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## Sensitivity Analysis of a Transit Bottleneck Model

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### Abstract

In urban mass transit, station platforms constitute waiting areas for incoming users willing to board service vehicles. In the transit bottleneck model of Leurent et al. (2014, 2015), the passenger stocks and average wait times according to station of destination are obtained as the solution of a fixed point problem (FPP) with respect to passenger stocks: multiple service routes are allowed, each of which with residual in-vehicle capacity that can be or not saturated by the flow of users incoming at the station of interest. The paper provides a full sensitivity analysis of all model outcomes to all model inputs, namely entry flows according to destination stations, residual in-vehicle capacities and route frequencies. The method consists in partial differentiation by formal calculus, since the FPP amounts to an implicit function that is differentiable almost everywhere. As instance of application, the case of a busy commuter rail line in Paris is studied, with emphasis on the marginal congestion costs of incoming as well as on-board users onto the users waiting for boarding.

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*Keywords:* sensitivity analysis; transit capacity; bottleneck model; traffic assignment

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### 1. Introduction

**Background:** Vehicle operations and passenger flows interact in many ways along a transit line. The interaction is a complex one since vehicle operations depend on passenger flows, while passenger flows are determined by supply conditions. This has led to the development of a scientific stream focused on traffic assignment models which incorporate traffic conditions in relation to passenger flows as well as passenger route choice – hence a major part of demand response to supply conditions: the textbook by (Gentile et al., 2016) stands as a reference on these issues. To go further in the analysis of interactions, sensitivity analysis of such a traffic model is relevant as it enables one to measure the quantitative impact of selected model inputs onto selected model outputs. This can yield useful insight into potential improvements to sub-systems (services, lines or networks) and pave the way to their optimal design (Clark and Watling, 2002). In economic theory, an important instance pertains to the evaluation of the impact of a marginal user on production costs or

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demand surplus or the system welfare function that combines them: analytical formulas are especially useful to gain insight on such issues and to derive optimal prices for congestion impacts (from Vickrey, 1969, to Yang and Tang, 2018). Other features of interest in transit systems encompass line frequency (Mohring, 1972, Gao et al., 2004) since the ability to deliver one additional run along a railway line in a busy period can yield many benefits to serve more users and relieve congestion.

Among the traffic assignment models that feature out on-board crowding and discomfort in relation to passenger flows and vehicle capacity, let us quote dynamic models (Poon et al., 2004, Hamdouch and Lawphongpanich, 2008, Sumalee et al., 2009) and the static model “CapTA” of Leurent et al. (2014), in which traffic performance along a given line is modeled with analytical formulas and specific algorithms to solve them in limited computational complexity. A specific sub-model in CapTA, called the “transit bottleneck” model, deals with passenger boarding on each station platform in relation to the residual capacity of the incoming vehicles and their service frequency (Leurent et al., 2015). It has three specific features as follows: (i) exit capacity is delivered in bulks at discrete instants, (ii) multiple stations of destination are available from the station of boarding, (iii) the line can include multiple service routes (also called missions), each with specific residual capacity, frequency and service route i.e. subset of destination stations down that of boarding.

**Objective:** The paper provides a full sensitivity analysis of the transit bottleneck model outcomes to several model inputs along that transit line, including mission characteristics and passenger flows from access to egress stations. Special interest is focused on platform waiting in terms of passenger stocks and average wait times according to destination stations. The sensitivity matrix is obtained in an analytical way. A computational scheme is put forward, with time complexity in cubic order of the station number.

**Method.** In the transit bottleneck model, the traffic variables are macroscopic; the passenger stocks satisfy a set of relations that are put under analytical form and they can be computed as the unique solution of a nonlinear Fixed Point Problem. This makes all model outcomes an implicit function of the model inputs and the passenger stocks. By formal calculus, we obtain the partial derivatives of all outcomes with respect to all factors: the influence of every input is both direct and indirect i.e. channeled by the passenger stocks that are intermediate variables.

**Structure.** The body of the paper is organized in four sections. Section 2 recalls the main features of the transit bottleneck model. Section 3 provides the partial differentiations and sets out the characteristic equation that involves the Jacobian matrix of the passenger stocks for every destination with respect to all model inputs. Section 4 deals with the instance of a busy commuter rail line in Paris. Section 5 concludes and points out to research directions.

## 2. The transit bottleneck model: a refresher

### 2.1. Definitions and notation

Let  $i$  be a given access station along line  $L$ ,  $N_i$  the set of egress stations  $n$  served by transit services  $z \in Z_i$ . The subset of services that dwell at  $n$  coming from  $i$  is denoted as  $\{z \in (i, n)\}$ .

