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Two linear conjugation problems, arising in the theory of the flow over an obstacle.

Evgeny V. Semenko, Tatiana I. Semenko¹

Abstract. The two linear conjugation problems on the real axis are considered. These problems additionally include the symmetry condition and the asymptotic condition at the infinity and arise in the hydrodynamics, in some models of the flow over an obstacle. The coefficients and right hand sides of the conjugation conditions (and, naturally, the solution) depend on the complex parameter (ω in the text). The specific feature of these problems is that the right hand sides of conjugation conditions include the values of the sought analytic functions in given points. Additionally the right hand side in the first problem includes the unknown function, that should be determined together with the solution.

The solutions of the problems are constructed and some their properties are discussed, in particular the asymptotics of the solutions, when the parameter ω tends to zero, is found.

Key words. Linear conjugation problem, complex variables, analytic function, Cauchy integral.

1 Introduction, problems statement

This article is organized as follows. The section 1 contains the statement of two problems (problem 1 and 2 respectively). In the key section 2, we construct the solution of problem 1 for complex parameters ω , $\text{Im}\omega > 0$. The following section 3 is devoted to the solution at the real value of the parameter ω , when the coefficient in the linear conjugation condition degenerates. In the short section 4, we construct the solution of problem 2. In the section 5, we find the asymptotics of the solutions at $\omega \rightarrow 0$. At last, in the appendix (section 6), all necessary subsidiary technical results are collected.

In the whole text, $\nu = \text{const} \in \mathbb{R}$, $\nu > 1$, and we use on default the notations ξ , ω for the complex numbers ($\text{Im}\omega \geq 0$) and χ , μ , s for the real numbers.

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1.1 The problem 1

At each fixed ω , $\text{Im } \omega \geq 0$, we consider the linear conjugation (Riemann) problem

$$\begin{aligned} & (\nu(\omega - \chi)^2 \cosh(\chi) - \chi \sinh(\chi))V^+(\chi, \omega) = \\ & -(\nu(\omega + \chi)^2 \cosh(\chi) - \chi \sinh(\chi))V^-(\chi, \omega) - \chi \sinh(\chi)V^+(\omega, \omega) - \\ & \frac{\sinh(\chi)(\omega^2 - \chi^2)}{\chi} \left(\frac{1}{2\pi^2} + h(\omega) \right) + \cosh(\chi)(\omega^2 - \chi^2)h(\omega), \quad \chi \in \mathbb{R}, \end{aligned} \quad (1)$$

with the symmetry condition

$$V^+(\xi, \omega) = V^(-(-\xi, \omega), \quad \xi \in \mathbb{C}, \quad (2)$$

and the asymptotic condition at the infinity

$$V^\pm(\xi, \omega) = -\frac{h(\omega)}{2\nu} + O\left(\frac{1}{\xi}\right), \quad \xi \rightarrow \infty, \quad (3)$$

where the unknowns $V^\pm(\chi, \omega)$ are the boundary values of the functions $V^\pm(\xi, \omega)$, analytic with respect to ξ at $\pm \text{Im } \xi > 0$ respectively. The right hand side of the equation (1) contains the value of unknown $V^+(\omega, \omega)$ and the function $h(\omega)$, which is unknown too and should be determined together with the solution $V^\pm(\xi, \omega)$.

The equation (1) we write in a form

$$V^+(\chi, \omega) = -G(\chi, \omega)V^-(\chi, \omega) + g(\chi, \omega), \quad \chi \in \mathbb{R}, \quad (4)$$

where

$$G(\chi, \omega) = \frac{\nu(\chi + \omega)^2 - \chi \tanh(\chi)}{\nu(\chi - \omega)^2 - \chi \tanh(\chi)}, \quad \text{or } G(\chi, \omega) = \frac{F(-\chi, \omega)}{F(\chi, \omega)}, \quad (5)$$

here

$$F(\xi, \omega) = \nu \cosh(\xi)(\xi - \omega)^2 - \xi \sinh(\xi) \quad (6)$$

is analytic in a whole plane (entire) function;

$$g(\chi, \omega) = -g_1(\chi, \omega) \left(\frac{1}{2\pi^2} + h(\omega) \right) + g_2(\chi, \omega)h(\omega) - g_3(\chi, \omega)V^+(\omega, \omega), \quad (7)$$

where the functions

$$\begin{aligned} g_1(\xi, \omega) &= \frac{\sinh(\xi)(\omega^2 - \xi^2)}{\xi F(\xi, \omega)}, \quad g_2(\xi, \omega) = \frac{\cosh(\xi)(\omega^2 - \xi^2)}{F(\xi, \omega)}, \\ g_3(\xi, \omega) &= \frac{\xi \sinh(\xi)}{F(\xi, \omega)} \end{aligned} \quad (8)$$

are meromorphic in a whole plane and their poles are the zeroes of $F(\xi, \omega)$.

1.2 The problem 2

Like in the previous subsection, we consider the linear conjugation problem for unknown analytic functions $V_0^\pm(\xi, \omega)$:

$$\begin{aligned} V_0^+(\chi, \omega) + V_0^-(\chi, \omega) - V_0^+(\omega, \omega) = \\ \left(\frac{1}{2\pi^2} + h(\omega) \right) \frac{(\cosh(\chi) - 1)(\omega^2 - \chi^2)}{\chi^2 \cosh(\chi)} + \\ \frac{V^+(\chi, \omega) + V^-(\chi, \omega) - V^+(\omega, \omega)}{\cosh(\chi)}, \end{aligned} \quad (9)$$

with the symmetry condition

$$V_0^+(\xi, \omega) = V_0^(-\xi, \omega), \quad (10)$$

and the asymptotic condition at the infinity

$$V_0^\pm(\xi, \omega) = V_{02}(\omega) + O\left(\frac{1}{\xi}\right), \quad \xi \rightarrow \infty, \quad (11)$$

where the function $V_{02}(\omega)$ is unknown and should be determined together with the solution $V_0^\pm(\xi, \omega)$. Here $V^\pm(\xi, \omega)$ and $h(\omega)$ is the solution of the problem 1.

2 The solution of the problem 1

In this section, we assume on default $\text{Im } \omega > 0$.

To construct the solution of linear conjugation problem, we use absolutely classical way, see e.g. [1]. As usual, we introduce the notation for the Cauchy integrals

$$\begin{aligned} S^\pm(f(s) | \xi) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s) ds}{s - \xi}, \quad \pm \text{Im } \xi > 0, \\ S(f(s) | \chi) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s) ds}{s - \chi}, \quad \chi \in \mathbb{R}, \end{aligned}$$

and use the well known Plemelj formulas (see e.g. [1, 2]):

$$S^\pm(f(s) | \chi) = \lim_{\xi \rightarrow \chi \in \mathbb{R}} S^\pm(f(s) | \xi) = S(f(s) | \chi) \pm \frac{f(\chi)}{2}. \quad (12)$$

In particular,

$$S^\pm(1 | \xi) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{ds}{s - \xi} = \pm \frac{1}{2}, \quad \pm \operatorname{Im} \xi > 0; \quad S(1 | \chi) = 0, \quad \chi \in \mathbb{R}. \quad (13)$$

We note more two evident properties of the Cauchy integrals for the even and odd functions:

$$\begin{aligned} \varphi(-s) = \varphi(s) &\implies S^+(\varphi | \xi) = -S^-(\varphi | -\xi), \\ S(\varphi | 0) = 0, \quad S^\pm(\varphi | 0) &= \pm \frac{\varphi(0)}{2}; \end{aligned} \quad (14)$$

$$\varphi(-s) = -\varphi(s) \implies S^+(\varphi | \xi) = S^-(\varphi | -\xi), \quad S^\pm(\varphi | 0) = S(\varphi | 0). \quad (15)$$

2.1 The solution of the homogeneous problem

Here we consider the homogeneous problem

$$X^+(\chi, \omega) = G(\chi, \omega)X^-(\chi, \omega), \quad \chi \in \mathbb{R}, \quad (16)$$

where $G(\chi, \omega)$ has a form (5), and $X^\pm(\xi, \omega)$ are unknown functions, analytic with respect to ξ at $\pm \operatorname{Im} \xi > 0$ respectively.

Lemma 1 *1. $F(\chi, \omega) \neq 0$ and so $G(\chi, \omega)$ is smooth, bounded and doesn't vanish, and*

$$\operatorname{ind} G(\chi, \omega) \Big|_{\chi \in \mathbb{R}} = \frac{1}{2\pi} \Delta \arg G(\chi, \omega) \Big|_{\chi \in \mathbb{R}} = -2.$$

- 2. In class of functions with no more than power-law growth with respect to ξ at the infinity, there exists only (up to an ω -dependent factor $C(\omega)$) analytic at $\pm \operatorname{Im} \xi > 0$ and non vanishing solution of problem (16). This solution $X^\pm(\xi, \omega)$ is analytic with respect to ω in the upper half plane.*
- 3. $X^+(-\xi, \omega) = X^-(\xi, \omega)$ and the solution X^\pm (i.e. the factor $C(\omega)$) may be chosen so that*

$$X^\pm(\xi, \omega) = \xi^2 \pm 2\omega\xi + O(1), \quad \xi \rightarrow \infty. \quad (17)$$

So, the additional condition

$$\lim_{\xi \rightarrow \infty} \frac{X^\pm(\xi, \omega)}{\xi^2} = 1 \quad (18)$$

determines the non vanishing solution of the homogeneous problem (16) unambiguously.

4. The function

$$Y(\xi, \omega) = \begin{cases} F(\xi, \omega)X^+(\xi, \omega), & \text{Im } \xi \geq 0, \\ F(-\xi, \omega)X^-(\xi, \omega), & \text{Im } \xi \leq 0, \end{cases} \quad (19)$$

is even and entire with respect to ξ and

$$\frac{1}{Y(\chi, \omega)} = \frac{1}{\nu\chi^4 \cosh(\chi)} \left(1 + O\left(\frac{1}{\chi}\right) \right), \quad \chi \rightarrow \infty, \quad \chi \in \mathbb{R}. \quad (20)$$

Proof. The function (6) $F(\chi, \omega)$ we represent in a form

$$F(\chi, \omega) = \nu \cosh(\chi) \left(\chi - \sqrt{\frac{\chi \tanh(\chi)}{\nu}} - \omega \right) \left(\chi + \sqrt{\frac{\chi \tanh(\chi)}{\nu}} - \omega \right). \quad (21)$$

Since $\chi \pm \sqrt{\chi \tanh(\chi)/\nu} \in \mathbb{R}$, then at $\text{Im } \omega \neq 0$ we have $F(\chi, \omega) \neq 0$. Besides, it follows from (5), (21), that

$$G(\chi, \omega) = \left(\frac{\chi - \sqrt{\frac{\chi \tanh(\chi)}{\nu}} + \omega}{\chi - \sqrt{\frac{\chi \tanh(\chi)}{\nu}} - \omega} \right) \cdot \left(\frac{\chi + \sqrt{\frac{\chi \tanh(\chi)}{\nu}} + \omega}{\chi + \sqrt{\frac{\chi \tanh(\chi)}{\nu}} - \omega} \right). \quad (22)$$

Let $x(\chi) = \chi - \sqrt{\chi \tanh(\chi)/\nu}$. The function $x(\chi)$ is real and continuous and $x(\pm\infty) = \pm\infty$, and so

$$\text{ind} \frac{\chi - \sqrt{\frac{\chi \tanh(\chi)}{\nu}} + \omega}{\chi - \sqrt{\frac{\chi \tanh(\chi)}{\nu}} - \omega} \Big|_{\chi \in \mathbb{R}} = \text{ind} \frac{x + \omega}{x - \omega} \Big|_{x \in \mathbb{R}} = -1,$$

since the function $(z + \omega)/(z - \omega)$, $z = x + iy$ is analytic in a whole plane, including $z = \infty$ and has one zero $z = -\omega$ in the lower half plane and one pole $z = \omega$ in the upper half plane. Similarly

$$\text{ind} \frac{\chi + \sqrt{\frac{\chi \tanh(\chi)}{\nu}} + \omega}{\chi + \sqrt{\frac{\chi \tanh(\chi)}{\nu}} - \omega} \Big|_{\chi \in \mathbb{R}} = -1,$$

and it follows from (22), that the index of $G(\chi, \omega)$ as a whole is equal to -2 . The first lemma statement proved.

Now we construct the solution $X^\pm(\xi, \omega)$. We choose arbitrary in the upper half plane two points ξ_1 and ξ_2 (they may be coincident and may be depended on ω) and let

$$G_0(\chi, \omega) = G(\chi, \omega) \frac{(\chi - \xi_1)(\chi - \xi_2)}{(\chi + \xi_1)(\chi + \xi_2)}. \quad (23)$$

Evidently, the index of $G_0(\chi, \omega)$ is equal to zero, i.e. one may choose the one-valued and continuous branch of $\ln G_0(\chi, \omega)$. Let

$$\begin{aligned} X_0^\pm(\xi, \omega) &= \exp(S^\pm(\ln G_0(s, \omega) | \xi)), \\ X^\pm(\xi, \omega) &= X_0^\pm(\xi, \omega)(\xi \pm \xi_1)(\xi \pm \xi_2). \end{aligned} \quad (24)$$

According to the Plemelj formulas (12) and (23), (24), we obviously have

$$\frac{X_0^+(\chi, \omega)}{X_0^-(\chi, \omega)} = G_0(\chi, \omega) \text{ and } \frac{X^+(\chi, \omega)}{X^-(\chi, \omega)} = G(\chi, \omega),$$

i.e. $X^\pm(\xi, \omega)$ is the solution of the homogeneous problem (16). The functions $X_0^\pm(\xi, \omega)$, and together with them the functions $X^\pm(\xi, \omega)$, are analytic with respect to ξ and don't vanish at $\pm \text{Im } \xi > 0$.

The function $F(\xi, \omega)$ and together with it the coefficient $G(\chi, \omega)$ are evidently analytic with respect to ω . So, if the chosen points ξ_1, ξ_2 are analytic with respect to ω , then the functions (24) $X_0^\pm(\xi, \omega)$ and $X^\pm(\xi, \omega)$ are obviously analytic with respect to ω .

Let's prove the uniqueness of $X^\pm(\xi)$. If $X_1^\pm(\xi, \omega)$ is the other non vanishing solution of the problem (16), i.e.

$$\frac{X_1^+(\chi, \omega)}{X_1^-(\chi, \omega)} = \frac{X^+(\chi, \omega)}{X^-(\chi, \omega)},$$

then for $\Phi^\pm(\xi, \omega) = X_1^\pm(\xi, \omega)/X^\pm(\xi, \omega)$ we have $\Phi^+(\chi, \omega) = \Phi^-(\chi, \omega)$, i.e. the function $\Phi(\xi, \omega)$ is analytic with respect to ξ in a whole plane, hasn't zeroes and has no more than power-law growth at the infinity, whence follows $\Phi^\pm(\xi, \omega) = C(\omega)$. The second lemma statement proved.

Now we go to the third statement. It evidently follows from (5) and (23), that

$$G(-\chi, \omega) = G^{-1}(\chi, \omega), \quad G_0(-\chi, \omega) = G_0^{-1}(\chi, \omega),$$

and so the function $\ln G_0(\chi, \omega)$ is odd with respect to χ , then according to (15) $S^\pm(\ln G_0 | \xi)$ is even: $S^+(\ln G_0 | -\xi) = S^-(\ln G_0 | \xi)$, and so the functions (24) $X_0^\pm(\xi, \omega)$, $X^\pm(\xi, \omega)$ are symmetric:

$$X_0^+(-\xi, \omega) = X_0^-(\xi, \omega), \quad X^+(-\xi, \omega) = X^-(\xi, \omega). \quad (25)$$

The representation (5) evidently follows that

$$G(\chi, \omega) = 1 + \frac{G_1(\omega)}{\chi} + O\left(\frac{1}{\chi^2}\right), \quad \chi \rightarrow \infty,$$

consequently according to the (23) we have the same representation for G_0 :

$$G_0(\chi, \omega) = 1 + \frac{G_1^0(\omega)}{\chi} + O\left(\frac{1}{\chi^2}\right), \quad \chi \rightarrow \infty.$$

Whence one may choose the branch of $\ln G_0(\chi)$ in such a way, that

$$\ln G_0(\chi) = \frac{G_1^0(\omega)}{\chi} + O\left(\frac{1}{\chi^2}\right), \quad \chi \rightarrow \infty.$$

So according to the formula (24) and lemma 2 (Appendix, subsection 6.1), we obtain

$$\ln X_0^\pm(\xi, \omega) = \frac{X_{01}^\pm(\omega)}{\xi} + O\left(\frac{1}{\xi^2}\right), \quad \xi \rightarrow \infty,$$

hence

$$X_0^\pm(\xi, \omega) = 1 + \frac{X_{01}^\pm(\omega)}{\xi} + O\left(\frac{1}{\xi^2}\right), \quad \xi \rightarrow \infty,$$

and so

$$X^\pm(\xi, \omega) = (\xi \pm \xi_1)(\xi \pm \xi_2)X_0^\pm(\xi, \omega) = \xi^2 + A_1^\pm(\omega)\xi + O(1), \quad \xi \rightarrow \infty, \quad (26)$$

and the symmetry condition $X^+(-\xi, \omega) = X^-(-\xi, \omega)$ follows that $A_1^\pm(\omega) = \pm A_1(\omega)$.

