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► **To cite this version:**

| Jordan Serres. Stability of Poincaré constant. 2021. hal-03234758v1

HAL Id: hal-03234758

<https://hal.science/hal-03234758v1>

Preprint submitted on 25 May 2021 (v1), last revised 2 Feb 2022 (v2)

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Stability of Poincaré constant

Jordan Serres

May 25, 2021

Abstract

We study stability of the sharp Poincaré constant of the invariant probability measure of a reversible diffusion process satisfying some natural conditions. The proof is based on the spectral interpretation of Poincaré inequalities and Stein's method. In particular, these results are applied to the gamma distributions and to strictly log-concave measures in dimension one, giving stability for Brascamp-Lieb inequalities.

1 Introduction

A probability measure μ on \mathbb{R}^d is said to satisfy a Poincaré inequality when there exists a positive finite constant C such that for all functions f in the Sobolev space $H^1(\mu)$,

$$\mathrm{Var}_\mu(f) \leq C \int |\nabla f|^2 d\mu. \quad (1)$$

We denote by $C_P(\mu)$ the smallest constant for which the above inequality holds. Poincaré inequalities have many applications (see for instance the survey [1]). For instance they can be seen as embeddings of weighted Sobolev spaces in L^2 , or as quantifying concentration of measure phenomenon (see e.g. [17]). The sharp Poincaré constant governs ergodicity of the underlying dynamic in the L^2 sense, and the convergence rate of some algorithms used for numerical simulations (see e.g. [10, 12]). When μ is reversible for a Markov process, a Poincaré inequality has a spectral interpretation: the infinitesimal generator L of the Markov process is symmetric on $L^2(\mu)$ and the quantity $\lambda_\mu := \frac{1}{C_P(\mu)}$ is then the spectral gap of the positive symmetric operator $-L$ (see [2, section 4.2.1]).

Stability results for Poincaré constant began to appear in the late 80's. Chen [5, Corollary 2.1] showed that all isotropic probability measures on \mathbb{R}^d have sharp Poincaré constant greater than 1. He proved furthermore that the standard Gaussian is the only one attaining 1. Then Utev [18] refined this result in dimension one, quantifying the difference between Poincaré constants in term of total variation distance:

$$C_P(\nu) \geq 1 + \frac{1}{3} d_{TV}(\nu, \gamma)^2$$

where ν is a normalized probability measure on \mathbb{R} , γ is the standard Gaussian and d_{TV} is the total variation distance.

More recently, Courtade, Fathi and Pananjady [8], extended it to the multidimensional case with the Wasserstein-2 distance:

$$C_P(\nu) \geq 1 + \frac{W_2(\nu, \gamma)^2}{d} \quad (2)$$

where ν is a centered probability measure on \mathbb{R}^d , normalized such that $\int |x|^2 d\nu = d$, γ denotes the Gaussian $\mathcal{N}(0, I_d)$ and W_2 is the 2-Wasserstein distance (see [19, chapter 6]). Our goal is to get such stability results in a more abstract framework, say for a general reference probability measure μ on a manifold instead of the Gaussian on \mathbb{R}^d . What we call stability results for Poincaré constant with respect to some distance d , are inequations of the form

$$d(\mu, \nu) \leq \phi(C_P(\mu), C_P(\nu)) \quad (3)$$

where d is a distance on the space of probability measures, $\phi : \{(x, y) \in \mathbb{R}^2 \mid y \geq x > 0\} \rightarrow \mathbb{R}_+$ is a continuous function such that $\forall x > 0$, $\phi(x, x) = 0$, and $C_P(\mu)$, $C_P(\nu)$ are respectively sharp Poincaré constants for μ a reference measure, and ν satisfying some constraints.