During period  $[0, H]$ , service  $z$  is operated at frequency  $\varphi_z$  by homogeneous vehicles, each of which supplies a residual capacity of  $k'_z$  at station  $i$  (after the egress of the passengers destined to  $i$ ), yielding available capacity of  $\kappa'_z = \varphi_z k'_z$  in the period. By assumption, passengers arrive at  $i$  under exogenous flow rates  $[x_{in}^+ : n \in N_i]$ , yielding entry volumes  $x_{in}^+ H$  by egress station.

Let  $\varphi_{in} \equiv \sum_{z \in (i, n)} \varphi_z$  be the combined frequency between  $i$  and  $n$  of the services in  $Z_i$ .

It is assumed that the platform is shared by the services and that no vehicle can overtake another one, meaning that a time-minimizing passenger is eager to board a relevant vehicle as soon as it has some space available to him. A line of operations consists in a set of services such that the routes and egress stations make up a connected component in the bipartite graph in  $Z_i \times N_i$  that links the services to the stations that they serve.

Table 1. Notation

$L$	A line, with set $N_L$ of stations and set $A_L$ of links
$z$	A transit service (mission, route) in $L$ , with set $N_z$ of stations and link sequence $P_z$ along route
$Z_L$	Set of services belonging to line $L$ of operations
$i, \ell, m, n$	Station along line, with set $N_i$ of down stations $n$ and set $Z_i$ of services in $Z_L$ visiting $i$
$k(i)z$	Residual passenger capacity at station $i$ in a vehicle running for service $z$
$\bar{\sigma}_n, \sigma_n^\#$	Average (resp. maximum) passenger stock waiting at $i$ and destined to egress station $n$
$v_{iz}$	Number of passengers candidate to board in a $z$ -vehicle at $i$
$\pi_{zi}$	Probability of immediate boarding in a $z$ -vehicle at $i$
$w_{in}, W_{in}$	Average (resp. Total) wait time of $n$ -destined passengers on platform at station $i$

2.2. Unsaturated Case

If capacity is available to each user in the first vehicle serving his destination to come after his instant of arrival, then he can board that vehicle. Denoting by  $\alpha$  a service regularity parameter ( $\frac{1}{2}$  if perfect, 1 if memoryless etc), the average wait time for egress station  $n$  is:

$$w_{in} = \alpha / \varphi_{in} . \tag{1}$$

Considering the statistical distribution of headways  $\eta$ , it holds that :

$$w_{in} = \frac{1}{2} E[\eta^2] / E[\eta] = \frac{1}{2} (1 + \gamma_\eta^2) / \varphi_{in} , \tag{2}$$

Wherein  $\gamma_\eta$  denotes the relative dispersion of that distribution. So  $\alpha = \frac{1}{2} (1 + \gamma_\eta^2)$ .

Then each service  $z$  gets a share  $\varphi_z / \varphi_{in}$  of trip volume  $V_n = x_{in}^+ H$ . The resulting passenger flow,  $x_{in}^+ H (\varphi_z / \varphi_{in})$ , yields a number  $x_{in}^+ / \varphi_{in}$  of passengers destined to  $n$  per vehicle running on service  $z$  since there are  $H\varphi_z$  such runs in the period. There remains available capacity if and only if, per vehicle run,

$$\sum_{n:(i,n) \ni z} \frac{x_{in}^+}{\varphi_{in}} \leq k'_z . \tag{3}$$

**Queue dynamics.** Denote by  $\sigma_{in}(h)$  the size of the passenger stock on platform  $i$  for exit station  $n$  with respect to time  $h$ , with average value  $\bar{\sigma}_n$ , minimum value  $\sigma_n^0$  and maximum value  $\sigma_n^\#$ . In the unsaturated case the stock vanishes periodically i.e.  $\sigma_n^0 = 0$  at any instant  $h_0$  of vehicle departure. A maximum is achieved just before that instant and the next maximum will be achieved at  $h_0 + 1 / \varphi_{in}$  with value  $x_{in}^+ / \varphi_{in}$ . Thus, it holds that:

$$\bar{\sigma}_n = \frac{1}{2} x_{in}^+ / \varphi_{in} , \quad \sigma_n^0 = 0 \quad \text{and} \quad \sigma_n^\# = x_{in}^+ / \varphi_{in} . \tag{4a,b,c}$$

During the period, total waiting time amounts to  $H\bar{\sigma}_n$ : individual wait time on average over the  $Hx_{in}^+$  users is

$$w_{in} = \frac{\bar{\sigma}_n}{x_{in}^+} = \frac{1}{2\varphi_{in}} . \tag{5}$$

2.3. Saturated Case

Similarly to (4), over a given cycle it holds that

$$\bar{\sigma}_n = \sigma_n^\# - \frac{1}{2} x_{in}^+ / \varphi_{in} \quad \text{and} \quad \bar{\sigma}_n = \sigma_n^0 + \frac{1}{2} x_{in}^+ / \varphi_{in} . \tag{6a,b}$$