Let's find A_1 . Up to the branch choice, we have at $\chi \rightarrow \infty$

$$\begin{aligned} \ln G(\chi, \omega) &= -4\pi i \theta(\chi) + \frac{4\omega}{\chi} + O\left(\frac{1}{\chi^2}\right), \\ \ln \frac{(\chi - \xi_1)(\chi - \xi_2)}{(\chi + \xi_1)(\chi + \xi_2)} &= 4\pi i \theta(\chi) - \frac{2(\xi_1 + \xi_2)}{\chi} + O\left(\frac{1}{\chi^2}\right), \end{aligned}$$

where $\theta(\chi)$ is a Heaviside function, whence

$$\ln G_0(\chi) = \frac{4\omega - 2(\xi_1 + \xi_2)}{\chi} + O\left(\frac{1}{\chi^2}\right),$$

and then according to the the formula (24) and lemma 2 we obtain

$$\ln X_0^\pm(\xi, \omega) = \pm \frac{2\omega - (\xi_1 + \xi_2)}{\xi} + O\left(\frac{1}{\xi^2}\right), \quad \xi \rightarrow \infty,$$

i.e.

$$X_0^\pm(\xi) = 1 \pm \frac{2\omega - (\xi_1 + \xi_2)}{\xi} + O\left(\frac{1}{\xi^2}\right), \quad \xi \rightarrow \infty,$$

and since

$$(\xi \pm \xi_1)(\xi \pm \xi_2) = \xi^2 \pm (\xi_1 + \xi_2)\xi + O(1), \quad \xi \rightarrow \infty,$$

then

$$X^\pm(\xi) = (\xi \pm \xi_1)(\xi \pm \xi_2)X_0^\pm(\xi) = \xi^2 \pm 2\omega\xi + O(1), \quad \xi \rightarrow \infty,$$

hence $A_1^\pm = \pm A_1 = \pm 2\omega$. Thus the third lemma statement follows from (25), (26).

Now we consider the last lemma statement. It follows from the equalities

$$G(\chi, \omega) = F(-\chi, \omega)/F(\chi, \omega) = X^+(\chi, \omega)/X^-(\chi, \omega),$$

that $F(\chi, \omega)X^+(\chi, \omega) = F(-\chi, \omega)X^-(\chi, \omega)$, $\chi \in \mathbb{R}$, i.e. the function (19) $Y(\xi, \omega)$ is entire with respect to ξ . The equality $Y(-\xi, \omega) = Y(\xi, \omega)$ obviously follows from (25). At last, from the representation $F(\xi, \omega)$ (6) it follows that

$$F(\chi, \omega) = \nu\chi^2 \cosh(\chi) \left(1 + O\left(\frac{1}{\chi}\right)\right), \quad \chi \rightarrow \infty, \quad \chi \in \mathbb{R},$$

together with the condition (17) it implies the condition (20). So, the lemma proved completely.

Remark. We emphasize, at the proving of the lemma, we directly construct the solution $X^\pm(\xi, \omega)$ of the homogeneous problem (16), satisfying to the condition (18). From the uniqueness of this solution (the third statement of the lemma) it follows that this solution doesn't depend on the choice of the points ξ_j , $j = \overline{1, 2}$.

2.2 The solution of the inhomogeneous problem

We construct the solution of the inhomogeneous problem (4) by the absolutely classical way: since the coefficient index $\text{ind } G = -2 < 0$ (the first statement of the lemma 1), then the only bounded solution of problem (4) must have a form

$$V^\pm(\xi, \omega) = \pm X^\pm(\xi, \omega)S^\pm \left(\frac{g(s, \omega)}{X^+(s, \omega)} \mid \xi \right), \quad (27)$$

and the solution (27) is bounded if and only if the right hand side $g(\chi, \omega)$ satisfies two solvability conditions, see e.g.[1].

So, we have to investigate the function (27). According to the representations (7), (8) of $g(\chi, \omega)$, it has a form

$$V^\pm(\xi, \omega) = \mp X^\pm(\xi, \omega) S^\pm(g_{01}(s, \omega) | \xi) \left(\frac{1}{2\pi^2} + h(\omega) \right) \pm \quad (28)$$

$$X^\pm(\xi, \omega) S^\pm(g_{02}(s, \omega) | \xi) h(\omega) \mp X^\pm(\xi, \omega) S^\pm(g_{03}(s, \omega) | \xi) V^\pm(\omega, \omega),$$

where

$$g_{01}(\chi, \omega) = \frac{g_1(\chi, \omega)}{X^+(\chi, \omega)} = \frac{\sinh(\chi)(\omega^2 - \chi^2)}{\chi Y(\chi, \omega)},$$

$$g_{02}(\chi, \omega) = \frac{g_2(\chi, \omega)}{X^+(\chi, \omega)} = \frac{\cosh(\chi)(\omega^2 - \chi^2)}{Y(\chi, \omega)}, \quad (29)$$

$$g_{03}(\chi, \omega) = \frac{g_3(\chi, \omega)}{X^+(\chi, \omega)} = \frac{\chi \sinh(\chi)}{Y(\chi, \omega)},$$

and $Y(\xi, \omega)$ have a form (19).

Since $Y(\chi, \omega)$ is even with respect to χ (the last statement of the lemma 1), then the functions (29) $g_{0j}(\chi, \omega)$, $j = \overline{1, 3}$ are even too, and so according to the Cauchy integral property (14), the functions $S(g_{0j} | \xi)$ are odd: $S^+(g_{0j} | \xi) = -S^-(g_{0j} | -\xi)$, $j = \overline{1, 3}$, therefore the solution (28) is symmetric: $V^+(\xi, \omega) = V^-(-\xi, \omega)$, i.e. the condition (2) fulfilled.

To construct the solution of the problem 1, it remains to fulfill the asymptotic condition (3) and to find the functions $h(\omega)$ and yet unknown $V^+(\omega, \omega)$.

The asymptotics of $1/Y(\chi, \omega)$ (20) evidently follows that the functions (29) g_{0j} have a representation

$$g_{01}(\chi, \omega) = O\left(\frac{1}{\chi^3}\right), \quad g_{02} = -\frac{1}{\nu\chi^2} + O\left(\frac{1}{\chi^3}\right), \quad g_{03} = O\left(\frac{1}{\chi^3}\right), \quad \chi \rightarrow \infty.$$

Using the lemma 2 (Appendix, subsection 6.1), we obtain

$$S^\pm(g_{0j}(s, \omega) | \xi) = -\frac{J_j(\omega)}{\xi} - \frac{K_j(\omega)}{\xi^2} \pm \frac{R_j}{2\xi^2} + O\left(\frac{1}{\xi^3}\right), \quad \xi \rightarrow \infty, \quad (30)$$

where

$$J_j(\omega) = \frac{1}{2\pi i} \int_{\mathbb{R}} g_{0j}(s, \omega) ds, \quad j = \overline{1, 3}; \quad (31)$$

$$K_j(\omega) = \frac{1}{2\pi i} \int_{\mathbb{R}} s g_{0j}(s, \omega) ds, \quad j = \overline{1, 3};$$

$$R_j = \begin{cases} 0, & j = 1, 3, \\ -1/\nu, & j = 2. \end{cases} \quad (32)$$

But the functions $g_{0j}(s, \omega)$ are even with respect to s , whence follows $K_j(\omega) \equiv 0$, $j = \overline{1, 3}$. Taking in mind the asymptotics $X^\pm(\xi, \omega)$ (17), the formula (30) with $K_j(\omega) \equiv 0$, $j = \overline{1, 3}$ and R_j of the form (32), we finally obtain the asymptotics of the solution (28):

$$V^\pm(\xi, \omega) = \pm(\xi \pm 2\omega)J(\omega) - \frac{h(\omega)}{2\nu} + O\left(\frac{1}{\xi}\right), \quad \xi \rightarrow \infty,$$

where

$$J(\omega) = J_1(\omega) \left(\frac{1}{2\pi^2} + h(\omega) \right) - h(\omega)J_2(\omega) + J_3(\omega)V^+(\omega, \omega),$$

and $J_j(\omega)$ have a form (31). So to satisfy the condition (3), we have to fulfill the equality $J(\omega) = 0$ or

$$J_1(\omega) \left(\frac{1}{2\pi^2} + h(\omega) \right) - h(\omega)J_2(\omega) + J_3(\omega)V^+(\omega, \omega) = 0. \quad (33)$$

To analyze the equation (33), we introduce the notations

$$P_j(\omega) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{s^j \cosh(s)}{Y(s, \omega)} ds, \quad Q_j(\omega) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{s^j \sinh(s)}{Y(s, \omega)} ds. \quad (34)$$

The analysis and calculation of these integrals we adduce in the appendix, subsection 6.3.

In the terms (34), we have (see (29))

$$J_1(\omega) = \omega^2 Q_{-1}(\omega) - Q_1(\omega), \quad J_2(\omega) = \omega^2 P_0(\omega) - P_2(\omega), \quad J_3(\omega) = Q_1(\omega),$$

and since $Q_1 = \nu P_2 + \nu \omega^2 P_0$ (Appendix, subsection 6.3, formula (145)), then the equation (33) takes a form

$$\begin{aligned} & \omega^2 \left(Q_{-1}(\omega) \left(\frac{1}{2\pi^2} + h(\omega) \right) - P_0(\omega)h(\omega) + \nu P_0(\omega)V^+(\omega, \omega) \right) = \\ & = Q_1(\omega) \left(\frac{1}{2\pi^2} + h(\omega) \right) - P_2(\omega)h(\omega) - \nu P_2(\omega)V^+(\omega, \omega). \end{aligned} \quad (35)$$

On the other hand, we have from the formula (28) the equation

$$\begin{aligned} & V^+(\omega, \omega) = -X^+(\omega, \omega)S^+(g_{01}(s, \omega) | \omega) \left(\frac{1}{2\pi^2} + h(\omega) \right) + \\ & X^+(\omega, \omega)S^+(g_{02}(s, \omega) | \omega)h(\omega) - X^+(\omega, \omega)S^+(g_{03}(s, \omega) | \omega)V^+(\omega, \omega). \end{aligned} \quad (36)$$

But for each even function $\varphi(s)$ we evidently have

$$\begin{aligned} S^+(\varphi(s)(\omega^2 - s^2) | \omega) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\varphi(s)(\omega^2 - s^2)}{s - \omega} ds = \\ &= -\frac{1}{2\pi i} \int_{\mathbb{R}} s\varphi(s) ds - \frac{\omega}{2\pi i} \int_{\mathbb{R}} \varphi(s) ds = -\frac{\omega}{2\pi i} \int_{\mathbb{R}} \varphi(s) ds, \end{aligned} \quad (37)$$

and from the representations (29) in accordance with (37) and in terms of (34) it follows

$$S^+(g_{01}(s, \omega) | \omega) = -\omega Q_{-1}(\omega), \quad S^+(g_{02}(s, \omega) | \omega) = -\omega P_0(\omega). \quad (38)$$

For $S^+(g_{03} | \omega)$ we use the formula (146) (Appendix, subsection 6.3), and thus the equation (36) takes a form

$$Q_{-1}(\omega) \left(\frac{1}{2\pi^2} + h(\omega) \right) - P_0(\omega)h(\omega) + \nu P_0(\omega)V^+(\omega, \omega) = 0. \quad (39)$$

Together with the equation (35), we obtain the system

$$\begin{cases} \left(\frac{Q_{-1}(\omega)}{\nu P_0(\omega)} - \frac{1}{\nu} \right) h(\omega) + V^+(\omega, \omega) = -\frac{Q_{-1}(\omega)}{2\pi^2 \nu P_0(\omega)} \\ \left(\frac{Q_1(\omega)}{\nu P_2(\omega)} - \frac{1}{\nu} \right) h(\omega) - V^+(\omega, \omega) = -\frac{Q_1(\omega)}{2\pi^2 \nu P_2(\omega)} \end{cases} \quad (40)$$

The solution of this system is

$$h(\omega) = -\frac{1}{2\pi^2} \frac{\Delta_h(\omega)}{\Delta(\omega)}, \quad V^+(\omega, \omega) = -\frac{1}{2\pi^2} \frac{\Delta_V(\omega)}{\Delta(\omega)}, \quad (41)$$

where

$$\begin{aligned} \Delta(\omega) &= \frac{Q_{-1}}{\nu P_0} + \frac{Q_1}{\nu P_2} - \frac{2}{\nu}, \quad \Delta_h = \frac{Q_{-1}}{\nu P_0} + \frac{Q_1}{\nu P_2} = \Delta(\omega) + \frac{2}{\nu}, \\ \Delta_V &= \frac{1}{\nu} \left(\frac{Q_1}{\nu P_2} - \frac{Q_{-1}}{\nu P_0} \right). \end{aligned} \quad (42)$$

In the appendix, subsection 6.4, we investigate the functions $\Delta(\omega)$, $\Delta_h(\omega)$, $\Delta_V(\omega)$ in (42), in particular we establish for them other representations, show the solvability of the system (40), i.e. $\Delta(\omega) \neq 0$, for each complex $\omega \neq 0$, $\text{Im} \omega \geq 0$, and find their asymptotics at $\omega \rightarrow 0$. Thus, the solution of the inhomogeneous problem 1 at $\text{Im} \omega > 0$ is constructed completely.

To formulate the final result, we should return to the representation (21) of $F(\chi, \omega)$. We introduce at $\chi \in \mathbb{R}$ the functions

$$\mu_{\pm}(\chi) = \chi \pm \frac{\psi(\chi)}{\sqrt{\nu}}, \quad \psi(\chi) = \text{sign}(\chi) \sqrt{\chi \tan(\chi)}. \quad (43)$$

The functions $\mu_{\pm}(\chi)$ are evidently odd, smooth, monotonically increase (at $\nu > 1$) and $\mu_{\pm}(\chi) \neq 0$, $\mu_+(\chi) \neq \mu_-(\chi)$, $\text{sign}(\mu_{\pm}(\chi)) = \text{sign}(\chi)$ if $\chi \neq 0$. Therefore the inverse functions $\chi_{\pm}(\mu) = \mu_{\pm}^{-1}$ exist, are smooth and monotonically increase too, and $\chi_{\pm}(\mu) \neq 0$, $\chi_+(\mu) \neq \chi_-(\mu)$, $\text{sign}(\chi_{\pm}(\mu)) = \text{sign}(\mu)$ if $\mu \neq 0$. The full description of the properties of the functions $\mu_{\pm}(\chi)$, $\chi_{\pm}(\mu)$ see in Appendix, subsection 6.2.

Just in terms of $\chi_{\pm}(s)$ we formulate the final result. Let's introduce the functions

$$h_0(s) = \ln \left(\frac{s^2}{\chi_+(s)\chi_-(s)} \right), \quad h_1(s) = \frac{1}{2} \left(\frac{s^2}{\chi_+^2(s)} + \frac{s^2}{\chi_-^2(s)} \right), \quad s \in \mathbb{R}; \quad (44)$$

$$H_j(\omega) = \frac{\omega}{\pi i} \int_{\mathbb{R}} \frac{h_j(s)}{s^2 - \omega^2} ds, \quad \text{Im} \omega > 0.$$

Using the representations (162), (165) (Appendix, subsection 6.4) for the functions $\Delta(\omega)$, $\Delta_h(\omega)$ and $\Delta_V(\omega)$ from (42), we finally formulate the solution of the problem 1 in the following theorem.

Theorem 1 *The solution of the problem 1 has a form*

$$h(\omega) = \frac{H_1(\omega) - e^{H_0(\omega)}}{H_1(\omega) - e^{H_0(\omega)} - 2/\nu}, \quad V^{\pm}(\xi, \omega) = \pm X^{\pm}(\xi, \omega) S^{\pm}(g_{00}(s, \omega) | \xi),$$

where $H_j(\omega)$, $j = \overline{0, 1}$ have a form (44), $X^{\pm}(\xi, \omega)$ is a solution of homogeneous problem, constructed in the lemma 1 (see formula (24)),

$$g_{00}(s, \omega) = -g_{01}(s, \omega) \left(\frac{1}{2\pi^2} + h(\omega) \right) + g_{02}(s, \omega) h(\omega) -$$

$$g_{03}(s, \omega) \frac{2 - H_1(\omega) - e^{H_0(\omega)}}{\nu(H_1(\omega) - e^{H_0(\omega)}) - 2},$$

$$g_{01}(s, \omega) = \frac{\sinh(s)(\omega^2 - s^2)}{sY(s, \omega)}, \quad g_{02}(s, \omega) = \frac{\cosh(s)(\omega^2 - s^2)}{Y(s, \omega)},$$

$$g_{03}(s, \omega) = \frac{s \sinh(s)}{Y(s, \omega)}, \quad Y(s, \omega) = F(s, \omega) X^+(s, \omega),$$

$$F(s, \omega) = \nu(s - \omega)^2 \cosh(s) - s \sinh(s).$$

3 The solution at real $\omega \neq 0$

Here we essentially use the introduced in the previous subsection functions $\chi_{\pm}(\mu)$. It follows from the representation (21): at real $\omega = \mu$ the function $F(\xi, \omega)$ has two real roots $\xi = \chi_{\pm}(\mu)$. So, the coefficient $G(\chi, \omega) = F(-\chi, \omega)/F(\chi, \omega)$ degenerates (vanishes or approaches infinity) at real ω and so the solution $V^{\pm}(\xi, \omega)$ may become singular. Here we investigate the limit value $V^{\pm}(\xi, \omega)$, when $\omega \rightarrow \mu \in \mathbb{R}$, $\mu \neq 0$.

