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# VIBRATIONAL BLACK HOLE AS A MATCHING STRUCTURE

Mikhail Mironov, Alexei Gladilin

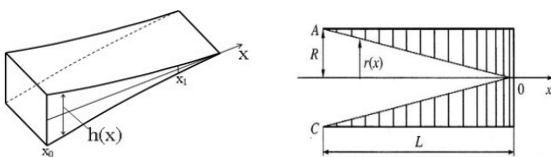
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## ABSTRACT

The wave decelerating structures with the propagation velocity gradually decreasing to zero at a finite interval theoretically provide complete absorption of the incoming wave without the use of absorbing substances. Due to the physical imperfections of the proposed vibrational black hole (VBH) designs, an absorbing material has to be added to effectively absorb waves. We consider the use of VBH as a matching structure that connects an element with high impedance to another element with low impedance and absorbing material. Using the set of exact solutions for parabolically sharpened rod, simple algorithm for recalculation of the elements of the impedance matrix of a transversal oscillating VBH rod, is suggested.

## 1. INTRODUCTION

Decelerating structures with a propagation velocity that gradually decreases to zero at a finite interval theoretically provide complete absorption of the incoming wave without the use of absorbing materials [1, 2]. The time of waves propagation over such a structure is equal to infinity. The wave that enters the slowing structure will not reach its end in any finite time and, therefore, will not be reflected from it. This property resembles the behavior of light in a strong gravitational field near a cosmological black hole. In this regard, in acoustics, structures that slow down the speed of wave propagation and capture waves due can be called, depending on the type of waves, acoustic black holes (ABH), or vibrational black holes (VBH) [3]. The initially suggested constructions of VBH and ABH are shown in Fig.1.



**Figure 1.** Two examples of construction with slowing down the propagation velocity. VBH – on the left [1]; ABH – on the right [2].

To date, there is an extensive literature on the theoretical and experimental study of A/VBH. Intensive research on various aspects of this problem began after the papers [4,5]. Many works published before 2018 are noted, for example, in the review [6]. More recent works are usually

also provided with a fairly clear description of the current state of the problem.

In this paper, the decelerating structure of the VBH is considered from the point of view of matching acoustically rigid and acoustically soft structures. There are different methods for calculating the vibration of rods of variable cross-section: WKB-approximation [7, 8, 9], impedance matrices with using Riccati equations [10, 11, 12], Rayleigh-Ritz approach with different sets of basic functions [13, 14, 15] and numerical FEM methods [16]. Below impedance matrix calculations for the standard VBH - rod with parabolic profile - are developed. Exact solutions in the form of power functions are used [17].

## 2. EQUATIONS, MATRIX OF INNER IMPEDANCE, ANALYTICAL SOLUTIONS

The equation of transverse vibrations of a rod with variables along its axis of linear mass and flexural stiffness has the form in the approximation of the Euler-Bernoulli model:

$$-\rho \cdot S(x)\omega^2 \eta + (E \cdot I(x)\eta''') = 0 \quad (1)$$

$\rho$  is the density of the rod material,  $S(x)$  is the cross-sectional area,  $\omega$  is the frequency,  $I(x)$  is the moment of inertia of the cross-section,  $\eta$  is the transverse displacement.

The moment and cutting force acting on the cross section inside the rod are equal:

$$\begin{aligned} M &= -EI\eta'' \\ F &= M' = -(EI\eta''') \end{aligned} \quad (2)$$

Then the internal impedance matrix is introduced  $\mathbf{Z}(x)$  in section  $x$ , connecting bending moment  $M(x)$  and force  $F(x)$  with displacement  $\eta$  and slope  $\eta'$  in the same section [10]:

$$M(x) = Z_{11}(x)\eta'(x) + Z_{12}(x)\eta(x) \quad (3)$$

$$F(x) = Z_{21}(x)\eta'(x) + Z_{22}(x)\eta(x) \quad (4)$$

Formulas (1-4) fully describe the problem of converting the impedance matrix from one section of a non-uniform rod to any other section.

Next, for certainty, we consider one of the possible distributions  $S(x)$  and  $I(x)$ , which provide solutions in the form of power functions, discussed earlier in [3, 17].

Specifically, we consider a parabolically sharpened rod (see left part of fig.1) [17] for which:

$$S(x) = h(x) = h_0 \left(\frac{x}{L}\right)^2 \quad (5)$$

$$I(x) = \frac{1}{12} h^3(x) = \frac{1}{12} h_0^3 \left(\frac{x}{L}\right)^6 \quad (6)$$

Substituting (5, 6) into (1), we get the following differential equation:

$$bx^2\eta - (x^6\eta''') = 0 \quad (7)$$

with nondimensional parameter

$$b = 12 \frac{\omega^2}{c_Y^2} \left(\frac{L^2}{h_0}\right)^2 = (\kappa(h_0) \cdot L)^4 \quad (8)$$

Here and further  $c_Y = \sqrt{E/\rho}$  is the Young velocity of the longitudinal wave propagation in the rod,

$$\kappa(h_0) = \left(12 \frac{\omega^2}{c_Y^2 h_0^2}\right)^{1/4} \quad (9)$$

The dimensionless parameter  $b$  is proportional to the square of the frequency and is equal to the fourth power of the wave size of the original segment of the homogeneous rod at  $x = L$ .