If some passengers cannot board the first vehicle relevant to them, then they have to wait for next vehicles. About the  $\sigma_n(h)$  users waiting at  $h$  for egress station  $n$ , assume for simplicity that  $\sigma_n(h) = \sigma_n^\#$  at every  $h$  of vehicle arrival. When a vehicle serving  $z$  arrives, the number of candidate riders amounts to

$$v_z^\# = \sum_{n \in z} \sigma_n^\# . \tag{7}$$

Assuming equity among them, from available capacity  $k'_z$  the individual probability to board is

$$\pi_z = \min\{1, k'_z / v_z^\#\} \quad (\text{with } \pi_z = 1 \text{ if } v_z^\# = 0). \tag{8}$$

Then, there are  $\pi_z \sigma_n^\#$  users boarding any vehicle serving  $z$  to exit at  $n$ . Flow rate during  $[0, H]$  to  $n$  via  $z$  is

$$x_{inz}^- = \varphi_z \pi_z \sigma_n^\#. \tag{9}$$

The total flow rate from  $i$  to  $n$  during  $[0, H]$  is the throughput rate as follows:

$$x_{in}^- = \sum_{z \ni n} x_{inz}^- = (\varphi \pi)_n \sigma_n^\#, \text{ wherein: } (\varphi \pi)_n \equiv \sum_{z \ni n} \varphi_z \pi_z. \tag{10}$$

In (7-10) we consider a notional maximum stock  $\sigma_n^\#$  that can be interpreted as a notional average over the period plus the increment between maximum and average that stems from (7). Queuing saturation occurs when at least one service has  $\pi_z < 1$ .

To characterize service availability  $\pi_z$  and passenger stock  $\sigma_n^\#$  by egress station  $n$ , let us consider a bottleneck model of user flow for that station served by successive relevant vehicles (Fig. 1). Arrival flow has rate  $x_{in}^+$  on  $[0, H]$ . Exit flow rate is  $x_{in}^-$  on  $\omega_n + [0, H_n]$ : parameter  $\omega_n$  is the time up to the first arrival at  $i$  of a run that will serve  $n$ , whereas  $H_n$  is the time required to clear flow  $x_{in}^+ H$ . If  $x^- < x^+$  then total wait time amounts to the area of polygon OADC on the figure. As  $A(OADC) = A(BED) - A(OEA) - A(BOC)$ .

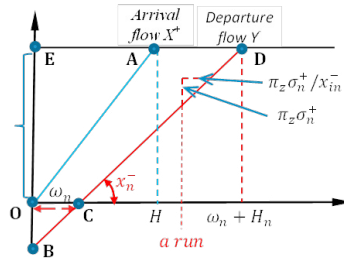


Fig. 1. Cumulative flows at bottleneck

Flow conservation implies that:  $V_n = H x_{in}^+ = H_n x_{in}^-$ . (11)

From formula  $A(OADC) = A(BED) - A(OEA) - A(BOC)$ , the total wait time amounts to:

$$W_{in}^0 = \frac{1}{2} \left( \frac{(V_n + \omega_n x_{in}^-)^2}{x_{in}^-} - \frac{V_n^2}{x_{in}^+} - \omega_n^2 x_{in}^- \right) = \frac{1}{2} \frac{V_n^2}{x_{in}^+} \left( \frac{x_{in}^+}{x_{in}^-} - 1 \right) \delta_n^\# + \omega_n V_n. \tag{12}$$

Indicator  $\delta_n^\#$  takes value 1 if  $x_{in}^+ > x_{in}^-$  or 0 otherwise, so as to encompass both cases either saturated or not. Time  $W_{in}^0$  needs be corrected to account for the discrete nature of capacity delivery by service runs. A  $z$  run at time  $h$  delivers an available capacity of  $\pi_z \sigma_n^\#$  to egress station  $n$ , yielding a vertical segment up from the  $Y^-$  curve of cumulative departures, followed by a horizontal segment of length  $\pi_z \sigma_n^\# / x_{in}^-$ . So the continuous model needs be corrected by subtracting the area of the corresponding small triangle, i.e.  $\pi_z^2 \sigma_n^{\#2} / (2 x_{in}^-)$ , as many times as there are runs serving the egress station during the queuing period  $H_n$ . Over services  $z \in (i, n)$  the total correction amounts to

$$C_n = \sum_{z \in (i, n)} \frac{1}{2} \frac{\pi_z^2 \sigma_n^{\#2}}{x_{in}^-} . H_n \varphi_z = \frac{1}{2} H_n (\varphi \pi^2)_n \frac{\sigma_n^{\#2}}{x_{in}^-}, \text{ wherein } (\varphi \pi^2)_n \equiv \sum_{z \in (i, n)} \varphi_z \pi_z^2. \tag{13}$$