The real roots $\xi = \chi_{\pm}(\mu)$ are evidently the values of the complex roots $\xi_{\pm}(\omega)$ for the real $\omega = \mu$: $\xi_{\pm}(\mu) = \chi_{\pm}(\mu)$. The functions $\xi_{\pm}(\omega)$ are defined and analytic with respect to ω at least when ω is situated near the real axis: $\text{Im } \omega \approx 0$. Since the index of $G(\chi, \omega) = F(-\chi, \omega)/F(\chi, \omega)$ is equal to -2 , then the roots $\xi_{\pm}(\omega)$ are situated in the upper half plane, when $\text{Im } \omega > 0$. We use exactly the points $\xi_1 = \xi_1(\omega) = \xi_-(\omega)$, $\xi_2 = \xi_2(\omega) = \xi_+(\omega)$ for the construction of the solution $X^{\pm}(\xi, \omega)$ of the homogeneous problem (16) (we remind, this solution doesn't depend on the choice of $\xi_{1,2}$). Then the function

$$F_0(\xi, \omega) = \frac{F(\xi, \omega)}{(\xi - \xi_1(\omega))(\xi - \xi_2(\omega))} \quad (45)$$

is entire (analytic in a whole plane) with respect to ξ and has a continuous limit, when $\omega \rightarrow \mu \in \mathbb{R}$, i.e. $F_0(\xi, \omega) \rightarrow F_0(\xi, \mu)$ and $F_0(\chi, \mu) \neq 0$, $\chi \in \mathbb{R}$. Hence the function (23) $G_0(\chi, \omega) = F_0(-\chi, \omega)/F_0(\chi, \omega)$ has a continuous limit too: $G_0(\chi, \omega) \rightarrow G_0(\chi, \mu) \neq 0$, and so for the function $X_0^{\pm}(\xi, \omega)$ in (24) we obtain

$$X_0^{\pm}(\xi, \omega) \xrightarrow{\omega \rightarrow \mu} \exp(S^{\pm}(\ln G_0(s, \mu) | \xi)) = X_0^{\pm}(\xi, \mu) \neq 0.$$

We introduce the function

$$Y_0(\xi, \omega) = \begin{cases} F_0(\xi, \omega)X_0^+(\xi, \omega), & \text{Im } \xi \geq 0, \\ F_0(-\xi, \omega)X_0^-(\xi, \omega), & \text{Im } \xi \leq 0. \end{cases} \quad (46)$$

Absolutely similar to the proof of the lemma 1, we conclude, the function $Y_0(\xi, \omega)$ is entire with respect to ξ , is even ($Y_0(-\xi, \omega) = Y_0(\xi, \omega)$) and the previous considerations imply, that

$$Y_0(\xi, \omega) \xrightarrow{\omega \rightarrow \mu} Y_0(\xi, \mu) \neq 0. \quad (47)$$

In terms of the introduced function $Y_0(\xi, \omega)$, we rewrite the solution (28) (see also (24)) in a form

$$V^{\pm}(\xi, \omega) = \frac{(\xi \pm \xi_1)(\xi \pm \xi_2)X_0^{\pm}(\xi, \omega)}{2\pi i} \int_{\mathbb{R}} \frac{(s^2 + 1)^2 g_0(s, \omega) ds}{(s - \xi)(s^2 - \xi_1^2)(s^2 - \xi_2^2)}, \quad (48)$$

$\xi_j = \xi_j(\omega)$, $j = \overline{1, 2}$, where, according to the (29),

$$g_0(\chi, \omega) = -\frac{\sinh(\chi)(\omega^2 - \chi^2)}{\chi(\chi^2 + 1)^2 Y_0(\chi, \omega)} \left(\frac{1}{2\pi^2} + h(\omega) \right) + \frac{\cosh(\chi)(\omega^2 - \chi^2)}{(\chi^2 + 1)^2 Y_0(\chi, \omega)} h(\omega) - \frac{\chi \sinh(\chi)}{(\chi^2 + 1)^2 Y_0(\chi, \omega)} V^+(\omega, \omega). \quad (49)$$

Since the function $Y_0(\chi, \omega)$ is even with respect to χ , then the function $g_0(\chi, \omega)$ is even with respect to χ too.

In the Appendix, subsection 6.4, it is shown: the functions $h(\omega)$, $V^+(\omega, \omega)$ are continuous at $\text{Im } \omega \geq 0$, $\omega \neq 0$, i.e. have a continuous limit at $\omega \rightarrow \mu \in \mathbb{R}$:

$$h(\omega) \xrightarrow{\omega \rightarrow \mu} h(\mu), \quad V^+(\omega, \omega) \xrightarrow{\omega \rightarrow \mu} V^+(\mu, \mu). \quad (50)$$

The representation (49) and the statements (47) and (50) evidently imply

$$g_0(\chi, \omega) \xrightarrow{\omega \rightarrow \mu} g_0(\chi, \mu) = -\frac{\sinh(\chi)(\mu^2 - \chi^2)}{\chi(\chi^2 + 1)^2 Y_0(\chi, \mu)} \left(\frac{1}{2\pi^2} + h(\mu) \right) + \frac{\cosh(\chi)(\mu^2 - \chi^2)}{(\chi^2 + 1)^2 Y_0(\chi, \mu)} h(\mu) - \frac{\chi \sinh(\chi)}{(\chi^2 + 1)^2 Y_0(\chi, \mu)} V^+(\mu, \mu). \quad (51)$$

In order to go to the limit in the representation (48), we use the expansions into a sum of simple fractions:

$$\frac{1}{(\xi \mp \xi_1)(\xi \mp \xi_2)} = \frac{\pm 1}{\xi_1 - \xi_2} \left(\frac{1}{\xi \mp \xi_1} - \frac{1}{\xi \mp \xi_2} \right), \quad (52)$$

$$\begin{aligned} \frac{(s^2 + 1)^2}{(s - \xi)(s^2 - \xi_1^2)(s^2 - \xi_2^2)} &= \frac{(\xi^2 + 1)^2}{(\xi^2 - \xi_1^2)(\xi^2 - \xi_2^2)} \cdot \frac{1}{s - \xi} + \\ &+ \frac{(\xi_1^2 + 1)^2}{2\xi_1(\xi_1^2 - \xi_2^2)} \left(\frac{1}{(\xi + \xi_1)(s + \xi_1)} - \frac{1}{(\xi - \xi_1)(s - \xi_1)} \right) + \\ &+ \frac{(\xi_2^2 + 1)^2}{2\xi_2(\xi_2^2 - \xi_1^2)} \left(\frac{1}{(\xi + \xi_2)(s + \xi_2)} - \frac{1}{(\xi - \xi_2)(s - \xi_2)} \right). \end{aligned} \quad (53)$$

Using the formula (53) for the solution (48), we obtain

$$\begin{aligned} V^\pm(\xi, \omega) &= \frac{X_0^\pm(\xi, \omega)(\xi^2 + 1)^2}{(\xi \mp \xi_1)(\xi \mp \xi_2)} S^\pm(g_0(s, \omega) | \xi) + \\ &\frac{(\xi \pm \xi_1)(\xi \pm \xi_2) X_0^\pm(\xi, \omega)(\xi_1^2 + 1)^2}{2\xi_1(\xi_1^2 - \xi_2^2)} \left(\frac{S^-(g_0(s, \omega) | -\xi_1)}{\xi + \xi_1} - \frac{S^+(g_0(s, \omega) | \xi_1)}{\xi - \xi_1} \right) + \\ &\frac{(\xi \pm \xi_1)(\xi \pm \xi_2) X_0^\pm(\xi, \omega)(\xi_2^2 + 1)^2}{2\xi_2(\xi_2^2 - \xi_1^2)} \left(\frac{S^-(g_0(s, \omega) | -\xi_2)}{\xi + \xi_2} - \frac{S^+(g_0(s, \omega) | \xi_2)}{\xi - \xi_2} \right). \end{aligned}$$

But the function $g_0(s, \omega)$ is even with respect to s , and so $S^+(g_0 | \xi_j) = -S^-(g_0 | -\xi_j)$, $j = \overline{1, 2}$ and the previous formula takes a form

$$\begin{aligned} V^\pm(\xi, \omega) &= \frac{X_0^\pm(\xi, \omega)(\xi^2 + 1)^2}{(\xi \mp \xi_1)(\xi \mp \xi_2)} S^\pm(g_0(s, \omega) | \xi) \mp \\ &\frac{\xi(\xi \pm \xi_2)X_0^\pm(\xi, \omega)(\xi_1^2 + 1)^2}{\xi_1(\xi \mp \xi_1)(\xi_1^2 - \xi_2^2)} S^\pm(g_0(s, \omega) | \pm \xi_1) \mp \\ &\frac{\xi(\xi \pm \xi_1)X_0^\pm(\xi, \omega)(\xi_2^2 + 1)^2}{\xi_2(\xi \mp \xi_2)(\xi_2^2 - \xi_1^2)} S^\pm(g_0(s, \omega) | \pm \xi_2). \end{aligned} \quad (54)$$

Using the formula (52) for the first summand in (54), we convert the expression (54) into the form

$$\begin{aligned} V^\pm(\xi, \omega) &= \frac{\pm X_0^\pm(\xi, \omega)}{(\xi_1 - \xi_2)(\xi \mp \xi_1)} \times \\ &\left((\xi^2 + 1)^2 S^\pm(g_0(s, \omega) | \xi) - \frac{\xi(\xi \pm \xi_2)(\xi_1^2 + 1)^2}{\xi_1(\xi_1 + \xi_2)} S^\pm(g_0(s, \omega) | \pm \xi_1) \right) \pm \\ &\frac{X_0^\pm(\xi, \omega)}{(\xi_2 - \xi_1)(\xi \mp \xi_2)} \times \\ &\left((\xi^2 + 1)^2 S^\pm(g_0(s, \omega) | \xi) - \frac{\xi(\xi \pm \xi_1)(\xi_2^2 + 1)^2}{\xi_2(\xi_1 + \xi_2)} S^\pm(g_0(s, \omega) | \pm \xi_2) \right). \end{aligned} \quad (55)$$

In (55), we go to the limit at $\omega \rightarrow \mu \neq 0$, $\mu \in \mathbb{R}$ and $\xi \rightarrow \chi \in \mathbb{R}$. Then $\xi_{1,2}(\omega) \rightarrow \chi_{1,2} = \chi_{1,2}(\mu)$, $\chi_1(\mu) = \chi_-(\mu)$, $\chi_2(\mu) = \chi_+(\mu)$, and according to the Plemelj formulas (12)

$$S^\pm(g_0(s, \omega) | \xi) \rightarrow S(g_0(s, \mu) | \chi) \pm \frac{g_0(\chi, \mu)}{2},$$

$$S^\pm(g_0(s, \omega) | \pm \xi_j) \rightarrow S(g_0(s, \mu) | \pm \chi_j) \pm \frac{g_0(\pm \chi_j, \mu)}{2}, \quad j = \overline{1, 2},$$

and so we obtain

$$V^\pm(\chi, \mu) = \Phi_1^\pm(\chi, \mu) + \Phi_2^\pm(\chi, \mu) + \Phi_3^\pm(\chi, \mu) + \Phi_4^\pm(\chi, \mu), \quad (56)$$

where

$$\begin{aligned} \Phi_1^\pm(\chi, \mu) &= \frac{\pm X_0^\pm(\chi, \mu)(\chi^2 + 1)^2}{(\chi_1 - \chi_2)(\chi \mp \chi_1)} \times \\ &\left(S(g_0(s, \mu) | \chi) - \frac{\chi(\chi \pm \chi_2)(\chi_1^2 + 1)^2}{\chi_1(\chi_1 + \chi_2)(\chi^2 + 1)^2} S(g_0(s, \mu) | \pm \chi_1) \right), \end{aligned}$$

$$\begin{aligned}\Phi_2^\pm(\chi, \mu) &= \frac{X_0^\pm(\chi, \mu)(\chi^2 + 1)^2}{2(\chi_1 - \chi_2)(\chi \mp \chi_1)} \left(g_0(\chi, \mu) - \frac{\chi(\chi \pm \chi_2)(\chi_1^2 + 1)^2 g_0(\pm\chi_1, \mu)}{\chi_1(\chi_1 + \chi_2)(\chi^2 + 1)^2} \right), \\ \Phi_3^\pm(\chi, \mu) &= \frac{\pm X_0^\pm(\chi, \mu)(\chi^2 + 1)^2}{(\chi_2 - \chi_1)(\chi \mp \chi_2)} \times \\ &\quad \left(S(g_0(s, \mu) | \chi) - \frac{\chi(\chi \pm \chi_1)(\chi_2^2 + 1)^2}{\chi_2(\chi_1 + \chi_2)(\chi^2 + 1)^2} S(g_0(s, \mu) | \pm\chi_2) \right), \\ \Phi_4^\pm(\chi, \mu) &= \frac{X_0^\pm(\chi, \mu)(\chi^2 + 1)^2}{2(\chi_2 - \chi_1)(\chi \mp \chi_2)} \left(g_0(\chi, \mu) - \frac{\chi(\chi \pm \chi_1)(\chi_2^2 + 1)^2 g_0(\pm\chi_2, \mu)}{\chi_2(\chi_1 + \chi_2)(\chi^2 + 1)^2} \right).\end{aligned}$$

The functions $\Phi_j^\pm(\chi, \mu)$, $j = \overline{1, 4}$ are continuous (and even smooth) for every real (χ, μ) . Indeed, the only discontinuity points can be $\pm\chi_1$ for Φ_j^\pm , $j = \overline{1, 2}$, and $\pm\chi_2$ for Φ_j^\pm , $j = \overline{3, 4}$. But obviously

$$\Phi_2^\pm(\pm\chi_1, \mu) = \frac{X_0^\pm(\pm\chi_1, \mu)(\chi_1^2 + 1)^2}{2(\chi_1 - \chi_2)} \left((g_0)_\chi'(\pm\chi_1, \mu) - (\varphi_1^\pm)'(\pm\chi_1)g_0(\pm\chi_1, \mu) \right),$$

where

$$\varphi_1^\pm(\chi) = \frac{\chi(\chi \pm \chi_2)(\chi_1^2 + 1)^2}{\chi_1(\chi_1 + \chi_2)(\chi^2 + 1)^2}.$$

Absolutely similarly with the using of the formula $S(1 | \chi) = 0$, see (13), we obtain

$$\begin{aligned}\Phi_1^\pm(\pm\chi_1, \mu) &= \frac{\pm X_0^\pm(\pm\chi_1, \mu)(\chi_1^2 + 1)^2}{(\chi_1 - \chi_2)} \times \\ &\quad \left(\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{g_0(s, \mu) - g_0(\pm\chi_1, \mu)}{(s \mp \chi_1)^2} ds - (\varphi_1^\pm)'(\pm\chi_1)S(g_0(s, \mu) | \pm\chi_1) \right).\end{aligned}$$

By the same way

$$\begin{aligned}\Phi_3^\pm(\pm\chi_2, \mu) &= \frac{\pm X_0^\pm(\pm\chi_2, \mu)(\chi_2^2 + 1)^2}{(\chi_2 - \chi_1)} \times \\ &\quad \left(\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{g_0(s, \mu) - g_0(\pm\chi_2, \mu)}{(s \mp \chi_2)^2} ds - (\varphi_2^\pm)'(\pm\chi_2)S(g_0(s, \mu) | \pm\chi_2) \right), \\ \Phi_4^\pm(\pm\chi_2, \mu) &= \frac{X_0^\pm(\pm\chi_2, \mu)(\chi_2^2 + 1)^2}{2(\chi_2 - \chi_1)} \left((g_0)_\chi'(\pm\chi_2, \mu) - (\varphi_2^\pm)'(\pm\chi_2)g_0(\pm\chi_2, \mu) \right),\end{aligned}$$

where

$$\varphi_2^\pm(\chi) = \frac{\chi(\chi \pm \chi_1)(\chi_2^2 + 1)^2}{\chi_2(\chi_1 + \chi_2)(\chi^2 + 1)^2}.$$

Thus, we established the continuity of $V^\pm(\chi, \mu)$ for every real χ and real $\mu \neq 0$. So, the solution $V^\pm(\xi, \omega)$ of the problem (1), (2), (3) is regular (continuous and even smooth) for every real ω , except the points, where $\chi_1 = \chi_2$ or $\chi_1 = 0$ or $\chi_2 = 0$, or $\chi_1 + \chi_2 = 0$, in any case it is an only point $\omega = \mu = 0$.

