Partition of Unity Finite Element Method applied to 2D vibroacoustic coupling

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PARTITION OF UNITY ELEMENT METHOD APPLIED TO 2D VIBROACOUSTIC COUPLING

Christophe Langlois[1] **Jean-Daniel Chazot**[1] **Emmanuel Perrey-Debain**[1]
Li Cheng[2]
[1] Université de technologie de Compiègne, CNRS, Roberval(Mechanics energy and electricity), Centre de recherche Royallieu -CS 60319 -60203 Compiègne Cedex
[2] Departement of Mechanical Engineering, The Hong Kong Polytechnic University, Hong Kong P.R. China
christophe.langlois@utc.fr, jean-daniel.chazot@utc.fr

ABSTRACT

The Finite Element Method is one of the most popular methods used to solve wave propagation problems. Yet, this method is computationally intensive as the number of wavelengths increases in the computational domain. As an alternative, the Partition of Unity Finite Element Method (PUFEM) emerges as a promising method to tackle short-wave problems for both acoustics and vibration problems. The method is an extension of the classical Finite Element Method in which enrichment basis functions are used. This entails a reduction in the number of degrees of freedom in the model while maintaining its accuracy. In the present work, the PUFEM method is extended and applied to 2D vibro-acoustic problems. The configuration under investigation consists of a Euler-Bernoulli beam coupled with an enclosed acoustic cavity. Wave functions combined with polynomials are used as the enrichment basis for the beam and plane waves for the acoustic domain. Computational performances measured in terms of accuracy and number of degrees of freedom are demonstrated via convergence analyses. It is shown that the method outperforms classical FEM, as evidenced by the highly satisfactory level of errors reached by using a very small number of degrees of freedom.

1. INTRODUCTION

The finite element method (FEM) is the most widely used method to solve wave-like problems. However, the method is best-suited to simulate waves in so-called low frequency regime. Indeed, assuming classical linear interpolation, the FEM requires at least 10 elements per wavelength. Hence as the wavelength shortens the model becomes heavier computer-wise. In addition this method suffers from dispersion and pollution errors [1]. This makes the classical FEM both inaccurate and expensive for solving short-wave problems. To overcome these issues, alternative methods have been developed. One very common technique widely used in commercial softwares for acoustics is the statistical energy analysis (SEA) [2]. This method is particularly well suited for quick system level response predictions at the early design stage of a product, and for predicting responses at higher frequencies but relies on many assumptions and is not deterministic. On the other hand, the inclusion of the wave character of the solution in the FEM formulation gave rise to new deterministic methods *i.e.*: the Partition of Unity Finite Element Method (PUFEM) [3], the Ultra-Weak formulation [4], Wave-Based Methods [5], the Discontinuous Enrichment Method and the Variational Theory of Complex Rays [6].
In the present work which deals with vibro-acoustic problems, the PUFEM is favored as it has the advantage of being very similar to classical FEM which facilitates its implementation. In addition it has already been developed with success for the numerical simulation of acoustic [7], [8] and structural waves [9].

2. FORMULATION

The aim of this work is to model the coupling between a beam and an acoustic cavity. Throughout this paper Γ_s refers to the beam domain and Ω_f refers to the acoustic domain. The time convention used is $\exp(-\mathrm{i}\omega t)$ where ω is the angular frequency.

2.1 Weak formulation

2.1.1 Beam

The beam is considered as a Euler-Bernoulli beam. So its motion is described by the transverse displacement w which satisfies :

$$EI\frac{\partial^4 w}{\partial x^4} - \omega^2 \rho_s S w = Q \quad \text{in} \quad \Gamma_s, \tag{1}$$

where E is the Young's modulus, $I = bh^3/12$ the second area moment, b the width of the beam, h its height, ρ_s its density, $S = bh$ its cross section area and Q an arbitrary load. The associated variational formulation writes

$$\begin{aligned}\int_{\Gamma_s} \left(EI\frac{\partial^2 \delta w}{\partial x^2}\frac{\partial^2 w}{\partial x^2} - \omega^2 \rho_s S w \delta w \right) \mathrm{d}\Gamma = \\ \int_{\Gamma_s} \delta w Q \mathrm{d}\Gamma + \delta w \lambda(0) + \delta w \lambda(L_x) \quad \text{in} \quad \Gamma_s,\end{aligned} \tag{2}$$

where δw is the test function and $\lambda(x)$ are Lagrange multipliers used to impose the *pinned* boundary conditions at both ends of the beam.

2.1.2 Acoustics

In the acoustic domain the pressure satisfies the Helmholtz equation :

$$\Delta p + k_f^2 p = 0, \tag{3}$$

where p is the pressure, $k_f = \omega/c$ is the wave number and c the speed of sound. The associated variational formulation is

$$\int_{\Omega_f} \left(\nabla \delta p \nabla p - k_f^2 p \delta p\right) \mathrm{d}\Omega - \int_{\Gamma} \delta p \frac{\partial p}{\partial n} \mathrm{d}\Gamma = 0, \tag{4}$$

where δp is the test function and Γ is the boundary of Ω_f. The boundary Γ can be split into two sub-boundaries as $\Gamma = \Gamma_f \cup \Gamma_s$ where Γ_f is a boundary on which a variety of boundary condition can be applied.