As  $H_n = H x_{in}^+ / x_{in}^-$  and  $x_{in}^- = (\varphi \pi)_n \sigma_n^\#$  it turns out that  $C_n = \frac{1}{2} H x_{in}^+ (\varphi \pi^2)_n / (\varphi \pi)_n^2$ . Thus the total wait time is

$$W_{in} = W_{in}^0 - C_n = \frac{1}{2} H^2 x_{in}^+ \left( \frac{x_{in}^+}{x_{in}^-} - 1 \right) \delta_n^\# + \omega_n V_n - \frac{1}{2} H x_{in}^+ \frac{(\varphi \pi^2)_n}{(\varphi \pi)_n^2}, \text{ hence}$$

$$W_{in} = \frac{1}{2} H x_{in}^+ \left( H \left( \frac{x_{in}^+}{x_{in}^-} - 1 \right) \delta_n^\# + 2 \omega_n - \frac{(\varphi \pi^2)_n}{(\varphi \pi)_n^2} \right). \tag{14}$$

Average passenger stock  $\bar{\sigma}_n$  satisfies:  $\bar{\sigma}_n = W_{in} / H_n$ . (15)

Combining (6a), (14) and (15) and rearranging, we get that

$$\sigma_n^\# = \frac{x_{in}^+ (H \delta_n^\# + 1 / \varphi_{in})}{2 + (H \delta_n^\# - 2\omega_n)(\varphi\pi)_n + (\varphi\pi^2)_n / (\varphi\pi)_n}$$
 (16)

As the  $\pi_z$  depend on the  $\sigma_n^\#$  through the  $v_z^\#$  (recall eqns (7) and (8)), the set of equations (16) for all stations  $n$  down  $i$  is a Fixed Point Problem (FPP) in  $\sigma_{i\bullet} = [\sigma_{(i)n} : n \in N_i]$ .

The average wait time by individual passenger,  $w_{in} \equiv W_{in} / (x_{in}^+ H)$ , satisfies that

$$w_{in} = \frac{[\omega_n - \frac{1}{2} H \delta_n^\# + H \delta_n^\# \varphi_{in} / (\varphi\pi)_n - \frac{1}{2} (\varphi\pi^2)_n / (\varphi\pi)_n^2]}{(1 + H \delta_n^\# \varphi_{in})}$$
 (17)

**Capacitated Regime.** The saturated and unsaturated cases join together when  $x_{in}^+ = x_{in}^-$ , i.e. when  $x_{in}^+ = \sigma_n^\# (\varphi\pi)_n$ . Then, (2.16) imposes  $(H \delta_n^\# + 1 / \varphi_{in})(\varphi\pi)_n = 2(1 - \omega_n(\varphi\pi)_n) + H \delta_n^\# (\varphi\pi)_n + (\varphi\pi^2)_n / (\varphi\pi)_n$ , which is equivalent to

$$(\varphi\pi)_n / \varphi_{in} = 2(1 - \omega_n(\varphi\pi)_n) + (\varphi\pi^2)_n / (\varphi\pi)_n$$

At that point, notionally  $\pi_z = 1$  for every  $z$ , yielding  $(\varphi\pi)_n = \varphi_{in}$ , so that the equality condition is reduced to  $\omega_n = 1 / \varphi_{in}$ . Under this set-up, formula (16) also applies to the unsaturated case, since replacing every  $\pi_z$  by 1 yields that  $\sigma_n^\# = x_{in}^+ / \varphi_{in}$  as in (4c).

### 3. Sensitivity Analysis of the Platform Bottleneck Model

Every outcome of the bottleneck model, from passenger stocks to waiting times passing by boarding probabilities, stems from the stock variables  $[\sigma_m^\# : m]$  that satisfy the FPP (16). Their respective sensitivities with respect to the model inputs taken as factors, namely the passenger flows  $x_{in}^+$ , service frequencies  $\varphi_z$  and vehicle residual capacities  $k'_z$ , involve the sensitivities of the stock variables to these factors. This Section provides principles and formulas to derive the sensitivity coefficient of every model outcome with respect to each model factor.

#### 3.1. Principles of sensitivity analysis

Let us index by  $\ell$  and denote as  $X_\ell$  a generic factor such as  $x_{in}^+$ ,  $\varphi_z$  or  $k'_z$ . Similarly, denote as  $f$  a generic model outcome. Assuming that  $f$  is a smooth function of  $X_\ell$ , its sensitivity to  $X_\ell$  can be measured by the ‘‘sensitivity coefficient’’ which amounts to the value of the partial derivative  $\partial f / \partial X_\ell$ .

