1.** *Let  $\tilde{\nu} \in \tilde{\mathcal{I}} \cap \tilde{\mathcal{S}}$ . Then (93) holds for every  $(x, y) \in V \times V$  such that  $x \neq y$ , and  $x \xrightarrow{k} y$  or  $y \xrightarrow{k} x$  for some  $k$ .*

*Proof of Lemma A.1.* We prove by induction on  $k$  that (93) holds for every  $(x, y) \in V \times V$  such that  $x \neq y$  and  $x \xrightarrow{k} y$ . Applying the statement to  $(\xi, \eta)$  then shows that it holds for  $(\eta, \xi)$  and  $y \xrightarrow{k} x$ .

We now use the computation done between (212) and (218). The sums in (214)–(215) are boundary contributions, that we denote respectively by  $\Gamma_N^i(\eta, \xi)$  and  $\Gamma_N^o(\eta, \xi)$ . Since  $\tilde{\nu} \in \tilde{\mathcal{I}}$ , we have

$$\int_{\mathcal{X} \times \mathcal{X}} \tilde{L}F_N(\eta, \xi) d\tilde{\nu}(\eta, \xi) = 0 \quad (228)$$

We have to exploit (228); for this we distinguish between the two assumptions:

*First case.* We assume that  $\tilde{\nu} \in \tilde{\mathcal{S}}$ . Since  $J_{((u+z, i), (v+z, j))} = \tau_z J_{(u, i), (v, j)}$  for all  $u, v, z \in \mathbb{Z}$ , we have

$$\int_{\mathcal{X} \times \mathcal{X}} \Gamma_N^i(\eta, \xi) d\tilde{\nu}(\eta, \xi) - \int_{\mathcal{X} \times \mathcal{X}} \Gamma_N^o(\eta, \xi) d\tilde{\nu}(\eta, \xi) = 0 \quad (229)$$

*Second case.* We assume (92). The latter with the inequalities

$$\begin{aligned} |\Gamma_N^i(\eta, \xi)| &\leq \sum_{i \in W} l_i (|\eta(-N-1, i) - \xi(-N-1, i)| + |\eta(-N, i) - \xi(-N, i)|) \\ |\Gamma_N^o(\eta, \xi)| &\leq \sum_{i \in W} d_i (|\eta(N, i) - \xi(N, i)| + |\eta(N+1, i) - \xi(N+1, i)|) \end{aligned}$$

leads to

$$\lim_{N \rightarrow +\infty} \left\{ \int_{\mathcal{X} \times \mathcal{X}} \Gamma_N^i(\eta, \xi) d\tilde{\nu}(\eta, \xi) - \int_{\mathcal{X} \times \mathcal{X}} \Gamma_N^o(\eta, \xi) d\tilde{\nu}(\eta, \xi) \right\} = 0 \quad (230)$$

Using (229) for all  $N$ , we obtain that for every  $(x, y) \in V^2$  such that  $p(x, y) > 0$ ,

$$\int_{\mathcal{X} \times \mathcal{X}} D_{x, y}(\eta, \xi) d\tilde{\nu}(\eta, \xi) = 0$$

This implies (93) for  $k = 1$ .

Now assume (93) holds for  $k - 1$ . If  $A$  is a subset of  $\mathcal{X} \times \mathcal{X}$  and  $(\eta, \xi) \in \mathcal{X} \times \mathcal{X}$ , we write  $(\eta, \xi) \xrightarrow{n} A$  if there exists a sequence of coupled configurations,  $(\eta_0, \xi_0) = (\eta, \xi), \dots, (\eta_n, \xi_n) = (\eta', \xi')$ , such that  $a[(\eta_i, \xi_i); (\eta_{i+1}, \xi_{i+1})] > 0$  for every  $i = 0, \dots, n-1$ , and  $(\eta', \xi') \in A$ . Assume  $A = A_0$  is a local set (that is, such that its indicator function is a local function) and

$$\begin{aligned} A_n &:= \{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : (\eta, \xi) \xrightarrow{n} A_0\} \\ A'_n &:= \{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : (\eta, \xi) \xrightarrow{i} A_0 \text{ for some } i \leq n\} \end{aligned}$$

Then (226)–(227) implies that there exist positive constants  $a_n, b_n$  such that

$$\tilde{L}\mathbf{1}_{A_n} \geq a_n \mathbf{1}_{A_{n+1}} - b_n \mathbf{1}_{A_n} \quad (231)$$

Iterating (231) shows that if  $\tilde{\nu} \in \tilde{\mathcal{I}}$  and  $\tilde{\nu}(A) = 0$ , then  $\tilde{\nu}(A_n) = 0$ , hence  $\tilde{\nu}(A'_n) = 0$ . For the induction step, we use this as follows. Let  $E^n$  denote the set of coupled configurations  $(\eta, \xi) \in \mathcal{X} \times \mathcal{X}$  such that there is no pair of opposite discrepancies at sites  $x, y \in V$  if  $x \xrightarrow{i} y$  or  $y \xrightarrow{i} x$  for any  $i \leq n$ . We choose  $A_0 = E^{k-1}$  so that  $\tilde{\nu}(A_0) = 0$  by the induction assumption. Then we claim that  $E^k$  is contained in  $A'_{k-1}$ . Indeed, assume  $x \xrightarrow{k} y$  and  $(\eta, \xi) \in E_{x,y}$ . Let  $(x = x_0, \dots, x_k = y)$  denote a  $p$ -path from  $x$  to  $y$ . By the induction assumption,  $\tilde{\nu}$ -almost surely, we have  $\eta(x_i) = \xi(x_i)$  for all  $i = 1, \dots, k-1$ . If  $\eta(x_1) = \xi(x_1) = 0$ , then  $(\eta^{x_0, x_1}, \xi^{x_0, x_1}) \in E_{x_1, y}$ . Otherwise let  $i^*$  be the maximum index  $i$  such that  $\eta(x_i) = \xi(x_i) = 1$ . Then one can find a sequence of at most  $k-1$  transitions leading from  $(\eta, \xi)$  to some  $(\eta', \xi') \in E_{x, x_{k-1}}$  as follows: (i) if  $i^* < k-1$ , the coupled particle at  $x_{i^*}$  jumps from  $x_{i^*}$  to  $x_{k-1}$  along the path; (ii) the coupled particle at  $x_{k-1}$  exchanges with the  $\xi$ -discrepancy at  $y$ .  $\square$

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