From now on, let us consider L an infinitesimal reversible Markov diffusion generator on a Riemannian manifold M with domain \mathcal{D} , see [2, Sect. 1.4] for further details. Let $\Gamma(f, g) := \frac{1}{2}(L(fg) - fLg - gLf)$ be the carré du champ operator. The diffusion property means that $\forall \phi \in C^2(\mathbb{R})$, $\Gamma(\phi(f)) = \phi'(f)^2 \Gamma(f)$ and also $L(\phi(f)) = \phi''(f) \Gamma(f) + \phi'(f) L(f)$, see [2, section 1.11]. Let us assume μ to be the only reversible probability measure on M for the Markovian process generated by L . This existence and uniqueness assumption is verified for instance if the process is irreducible and strongly Feller (see [11, Chapter 4]).

Assume that μ satisfies the following Poincaré inequality:

$$\forall f \in H^1(\mu), \text{Var}_\mu(f) \leq C_P(\mu) \int \Gamma(f) d\mu \quad (4)$$

where

$$H^1(\mu) := \{f \in L^2(\mu) \mid \int f d\mu = 0 \text{ and } \int \Gamma(f) d\mu < \infty\} \quad (5)$$

and $C_P(\mu) = \sup_{f \in H^1(\mu)} \frac{\text{Var}_\nu(f)}{\int \Gamma(f) d\nu} < \infty$ is the sharp Poincaré constant.

Assumptions on (L, μ) .

1. *There exists an eigenfunction f_0 attaining the spectral gap: $-Lf_0 = \lambda_\mu f_0$. We will always choose it to be normalized with respect to μ .*
2. *$\Gamma(f_0)$ can be written as $h \circ f_0$ for some smooth function $h : I \rightarrow \mathbb{R}_+$ with $I := f_0(M)$.*
3. *The function h does not vanish at the interior of I .*

Let ν be another measure on M satisfying the same Poincaré inequality with sharp constant $C_P(\nu)$:

$$\forall f \in H^1(\nu), \text{Var}_\nu(f) \leq C_P(\nu) \int \Gamma(f) d\nu.$$

where $H^1(\nu)$ is defined similarly as above. We also ask ν to be f_0 -normalized, that is:

$$\int f_0 d\nu = 0, \int f_0^2 d\nu = 1, \text{ and } \int \Gamma(f_0) d\nu = \frac{1}{C_P(\mu)}. \quad (6)$$

Applying Poincaré inequality for ν to f_0 , one immediately gets $C_P(\nu) \geq C_P(\mu)$. We refine this minoration by proving the following stability theorem:

Theorem 1. *Under Assumptions 1, 2 and 3,*

$$W_1(\mu^*, \nu^*) \leq C_h \left(\sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right), \quad (7)$$

where μ^* (resp. ν^*) is the pushforward of μ (resp. ν) by f_0 , W_1 is the 1-Wasserstein distance (Cf. Definition 5), $\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu) C_P(\mu)}$, and C_h is a constant defined in Proposition 18 whose finiteness only depends on the behavior of h at the boundary of I .

Assumption 1 is used to push forward the diffusion from M to $I \subset \mathbb{R}$ through f_0 (Cf. Section 2). This assumption is necessary. Indeed for $M = \mathbb{R}$ and $Lf(x) = f''(x) - \text{sgn}(x)f'(x)$ where $\text{sgn}(x)$ denotes the sign of x , the spectral gap is not attained (Cf. [2, Section 4.4.1]). Assumption 2 guarantees the push forward process to be Markovian (Cf. Section 2). This assumption would obviously be verified as soon as f_0 is injective. In dimension 1, it is always the case (Cf. Lemma 3). Assumption 3 guarantees the diffusion on I to be irreducible (Cf. Section 2.1).

The method followed in this article is to derive some approximate integration by part formula for ν (Cf. Section 3) and then to use Stein's method (Cf. Section 4). The problem is then reduced to bound the carré du champ operator of the solution (Cf. Section 4.1).