**Chain rule.** If  $f$  is a function of both the  $\sigma_m^\#$  and  $X_\ell$ , then the sensitivity coefficient of  $f$  with respect to  $X_\ell$  involves the direct influence together with indirect influences via the  $\sigma_m^\#$ :

$$\frac{df}{dX_\ell} = \frac{\partial f}{\partial X_\ell} + \sum_m \frac{\partial f}{\partial \sigma_m^\#} \frac{\partial \sigma_m^\#}{\partial X_\ell}$$
 (18)

**Implicit function.** As  $\sigma \equiv [\sigma_m^\# : m]$  is determined by the FPP, it is an implicit function of  $\mathbf{X} \equiv [X_\ell : \ell]$ . Let us denote  $F_m(\sigma, \mathbf{X}) \equiv S_m - \sigma_m^\#$ , with

$$S_m(\sigma, \mathbf{X}) \equiv \frac{x_{im}^+ (H \delta_m^\# + 1 / \varphi_{im})}{2 + (H \delta_m^\# - 2\omega_m)(\varphi\pi)_m + (\varphi\pi^2)_m (\varphi\pi)_m^{-1}}$$
 (19)

In vector form, the FPP amounts to  $\mathbf{F}(\sigma, \mathbf{X}) = \mathbf{0}$ . Then, if  $\mathbf{F}$  is sufficiently smooth with respect to both  $\sigma$  and  $\mathbf{X}$ , it must hold that the overall derivative with respect to  $\mathbf{X}$  is null:

$$\nabla_\sigma \mathbf{F} \cdot \nabla_{\mathbf{X}} \sigma + \nabla_{\mathbf{X}} \mathbf{F} = \mathbf{0},$$
 (20)

Wherein:  $\nabla_\sigma \mathbf{F} = [\partial F_m / \partial \sigma_n^\# : m, n]$ ,  $\nabla_{\mathbf{X}} \sigma = [\partial \sigma_n^\# / \partial X_\ell : n, \ell]$  and  $\nabla_{\mathbf{X}} \mathbf{F} = [\partial F_m / \partial X_\ell : m, \ell]$ . As  $\mathbf{F} = \mathbf{S} - \sigma$ , we have that  $\nabla_\sigma \mathbf{F} = \nabla_\sigma \mathbf{S} - \mathbf{I}$  with  $\mathbf{I}$  the identity matrix and  $\nabla_{\mathbf{X}} \mathbf{F} = \nabla_{\mathbf{X}} \mathbf{S}$ . Thus,

$$\nabla_{\mathbf{X}} \sigma = (\mathbf{I} - \nabla_\sigma \mathbf{S})^{-1} \cdot \nabla_{\mathbf{X}} \mathbf{S}$$
 (21)

We are now fully equipped to analyze the sensitivities of model outcomes to model factors. We need first evaluate  $\nabla_\sigma \mathbf{S}$  and invert matrix  $\mathbf{I} - \nabla_\sigma \mathbf{S}$ , then evaluate  $\nabla_{\mathbf{X}} \mathbf{S}$ , next compute the matrix product between  $(\mathbf{I} - \nabla_\sigma \mathbf{S})^{-1}$  and

$\nabla_{\mathbf{X}} \mathbf{S}$  to recover the full array of sensitivity coefficients for the  $\sigma_m^\#$  with respect to the  $X_\ell$ , i.e.  $\nabla_{\mathbf{X}} \sigma$ . Furthermore, the sensitivity of any model outcome  $f$  to any factor  $X_\ell$  can be recovered by combining the  $\partial \sigma_m^\# / \partial X_\ell$  to the partial derivatives  $\partial f / \partial \sigma_m^\#$  and  $\partial f / \partial X_\ell$  on the basis of the chain formula (18).

3.2. Detailed formulas

To get matrix  $\nabla_{\sigma} \mathbf{S} = [\partial S_m / \partial \sigma_n^\# : m, n]$ , we only need to trace out the influence of each  $\sigma_n^\#$  on every  $\pi_z$ , since each function  $S_m$  involves the  $\sigma_n^\#$  only through the boarding probabilities that depend on  $k'_z$  and  $v_z$  hence the  $\sigma_n^\#$ . Denoting by  $D_m$  the denominator in (19), we have that

$$\frac{\partial S_m}{\partial \sigma_n^\#} = - \frac{S_m^2}{x_{im}^+ (H \delta_m^\# + 1 / \varphi_{im})} \frac{\partial D_m}{\partial \sigma_n^\#} . \tag{22}$$