4 The solution of the problem 2

We proceed to the problem (9), (10), (11). The conjugation condition (9) has a form

$$V_0^+(\chi, \omega) + V_0^-(\chi, \omega) = V_0^+(\omega, \omega) - \left(\frac{1}{2\pi^2} + h(\omega) \right) + g_0(\chi, \omega), \quad (57)$$

where

$$g_0(\chi, \omega) = \left(\frac{1}{2\pi^2} + h(\omega) \right) \frac{\omega^2(\cosh(\chi) - 1) + \chi^2}{\chi^2 \cosh(\chi)} + \frac{V^+(\chi, \omega) + V^-(\chi, \omega) - V^+(\omega, \omega)}{\cosh(\chi)}, \quad (58)$$

and $V^\pm(\chi, \omega)$, $h(\omega)$ is a solution of the problem 1. In accordance with the conditions (2), (3), the function $g_0(\chi, \omega)$ (58) is even with respect to χ and

$$g_0(\chi, \omega) = O\left(\frac{1}{\chi^2}\right), \quad \chi \rightarrow \infty. \quad (59)$$

The bounded solution of the problem (57) has evidently form (see e.g. [1])

$$V_0^\pm(\xi, \omega) = \pm C(\omega) + \frac{V_0^+(\omega, \omega)}{2} - \frac{1}{2} \left(\frac{1}{2\pi^2} + h(\omega) \right) \pm S^\pm(g_0 | \xi), \quad (60)$$

and since the function g_0 is even, then $S^+(g_0 | -\xi) = -S^-(g_0 | \xi)$. Then the symmetry condition (10) gives $C(\omega) = 0$, i.e. the solution is unique. At that, by virtue of the condition (59) and lemma 2 (Appendix, subsection 6.1), we obtain

$$V_0^\pm(\xi, \omega) = \frac{V_0^+(\omega, \omega)}{2} - \frac{1}{2} \left(\frac{1}{2\pi^2} + h(\omega) \right) + O\left(\frac{1}{\xi}\right), \quad \xi \rightarrow \infty,$$

i.e. we've find the unknown $V_{02}(\omega)$ in the asymptotic condition (11):

$$V_{02}(\omega) = \frac{V_0^+(\omega, \omega)}{2} - \frac{1}{2} \left(\frac{1}{2\pi^2} + h(\omega) \right).$$

It remains to determine $V_0^+(\omega, \omega)$.

We have from the representation (60), where $C(\omega) = 0$, that

$$V_0^+(\omega, \omega) = \frac{V_0^+(\omega, \omega)}{2} - \frac{1}{2} \left(\frac{1}{2\pi^2} + h(\omega) \right) + S^+(g_0 | \omega),$$

i.e.

$$V_0^+(\omega, \omega) = 2S^+(g_0 | \omega) - \left(\frac{1}{2\pi^2} + h(\omega) \right).$$

Theorem 2 *The solution of the problem 2 has a form*

$$V_0^\pm(\xi, \omega) = S^+(g_0(s, \omega) | \omega) - \left(\frac{1}{2\pi^2} + h(\omega) \right) \pm S^\pm(g_0(s, \omega) | \xi),$$

$$V_{02}(\omega) = S^+(g_0(s, \omega) | \omega) - \left(\frac{1}{2\pi^2} + h(\omega) \right),$$

where

$$g_0(s, \omega) = \left(\frac{1}{2\pi^2} + h(\omega) \right) \frac{\omega^2(\cosh(s) - 1) + s^2}{s^2 \cosh(s)} + \frac{V^+(s, \omega) + V^-(s, \omega) - V^+(\omega, \omega)}{\cosh(s)},$$

and $h(\omega)$, $V^\pm(\xi, \omega)$ is a solution of the problem 1.

Obviously, the solution of problem 2 is continues with respect to ξ at $\pm \text{Im} \xi \geq 0$, continues with respect to ω at $\text{Im} \omega \geq 0$, $\omega \neq 0$ and analytic at $\text{Im} \omega > 0$.

5 The asymptotics of the solutions at $\omega \rightarrow 0$

5.1 The asymptotics of the solution of problem 1

We start from the asymptotics of the functions $h(\omega)$ and $V^+(\omega, \omega)$. The representation (41) and the asymptotic formulas (166) (Appendix, subsection (6.4)) imply

$$h(\omega) = \frac{iA_0}{\omega} + A_1 + O(\omega), \quad V^+(\omega, \omega) = -A_2 + O(\omega), \quad \omega \rightarrow 0, \quad (61)$$

where

$$A_0 = \frac{1}{2\pi\nu A}, \quad A_1 = \frac{\nu + 3}{24\nu(\nu - 1)^2 A^2} - \frac{\nu + 1}{2\pi^2(\nu - 1)}, \quad A_2 = \frac{1}{2\pi^2\nu},$$

$$A = A(\nu) = \int_{\mathbb{R}} \frac{\cosh(s) - \sinh(s)/s}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds. \quad (62)$$

Now we go to the asymptotics of the solution $V^\pm(\xi, \omega)$. Let $U^\pm(\xi, \omega) = \omega V^\pm(\xi, \omega)$, then (see (1), (2), (3)) the function U^\pm is a solution of the linear conjugation problem

$$\begin{aligned} & (\nu(\omega - \chi)^2 \cosh(\chi) - \chi \sinh(\chi))U^+(\chi, \omega) = \\ & -(\nu(\omega + \chi)^2 \cosh(\chi) - \chi \sinh(\chi))U^-(\chi, \omega) - \\ & \chi \sinh(\chi)\omega V^+(\omega, \omega) - \frac{\sinh(\chi)(\omega^2 - \chi^2)}{\chi} \left(\frac{\omega}{2\pi^2} + \omega h(\omega) \right) + \\ & \cosh(\chi)(\omega^2 - \chi^2)\omega h(\omega). \end{aligned} \quad (63)$$

Whence, based on the asymptotics (61), we conclude: at $\omega \rightarrow 0$ the function U^\pm have a form $U^\pm(\xi, \omega) = U_0^\pm(\xi) + U_1^\pm(\xi)\omega + O(\omega^2)$, where the functions $U_j^\pm(\xi)$, $j = \overline{0, 1}$ satisfy the conjugation conditions

$$U_0^+(\chi) = -U_0^-(\chi) - iA_0 \frac{\cosh(\chi) - \sinh(\chi)/\chi}{\nu \cosh(\chi) - \sinh(\chi)/\chi}, \quad (64)$$

$$\begin{aligned} & (\nu \cosh(\chi) - \sinh(\chi)/\chi)U_1^+(\chi) = -(\nu \cosh(\chi) - \sinh(\chi)/\chi)U_1^-(\chi) + \\ & \cosh(\chi) \left(2\nu \frac{U_0^+(\chi) - U_0^-(\chi)}{\chi} - A_1 \right) + \frac{\sinh(\chi)}{\chi} \left(\frac{1}{2\pi^2} + A_1 + A_2 \right). \end{aligned} \quad (65)$$

At that the symmetry condition (2) evidently implies, that the functions $U_j^\pm(\xi)$ are symmetric too: $U_j^+(\xi) = U_j^-(\xi)$, $j = \overline{0, 1}$.

The bounded solution of the problem (64) has a form

$$U_0^\pm(\xi) = \pm C \mp iA_0 S^\pm(W_0(s) | \xi), \quad W_0(s) = \frac{\cosh(\chi) - \sinh(\chi)/\chi}{\nu \cosh(\chi) - \sinh(\chi)/\chi},$$

where $C = \text{const}$. But the function $W_0(s)$ is even, therefore $S^+(W_0 | -\xi) = -S^-(W_0 | \xi)$ and the symmetry condition $U_0^+(\xi) = U_0^-(\xi)$ implies $C = 0$. Thus, we have the representation

$$U_0^\pm(\xi) = \mp iA_0 S^\pm(W_0(s) | \xi), \quad W_0(\chi) = \frac{\cosh(\chi) - \sinh(\chi)/\chi}{\nu \cosh(\chi) - \sinh(\chi)/\chi}, \quad (66)$$

where A_0 has a form (62).

Now let's go to the problem (65). We note, the function $W_0(s)/s$ (see (66)) is smooth and odd, in particular $S(W_0(s) | 0) = 0$, and therefore, taking in mind the Plemelj formulas (12), we have

$$\frac{U_0^+(\chi) - U_0^-(\chi)}{\chi} = -\frac{2iA_0}{\chi}(S(W_0 | \chi) - S(W_0 | 0)) = -2iA_0S\left(\frac{W_0(s)}{s} | \chi\right).$$

Thus the equation (65) takes a form

$$U_1^+(\chi) = -U_1^-(\chi) + W_1(\chi), \quad W_1(\chi) = \frac{1}{\nu \cosh(\chi) - \sinh(\chi)/\chi} \times \\ \left(\cosh(\chi) \left(-4i\nu A_0 S\left(\frac{W_0(s)}{s} | \chi\right) - A_1 \right) + \left(\frac{1}{2\pi^2} + A_1 + A_2 \right) \frac{\sinh(\chi)}{\chi} \right),$$

and absolutely similar to the previous considerations, we obtain the representation $U_1^\pm(\xi) = \pm S^\pm(W_1 | \xi)$.

We summarize all results in the next theorem.

Theorem 3 *The solution of the problem 1 has the following asymptotics at $\omega \rightarrow 0$:*

$$h(\omega) = \frac{iA_0}{\omega} + A_1 + O(\omega), \quad V^\pm(\xi, \omega) = \frac{U_0^\pm(\xi)}{\omega} + U_1^\pm(\xi) + O(\omega),$$

where

$$U_0^\pm(\xi) = \mp iA_0 S^\pm(W_0(s) | \xi), \quad U_1^\pm(\xi) = \pm S^\pm(W_1(s) | \xi), \\ W_0(\chi) = \frac{\cosh(\chi) - \sinh(\chi)/\chi}{\nu \cosh(\chi) - \sinh(\chi)/\chi}, \\ W_1(\chi) = \frac{1}{\nu \cosh(\chi) - \sinh(\chi)/\chi} \times \\ \left(\cosh(\chi) \left(-4i\nu A_0 S\left(\frac{W_0(s)}{s} | \chi\right) - A_1 \right) + \left(\frac{1}{2\pi^2} + A_1 + A_2 \right) \frac{\sinh(\chi)}{\chi} \right), \\ A_0 = \frac{1}{2\pi\nu A}, \quad A_1 = \frac{\nu + 3}{24\nu(\nu - 1)^2 A} - \frac{\nu + 1}{2\pi^2(\nu - 1)}, \quad A_2 = \frac{1}{2\pi^2\nu}, \\ A = A(\nu) = \int_{\mathbb{R}} \frac{\cosh(s) - \sinh(s)/s}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds.$$

5.2 The asymptotics of the solution of problem 2

In this subsection, we use on default the notations of the theorem 3.

By virtue of the theorem 3 and formula (61), the function $g_0(s, \omega)$ (58) has the following asymptotics at $\omega \rightarrow 0$:

$$g_0(s, \omega) = \frac{iA_0 + U_0^+(s) + U_0^-(s)}{\omega \cosh(s)} + \frac{1/(2\pi^2) + A_1 + U_1^+(s) + U_1^-(s) + A_2}{\cosh(s)} + O(\omega),$$

and since

$$iA_0 + U_0^+(s) + U_0^-(s) = iA_0 - iA_0 W_0(s) = \frac{iA_0(\nu - 1) \cosh(s)}{\nu \cosh(s) - \sinh(s)/s},$$

$$U_1^+(s) + U_1^-(s) = W_1(s),$$

then we obtain

$$g_0(s, \omega) = \frac{q_{-1}(s)}{\omega} + q_0(s) + O(\omega), \quad (67)$$

where

$$q_{-1}(s) = \frac{iA_0(\nu - 1)}{\nu \cosh(s) - \sinh(s)/s}, \quad q_0(s) = \frac{1/(2\pi^2) + A_1 + W_1(s) + A_2}{\cosh(s)}. \quad (68)$$

Now we should consider the asymptotics $S^+(g_0 | \omega)$, see theorem 2. Absolutely similar to the proof of the lemma 3 (Appendix, subsection 6.1), we easy obtain

$$S^+(g(s) | \omega) = S(g(s) | 0) + \frac{g(0)}{2} + \frac{\omega}{2\pi i} \int_{\mathbb{R}} \frac{g(s) - g(0)}{s^2} ds + O(\omega^2). \quad (69)$$

But the functions $q_j(s)$ are obviously even, i.e. $S(q_j | 0) = 0$, $j = \overline{-1, 0}$, whence the formulas (67) and (69) imply

$$S^+(g_0(s, \omega) | \omega) = \frac{q_{-1}(0)}{2\omega} + \frac{q_0(0)}{2} - A_3 + O(\omega),$$

$$A_3 = -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{q_{-1}(s) - q_{-1}(0)}{s^2} ds. \quad (70)$$

We evidently have from the formula (68), that $q_{-1}(0) = iA_0$. Next,

$$\begin{aligned}
A_3 &= -\frac{A_0}{2\pi} \int_{\mathbb{R}} \frac{\nu(1 - \cosh(s)) + \sinh(s)/s - 1}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds = \\
&= -\frac{A_0}{2\pi} \left[\int_{\mathbb{R}} \frac{(\nu - 1)(1 - \cosh(s))}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds + \int_{\mathbb{R}} \frac{\sinh(s)/s - \cosh(s)}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds \right] = \\
&= -\frac{1}{4\pi^2\nu A} \left[-A - (\nu - 1) \int_{\mathbb{R}} \frac{\cosh(s) - 1}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds \right],
\end{aligned}$$

and finally

$$A_3 = \frac{1}{4\pi^2\nu} + \frac{\nu - 1}{4\pi^2\nu A} \int_{\mathbb{R}} \frac{\cosh(s) - 1}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds. \quad (71)$$

Then, using the representation $W_1(s)$ in the theorem 3, and taking in mind, that $A(\nu) = 2\pi i S(W_0(s)/s | 0)$, we have $W_1(0) = -1/(2\pi^2\nu) = -A_2$ (see theorem 3), therefore by virtue of the (68) $q_0(0) = 1/(2\pi^2) + A_1$, and so, according to the formula (70),

$$S^+(g_0(s, \omega | \omega)) = \frac{iA_0}{2\omega} + \frac{A_1}{2} + \frac{1}{4\pi^2} - A_3 + O(\omega), \quad (72)$$

where $A_3 = A_3(\nu)$ have a form (71).

Substituting the asymptotics of $h(\omega)$ (61), $g_0(s, \omega)$ (67), (68), and $S^+(g_0 | \omega)$ (72) in the formulas for the solution of problem 2, given in the theorem 2, we obtain the asymptotics of the mentioned solution.

Theorem 4 *At $\omega \rightarrow 0$*

$$V_0^\pm(\xi, \omega) = \frac{V_{00}^\pm(\xi)}{\omega} + V_{10}^\pm(\xi) + O(\omega), \quad V_{02}(\omega) = -\frac{iA_0}{2\omega} - A_4 + O(\omega),$$

where

$$\begin{aligned}
V_{00}^\pm(\xi) &= -\frac{iA_0}{2} \pm S^\pm(q_{-1}(s) | \xi), \\
V_{10}^\pm(\xi) &= -\frac{A_1}{2} - \frac{1}{4\pi^2} - A_3 \pm S^\pm(q_0(s) | \xi), \\
q_{-1}(s) &= \frac{iA_0(\nu - 1)}{\nu \cosh(s) - \sinh(s)/s}, \quad q_0(s) = \frac{1/(2\pi^2) + A_1 + W_1(s) + A_2}{\cosh(s)}, \\
A_4 &= \frac{A_1}{2} + \frac{1}{4\pi^2} + A_3,
\end{aligned}$$

and the constants A_0, A_1, A_3 have a form (62) and (71) respectively.