A particular case, treated by Courtade and Fathi [8, 7], is when $L = \Delta - x \cdot \nabla$ is the Ornstein-Uhlenbeck process on \mathbb{R}^d . The invariant measure is the standard Gaussian measure γ and the d coordinate projections x_1, \dots, x_d are orthogonal eigenfunctions satisfying Assumption 1, 2 and 3. Choosing any of these projections gives the one dimensional Gaussian $\gamma(0, 1)$ for the pushforward measure. In this case, the approach described above works using only classical results of Stein's method for the Gaussian (Cf. [14, Section 2]). Moreover, if we use the same approach with $f_0 = (x_1, \dots, x_d)$ a vector valued function, it works thanks to technical bounds obtained in [13, Lemma 3.3] and it gives similar results as (2). Taking vector valued eigenfunction when the dimension of the eigenspace is greater than 1 appears to be the natural extension of this method. Technical issues come here from the fact that Stein's method involves control of the Hessian of solutions to a family of PDE.

In Section 6.2, we obtain stability result for the Laguerre process as Corollary of Theorem 1. As far as we know, stability results had not yet been investigated in this case.

One can ask for an other distance than the 1-Wasserstein in the stability inequation (7). The main difference lies in the set of target function chosen using Stein's method. We will see in Section 5 that replacing the condition $C_S < \infty$ by $h > \kappa > 0$, where κ is a constant, gives a stability result in total variation distance (and hence in Kolmogorov distance). We illustrate this result in Section 5 with uniformly log-concave measures.

A natural question arising is then to compare stability results involving different distances. We will say that a d_1 stability result

$$d_1(\mu, \nu) \leq \phi_1(C_P(\mu), C_P(\nu)) \quad (8)$$

is stronger than a d_2 one

$$d_2(\mu, \nu) \leq \phi_2(C_P(\mu), C_P(\nu)) \quad (9)$$

if 8 implies the existence of ϕ_2 in 9. Comparison between these different types of stability results are discussed in Section 7.

2 The quotient process

In the Ornstein-Uhlenbeck process, the first non zero eigenfunctions are injective. So one may expect the important properties of L to be preserved when mapping it onto I through such eigenfunctions. In a broader context, according to [2, page 60], Assumption 2 implies that all of the Markovian structure on the manifold is mapped onto a Markovian structure on $I := f_0(M) \subset \mathbb{R}$. Assuming that M is connected, I is an interval since f_0 is continuous. We define $a := \inf I$ and $b := \sup I$. Moreover, the diffusion property allows to write that

$$L(\phi(f_0)) = \phi''(f_0)\Gamma f_0 + \phi'(f_0)Lf_0 = \phi''(f_0)h(f_0) - \phi'(f_0)\frac{1}{C_P(\mu)}f_0.$$

Hence, the induced Markov process has generator

$$L^*(\phi)(x) := h(x)\phi''(x) - \frac{x}{C_P(\mu)}\phi'(x) \quad (10)$$

and reversible measure $\mu^* := f_0^\# \mu$. This one dimensional Markov process will be called the quotient process.

If h is constant, then it is equal to λ_μ . Indeed, one gets that $\Gamma(f_0) = h$, but we know that $\int \Gamma(f_0)d\mu = \lambda_\mu$. In this case, L^* is reduced to the Ornstein-Uhlenbeck generator with reversible measure $\mathcal{N}(0, \lambda_\mu)$. Therefore, $\mu^* = \mathcal{N}(0, \lambda_\mu)$. Similarly, $\nu^* := f_0^\# \nu$ will satisfy Poincaré inequality with sharp constant $C_P(\nu^*) \leq \frac{C_P(\nu)}{C_P(\mu)}$. Indeed, using Poincaré inequality for ν ,

$$\text{var}_{\nu^*}(\phi) = \text{var}_\nu(\phi \circ f_0) \leq C_P(\nu) \int \Gamma(\phi \circ f_0)d\nu = C_P(\nu) \int \phi'(f_0)^2 h(f_0)d\nu = \frac{C_P(\nu)}{C_P(\mu)} \int \phi'^2 d\nu^*.$$

Of course the carré du champ operator used in Poincaré inequalities on \mathbb{R} is the square of the first derivative. At that point, we can apply known stability results (Cf. [8]) and obtain:

$$\frac{C_P(\nu)}{C_P(\mu)} \geq C_P(\nu^*) \geq C_P(\mu) + W_2(\nu^*, \mu^*)^2.$$

In the sequel, the goal will be to prove such stability inequalities in a broader context, that is without assuming h to be constant. Let us now compute the carré du champ of the quotient process.