We are looking for a solution (7) in the form of a power function:

$$\eta(x) = (x/L)^\nu \quad (10)$$

After substituting (10) into (7), an algebraic equation for the exponent  $\nu$  is obtained:

$$\nu(\nu-1)(\nu+4)(\nu+3) - b = 0 \quad (11)$$

The solutions to this equation are as follows [17]:

$$\nu_{1,2,3,4} = -\frac{3}{2} \pm \left\{ \frac{17}{4} \pm \sqrt{4+b} \right\}^{1/2} \quad (12)$$

At  $b \rightarrow 0$  (low frequencies), all roots (12) are real. Vibrations of the rod are in-phase at all points, there are no waves. In the other extreme case  $b \rightarrow \infty$  (high frequencies), (12) gives a pair of purely real (with a plus sign before the square root in (12)) and a pair of complex – conjugated (with a minus sign before the square root in (12)) roots. Complex roots correspond to propagating waves; real roots correspond to non-propagating waves. Thus, there is a complete analogy with a constant cross-section rod, for which the complete set of solutions consists of two propagating and two exponentially damped waves. The critical value  $b$  at which the imaginary component appears is equal to:

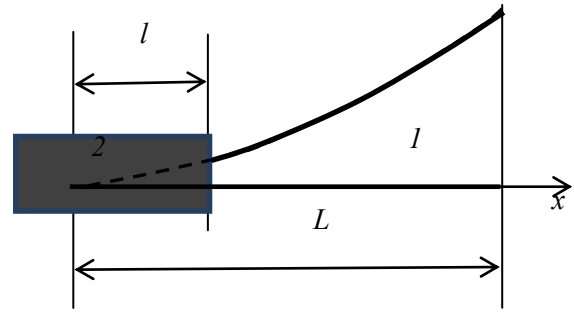
$$b^* = \left(\frac{3}{2}\right)^2 \left(\frac{5}{2}\right)^2 \quad (13)$$

Accordingly, the critical frequency is equal to:

$$\omega^* = \sqrt{\frac{c_Y^2}{12} b^*} \cdot \frac{h_0}{x_0^2} = \frac{1}{\sqrt{12}} c_Y \frac{15}{4} \cdot \frac{h_0}{x_0^2} \quad (14)$$

### 3. CALCULATION OF INPUT IMPEDANCE MATRIX

Next, we solve the boundary value problem (fig.2).



**Figure 2.** Truncated vibrational black hole 1 with impedance load at the end 2.  $L$  – the length of full VBH,  $l$  – the length of truncation.

Parabolically sharpened rod 1 with the length  $L$  is cut in the section  $x=l$ . Some load 2 with impedance matrix  $Z(l)$  is attached to this section (shaded rectangle in fig.2). It is necessary to calculate the impedance matrix  $Z(L)$  in the section  $x=L$ .

Let's set the displacement and slope in the section  $x=L$ :

$$\eta(L) = \xi, \quad \eta'(L) = \xi'.$$

We search the displacement field as the sum of four solutions - power functions (12) with exponents (12) [17]:

$$\eta(x) = \sum_{j=1..4} A_j (x/L)^{\nu_j} \quad (15)$$

Let's rewrite all the exponents of the power functions (12) separately:

$$\nu_2 = -\frac{3}{2} - \left\{ \frac{17}{4} - \sqrt{4+b} \right\}^{1/2}; \quad \nu_4 = -\frac{3}{2} + \left\{ \frac{17}{4} + \sqrt{4+b} \right\}^{1/2};$$

$$\nu_1 = -\frac{3}{2} + \left\{ \frac{17}{4} - \sqrt{4+b} \right\}^{1/2}; \quad \nu_3 = -\frac{3}{2} - \left\{ \frac{17}{4} + \sqrt{4+b} \right\}^{1/2}.$$

The exponent  $\nu_2$  (complex) corresponds to propagation in the negative axis direction (to the end of BH) wave, the exponent  $\nu_4$  (purely real) corresponds to a wave, decaying in this direction. The exponent  $\nu_1$  (complex) corresponds to a wave propagating in the positive direction of the axis (from the end of the BH), the exponent  $\nu_3$  (purely real) corresponds to a wave decaying in this direction.