6 Appendix

6.1 The asymptotics of Cauchy integrals

Lemma 2 *Let the function $\varphi(\chi)$ be smooth at $\chi \in \mathbb{R}$ and*

$$\varphi(\chi) = \frac{\varphi_1}{\chi} + \frac{\varphi_2}{\chi^2} + O\left(\frac{1}{\chi^3}\right), \quad \chi \rightarrow \infty, \quad \varphi_j = \text{const}, \quad j = \overline{1, 2}. \quad (73)$$

Then the Cauchy integral $S^\pm(\varphi | \xi)$ has an asymptotics

$$S^\pm(\varphi | \xi) = \frac{\varphi_1^\pm}{\xi} + \frac{\varphi_2^\pm}{\xi^2} + O\left(\frac{1}{\xi^3}\right), \quad \xi \rightarrow \infty, \quad (74)$$

where

$$\varphi_1^\pm = \pm \frac{\varphi_1}{2} - \frac{1}{2\pi i} \int_{\mathbb{R}} \varphi(s) ds, \quad \varphi_2^\pm = \pm \frac{\varphi_2}{2} - \frac{1}{2\pi i} \int_{\mathbb{R}} (s\varphi(s) - \varphi_1) ds. \quad (75)$$

Proof. We mean $\xi \rightarrow \infty$ on default. The condition (73) obviously implies the representation

$$\varphi(\chi) = \frac{\varphi_1}{\chi} + \frac{\varphi_2}{\chi^2} + \frac{\varphi_3(\chi)}{\chi^3}, \quad |\chi| > 1, \quad (76)$$

where the function $\varphi_3(\chi)$ is bounded. Let's denote $S^\pm(\varphi | \xi) = \Phi^\pm(\xi)$ for short. We have

$$\Phi^\pm(\xi) = \frac{1}{2\pi i} \int_{-1}^1 \frac{\varphi(s) ds}{s - \xi} + \frac{1}{2\pi i} \int_{|s|>1} \frac{\varphi(s) ds}{s - \xi} = \Phi_1^\pm(\xi) + \Phi_2^\pm(\xi).$$

At $|s| < 1$, we use the decomposition

$$\frac{1}{s - \xi} = -\frac{1}{\xi} - \frac{s}{\xi^2} + O\left(\frac{1}{\xi^3}\right),$$

then

$$\Phi_1^\pm(\xi) = -\frac{1}{2\pi i \xi} \int_{-1}^1 \varphi(s) ds - \frac{1}{2\pi i \xi^2} \int_{-1}^1 s\varphi(s) ds + O\left(\frac{1}{\xi^3}\right). \quad (77)$$

At $|s| > 1$, we use the representation

$$\begin{aligned} \frac{1}{2\pi i} \int_{|s|>1} \frac{ds}{s - \xi} &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{ds}{s - \xi} - \frac{1}{2\pi i} \int_{-1}^1 \frac{ds}{s - \xi} = \\ &= S^\pm(1 | \xi) - \frac{1}{2\pi i} \ln \frac{\xi - 1}{\xi + 1}, \end{aligned} \quad (78)$$

and, having in mind the formula (13) and the asymptotic decomposition

$$\ln \frac{\xi - 1}{\xi + 1} = -\frac{2}{\xi} + O\left(\frac{1}{\xi^3}\right), \quad \xi \rightarrow \infty,$$

we rewrite (78) it in a form

$$\frac{1}{2\pi i} \int_{|s|>1} \frac{ds}{s - \xi} = \frac{\text{sign}(\text{Im } \xi)}{2} + \frac{1}{2\pi i \xi} \int_{-1}^1 ds + O\left(\frac{1}{\xi^3}\right). \quad (79)$$

Using the expansion into a sum of simple fractions

$$\frac{1}{s^j(s - \xi)} = \frac{1}{\xi^j(s - \xi)} - \sum_{k=1}^j \frac{1}{\xi^k s^{j-k+1}}, \quad j \geq 1,$$

and the formula (79), we obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_{|s|>1} \frac{ds}{s(s - \xi)} &= \frac{\text{sign}(\text{Im } \xi)}{2\xi} - \frac{1}{2\pi i \xi^2} \int_{-1}^1 ds - \frac{1}{2\pi i \xi} \int_{|s|>1} \frac{ds}{s} + O\left(\frac{1}{\xi^3}\right); \\ \frac{1}{2\pi i} \int_{|s|>1} \frac{ds}{s^2(s - \xi)} &= \frac{\text{sign}(\text{Im } \xi)}{2\xi^2} - \frac{1}{2\pi i \xi^2} \int_{|s|>1} \frac{ds}{s} - \frac{1}{2\pi i \xi} \int_{|s|>1} \frac{ds}{s^2} + O\left(\frac{1}{\xi^3}\right); \\ \frac{1}{2\pi i} \int_{|s|>1} \frac{\varphi_3(s) ds}{s^3(s - \xi)} &= -\frac{1}{2\pi i \xi} \int_{|s|>1} \frac{\varphi_3(s) ds}{s^3} - \frac{1}{2\pi i \xi^2} \int_{|s|>1} \frac{\varphi_3(s) ds}{s^2} + O\left(\frac{1}{\xi^3}\right); \end{aligned}$$

and in total according to the representation (76)

$$\begin{aligned} \Phi_2^\pm(\xi) &= \frac{1}{\xi} \left(\frac{\varphi_1 \text{sign}(\text{Im } \xi)}{2} - \frac{1}{2\pi i} \int_{|s|>1} \left(\frac{\varphi_1}{s} + \frac{\varphi_2}{s^2} + \frac{\varphi_3(s)}{s^3} \right) ds \right) + \\ &\frac{1}{\xi^2} \left(\frac{1}{2\pi i} \int_{-1}^1 \varphi_1 ds + \frac{\varphi_2 \text{sign}(\text{Im } \xi)}{2} - \frac{1}{2\pi i} \int_{|s|>1} \left(\frac{\varphi_2}{s} + \frac{\varphi_3(s)}{s^2} \right) ds \right) + O\left(\frac{1}{\xi^3}\right). \end{aligned}$$

Together with the formula (77), we obtain

$$\begin{aligned} \Phi^\pm(\xi) &= \Phi_1^\pm(\xi) + \Phi_2^\pm(\xi) = \frac{\text{sign}(\text{Im } \xi)}{2} \left(\frac{\varphi_1}{\xi} + \frac{\varphi_2}{\xi^2} \right) - \\ &\frac{1}{2\pi i \xi} \int_{\mathbb{R}} \varphi(s) ds - \frac{1}{2\pi i \xi^2} \int_{\mathbb{R}} (s\varphi(s) - \varphi_1) ds + O\left(\frac{1}{\xi^3}\right). \end{aligned} \quad (80)$$

The lemma statement (the formulas (74), (75)) obviously follows from the formula (80).

Remark. Absolutely similar if $\varphi(\chi) = \varphi_1/\chi + O(1/\chi^2)$ at $\chi \rightarrow \infty$, where $\varphi_1 = \text{const}$, then $S^\pm(\varphi | \xi) = \varphi_1^\pm/\xi + O(1/\xi^2)$, $\xi \rightarrow \infty$ where φ_1^\pm has a form (75).

Now we proceed to the specific Cauchy integrals, which we use to investigate the function, depending on one variable ω only, and accordingly we use the notations

$$S_0^+(\varphi(s) | \omega) = \frac{\omega}{2\pi i} \int_{\mathbb{R}} \frac{\varphi(s) ds}{s^2 - \omega^2}, \quad \text{Im } \omega > 0; \quad (81)$$

$$S_0(\varphi(s) | \mu) = \frac{\mu}{2\pi i} \int_{\mathbb{R}} \frac{\varphi(s) ds}{s^2 - \mu^2}, \quad \mu \in \mathbb{R}. \quad (82)$$

We obviously have the representations

$$\begin{aligned} S_0^+(\varphi | \omega) &= S^+(\varphi | \omega) - S^-(\varphi | -\omega), \quad \text{Im } \omega > 0; \\ S_0(\varphi | \mu) &= S(\varphi | \mu) - S(\varphi | -\mu), \quad \mu \in \mathbb{R}, \end{aligned} \quad (83)$$

and, in particular (see (13))

$$S_0^+(1 | \omega) = 1, \quad \text{Im } \omega > 0; \quad S_0(1 | \mu) = 0, \quad \mu \in \mathbb{R}. \quad (84)$$

If the function $\varphi(s)$ is even, then according to the Plemelj formulas (12)

$$S_0^+(\varphi | \mu) = S_0(\varphi | \mu) + \varphi(\mu), \quad \mu \in \mathbb{R}. \quad (85)$$

Lemma 3 *Let the function $\varphi(s)$ be real, even and smooth and*

$$\varphi(s) = A + \frac{B}{|s|} + O\left(\frac{1}{s^2}\right), \quad s \rightarrow \infty, \quad A, B = \text{const}.$$

Then:

1. At $\mu \in \mathbb{R}$

$$\text{Re } S_0^+(\varphi | \mu) = \varphi(\mu), \quad \text{Im } S_0^+(\varphi | \mu) = -iS_0(\varphi | \mu).$$

2. At $\omega \rightarrow 0$

$$S_0^+(\varphi | \omega) = \varphi(0) + \frac{\omega}{\pi i} \int_{\mathbb{R}} \frac{\varphi(s) - \varphi(0)}{s^2} ds + \frac{\varphi''(0)\omega^2}{2} + O(\omega^3).$$

3. At $\omega \rightarrow \infty$ $S_0^+(\varphi | \omega) = A + o(1)$.

Proof. The first lemma statement evidently follows from (85).

We proceed to the second statement. According to the (84), we have

$$S_0^+(\varphi | \omega) = \varphi(0) + S_0^+(\varphi(s) - \varphi(0) | \omega) = \varphi(0) + \frac{\omega}{\pi i} \int_{\mathbb{R}} \frac{s^2 \varphi_1(s) ds}{s^2 - \omega^2}, \quad (86)$$

where $\varphi_1(s) = (\varphi(s) - \varphi(0))/s^2$. But since $\varphi(s)$ is even and smooth, and so $\varphi'(0) = 0$, then $\varphi_1(s)$ is even and smooth too, therefore

$$\lim_{\omega \rightarrow 0} \frac{1}{\pi i} \int_{\mathbb{R}} \frac{s^2 \varphi_1(s) ds}{s^2 - \omega^2} = \frac{1}{\pi i} \int_{\mathbb{R}} \varphi_1(s) ds.$$

Whence

$$S_0^+(\varphi | \omega) = \varphi(0) + O(\omega),$$

and so

$$S_0^+(\varphi_1 | \omega) = \varphi_1(0) + O(\omega) = \frac{\varphi''(0)}{2} + O(\omega).$$

Then it follows from (86), that

$$\begin{aligned} S_0^+(\varphi | \omega) &= \varphi(0) + \frac{\omega}{\pi i} \int_{\mathbb{R}} \frac{(s^2 - \omega^2 + \omega^2) \varphi_1(s) ds}{s^2 - \omega^2} = \\ &= \varphi(0) + \frac{\omega}{\pi i} \int_{\mathbb{R}} \varphi_1(s) ds + \omega^2 S_0^+(\varphi_1(s) | \omega) = \\ &= \varphi(0) + \frac{\omega}{\pi i} \int_{\mathbb{R}} \frac{\varphi(s) - \varphi(0)}{s^2} ds + \frac{\omega^2 \varphi''(0)}{2} + O(\omega^3). \end{aligned}$$

We've proved the second lemma statement.

Let now $\omega \rightarrow \infty$. First (see (84)),

$$S_0^+(\varphi | \omega) = A + S_0^+(\varphi(s) - A | \omega). \quad (87)$$

We represent

$$S_0^+(\varphi(s) - A | \omega) = S_0^+(\varphi_0(s) | \omega) + S_0^+(\varphi_1(s) | \omega), \quad (88)$$

where

$$\varphi_0(s) = \begin{cases} \varphi(s) - A, & |s| < 1, \\ 0, & |s| > 1, \end{cases} \quad \varphi_1(s) = \begin{cases} 0, & |s| < 1, \\ \varphi(s) - A, & |s| > 1. \end{cases}$$

Obviously,

$$S_0^+(\varphi_0 | \omega) = o(1), \quad \omega \rightarrow \infty. \quad (89)$$

In turn, at $|s| > 1$ we have the lemma condition

$$\varphi_1(s) = \frac{B}{|s|} + \frac{\varphi_2(s)}{s^2}, \quad (90)$$

the function φ_2 is bounded and we can assume, that $\varphi_2(s) \equiv 0$ at $|s| < 1$. As it was shown at the proof of the previous lemma,

$$S^\pm \left(\frac{\varphi_2(s)}{s^2} | \omega \right) = -\frac{1}{2\pi i \omega} \int_{|s|>1} \frac{\varphi_2(s) ds}{s^2} + O \left(\frac{1}{\omega^2} \right) = o(1),$$

whence, according to the formula (83),

$$S_0^+ \left(\frac{\varphi_2(s)}{s^2} | \omega \right) = S^+ \left(\frac{\varphi_2(s)}{s^2} | \omega \right) - S^- \left(\frac{\varphi_2(s)}{s^2} | -\omega \right) = o(1). \quad (91)$$

We now turn to the integral from $1/|s|$. Similar to the previous considerations,

$$\frac{\omega}{\pi i} \int_{|s|>1} \frac{ds}{|s|(s^2 - \omega^2)} = \frac{1}{2\pi i} \int_{|s|>1} \frac{ds}{|s|(s - \omega)} - \frac{1}{2\pi i} \int_{|s|>1} \frac{ds}{|s|(s + \omega)} = S_1^+(\omega) - S_1^-(-\omega).$$

We directly compute

$$S_1^\pm(\omega) = -\frac{\ln(1 - \omega^2)}{2\pi i \omega},$$

and so, independently on the choice of a branch of the logarithm,

$$\frac{\omega}{\pi i} \int_{|s|>1} \frac{ds}{|s|(s^2 - \omega^2)} = o(1), \quad \omega \rightarrow \infty. \quad (92)$$

Combining the formulas (87), (88), (89), (90), (91) and (92), we obtain the last lemma statement. The lemma is proved completely.

6.2 The functions $\chi_\pm(\mu)$.

These functions are the inverse functions for the $\mu_\pm(\chi)$ of the form (43), i.e.

$$\frac{\mu_\pm(\chi)}{\chi} = 1 \pm \sqrt{\frac{\tan(\chi)}{\nu\chi}}. \quad (93)$$

We list several obvious properties of these functions, which follow from formulas (43) or (93):

- the functions $\mu_{\pm}(\chi)$ and, respectively, $\chi_{\pm}(\mu)$ are odd, smooth, monotonically increase, $\mu_{\pm}(0) = \chi_{\pm}(0) = 0$;
- $\mu_{\pm}(\chi)/\chi$ and, respectively, $\mu/\chi_{\pm}(\mu)$ are even and smooth;
- $\mu_{1,2}(\chi)/\chi > 0$ and, respectively, $\mu/\chi_{1,2}(\mu) > 0$;
- at $s \in (0, +\infty)$ the functions $\mu_{-}(s)/s$ and $s/\chi_{-}(s)$ monotonically increase from the value $1 - 1/\sqrt{\nu} \in (0, 1)$ at $s = 0$ till the value 1 at $s = +\infty$;
- at $s \in (0, +\infty)$ the functions $\mu_{+}(s)/s$ and $s/\chi_{+}(s)$ monotonically decrease from the value $1 + 1/\sqrt{\nu} > 1$ at $s = 0$ till the value 1 at $s = +\infty$.

In the next lemma, we establish the asymptotics of the functions $\chi_{\pm}(s)$ at $s \rightarrow 0$ and $s \rightarrow \infty$. We use the notation $\beta = 1/\sqrt{\nu}$.

Lemma 4 1. At $s \rightarrow 0$

$$\frac{s}{\chi_{\pm}(s)} = 1 \mp \beta \pm \frac{\beta s^2}{6(1 \mp \beta)^2} + O(s^4), \quad (94)$$

$$\ln \left(\frac{s}{\chi_{\pm}(s)} \right) = \ln(1 \mp \beta) \pm \frac{\beta s^2}{6(1 \mp \beta)^3} + O(s^4), \quad (95)$$

$$\left(\frac{s}{\chi_{\pm}(s)} \right)^2 = (1 \mp \beta)^2 \pm \frac{2\beta s^2}{6(1 \mp \beta)} + O(s^4). \quad (96)$$

2. At $s \rightarrow \infty$

$$\frac{s}{\chi_{\pm}(s)} = 1 \pm \frac{A_1\beta}{\sqrt{|s|}} + \frac{A_2\beta^2}{|s|} \pm \frac{A_3\beta^3}{|s|\sqrt{|s|}} + O\left(\frac{1}{s^2}\right), \quad (97)$$

$$\ln \left(\frac{s}{\chi_{\pm}(s)} \right) = \pm \frac{A_4\beta}{\sqrt{|s|}} + \frac{A_5\beta^2}{|s|} \pm \frac{A_6\beta^3}{|s|\sqrt{|s|}} + O\left(\frac{1}{s^2}\right), \quad (98)$$

$$\left(\frac{s}{\chi_{\pm}(s)} \right)^2 = 1 \pm \frac{A_7\beta}{\sqrt{|s|}} + \frac{A_8\beta^2}{|s|} \pm \frac{A_9\beta^3}{|s|\sqrt{|s|}} + O\left(\frac{1}{s^2}\right), \quad (99)$$

here $A_j = \text{const}$, $j = \overline{1, 9}$.