Proposition 2. *The carré du champ operator associated to (L^*, μ^*) is*

$$(\Gamma^* \psi)(t) = h(t)\psi'(t)^2.$$

Moreover, with this operator, μ^* satisfies a Poincaré inequality with sharp constant $C_P(\mu)$ and the inequality becomes an equality for $\psi = Id$.

Proof. It is simply a computation using the basic definition of $\Gamma f := \frac{1}{2}[L(f^2) - 2fLf]$. The Poincaré inequality for μ^* follows from the Poincaré inequality for μ :

$$\text{var}_{\mu^*}(\psi) = \text{var}_\mu(\psi \circ f_0) \leq C_P(\mu) \int \Gamma(\psi \circ f_0)d\mu = C_P(\mu) \int \psi'(f_0)^2 h(f_0)d\mu = C_P(\mu) \int \Gamma^* \psi d\mu^*.$$

Now taking $\psi = Id$, the inequality becomes an equality because of the definition of f_0 , showing that $C_P(\mu)$ is sharp. \square

Replacing μ^* by ν^* in the proof above, we get that ν^* satisfies a Poincaré inequality with constant $C_P(\nu)$. However in this case, $C_P(\nu)$ may not be the sharp constant for ν^* .

2.1 The non vanishing assumption

Let us consider the case where $Lf := \alpha f'' + \beta f'$ is a diffusion operator on an interval $M = J := (c, d) \subset \mathbb{R}$, with α and β two continuous functions on J such that $\alpha > 0$ and

$$\rho(x) := \frac{1}{\alpha(x)} \exp\left(\int_c^x \frac{\beta(t)}{\alpha(t)} dt\right)$$

is well defined on J . So $d\mu := \rho(x)dx$ is reversible for L . Assume the operator L satisfies both a Poincaré inequality with sharp constant C_P and Assumption 1. Then the following holds.

Lemma 3. *The first eigenfunction f_0 is strictly monotone.*

Proof. We extend the proof in e.g. [15, Section 2] which treats the case $\alpha \equiv 1$. First, recall that f_0 is a minimizer of the Rayleigh ratio $\frac{\int_J \Gamma(f_0) d\mu}{\text{var}_\mu(f_0)}$, and introduce $g(x) := \int_c^x |f_0'(t)| dt$. Then $g'(x) = |f_0'(x)|$ hence $\Gamma(g) = \Gamma(f_0)$. So g has same energy than the eigenfunction f_0 :

$$\int_J \Gamma(g) d\mu = \int_J \Gamma(f_0) d\mu$$

On the other hand, g has greater variance than f_0 . Indeed,

$$\text{Var}_\mu(g) = \frac{1}{2} \int_J \int_J \left(\int_x^y |f_0'(t)| dt \right)^2 d\mu(x) d\mu(y) \geq \frac{1}{2} \int_J \int_J \left(\int_x^y f_0'(t) dt \right)^2 d\mu(x) d\mu(y) = \text{Var}_\mu(f_0)$$

But f_0 being a minimizer of the Rayleigh ratio, one can infer, since $f_0 \in C^1(J)$, that f_0' has same sign on the interval J , hence f_0 is monotone. Assume now $f_0' \geq 0$ and let us show that $f_0' > 0$. The eigenfunction f_0 being centered and continuous, it is negative in a neighborhood of c , then it is zero, and then positive up to d . Using classical ODE tools for the equation $\alpha f_0'' + \beta f_0' = \frac{-1}{C_P} f_0$, one gets the formulas:

$$f_0'(x) = \frac{1}{C_P \alpha(x) \rho(x)} \int_x^d f_0(t) d\mu(t) \quad (11)$$

$$= \frac{-1}{C_P \alpha(x) \rho(x)} \int_c^x f_0(t) d\mu(t) \quad (12)$$

Hence if $f_0'(x) = 0$ with $x \in (c, d)$, then either $f_0(x) < 0$ but Formula 12 implies $f_0'(x) > 0$, or $f_0(x) > 0$ but Formula 11 implies $f_0'(x) > 0$, or $f_0(x) = 0$ but both formulas give then again $f_0'(x) > 0$.