To obtain  $\partial D_m / \partial \sigma_n^\#$ , let us consider a more generic sensitivity  $\partial D_m / \partial y_\ell$  for any factor  $y_\ell$  that is channeled by the  $\pi_z$  only:

$$\frac{\partial D_m}{\partial y_\ell} = (H \delta_m^\# - 2\omega_m) \left( \sum_{z \ni m} \varphi_z \frac{\partial \pi_z}{\partial y_\ell} \right) + \left( \sum_{z \ni m} 2\varphi_z \pi_z \frac{\partial \pi_z}{\partial y_\ell} \right) (\varphi \pi)_m^{-1} - \frac{(\varphi \pi^2)_m}{(\varphi \pi)_m^2} \left( \sum_{z \ni m} \varphi_z \frac{\partial \pi_z}{\partial y_\ell} \right) .$$

Denoting  $h_{mz} \equiv H \delta_m^\# - 2\omega_m + 2\pi_z (\varphi \pi)_m^{-1} - (\varphi \pi^2)_m (\varphi \pi)_m^{-2}$ , we obtain that

$$\frac{\partial D_m}{\partial y_\ell} = \sum_{z \ni m} \varphi_z h_{mz} \frac{\partial \pi_z}{\partial y_\ell} . \tag{23}$$

As  $\partial \pi_z / \partial \sigma_n^\# = -1_z(n) \delta_z^\# \pi_z / v_z$ , by combining (24) and (23) and taking that  $\sigma_m^\# = S_m$ , we get

$$\frac{\partial S_m}{\partial \sigma_n^\#} = \frac{\sigma_m^{\#2}}{x_{im}^+ (H \delta_m^\# + 1 / \varphi_{im})} \sum_z 1_z(m) 1_z(n) \varphi_z h_{mz} \frac{\pi_z \delta_z^\#}{v_z} . \tag{24}$$

Let us come to the  $\partial S_m / \partial X_\ell$  by considering in turn  $x_{in}^+$ ,  $k'_z$  and  $\varphi_z$  as  $X_\ell$ .

$$\frac{\partial S_m}{\partial x_{in}^+} = \delta_{mn} \frac{\sigma_m^\#}{x_{im}^+} . \tag{25}$$

As for the sensitivity of  $S_m$  to  $k'_z$ , it is channeled by  $\pi_z$  only, with  $\frac{\partial \pi_z}{\partial k'_z} = \delta_z^\# \frac{\pi_z}{k'_z} = \frac{\delta_z^\#}{v_z}$ .

From (24) applied to  $y_\ell = k'_z$ , it holds that  $\frac{\partial D_m}{\partial k'_z} = 1_z(m) \varphi_z h_{mz} \frac{\delta_z^\#}{v_z}$ .

Then, as  $\frac{\partial S_m}{\partial k'_z} = - \frac{S_m^2}{x_{im}^+ (H \delta_m^\# + 1 / \varphi_{im})} \frac{\partial D_m}{\partial k'_z}$ , we obtain that

$$\frac{\partial S_m}{\partial k'_z} = - \frac{\sigma_m^{\#2}}{x_{im}^+ (H \delta_m^\# + 1 / \varphi_{im})} 1_z(m) \delta_z^\# \frac{\varphi_z h_{mz}}{v_z} . \tag{26}$$

The sensitivity of  $S_m$  to  $\varphi_z$  is more involved as it passes through  $\varphi_{im}$ ,  $\omega_m = 1 / \varphi_{im}$ ,  $(\varphi \pi)_m$  and  $(\varphi \pi^2)_m$ :

$$\frac{\partial S_m}{\partial \varphi_z} = -1_z(m) \frac{S_m}{(H \delta_m^\# + 1 / \varphi_{im}) \varphi_{im}^2} - \frac{S_m^2}{x_{im}^+ (H \delta_m^\# + 1 / \varphi_{im})} \frac{\partial D_m}{\partial \varphi_z} , \tag{27}$$

Wherein: 
$$\frac{\partial D_m}{\partial \varphi_z} = 1_z(m) \left\{ 2 \frac{(\varphi \pi)_m}{\varphi_{im}^2} + (H \delta_m^\# - 2\omega_m - \frac{(\varphi \pi^2)_m}{(\varphi \pi)_m^2}) \pi_z + \frac{\pi_z^2}{(\varphi \pi)_m} \right\} . \tag{28}$$

By inserting (28) into (27), we obtain  $\partial S_m / \partial \varphi_z$ .