Remark. One may compute the values of constants A_j , but, as we see below, there is no need.

Proof. By the definition, the functions $\chi_{\pm}(s)$ are the solutions of the equations

$$\chi_{\pm}(s) = s \pm \beta\psi(\chi_{\pm}(s)), \quad \beta = \frac{1}{\sqrt{\nu}}, \quad \psi(s) = \text{sign}(s)\sqrt{s \tanh(s)}. \quad (100)$$

Let $s \rightarrow 0$. It's easy to see, the function $\psi(s)$ is odd and

$$\psi(0) = \psi''(0) = \psi^{(4)}(0) = 0, \quad \psi'(0) = 1, \quad \psi'''(0) = -1. \quad (101)$$

Since the functions $\chi_{\pm}(s)$ are odd, then $\chi_{\pm}''(0) = \chi_{\pm}^{(4)}(0) = 0$. The equation (100) and the formula (101) implies

$$\chi'_{\pm}(s) = 1 \pm \beta\psi'(\chi_{\pm})\chi'_{\pm} \implies \chi'_{\pm}(0) = \frac{1}{1 \mp \beta\psi'(0)} = \frac{1}{1 \mp \beta},$$

and similarly

$$\begin{aligned} \chi_{\pm}''' &= \pm\beta\psi'''(\chi_{\pm})(\chi'_{\pm})^3 \pm 3\beta\psi''(\chi_{\pm})\chi'_{\pm}\chi''_{\pm} \pm \beta\psi'(\chi_{\pm})\chi_{\pm}''' \implies \\ \chi_{\pm}'''(0) &= \frac{\pm\beta\psi'''(0)(\chi'_{\pm}(0))^3 \pm 3\beta\psi''(0)\chi'_{\pm}(0)\chi''_{\pm}(0)}{1 \mp \beta\psi'(0)} = \mp \frac{\beta}{(1 \mp \beta)^4}. \end{aligned}$$

Hence

$$\frac{\chi_{\pm}(s)}{s} = \chi'_{\pm}(0) + \frac{\chi_{\pm}'''(0)s^2}{6} + O(s^4) = \frac{1}{1 \mp \beta} \mp \frac{\beta s^2}{6(1 \mp \beta)^4} + O(s^4),$$

and

$$\frac{s}{\chi_{\pm}(s)} = 1 \mp \beta \pm \frac{\beta s^2}{6(1 \mp \beta)^2} + O(s^4),$$

i.e. we proved the formula (94). The formulas (95) and (96) obviously follows from (94). The first lemma statement proved.

Now let $s \rightarrow \infty$. Then $\tanh(s) = \text{sign}(s) + O(e^{-|s|})$, whence $\sqrt{s \tanh(s)} = \sqrt{|s|} + O(e^{-|s|})$, and so

$$\mu_{\pm}(\chi) = \chi \pm \frac{\text{sign}(\chi)}{\sqrt{\nu}} \sqrt{|\chi|} + O(e^{-|\chi|}),$$

or

$$s = \chi_{\pm}(s) \pm \frac{\text{sign}(s)}{\sqrt{\nu}} \sqrt{|\chi_{\pm}(s)|} + O(e^{-|s|}).$$

This equation evidently implies the statement (97). The statements (98) and (99) obviously follows from (97). The lemma proved completely.

Now we return to the function $h_j(s)$, $H_j(\omega)$, $j = \overline{0, 1}$ of the form (44), i.e.

$$h_0(s) = \ln \left(\frac{s^2}{\chi_+(s)\chi_-(s)} \right), \quad H_0(\omega) = S_0^+(h_0(s) | \omega), \quad (102)$$

$$h_1(s) = \frac{1}{2} \left(\frac{s^2}{\chi_+^2(s)} + \frac{s^2}{\chi_-^2(s)} \right), \quad H_1(\omega) = S_0^+(h_1(s) | \omega), \quad (103)$$

where the Cauchy integral S_0^+ we've introduced in the previous subsection, the formula (81).

Theorem 5

1. The functions $h_j(s)$ are even and smooth and at real $\omega = \mu$
 $H_j(-\mu) = \overline{H_j(\mu)}$, $j = \overline{0, 1}$.

2. At $s \rightarrow 0$

$$h_0(s) = \ln \frac{\nu - 1}{\nu} + \frac{\nu(3\nu + 1)s^2}{6(\nu - 1)^3} + O(s^4), \quad (104)$$

$$h_1(s) = 1 + \frac{1}{\nu} + \frac{s^2}{3(\nu - 1)} + O(s^4).$$

3. At $s \rightarrow \infty$

$$h_0(s) = \frac{A_0}{|s|} + O\left(\frac{1}{s^2}\right), \quad h_1(s) = 1 + \frac{A_1}{|s|} + O\left(\frac{1}{s^2}\right), \quad (105)$$

$A_0, A_1 = \text{const.}$

4. At $\omega \rightarrow 0$

$$H_0(\omega) = \ln \frac{\nu - 1}{\nu} - i\omega B \frac{\nu}{\nu - 1} + \frac{\nu(3\nu + 1)\omega^2}{6(\nu - 1)^3} + O(\omega^3),$$

$$e^{H_0(\omega)} = \frac{\nu - 1}{\nu} - i\omega B - \omega^2 \left(\frac{\nu B^2}{\nu - 1} - \frac{3\nu + 1}{6(\nu - 1)^2} \right) + O(\omega^3), \quad (106)$$

$$H_1(\omega) = 1 + \frac{1}{\nu} + \frac{\omega^2}{3(\nu - 1)} + O(\omega^3),$$

where

$$B = \frac{2}{\pi} \int_{\mathbb{R}} \frac{\cosh(s) - \sinh(s)/s}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds. \quad (107)$$

5. At $\omega \rightarrow \infty$

$$H_0(\omega) = o(1), \quad \exp(H_0(\omega)) = 1 + o(1), \quad H_1(\omega) = 1 + o(1). \quad (108)$$

Proof. The first theorem statement evidently follows from the definitions $h_j(s)$, $j = \overline{0, 1}$ (102), (103), the simplest properties of $\chi_{\pm}(\mu)$ (see the beginning of this subsection) and the first statement of the lemma 3.

We go to the second statement. The definition $h_0(s)$ (102) and the formula (95) from the previous lemma immediately imply

$$h_0(s) = \ln(1 - \beta^2) + \frac{\beta^2(3 + \beta^2)s^2}{3(1 - \beta^2)^3} + O(s^4), \quad s \rightarrow 0,$$

and taking into account $\beta = 1/\sqrt{\nu}$, we obtain the formula (104) for $h_0(s)$. Absolutely similar we obtain from (96) the second part of (104).

We proceed to the $H_j(\omega)$, $j = \overline{0, 1}$. It follows from the lemma 3 (previous subsection), that

$$H_j(\omega) = h_j(0) + \frac{\omega}{\pi i} \int_{\mathbb{R}} \frac{h_j(s) - h_j(0)}{s^2} ds + \frac{h_j''(0)\omega^2}{2} + O(\omega^3), \quad \omega \rightarrow 0, \quad (109)$$

$$H_0(\omega) = o(1), \quad H_1(\omega) = 1 + o(1), \quad \omega \rightarrow \infty.$$

But (104) and (105) imply

$$h_0(0) = \ln \frac{\nu - 1}{\nu}, \quad h_0''(0) = \frac{\nu(3\nu + 1)}{3(\nu - 1)^3}, \quad h_1(0) = 1 + \frac{1}{\nu}, \quad h_1''(0) = \frac{2}{3(\nu - 1)},$$

and so to prove (106), (107) and (108) we only need to find the integrals

$$B_j = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{h_j(s) - h_j(0)}{s^2} ds, \quad j = \overline{0, 1}. \quad (110)$$

Let's start by the B_1 calculating. According to the $h_1(s)$ definition (103) and $h_1(0) = 1 + 1/\nu = 1 + \beta^2$, we represent

$$B_1 = -\frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{1}{\chi_+^2(s)} + \frac{1}{\chi_-^2(s)} - \frac{(1 - \beta)^2 + (1 + \beta)^2}{s^2} \right) ds = J_+ + J_-, \quad (111)$$

where

$$J_{\pm} = -\frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{1}{\chi_{\pm}^2(s)} - \frac{(1 \mp \beta)^2}{s^2} \right) ds.$$

We make the change of variable $s = \mu_{\pm}(p)$ in the integrals J_{\pm} respectively and obtain

$$J_{\pm} = -\frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\mu'_{\pm}(p)}{p^2} - \frac{(1 \mp \beta)^2 \mu'_{\pm}(p)}{\mu_{\pm}^2(p)} \right) dp,$$

and the representation (111) takes a form

$$B_1 = -\frac{1}{2\pi} \int_{\mathbb{R}} b(p) dp, \quad (112)$$

$$b(p) = \frac{\mu'_+(p) + \mu'_-(p)}{p^2} - \frac{(1 - \beta)^2 \mu'_+(p)}{\mu_+^2(p)} - \frac{(1 + \beta)^2 \mu'_-(p)}{\mu_-^2(p)}.$$

But (see (43)) $\mu_+(p) + \mu_-(p) = 2p$, i.e. $\mu'_+(p) + \mu'_-(p) = 2$. Using that, and taking in mind, that the integrand $b(p)$ in (112) is even, we calculate the integral B_1 in (112) in the sense of principal value:

$$B_1 = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{\varepsilon}^{\infty} \frac{2 dp}{p^2} - \int_{\varepsilon}^{\infty} \frac{(1 - \beta)^2 \mu'_+(p)}{\mu_+^2(p)} dp - \int_{\varepsilon}^{\infty} \frac{(1 + \beta)^2 \mu'_-(p)}{\mu_-^2(p)} dp \right) =$$

$$-\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(-\frac{2}{p} \Big|_{\varepsilon}^{\infty} + \frac{(1 - \beta)^2}{\mu_+(p)} \Big|_{\varepsilon}^{\infty} + \frac{(1 + \beta)^2}{\mu_-(p)} \Big|_{\varepsilon}^{\infty} \right),$$

or

$$B_1 = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\frac{2}{\varepsilon} - \frac{(1 - \beta)^2}{\mu_+(\varepsilon)} - \frac{(1 + \beta)^2}{\mu_-(\varepsilon)} \right). \quad (113)$$

It is easy follows from (43), that $\mu_{\pm}(s) = s(1 \mp \beta) + O(s^3)$, $s \rightarrow 0$ and therefore

$$\frac{1}{\mu_{\pm}(\varepsilon)} = \frac{1}{\varepsilon(1 \mp \beta)} + O(\varepsilon), \quad \varepsilon \rightarrow 0. \quad (114)$$

Substituting the expressions (114) in (113), we finally obtain

$$B_1 = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\frac{2}{\varepsilon} - \frac{1 - \beta}{\varepsilon} - \frac{1 + \beta}{\varepsilon} + O(\varepsilon) \right) = 0.$$

Thus we actually proved the formula (106) for $H_1(\omega)$.

Now we proceed to the B_0 calculating. We at once integrate its definition (110) by part and, using the definition of $h_0(s)$ (102), obtain

$$B_0 = -\frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{2}{s} - \frac{\chi'_+(s)}{\chi_+(s)} - \frac{\chi'_-(s)}{\chi_-(s)} \right) \frac{ds}{s} = J_+ + J_-, \quad (115)$$

where

$$J_{\pm} = -\frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{s} - \frac{\chi'_{\pm}(s)}{\chi_{\pm}(s)} \right) \frac{ds}{s}.$$

We make again the change of variable $s = \mu_{\pm}(p)$ in the integrals J_{\pm} respectively and the formula (115) takes a form

$$B_0 = \frac{1}{\pi} \left(\int_{\mathbb{R}} \left(\frac{1}{p} - \frac{\mu'_+(p)}{\mu_+(p)} \right) \frac{dp}{\mu_+(p)} + \int_{\mathbb{R}} \left(\frac{1}{p} - \frac{\mu'_-(p)}{\mu_-(p)} \right) \frac{dp}{\mu_-(p)} \right). \quad (116)$$

Calculating the integral (116) in the sense of the principal value, and using the obvious formulas (see (43))

$$\mu_+(p) + \mu_-(p) = 2p, \quad \mu_+(p)\mu_-(p) = p^2 - \frac{p \tanh(p)}{\nu},$$

and the formula (114), we obtain

$$\begin{aligned} B_0 &= \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{|s|>\varepsilon} \frac{\mu_+(s) + \mu_-(s)}{s\mu_+(s)\mu_-(s)} ds - \int_{|s|>\varepsilon} \frac{\mu'_+(s)}{\mu_+^2(s)} ds - \int_{|s|>\varepsilon} \frac{\mu'_-(s)}{\mu_-^2(s)} ds \right) = \\ &= \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{|s|>\varepsilon} \frac{2\nu \cosh(s)}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds - \frac{2}{\mu_+(\varepsilon)} - \frac{2}{\mu_-(\varepsilon)} \right) = \\ &= \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{|s|>\varepsilon} \frac{2\nu \cosh(s)}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds - \frac{4\nu}{(\nu - 1)\varepsilon} + O(\varepsilon) \right) = \\ &= \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{|s|>\varepsilon} \frac{2\nu \cosh(s)}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds - \int_{|s|>\varepsilon} \frac{2\nu ds}{(\nu - 1)s^2} \right) = \\ &= -\frac{2\nu}{\pi(\nu - 1)} \lim_{\varepsilon \rightarrow 0} \int_{|s|>\varepsilon} \frac{\cosh(s) - \sinh(s)/s}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds, \end{aligned}$$

and finally

$$B_0 = -\frac{2\nu}{\pi(\nu - 1)} \int_{\mathbb{R}} \frac{\cosh(s) - \sinh(s)/s}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds. \quad (117)$$

Substituting the expressions for B_0 (117) and (110) in the expression for $H_0(\omega)$ (109), we obviously obtain the representation (106), (107) for $H_0(\omega)$ at $\omega \rightarrow 0$. The theorem proved completely.

6.3 The analysis and calculation of some integrals

Here we analyze some important for the problems 1 and 2 functions and express them in terms of the introduced above functions $H_0(\omega)$ and $H_1(\omega)$.

We start with the values $\ln X^+(0, \omega)$ and $(\ln X^+)''_{\xi\xi}(0, \omega)$ for the introduced in the subsection 2.1 solution of the homogeneous problem $X^+(\xi, \omega)$. We take the representation (24) and choose $\xi_1 = \xi_2 = \omega$ (we remind, the solution $X^+(\xi, \omega)$ doesn't depend on the choice of points $\xi_{1,2}$, located in the upper half plane). Then the representation (23), (24) gives

$$\begin{aligned}\ln X^+(0, \omega) &= 2 \ln \omega + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\varphi(s, \omega)}{s} ds, \\ (\ln X^+)''_{\xi\xi}(0, \omega) &= -\frac{2}{\omega^2} + \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\varphi(s, \omega)}{s^3} ds,\end{aligned}\tag{118}$$

where (see (23), (5))

$$\varphi(s, \omega) = \ln \frac{F(-s, \omega)(s - \omega)^2}{F(s, \omega)(s + \omega)^2}.\tag{119}$$

The integrals converge because

$$\varphi(s, \omega) = O\left(\frac{1}{s}\right), \quad s \rightarrow \infty \quad \text{and} \quad \varphi(0, \omega) = \varphi'_s(0, \omega) = \varphi''_{ss}(0, \omega) = 0.$$

We integrate the integrals in (118) by part:

$$\begin{aligned}\ln X^+(0, \omega) &= 2 \ln \omega - \frac{1}{2\pi i} \int_{\mathbb{R}} \varphi'_s(s, \omega) \ln |s| ds, \\ (\ln X^+)''_{\xi\xi}(0, \omega) &= -\frac{2}{\omega^2} + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\varphi'_s(s, \omega) ds}{s^2}.\end{aligned}\tag{120}$$