Next, we use a dimensionless coordinate, normalized by length  $L$ :  $X = x/L$ . The dimensionless coordinate of the end section is denoted by  $\delta = l/L$ . After making the necessary differentiations in (2) and taking into account the power dependencies (5, 6) and (15), we obtain expressions for the moment and force through the wave amplitudes  $A_j$ :

$$M(x) = -EI\eta''(x) = -\frac{1}{12} \frac{E}{h_0^3 L^2} \sum_{j=1..4} A_j \nu_j (\nu_j - 1) (X)^{\nu_j + 4} \quad (16)$$

$$F(X) = -(EI\eta''')' = -\frac{1}{12} \frac{E}{h_0^3 L^3} \sum_{j=1..4} A_j \nu_j (\nu_j - 1) (\nu_j + 4) (X)^{\nu_j + 3} \quad (17)$$

Next, we solve the following boundary value problem. In the initial section  $X = 1$ , we set the displacement and the slope  $\eta(L) = \xi$ ,  $\eta'(L) = \xi'$ :

$$\sum_{j=1..4} A_j = \xi \quad (18)$$

$$\frac{1}{L} \sum_{j=1..4} A_j v_j = \xi' \quad (19)$$

In the end section  $X = \delta$  we set the impedance condition (3, 4) with impedance matrix  $\mathbf{Z}(\delta)$ :

$$\sum_{j=1..4} A_j \left[ -\frac{1}{12} \frac{E}{h_0^3} \frac{1}{L^2} v_j (v_j - 1) \delta^{v_j+4} - \right. \\ \left. - \frac{1}{L} Z_{11}(\delta) v_j \delta^{v_j-1} - Z_{12}(\delta) \delta^{v_j} \right] = 0 \quad (20)$$

$$\sum_{j=1..4} A_j \left[ -\frac{1}{12} \frac{E}{h_0^3} \frac{1}{L^3} v_j (v_j - 1) (v_j + 4) \delta^{v_j+3} - \right. \\ \left. - \frac{1}{L} Z_{21}(\delta) v_j \delta^{v_j-1} - Z_{22}(\delta) \delta^{v_j} \right] = 0 \quad (21)$$

The system of equations (18-21) for wave amplitudes  $A_j$  inside the BH is written in matrix form:

$$\mathbf{U} \cdot \mathbf{A} = \xi \quad (22)$$

Here, the matrix  $\mathbf{U}$  is a set of coefficients at the amplitudes in the right part of the equations (18 – 21),

$\mathbf{A}$  - the vector of wave amplitudes,  $\xi = (\xi, L\xi', 0, 0)^T$  - the vector of the right parts of the equations (18-21) (the upper index T is the transposition sign).

The matrix elements of  $\mathbf{U}$  are written below (the first subscript is the row number, the second subscript is the column number):

$$U_{1k} = 1; U_{2k} = v_k; \\ U_{3k} = -\frac{1}{12} \frac{E}{h_0^3} \frac{1}{L^2} v_k (v_k - 1) \delta^{v_k+4} - \\ - \frac{1}{L} Z_{11}(\delta) v_k \delta^{v_k-1} - Z_{12}(\delta) \delta^{v_k}; \\ U_{4k} = -\frac{1}{12} \frac{E}{h_0^3} \frac{1}{L^3} v_k (v_k - 1) (v_k + 4) \delta^{v_k+3} - \\ - \frac{1}{L} Z_{21}(\delta) v_k \delta^{v_k-1} - Z_{22}(\delta) \delta^{v_k}.$$

$k = 1, 2, 3, 4$

Equation (22) is solved by inverting the matrix  $\mathbf{U}$ :

$$\mathbf{A} = \mathbf{U}^{-1} \cdot \xi \quad (23)$$

After determining the wave amplitudes  $A_j$ , we are able to write an expression for the displacement field in the form (15) with known amplitudes and exponents. Then, substituting (15) in the expressions for the moment and force (2), we define them at any point in the BH, including in the initial section  $x = L$ . Next, the impedance matrix in this section is calculated using the formulas (3, 4):

$$M(L) = Z_{11}(L)\xi' + Z_{12}(L)\xi \quad (24)$$

$$F(L) = Z_{21}(L)\xi' + Z_{22}(L)\xi \quad (25)$$

Calculating the moment and force in the initial section at the specified values of displacement  $\xi$  and slope  $\xi'$  in the

same section, we get an elements of the impedance matrix in the initial section:

$$Z_{11}(L) = M(L) / \xi' \Big|_{\xi=0}; Z_{12}(L) = M(L) / \xi \Big|_{\xi'=0}; \\ Z_{21}(L) = F(L) / \xi' \Big|_{\xi=0}; Z_{22}(L) = F(L) / \xi \Big|_{\xi'=0}. \quad (26)$$

The right-hand side of these equalities includes the values of the impedances in the section  $x=l=\delta \cdot L$  through the values of momentum and force at  $x=L$ . As a consequence, this determines the relationship between the elements of the impedance matrix in the sections  $L$  and  $l=\delta \cdot L$ .

#### 4. CONCLUSION

The algorithm proposed above for recalculating the impedance matrix of a non-uniform rod with a special parabolic profile – VBH - includes only simple algebraic operations: calculation of power functions, inversion of the 4x4 matrix, and multiplication. This allows one to make quick calculations for selecting parameters of connected and connecting element without large computational costs. This algorithm can be useful for investigations of both vibration absorption and energy storage problems, where the general goal consists in efficient connecting the objects with very different impedances.

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