As for computational complexity, the main parameter is  $\bar{n}' = \bar{n} + 2\bar{z}$  with  $\bar{n}$  the number of stations and  $\bar{z}$  that of services. Matrices  $\nabla_{\sigma} \mathbf{S}$  and  $\nabla_{\sigma} \mathbf{F}$  have  $\bar{n}^2$  terms, while  $\nabla_{\mathbf{X}} \mathbf{S}$  and  $\nabla_{\mathbf{X}} \mathbf{F}$  have  $\bar{n} \cdot \bar{n}'$  terms. Then the computation of (21) is in  $O(\bar{n}'^3)$ , as is that of its operands since the computation of  $\partial D_m / \partial y_\ell$ , (23) and (24) are in  $O(\bar{n}')$ .

3.3. Related sensitivities

As concerns passenger stock  $v_z$  :

$$\frac{\partial v_z}{\partial X_\ell} = \sum_{m \in z} \frac{\partial S_m}{\partial X_\ell} . \tag{29}$$

Application to  $X_\ell = x_{in}^+$  yields:

$$\frac{\partial v_z}{\partial x_{in}^+} = 1_z(n) \frac{S_n}{x_{in}^+} . \tag{30}$$

As for boarding probability  $\pi_z$  :

$$\frac{\partial \pi_z}{\partial X_\ell} = \delta_z^\# \left\{ \frac{1}{v_z} \frac{\partial k'_z}{\partial X_\ell} - \frac{k'_z}{v_z^2} \frac{\partial v_z}{\partial X_\ell} \right\} . \tag{31}$$

As for exit flow  $x_{im}^- = (\varphi\pi)_m S_m$  :

$$\frac{\partial x_{im}^-}{\partial X_\ell} = (\varphi\pi)_m \frac{\partial S_m}{\partial X_\ell} + S_m \sum_{z \ni m} \left\{ \pi_z \frac{\partial \varphi_z}{\partial X_\ell} + \varphi_z \frac{\partial \pi_z}{\partial X_\ell} \right\} . \tag{32}$$

As for the average individual wait time  $w_{im}$ , the influence of  $X_\ell$  can pass through  $\varphi_z$  and  $\pi_z$  :

$$\begin{aligned} \frac{\partial w_{im}}{\partial X_\ell} = & \sum_{z \ni m} \frac{\partial \varphi_z / \partial X_\ell}{1 + H \delta_m^\# \varphi_{im}} \left\{ -H \delta_m^\# w_{im} - \varphi_{im}^{-2} + \frac{H \delta_m^\#}{(\varphi\pi)_m} - \frac{H \delta_m^\# \varphi_{im} \pi_z}{(\varphi\pi)_m^2} - \frac{1}{2} \frac{\pi_z^2}{(\varphi\pi)_m^2} + \pi_z \frac{(\varphi\pi^2)_m}{(\varphi\pi)_m^3} \right\} \\ & + \sum_{z \ni m} \frac{\partial \pi_z}{\partial X_\ell} \frac{1}{1 + H \delta_m^\# \varphi_{im}} \left\{ -\frac{H \delta_m^\# \varphi_{im} \varphi_z}{(\varphi\pi)_m^2} - \frac{\varphi_z \pi_z}{(\varphi\pi)_m^2} + \varphi_z \frac{(\varphi\pi^2)_m}{(\varphi\pi)_m^3} \right\} . \end{aligned} \tag{33}$$

Lastly, for  $W_{im} = H x_{im}^+ w_{im}$  :

$$\frac{\partial W_{im}}{\partial X_\ell} = H w_{im} \frac{\partial x_{im}^+}{\partial X_\ell} + H x_{im}^+ \frac{\partial w_{im}}{\partial X_\ell} . \tag{34}$$

4. Case study

The RER line A is a commuter rail line that runs along an East-West axis in the Paris area, with several branches connecting a central trunk. It serves more than one fourth of rail passenger traffic between the suburbs and the center of the agglomeration. Let us focus of the central trunk which that 8 stations (figure 2).