We present (see (21), (43))

$$F(s, \omega) = \nu \cosh(s)(\mu_+(s) - \omega)(\mu_-(s) - \omega),$$

then, according to the (119)

$$\varphi'_s(s, \omega) = -2\omega \left(\frac{\mu'_+(s)}{\mu_+^2(s) - \omega^2} + \frac{\mu'_-(s)}{\mu_-^2(s) - \omega^2} - \frac{2}{s^2 - \omega^2} \right), \quad (121)$$

and we obtain the representation for $\ln X^+(0, \omega)$:

$$\begin{aligned} \ln X^+(0, \omega) &= 2 \ln \omega + \frac{\omega}{\pi i} \times \\ &\left[\int_{\mathbb{R}} \frac{\mu'_+(s) \ln |s| ds}{\mu_+^2(s) - \omega^2} + \int_{\mathbb{R}} \frac{\mu'_-(s) \ln |s| ds}{\mu_-^2(s) - \omega^2} - \int_{\mathbb{R}} \frac{2 \ln |s| ds}{s^2 - \omega^2} \right] = \\ &2 \ln \omega + \frac{\omega}{\pi i} \left[\int_{\mathbb{R}} \frac{\ln |\chi_+(s)| ds}{s^2 - \omega^2} + \int_{\mathbb{R}} \frac{\ln |\chi_-(s)| ds}{s^2 - \omega^2} - \int_{\mathbb{R}} \frac{2 \ln |s| ds}{s^2 - \omega^2} \right], \end{aligned}$$

and finally, according to the definition S_0^+ (81) and the formula (102), we have the representation

$$\ln X^+(0, \omega) = 2 \ln \omega - H_0(\omega). \quad (122)$$

Now we consider $(\ln X^+)''_{\xi\xi}(0, \omega)$: according to the formulas (120), (121), we present

$$\begin{aligned} (\ln X^+)''_{\xi\xi}(0, \omega) &= -\frac{2}{\omega^2} - \frac{\omega}{\pi i} \int_{\mathbb{R}} \left(\frac{\mu'_+(s)}{\mu_+^2(s) - \omega^2} + \frac{\mu'_-(s)}{\mu_-^2(s) - \omega^2} + \frac{2}{\omega^2} \right) \frac{ds}{s^2} + \\ &\frac{\omega}{\pi i} \int_{\mathbb{R}} \left(\frac{2}{\omega^2} + \frac{2}{s^2 - \omega^2} \right) \frac{ds}{s^2}, \end{aligned}$$

and since (see (43)) $\mu_+(s) + \mu_-(s) = 2s$, i.e. $\mu'_+(s) + \mu'_-(s) = 2$, then, according to the definition S_0^+ (81), we can write

$$\begin{aligned} (\ln X^+)''_{\xi\xi}(0, \omega) &= -\frac{2}{\omega^2} - \frac{\omega}{\pi i} \int_{\mathbb{R}} \left(\frac{\mu'_1(s)}{\mu_+^2(s) - \omega^2} + \frac{\mu'_+(s)}{\omega^2} \right) \frac{ds}{s^2} - \\ &\frac{\omega}{\pi i} \int_{\mathbb{R}} \left(\frac{\mu'_-(s)}{\mu_-^2(s) - \omega^2} + \frac{\mu'_-(s)}{\omega^2} \right) \frac{ds}{s^2} + \frac{2}{\pi i \omega} \int_{\mathbb{R}} \frac{ds}{s^2 - \omega^2} = \\ &-\frac{2}{\omega^2} - \frac{1}{\omega^2} S_0^+ \left(\frac{s^2}{\chi_+^2(s)} \mid \omega \right) - \frac{1}{\omega^2} S_0^+ \left(\frac{s^2}{\chi_-^2(s)} \mid \omega \right) + \frac{2}{\omega^2} S_0^+(1 \mid \omega), \end{aligned}$$

and finally, using the formulas (84) and (103), we obtain the representation

$$(\ln X^+)''_{\xi\xi}(0, \omega) = -\frac{2H_1(\omega)}{\omega^2}. \quad (123)$$

Now we proceed to the analysis and computation of the introduced in the subsection 2.2 (formula (34)) integrals

$$P_j(\omega) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{s^j \cosh(s)}{Y(s, \omega)} ds, \quad Q_j(\omega) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{s^j \sinh(s)}{Y(s, \omega)} ds, \quad (124)$$

where on default $\text{Im } \omega > 0$. As we know (the lemma 1), $Y(s, \omega) \neq 0$ and the only singular points of integrals (124) are $s = \infty$ and $s = 0$ at $j < 0$. We consider the integrals (124) in the sense of the principal value, i.e.

$$\int_{\mathbb{R}} \cdots ds = \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_{\varepsilon < |s| < R} \cdots ds.$$

The asymptotics of $Y(\chi, \omega)$ (20) shows, the integrals $P_j(\omega)$ converge at $j = \overline{-1, 3}$ and $Q_j(\omega)$ converge at $j = \overline{-2, 2}$. We define

$$P_j(\omega) = \frac{1}{2\pi i} \int_{\mathbb{R}} s^j \left(\frac{\cosh(s)}{Y(s, \omega)} - \frac{1}{Y(0, \omega)} \right) ds, \quad j = \overline{-3, -2}. \quad (125)$$

Since the function $\varphi(s) = \cosh(s)/Y(s, \omega)$ is even, then $\varphi(s) - \varphi(0) = As^2 + O(s^4)$, $s \rightarrow 0$, $A = \text{const}$, i.e. the integrals (125) converge in the sense of the principal value too, see e.g. [1, 2]. We note, the integral from the odd function is equal to zero, and so

$$P_{-3}(\omega) = P_{-1}(\omega) = P_1(\omega) = P_3(\omega) = Q_{-2}(\omega) = Q_0(\omega) = Q_2(\omega) = 0. \quad (126)$$

Before calculating the integrals, we establish several auxiliary statements. We introduce the curves L_m , L_ε and the region Π_m :

$$L_m = \{ \xi = \pi m + i\beta, \quad \beta \in [0, \pi m] \} \cup \{ \xi = -\pi m + i\beta, \quad \beta \in [0, \pi m] \} \cup \{ \xi = \alpha + i\pi m, \quad |\alpha| \leq \pi m \}, \quad m \in \mathbb{N}; \quad (127)$$

$$\Pi_m = \{ \xi \mid \text{Re } \xi \in (-\pi m, \pi m), \quad \text{Im } \xi \in (0, \pi m) \}, \quad m \in \mathbb{N}; \quad (128)$$

$$L_\varepsilon = \{ \xi \mid |\xi| = \varepsilon, \quad \text{Im } \xi > 0 \}, \quad \varepsilon > 0; \quad (129)$$

we go around the curves L_m and L_ε counterclockwise.

Lemma 5 $\tanh(\xi) = O(1)$ at $\xi \in L_m$, $m \rightarrow \infty$.

Proof. If $\xi = \alpha + \pi im$, $\alpha \in \mathbb{R}$, then $\tanh(\xi) = \tanh(\alpha) = O(1)$, in its turn at $\xi = \pi m + i\beta$ we have

$$\tanh(\xi) = \frac{1 - e^{-2\pi m} e^{-2i\beta}}{1 + e^{-2\pi m} e^{-2i\beta}} = 1 + O(e^{-2\pi m}),$$

and similarly $\tanh(\xi) = -1 + O(e^{-2\pi m})$ at $\xi = -\pi m + i\beta$, as a whole we obtain the lemma statement.

Lemma 6 Let the function $\varphi(\xi)$ is defined in the upper half plane and

$$\frac{\varphi(\xi)}{\xi^4 \cosh(\xi)} \Big|_{\xi \in L_m} = \frac{\alpha}{\xi} + O\left(\frac{1}{\xi^2}\right), \quad m \rightarrow \infty. \quad (130)$$

Then

$$\lim_{m \rightarrow \infty} \frac{1}{2\pi i} \int_{L_m} \frac{\varphi(\xi) d\xi}{Y(\xi, \omega)} = \frac{\alpha}{2\nu}. \quad (131)$$

Proof. It follows from the lemma 5, that at $\xi \in L_m$ and $m \rightarrow \infty$

$$\frac{F(\xi, \omega)}{\nu \xi^2 \cosh(\xi)} = \left(1 - \frac{\omega}{\xi}\right)^2 - \frac{\tanh(\xi)}{\xi} = 1 + O\left(\frac{1}{\xi}\right),$$

and therefore

$$\frac{\nu \xi^2 \cosh(\xi)}{F(\xi, \omega)} = 1 + O\left(\frac{1}{\xi}\right), \quad \xi \in L_m, \quad m \rightarrow \infty. \quad (132)$$

Using the representation (132) and the asymptotic of $X^\pm(\xi, \omega)$ (17) (see lemma 1), we obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_{L_m} \frac{\varphi(\xi) d\xi}{Y(\xi, \omega)} &= \frac{1}{2\pi i} \int_{L_m} \frac{\varphi(\xi)}{\xi^4 \cosh(\xi)} \cdot \frac{\xi^2 \cosh(\xi)}{F(\xi, \omega)} \cdot \frac{\xi^2}{X^+(\xi, \omega)} d\xi = \\ &= \frac{1}{2\pi i} \int_{L_m} \left(\frac{\alpha}{\xi} + O\left(\frac{1}{\xi^2}\right)\right) \left(\frac{1}{\nu} + O\left(\frac{1}{\xi}\right)\right) \left(1 + O\left(\frac{1}{\xi}\right)\right) d\xi, \end{aligned}$$

and finally

$$\frac{1}{2\pi i} \int_{L_m} \frac{\varphi(\xi) d\xi}{Y(\xi, \omega)} = \frac{\alpha}{2\pi i \nu} \int_{L_m} \frac{d\xi}{\xi} + \int_{L_m} O\left(\frac{1}{\xi^2}\right) d\xi = I_1 + I_2. \quad (133)$$

For the first summand, we may deform the curve L_m into a semicircle and so calculate I_1 :

$$I_1 = \frac{\alpha}{2\pi i\nu} \int_{\substack{|\xi|=\pi m \\ \operatorname{Im}\xi>0}} \frac{d\xi}{\xi} = \frac{\alpha}{2\nu}. \quad (134)$$

Passing to the second summand, we note that the length of L_m is $L(L_m) = 4\pi m$, and at $\xi \in L_m$ we have $|\xi| \geq \pi m$, whence follows

$$|I_2| = \left| \int_{L_m} O\left(\frac{1}{\xi^2}\right) d\xi \right| \leq \frac{\operatorname{const} \cdot L(L_m)}{|\xi|^2} = o(1), \quad m \rightarrow \infty. \quad (135)$$

The lemma statement evidently follows from (133), (134) and (135).

Corollary. The lemma condition (130) is satisfied, and so the lemma statement is valid, for the functions $\xi^j \cosh(\xi)$, $j = \overline{-3, 3}$ with $\alpha = 1$ at $j = 3$ and $\alpha = 0$ at $j < 3$, and, according to the lemma 5, for the functions $\xi^j \sinh(\xi)$, $j = \overline{-2, 2}$ with $\alpha = 0$.

We calculate the integrals P_j , Q_j with the help of residues. Let $\xi_n = \xi_n(\omega)$, $n = \overline{1, \infty}$ be the zeroes of $Y(\xi, \omega)$ (i.e. the zeroes of $F(\xi, \omega)$, see the formulas (19), (6)) in the upper half plane and

$$\psi_n = \operatorname{Res} \frac{1}{Y(\xi_n, \omega)}. \quad (136)$$

Lemma 7 *Let the function $\varphi(\xi)$ be analytic in the upper half plane and*

$$\frac{\varphi(\xi)}{\xi^5 \cosh(\xi)} \Big|_{\xi \in L_m} = \frac{\alpha}{\xi} + O\left(\frac{1}{\xi^2}\right), \quad (137)$$

where L_m have a form (127). Then

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\varphi(s) ds}{sY(s, \omega)} = \frac{\varphi(0)}{2Y(0, \omega)} - \frac{\alpha}{2\nu} + \sum_n \frac{\varphi(\xi_n)}{\xi_n} \psi_n, \quad (138)$$

here ψ_n are the residues (136).

Proof. By the definition

$$\int_{\mathbb{R}} \frac{\varphi(s) ds}{sY(s, \omega)} = \lim_{\substack{\varepsilon \rightarrow 0 \\ m \rightarrow \infty}} \int_{\varepsilon < |s| < m} \frac{\varphi(s) ds}{sY(s, \omega)}.$$

On the other hand,

$$\int_{\varepsilon < |s| < m} \frac{\varphi(s) ds}{sY(s, \omega)} + \int_{L_m} \frac{\varphi(\xi) d\xi}{\xi Y(\xi, \omega)} - \int_{L_\varepsilon} \frac{\varphi(\xi) d\xi}{\xi Y(\xi, \omega)} = 2\pi i \sum_{\xi_n \in \Pi_m} \frac{\varphi(\xi_n)}{\xi_n} \psi_n,$$

where the region Π_m and curve L_ε have a form (128) and (129) respectively. Hence,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\varphi(s) ds}{sY(s, \omega)} = \\ & \lim_{\substack{\varepsilon \rightarrow 0 \\ m \rightarrow \infty}} \left(\sum_{\xi_n \in \Pi_m} \frac{\varphi(\xi_n)}{\xi_n} \psi_n - \frac{1}{2\pi i} \int_{L_m} \frac{\varphi(\xi) d\xi}{\xi Y(\xi, \omega)} + \frac{1}{2\pi i} \int_{L_\varepsilon} \frac{\varphi(\xi) d\xi}{\xi Y(\xi, \omega)} \right). \end{aligned} \quad (139)$$

But for any function $f(\xi)$, analytic in a neighborhood of zero (see e.g. [1, 2])

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{L_\varepsilon} \frac{f(\xi)}{\xi} d\xi = \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{2\pi i} \int_{L_\varepsilon} \frac{f(\xi) - f(0)}{\xi} d\xi + \frac{f(0)}{2\pi i} \int_{L_\varepsilon} \frac{d\xi}{\xi} \right) = \frac{f(0)}{2},$$

and so

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{L_\varepsilon} \frac{\varphi(\xi) d\xi}{\xi Y(\xi, \omega)} = \frac{\varphi(0)}{2Y(0, \omega)}. \quad (140)$$

The lemma statement (138) obviously follows from (139), (140) and lemma 6 (formula (131)).

Theorem 6

$$P_0(\omega) = -\frac{\omega}{4Y(0, \omega)}; \quad (141)$$

$$P_2(\omega) = \frac{1}{4\nu\omega}; \quad (142)$$

$$Q_{-1}(\omega) = \nu P_0(\omega) \left(-2 - \frac{1}{\nu} + \frac{\omega^2(1 - Z(\omega))}{2} \right), \quad (143)$$

where

$$Z(\omega) = \frac{Y''_{\xi\xi}(0, \omega)}{Y(0, \omega)}; \quad (144)$$

$$Q_1(\omega) = \nu(P_2(\omega) + \omega^2 P_0(\omega)); \quad (145)$$

$$S^+ \left(\frac{s \sinh(s)}{Y(s, \omega)} \mid \omega \right) = -\frac{1}{X^+(\omega, \omega)} - \nu\omega P_0(\omega). \quad (146)$$

Proof. We apply the lemma 7 sequentially to the integrals $P_j(\omega)$, $j = \overline{-3, 3}$ and $Q_j(\omega)$, $j = \overline{-2, 2}$ of the form (124) and (125). As follows from the corollary after the lemma 6, the lemma 6, and hence the lemma 7, is true for all of them.

For convenience, we introduce the notation

$$I_j(\omega) = \sum_n \xi_n^j \cosh(\xi_n) \psi_n, \quad (147)$$

where ψ_n are the residues of $1/Y(\xi, \omega)$, see (136).

We start from $P_{-3}(\omega)$ of the form (125). Since the function $f(\xi) = \cosh(\xi)/Y(\xi, \omega)$ is even, then $f(\xi) = f(0) + f''(0)\xi^2/2 + O(\xi^4)$, $\xi \rightarrow 0$, and so (see the proof of the previous lemma)

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{L_\varepsilon} \frac{f(\xi) - f(0)}{\xi^3} d\xi = \frac{f''(0)}{4}.$$

So, having in mind the formula (126), the statement of lemma 7 for P_{-3} takes a form

$$0 = P_{-3}(\omega) = I_{-3}(\omega) + \frac{1}{4} \left(\left(\frac{\cosh(\xi)}{Y(\xi, \omega)} \right)'' \right) \Big|_{\xi=0},$$

and so

$$I_{-3}(\omega) = \frac{1}{4} \left(\frac{Y''_{\xi\xi}(0, \omega)}{Y^2(0, \omega)} - \frac{1}{Y(0, \omega)} \right) = \frac{Z(\omega) - 1}{4Y(0, \omega)}, \quad (148)$$

where $Z(\omega)$ has a form (144).