2.1.3 Coupling

Applying the continuity of displacement leads to

$$\begin{cases} \dfrac{\partial p}{\partial n} = \rho_a \omega^2 w, & (5) \\ -pb + f_{ext} = Q, & (6) \end{cases}$$

where f_{ext} is a known force applied on the beam as illustrated in Fig. 1. Finally Eqn. (6) and Eqn. (5) are injected in Eqn. (4) and Eqn. (2) respectively. The following coupled system is obtained :

$$\begin{cases} \displaystyle\int_{\Omega_f} \left(\nabla \delta p \nabla p - k_f^2 p \delta p\right) \mathrm{d}\Omega = \int_{\Gamma_f} \delta p \frac{\partial p}{\partial n} \mathrm{d}\Gamma \\ \displaystyle + \int_{\Gamma_s} \delta p \rho_f \omega^2 w \mathrm{d}\Gamma \quad \text{in} \quad \Omega_f, \\ \displaystyle\int_{\Gamma_s} \left(EI \frac{\partial^2 \delta w}{\partial x^2} \frac{\partial^2 w}{\partial x^2} - \omega^2 \rho_s S w \delta w \right) \mathrm{d}\Gamma = +\delta w \lambda(0) \\ \displaystyle + \delta w \lambda(L_x) + \int_{\Gamma_s} \delta w (f_{ext} - pb) \mathrm{d}\Gamma \quad \text{in} \quad \Gamma_s. \end{cases}$$

2.2 PUFEM formulation

In [10] authors showed that the *'wave enrichment'* combined with a family of polynomial functions provide a highly satisfactory convergence rate for the Euler-Bernoulli beam. So the transverse displacement is expressed as follows :

$$w(x) = \sum_{i=1}^{2} \sum_{k=1}^{6} N_i^h(\mu) \Psi_{ik}(x) W_{ik} \tag{7}$$

where N^h are Hermite shape function for a 2-nodes line element, the unknown W_{ik} is the amplitude of the k^{th} function of the enrichment basis attached to node i. The enrichment basis is :

$$\Psi_{ik} = \left\{1, \tilde{x}_i, \tilde{x}_i^2, \tilde{x}_i^3, \cos(k_s \tilde{x}_i), \sin(k_s \tilde{x}_i)\right\}, \tag{8}$$

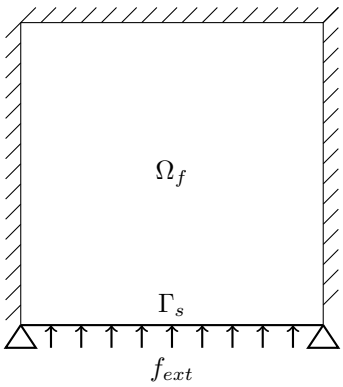

Figure 1: Configuration.

where $\tilde{x}_i = x - x_i$ and $k_s = \sqrt[4]{\frac{\rho_s S \omega^2}{EI}}$ is the the wave number for an infinite beam. For more details see [11]. The geometry is mapped using classical linear shape functions.

In [7] authors showed the efficiency of the plane wave basis to solve Helmholtz problems. Here, the pressure is expressed as a combination of plane waves travelling in different directions:

$$p(\boldsymbol{r}) = \sum_{j=1}^{3} \sum_{q=1}^{\mathcal{Q}_j} N_j(\xi, \eta) \exp(\mathrm{i}k \boldsymbol{d}_{jq}(\boldsymbol{r} - \boldsymbol{r}_j)) A_{jq} \tag{9}$$

where N_j are linear shape function of a triangular element, $\boldsymbol{d}_{jq}$ is the direction of the q^{th} plane wave attached to node j. $\boldsymbol{r}_j$ is the coordinate vector of node j. A_{jq} is the amplitude of the q^{th} plane wave attached to node j and $\mathcal{Q}_j$ is the number of plane waves at node j. At node, plane waves are evenly distributed on the unit circle like so :

$$\boldsymbol{d}_{jq} = \begin{pmatrix} \cos \alpha_{j,q} \\ \sin \alpha_{j,q} \end{pmatrix} \text{ where } \alpha_{j,q} = \frac{2\pi q}{\mathcal{Q}_j},\ q = 1, ..., \mathcal{Q}_j. \tag{10}$$

The number of plane waves at each node depends on the frequency and the element size according to the following criteria [8, 12] :

$$\mathcal{Q}_j = \text{round}[kh + C(kh)^{\frac{1}{3}}]. \tag{11}$$

Here, h is the largest element edge length connected to the node j within the acoustic domain, and C is a constant usually chosen in the interval $C \in [2, 20]$.