Hence f_0' is positive on J , justifying the claim that f_0 is strictly monotone. \square

In this case the monotonicity implies that h cannot vanish in the interior of I . This motivates Assumption 3 already mentioned in the introduction.

Assumption 3. *All $x_0 \in M$ such that $\Gamma(f_0)(x_0) = 0$ are global extrema of f_0 .*

If $\Gamma = |\nabla|^2$ for the metric of M , then the assumption can be reformulated: all critical points of f_0 are global extrema. The following is immediate and justify why we often write I instead of its interior $\overset{\circ}{I}$ in the sequel.

Proposition 4. *The eigenfunction f_0 satisfies Assumption 3 if, and only if, h does not vanish outside ∂I .*

Proof. Assume that Assumption 3 holds, and let $t \in I$ such that $h(t) = 0$. Then there exists $x \in M$ such that $t = f_0(x)$ and so $\Gamma(f_0)(x) = 0$. Hence x is a global extremum of f_0 . Hence t is on the boundary of I . Conversely let $x \in M$ such that $\Gamma(f_0)(x) = 0$. Then $f_0(x)$ is a zero of h . Hence by assumption it is on the boundary of I , so it is a global extremum of f_0 . \square

In the sequel we often work with Wasserstein distance. Let us recall the Kantorovitch-Rubinstein formula.

Definition 5. (see [19, Chapter 5]) *The 1-Wasserstein distance between two measures α, β on I is*

$$W_1(\alpha, \beta) = \sup_{f: (I, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)} \left| \int f d\alpha - \int f d\beta \right|$$

1-Lipschitz

2.2 Density of μ^*

First, we point out that $\int f_0 d\mu \in I$. Indeed f_0 is continuous, so $I = f_0(M)$ is connected in \mathbb{R} , hence convex. Therefore its expectation belongs to I . Recall that $\lambda_\mu := \frac{1}{C_P(\mu)}$. We now define the function $v : I \rightarrow \mathbb{R}_+$ by

$$v(t) := \exp\left(-\lambda_\mu \int_0^t \frac{u du}{h(u)}\right) \quad (13)$$

Taking 0 as reference point in the integral is justified by the above observation since f_0 was chosen to be centered. Moreover Assumption 3 guarantees that dividing by h makes sense.

Proposition 6. *The measure*

$$\rho(x)dx := \frac{1}{h(x)} \exp\left(-\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) \mathbb{1}_I dx$$

is invariant for the quotient process.

Proof. It is enough to show that

$$\int_I L^*(f) \frac{1}{h(x)} \exp\left(-\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) dx = 0$$

for all functions f compactly supported in I . Let f be such a function. Recall that $L^*f = hf'' - \frac{x}{C_P(\mu)}f'$. We first compute by an integration by parts:

$$\int_I f''(x)v(x)dx = \underbrace{[f'(x)v(x)]_{\partial I}}_{=0} + \lambda_\mu \int_I \frac{xf'(x)}{h(x)}v(x)dx,$$

and this allows us to conclude. \square

We know that the quotient process admits $f_0^\# \mu$ as invariant probability measure. Since we assumed μ to be the only invariant probability measure for L , Proposition 6 gives that the measure $\rho(x)dx$ is finite and then $f_0^\# \mu = \frac{1}{Z}\rho(x)dx$ where Z is a normalization constant. Let us conclude this section with a control of the tail of μ^* . We denote the cumulative distribution of μ^* by

$$q(t) := \mu^*(]-\infty, t]) = \int_{-\infty}^t \frac{\mathbb{1}_I}{Zh(y)} \exp\left(-\lambda_\mu \int_0^y \frac{u du}{h(u)}\right) dy.$$

Recall that $a := \inf I$, $b := \sup I$ and let us denote $I_- := I \cap \mathbb{R}_-$ and $I_+ := I \cap \mathbb{R}_+$.