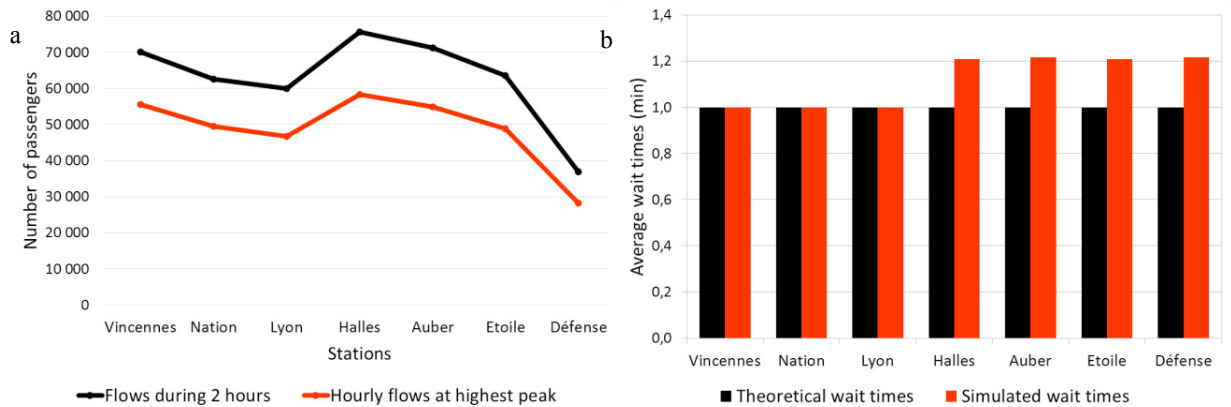


Fig. 2: Morning peak on RER A central trunk: (a) load profile; (b) average wait times.

On the morning peak from 7.30 am to 9.30 am, service frequency is scheduled at about 30 runs per hour in each direction. The service is provided by 5 routes, each with a frequency of 6 trains/h. Three routes (code names ZARA, YCAR, XUTI) are equipped with trains that can contain 1884 passengers (reference capacity at 4 p/m<sup>2</sup> in standing areas), the other two (code names UVAR and TNOR) with trains that can contain 2600 passengers.

To mimic the highest peak period at morning on working days, 80% of the 2 hours origin-destination trip matrix were taken as 1 hour flows. Then, traffic simulation using the CapTA model was performed. Boarding saturation occurs at station Halles, where 3 routes have insufficient residual capacity in their trains to accommodate all boarding candidates, implying that some users fail to board the first train serving the station after their own arrival.



Tab. 2: Sensitivity analysis for two stations along central trunk of RER A, morning peak.

	Factors	Station Lyon		Station Halles	
		Total stock	Average wait time	Total stock	Average wait time
Entry flows by destination	Halles	0,03	0	-	-
	Auber	0,03	0	0,08	1,6E-06
	Etoile	0,03	0	0,08	1,6E-06
	Défense	0,03	0	0,08	1,6E-06
Residual capacities by service route	ZARA / XUTI	0	0	-0,42	-1,7E-05
	YCAR	0	0	-0,42	-1,7E-05
	UVAR /TNOR	0	0	0	0
Frequencies by service route	ZARA / XUTI	-8,78	-0,0006	-60,47	-0,0018
	YCAR	-8,78	-0,0006	-63,09	-0,0019
	UVAR /TNOR	-8,78	-0,0006	-72,49	-0,0023

Let us analyze the sensitivity of passenger stocks (eqn. 21) and of individual average wait times (eqn. 33) at stations Lyon and Halles respectively. As station Lyon is not saturated, a marginal incoming user does not change the conditions of boarding and waiting for the other passengers: so the increase in passenger stock solely consists in the additional user himself - hence the  $.03 = 1 / \text{Frequency}$ . Then, for every route the train residual capacity is sufficient and a marginal variation in it exerts no effect; frequency has quite large marginal effect on both passenger stocks and wait times. At station Halles where three routes exhibit boarding saturation, a marginal user increases the overall stock by  $+0.08$  and the wait time by  $1.6 \cdot 10^{-6}$ . Routes ZARA, YCAR and XUTI have saturated trains: an additional place on-board would yield a reduction in passenger stock as well as in wait times, of about one more order of magnitude than the marginal user (since there are 6 trains per route and hourly period). As for route frequencies, their marginal effects on passenger stocks as well as on wait times are much larger than at station Lyon: boarding saturation acts as a magnifier. Taking a time value of 10 €/h, the congestion cost of a marginal user incoming at station Halles is valued at 0.4€ (using eqn. 34 and as there are 26.300 nass/h boarding at Halles). That of a marginal user boarded up Halles to alight down there onto the same candidate boarders is of €0.15 (using eqn. 34 w.r.t. residual capacity and  $\partial k' / H \partial x^+ = -1 / \varphi$ ).

## 5. Conclusion

The transit bottleneck of a station platform captures the interaction of boarding flows, vehicles of limited capacity and route frequencies. From simple postulates of homogenous vehicles for each route, homogenous headways, mingled waiting and users' compliance to vehicle capacity, the passenger stocks are cast into analytical formulas. These provide the basis for sensitivity analysis by formal differentiation of all model outcomes to all model inputs. Computation is quite tractable, with temporal complexity in the cubic of the number of stations. The application instance shows that congestion impacts arise when one or several services are saturated, with magnitude order to consider in the design of congestion pricing schemes.

Further research may be invested in two directions. First, to extend sensitivity analysis to the whole CapTA model, passing by the "Line model" which involves the station platform sub-model among other components. Second, to analyze uncertainty in model outcomes by propagating uncertainty in model inputs using the sensitivity coefficients established here.

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