Absolutely similarly

$$P_{-2}(\omega) = I_{-2}(\omega). \quad (149)$$

Now we apply the lemma 7 for $P_{-1}(\omega)$ of a form (124). Here $\varphi(\xi) = \cosh(\xi)$ and, again taking into account the formula (126), we obtain

$$0 = P_{-1}(\omega) = I_{-1}(\omega) + \frac{1}{2Y(0, \omega)},$$

and so together with the formula (148) we have

$$I_{-1}(\omega) = -\frac{1}{2Y(0, \omega)}, \quad I_{-3}(\omega) = \frac{I_{-1}(\omega)(1 - Z(\omega))}{2}. \quad (150)$$

Next in the same way we obtain

$$P_0(\omega) = I_0(\omega), \quad (151)$$

$$0 = P_1(\omega) = I_1(\omega), \quad (152)$$

$$P_2(\omega) = I_2(\omega), \quad (153)$$

and at last

$$0 = P_3(\omega) = I_3(\omega) - \frac{1}{2\nu},$$

and so

$$I_3(\omega) = \frac{1}{2\nu}. \quad (154)$$

Now we proceed to the integrals $Q_j(\omega)$, $j = \overline{-2, 2}$ of the form (124). According to the form of $F(\xi, \omega)$ (6), the equation

$$0 = F(\xi_n, \omega) = \nu(\xi_n - \omega)^2 \cosh(\xi_n) - \xi_n \sinh(\xi_n)$$

implies

$$\sinh(\xi_n) = \frac{\nu \cosh(\xi_n)(\xi_n - \omega)^2}{\xi_n}. \quad (155)$$

Using the formula (126), lemma 7, formulas (155) and (148), we obtain

$$0 = Q_{-2} = \sum_n \frac{\sinh(\xi_n)}{\xi_n^2} \psi_n + \frac{1}{2Y(0, \omega)} =$$

$$\nu \sum_n \frac{\cosh(\xi_n)(\xi_n - \omega)^2}{\xi_n^3} \psi_n - I_{-1} = \nu(I_{-1} - 2\omega I_{-2} + \omega^2 I_{-3}) - I_{-1},$$

whence (see (153), (150))

$$P_{-2} = I_{-2} = \frac{\nu - 1}{2\nu\omega} I_{-1} + \frac{\omega}{2} I_{-3} = I_{-1} \left(\frac{\nu - 1}{2\nu\omega} + \frac{1 - Z(\omega)}{2} \right). \quad (156)$$

Absolutely similarly

$$Q_{-1} = \nu \sum_n \frac{\cosh(\xi_n)(\xi_n - \omega)^2}{\xi_n^2} \psi_n = \nu(I_0 - 2\omega I_{-1} + \omega^2 I_{-2}); \quad (157)$$

then using (126) and (155)

$$0 = Q_0 = \sum_n \sinh(\xi_n) \psi_n = \nu(I_1 - 2\omega I_0 + \omega^2 I_{-1}),$$

and so according to the formulas (150), (151), (152)

$$P_0 = I_0 = \frac{\omega}{2} I_{-1} = -\frac{\omega}{4Y(0, \omega)}, \quad (158)$$

i.e. we prove the formula (141). At the same time, the formula (157), taking into account the formulas (158) and (156), takes a form (143).

Now we proceed to Q_1 . Using (155), (151), (152) and (153), we have

$$Q_1 = \nu(I_2 - 2\omega I_1 + \omega^2 I_0) = \nu(P_2 + \omega^2 P_0),$$

i.e. we obtain the formula (145).

We go to Q_2 . Using once again (126) and (155), we have

$$0 = Q_2 = \nu(I_3 - 2\omega I_2 + \omega^2 I_1),$$

and so according to the formulas (152), (153), (154)

$$P_2 = I_2 = \frac{I_3}{2\omega} = \frac{1}{4\nu\omega},$$

i.e. we obtain (142).

It remains to prove the formula (146). Absolutely similar to the proof of the lemma 7, and taking into account the equality (see (19), (6))

$$Y(\omega, \omega) = F(\omega, \omega)X^+(\omega, \omega) = -\omega \sinh(\omega)X^+(\omega, \omega),$$

we have

$$\begin{aligned} S^+ \left(\frac{s \sinh(s)}{Y(s, \omega)} \Big| \omega \right) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{s \sinh(s)}{(s - \omega)Y(s, \omega)} ds = \\ \frac{\omega \sinh(\omega)}{Y(\omega, \omega)} + \sum_n \frac{\xi_n \sinh(\xi_n)}{\xi_n - \omega} \psi_n &= -\frac{1}{X^+(\omega, \omega)} + \sum_n \frac{\nu(\xi_n - \omega)^2 \cosh(\xi_n)}{\xi_n - \omega} \psi_n = \\ -\frac{1}{X^+(\omega, \omega)} + \frac{\nu}{2\pi i} \int_{\mathbb{R}} \frac{(s - \omega) \cosh(s)}{Y(s, \omega)} ds, \end{aligned}$$

and since the function $\cosh(s)/Y(s, \omega)$ is even with respect to s , i.e.

$$\int_{\mathbb{R}} \frac{s \cosh(s)}{Y(s, \omega)} ds = 0,$$

then

$$S^+ \left(\frac{\chi \sinh(\chi)}{Y(\chi, \omega)} \Big| \omega \right) = -\frac{1}{X^+(\omega, \omega)} - \nu\omega P_0(\omega),$$

i.e. the theorem proved completely.

Now we express the results of the theorem 6 in terms of the functions $H_j(\omega)$, $j = \overline{0, 1}$ of the form (44) or (102), (103). According to the (19), (6) and (122), we have

$$Y(0, \omega) = F(0, \omega)X^+(0, \omega) = \nu\omega^4 e^{-H_0(\omega)}. \quad (159)$$

We go to the function $Z(\omega)$ of the form (144). Since $Y(\xi, \omega)$ is even with respect to ξ (see the lemma 1), i.e. $Y'_\xi(0, \omega) = 0$, then, taking in mind the formulas (19) and (6) again, we obtain

$$\begin{aligned} Z(\omega) &= (\ln Y)''_{\xi\xi}(0, \omega) + \left(\frac{Y'_\xi(0, \omega)}{Y(0, \omega)} \right)^2 = (\ln F)''_{\xi\xi}(0, \omega) + (\ln X^+)''_{\xi\xi}(0, \omega) = \\ &= 1 - \frac{2(\nu + 1)}{\nu\omega^2} + (\ln X^+)''_{\xi\xi}(0, \omega), \end{aligned}$$

and finally, according to the representation (123),

$$Z(\omega) = 1 - \frac{2(\nu + 1)}{\nu\omega^2} - \frac{2H_1(\omega)}{\omega^2}. \quad (160)$$

Thus, we can supplement the statements of the theorem 6 with the following ones:

Theorem 7

$$\frac{Q_{-1}(\omega)}{\nu P_0(\omega)} = -1 + H_1(\omega), \quad \frac{Q_1(\omega)}{\nu P_2(\omega)} = 1 - e^{H_0(\omega)}. \quad (161)$$

Indeed, both formulas evidently follow from the statements of the theorem 6 and the formulas (159) and (160).

6.4 On the functions $\Delta(\omega)$, $\Delta_h(\omega)$, $\Delta_V(\omega)$

In the subsection 2.2, we obtained the system of equations (40) for the unknowns $h(\omega)$ and $V^+(\omega, \omega)$ and, respectively, the expressions for them (41) and (42). In this subsection, we establish other representations for the functions $\Delta(\omega)$, $\Delta_h(\omega)$, $\Delta_V(\omega)$ in (42), show the solvability of the system (40) at $\omega \neq 0$, i.e. the condition $\Delta(\omega) \neq 0$, and find the asymptotics $\Delta(\omega)$, $\Delta_h(\omega)$, $\Delta_V(\omega)$ at $\omega \rightarrow 0$.

First, the formulas (42) and (161) give the representation

$$\Delta(\omega) = -\frac{2}{\nu} + H_1(\omega) - e^{H_0(\omega)}. \quad (162)$$

Theorem 8 $\Delta(\omega) \neq 0$ at $\text{Im } \omega \geq 0$, $\omega \neq 0$.

Proof. It follows from the theorem 5, formulas (106), (107) and (108), that

$$\Delta(\omega) = i\omega B + O(\omega^2), \quad \omega \rightarrow 0; \quad \Delta(\omega) = -\frac{2}{\nu} + o(1), \quad \omega \rightarrow \infty, \quad (163)$$

what in particular means $\Delta(\omega) \neq 0$ at $|\omega| = \varepsilon \ll 1$ or $|\omega| = R \gg 1$. Additionally the first statement of the theorem 5 implies $\Delta(-\mu) = \overline{\Delta(\mu)}$, $\mu \in \mathbb{R}$.

To prove this theorem, we consider the index (the increment of the argument) of $\Delta(\omega)$:

$$\text{ind } \Delta = \frac{1}{2\pi} \Delta \arg \Delta(\omega) \Big|_L$$

along the curve

$$L = \{ \omega \mid |\omega| = R, \text{Im } \omega > 0 \} \cup [-R, -\varepsilon] \cup \{ \omega \mid |\omega| = \varepsilon, \text{Im } \omega > 0 \} \cup [\varepsilon, R],$$

traversed counterclockwise and at $R \gg 1$, $\varepsilon \ll 1$. If $\Delta(\mu) \neq 0$ at $\mu \in [\varepsilon, R]$, then (see (163)) $\Delta(\omega) \neq 0$ at $\omega \in L$ and $\text{ind } \Delta$ is defined correctly, and at that if $\text{ind } \Delta = 0$, then $\Delta(\omega) \neq 0$ at $|\omega| \in [\varepsilon, R]$, $\text{Im } \omega \geq 0$, what means $\Delta(\omega) \neq 0$ in a whole upper half plane, excluding the point $\omega = 0$. In turn, to prove the last statement we consider the dependance of $\Delta(\omega)$ on the parameter $\nu > 1$, i.e. on $\beta = 1/\sqrt{\nu} \in (0, 1)$. Namely, if $\Delta(\mu) \neq 0$ at $\mu \in [\varepsilon, R]$, then $\text{ind } \Delta$ doesn't depend on β , and so if $\text{ind } \Delta = 0$ at least for only one $\beta \in (0, 1)$, then $\text{Ind } \Delta = 0$ for each $\beta \in (0, 1)$, and so $\Delta(\omega) \neq 0$ at $\omega \neq 0$.

First we consider the dependance of $\chi_{\pm}(s)$ on β at $\beta \rightarrow 0$. For $\chi_{\pm}(s)$ we have the equation (100) $\chi_{\pm}(s) = s \pm \beta\psi(\chi_{\pm}(s))$, where $\psi(s) = \text{sign}(s)\sqrt{s \tanh(s)}$, and so

$$\chi_{\pm}(s) \Big|_{\beta=0} = s;$$

$$\frac{\partial \chi_{\pm}}{\partial \beta} = \pm \psi(\chi_{\pm}) \pm \beta \psi'(\chi_{\pm}) \frac{\partial \chi_{\pm}}{\partial \beta} \implies \frac{\partial \chi_{\pm}}{\partial \beta} \Big|_{\beta=0} = \frac{\pm \psi(\chi_{\pm})}{1 \mp \beta \psi'(\chi_{\pm})} \Big|_{\beta=0} = \pm \psi(s);$$

absolutely similarly

$$\frac{\partial^2 \chi_{\pm}}{\partial \beta^2} \Big|_{\beta=0} = 2\psi(s)\psi'(s);$$

and in total

$$\begin{aligned} \chi_{\pm} &= \chi_{\pm} \Big|_{\beta=0} + \beta \frac{\partial \chi_{\pm}}{\partial \beta} \Big|_{\beta=0} + \frac{\beta^2}{2} \frac{\partial^2 \chi_{\pm}}{\partial \beta^2} \Big|_{\beta=0} + O(\beta^3) = \\ &= s \pm \beta\psi(s) + \beta^2\psi(s)\psi'(s) + O(\beta^3). \end{aligned}$$

Then we successively obtain

$$\begin{aligned}\frac{\chi_{\pm}(s)}{s} &= 1 \pm \beta \frac{\psi(s)}{s} + \beta^2 \frac{\psi(s)\psi'(s)}{s} + O(\beta^3), \\ \frac{s}{\chi_{\pm}(s)} &= 1 \mp \beta \frac{\psi(s)}{s} + \beta^2 \left(\frac{\psi^2(s)}{s^2} - \frac{\psi(s)\psi'(s)}{s} \right) + O(\beta^3), \\ \ln \frac{s}{\chi_{\pm}(s)} &= \mp \beta \frac{\psi(s)}{s} + \frac{\beta^2}{2} \left(\frac{\psi^2(s)}{s^2} - \frac{2\psi(s)\psi'(s)}{s} \right) + O(\beta^3), \\ \left(\frac{s}{\chi_{\pm}(s)} \right)^2 &= 1 \mp \beta \frac{2\psi(s)}{s} + \beta^2 \left(\frac{3\psi^2(s)}{s^2} - \frac{2\psi(s)\psi'(s)}{s} \right) + O(\beta^3),\end{aligned}$$

so, using the definitions (102), (103) and $\psi(s) = \text{sign}(s)\sqrt{s \tanh(s)}$, we have

$$\begin{aligned}h_0(s) &= \beta^2 \left(\frac{\psi^2(s)}{s^2} - \frac{2\psi(s)\psi'(s)}{s} \right) + O(\beta^3) = -\beta^2 \left(\frac{\psi^2(s)}{s} \right)' + O(\beta^3) = \\ &= -\frac{\beta^2}{\cosh^2(s)} + O(\beta^3),\end{aligned}$$

$$\begin{aligned}h_1(s) &= 1 + \beta^2 \left(\frac{3\psi^2(s)}{s^2} - \frac{2\psi(s)\psi'(s)}{s} \right) + O(\beta^3) = \\ &= 1 + \beta^2 \left(\frac{2\psi^2(s)}{s^2} - \left(\frac{\psi^2(s)}{s} \right)' \right) + O(\beta^3) = \\ &= 1 + \beta^2 \left(\frac{2 \tanh(s)}{s} - \frac{1}{\cosh^2(s)} \right) + O(\beta^3),\end{aligned}$$

then according to the (102), (103) and (162)

$$\begin{aligned}H_0(\omega) &= -\beta^2 S_0^+ \left(\frac{1}{\cosh^2(s)} \mid \omega \right) + O(\beta^3), \\ e^{H_0(\omega)} &= 1 - \beta^2 S_0^+ \left(\frac{1}{\cosh^2(s)} \mid \omega \right) + O(\beta^3), \\ H_1(\omega) &= 1 + \beta^2 S_0^+ \left(\frac{2 \tanh(\mu)}{\mu} - \frac{1}{\cosh^2(\mu)} \mid \omega \right) + O(\beta^3), \\ \Delta(\omega) &= -2\beta^2 + \beta^2 S_0^+ \left(\frac{2 \tanh(\mu)}{\mu} \mid \omega \right) + O(\beta^3),\end{aligned}$$

and finally, with the help of the (85), we obtain the boundary values of $\Delta(\omega)$ at $\omega = \mu \in \mathbb{R}$:

$$\Delta(\mu) = 2\beta^2 \left(S_0 \left(\frac{\tanh(\mu)}{\mu} \mid \mu \right) + \frac{\tanh(\mu)}{\mu} - 1 \right) + O(\beta^3),$$

therefore

$$\operatorname{Re} \Delta(\mu) = 2\beta^2 \left(\frac{\tanh(\mu)}{\mu} - 1 \right) + O(\beta^3) < 0, \quad \mu \neq 0. \quad (164)$$

First, the formula (164) implies $\Delta(\mu) \neq 0$ at $\mu \in [\varepsilon, R]$. But then, taking into account the asymptotics (163), we have $\operatorname{Re} \Delta(\omega) < 0$ at $\omega \in L$, which automatically follows in $\Delta = 0$. Thus, in $\Delta = 0$, when $\beta \approx 0$. On the other hand, similarly to the previous considerations one can show, $\operatorname{Re} \Delta(\mu) < 0$ at all $\beta \in (0, 1)$, and so $\Delta(\mu) \neq 0$, $\mu \in [\varepsilon, R]$, what finally implies the theorem statement. The theorem proved.

Remark. Since $\Delta(\omega) \neq 0$, then the unknowns $h(\omega)$ and $V^+(\omega, \omega)$ of a form (41) are analytic at $\operatorname{Im} \omega > 0$ and continues at $\operatorname{Im} \omega \geq 0$, $\omega \neq 0$.

Now we express the functions $\Delta_h(\omega)$, $\Delta_V(\omega)$ of the form (42) through the functions $H_j(\omega)$, $j = \overline{0, 1}$. The formulas (42) and (161) have already given the representation for $\Delta(\omega)$ (162) and also give the representations

$$\Delta_h(\omega) = H_1(\omega) - e^{H_0(\omega)}, \quad \Delta_V(\omega) = \frac{1}{\nu} (2 - H_1(\omega) - e^{H_0(\omega)}). \quad (165)$$

At last, we establish the asymptotics of $\Delta(\omega)$, $\Delta_h(\omega)$, $\Delta_V(\omega)$ at $\omega \rightarrow 0$. The statements of the theorem 5, formulas (106) and (107), evidently imply

$$\begin{aligned} \Delta(\omega) &= i\omega B(1 - i\omega B_1 + O(\omega^2)), & \Delta_h(\omega) &= \frac{2}{\nu} + i\omega B + O(\omega^2), \\ \Delta_V(\omega) &= \frac{i\omega B}{\nu}(1 - i\omega B_2 + O(\omega^2)), \end{aligned} \quad (166)$$

where

$$B_1 = \frac{\nu B}{\nu - 1} - \frac{\nu + 3}{6B(\nu - 1)^2}, \quad B_2 = \frac{\nu B}{\nu - 1} - \frac{5\nu - 1}{6B(\nu - 1)^2},$$

$$B = \frac{2}{\pi} \int_{\mathbb{R}} \frac{\cosh(s) - \sinh(s)/s}{s^2(\nu \cosh(s) - \sinh(s)/s)} ds.$$

References

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