3. NUMERICAL RESULTS

To test the accuracy of the method, it is applied to an academic case described in Fig. 1. This case consists of a beam Γ_s, simply supported at both ends on which a uniform loading is applied. The beam is coupled to a rectangular acoustic cavity Ω_f. The cavity is bounded by three rigid walls and the beam. These two domains are defined as $\Gamma_s = \{(x, z) \in \mathbb{R}^2 / \ 0 < x < 1, z = 0\}$, $\Omega_f = \{(x, z) \in \mathbb{R}^2 / \ 0 < x < 1, 0 < z < 1\}$. Fig. 2 and Fig. 3 represent the real part of the pressure field and the transverse displacement of the beam at $f = 3000$Hz. One can observe that elements span over several wavelengths which highlights the efficiency of the method.

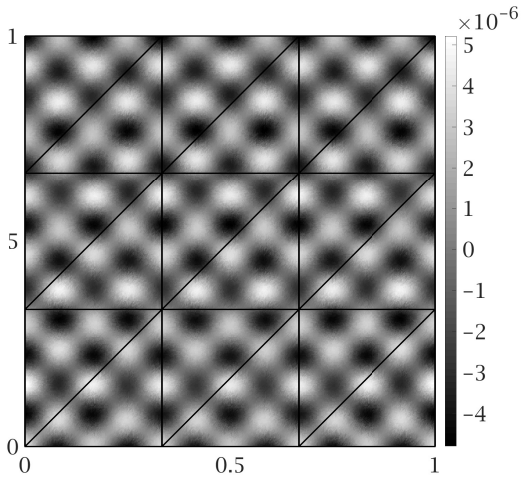

Figure 2: Real part of the pressure (Pa) field at 3000Hz.

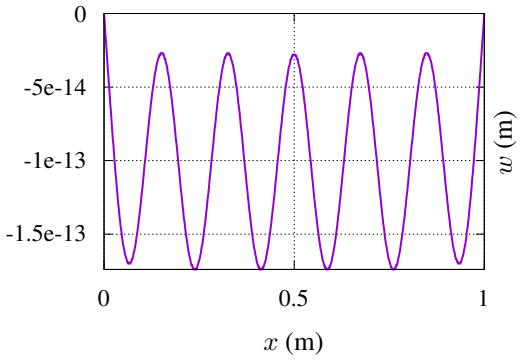

Figure 3: Transverse displacement of the beam at 3000Hz. 3 elements along the beam.

To further assess the quality of the model the $\mathcal{L}_2$ error is computed in each domain as follows :

$$\epsilon_2 = \frac{\sqrt{\int_\Omega |u_{comp} - u_{ref}|^2 \, \mathrm{d}\Omega}}{\sqrt{\int_\Omega |u_{ref}|^2 \, \mathrm{d}\Omega}}, \tag{12}$$

where u stands for either p or w whether the error is assessed in Ω_a or Γ_s. Subscript $_{comp}$ refers to results from PUFEM and $_{ref}$ refers to results of the reference solution. The reference solution obtained using modal expansion with coupled modes along the z-direction as explained in [13]. The error associated with Fig. 2 is 0.15% and with Fig. 3 is 0.18%.

Convergence of the model is shown in Fig. 4. It is obtained by performing h-refinement. It appears that highly satisfactory level of errors are reached quickly. Indeed with 1600 degrees of freedom $\epsilon_2 \approx 10^{-3}\%$. If one were to model the same problem using classical linear FEM, using the rule of 10 elements per wavelength, 8300 quadrangular elements would be necessary, leading to a minimum of $33,000$ degrees of freedom.

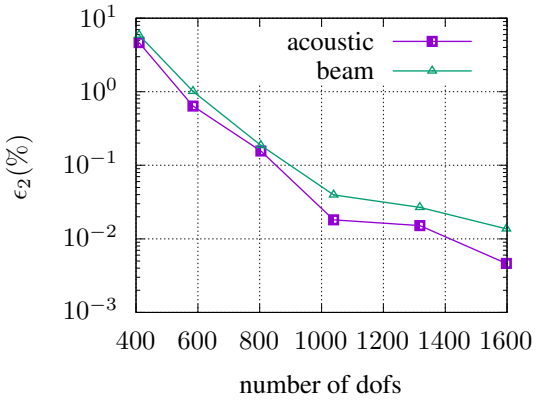

Figure 4: Convergence at $f = 3000$Hz.

4. CONCLUSION

The method presented in this paper couples a Euler-Bernoulli beam with an acoustic domain using the PUFEM. It is shown that it produces highly satisfactory results. Indeed low level of error are obtained around $10^{-3}\%$. The efficiency of the method is proven since this level of error is obtained with very low numbers of degrees of freedom. To push the analysis further authors are investigating other coupling strategies involving PUFEM elements of different sizes which permits to consider situations where structural and acoustic wavelengths differ notably.

5. ACKNOWLEDGEMENTS

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