Lemma 7. *One can bound the tail q of μ^* as follows:*

- $\forall t \in I_-$,

$$q(t) \leq \min\left(q(0), \frac{-C_P(\mu)}{Zt}\right) v(t).$$

- $\forall t \in I_+$,

$$1 - q(t) \leq \min\left(1 - q(0), \frac{C_P(\mu)}{Zt}\right) v(t).$$

Proof. We only process on I_- . The proof is similar on I_+ .

1. Let us define $f : I_- \rightarrow \mathbb{R}$, by $f(t) = q(t) - q(0)v(t)$. We want to show that $f \leq 0$. Compute

$$f'(t) = \frac{1}{Zh(t)} \exp\left(-\lambda_\mu \int_a^t \frac{u du}{h(u)}\right) + \lambda_\mu q(0) \frac{t}{h(t)} v(t) = \frac{v(t)}{h(t)} \left(\frac{1}{Z} + \lambda_\mu q(0)t\right).$$

If $-\frac{C_P(\mu)}{q(0)Z} < a$, then $f' \geq 0$ so $f \leq f(0) = 0$.

Else, f decreases on $(a, -\frac{C_P(\mu)}{q(0)Z})$ and increases on $(-\frac{C_P(\mu)}{q(0)Z}, 0)$ but $\lim_{t \rightarrow a} f(t) \leq 0$ and $f(0) = 0$, so we have the first claim.

2. Let us define $f : I_- \rightarrow \mathbb{R}$, by $f(t) = q(t) + \frac{C_P(\mu)}{Zt}v(t)$. We want to show that $f \leq 0$. Compute:

$$f'(t) = \frac{1}{Zh(t)} \exp\left(-\lambda_\mu \int_0^t \frac{u du}{h(u)}\right) - \frac{C_P(\mu)}{Zt^2}v(t) - \lambda_\mu \frac{tC_P(\mu)}{Zh(t)t}v(t) = -\frac{v(t)}{Z} \frac{C_P(\mu)}{t^2} < 0.$$

Hence $f \leq \lim_{t \rightarrow a} f(t) \leq 0$.

\square

3 Exact and approximate integration by parts formulas

Let us recall a classical result in Γ -calculus which we will often use in the sequel. For all $f, g \in H^1(\mu)$, the following integration by parts formula holds (Cf. formula (5) for the definition of $H^1(\mu)$).

$$\int \Gamma(f, g) d\mu = - \int f Lg d\mu \quad (14)$$

Indeed, since μ is the invariant measure of the process generated by L , $\int L\phi d\mu = 0$ for all functions $\phi \in \mathcal{D}$. Hence, integrating the definition of the carré du champ operator $\Gamma(f, g) := \frac{1}{2}(L(fg) - fLg - gLf)$ and using the reversibility of μ , one gets the result. Let us now state an integration by parts formula for the quotient process.

Proposition 8. *For all smooth enough $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi \circ f_0 \in H^1(\mu)$, it holds that:*

$$\int_{\mathbb{R}} x\phi(x) d\mu^*(x) = C_P(\mu) \int_{\mathbb{R}} h(x)\phi'(x) d\mu^*(x) \quad (15)$$

Proof. As a consequence of the integration by part formula (14) for the initial diffusion process, one gets

$$\int \Gamma(f_0, g) d\mu = - \int g Lf_0 d\mu = \frac{1}{C_P(\mu)} \int g f_0 d\mu.$$

We then use this equality with $g := \phi \circ f_0$ and use the diffusion property of Γ and the definition of h . \square
Let us state now an extension of the approximate integration by part formula which extend a Lemma from Courtade and Fathi [7, Lemma 2.3]. Let us recall that

$$H^1(\nu) := \{f \in L^2(\nu) \cap \mathcal{D} \mid \int f d\nu = 0 \text{ and } \int \Gamma(f) d\nu < \infty\}.$$

Theorem 9. *The following inequality holds for all $g \in H^1(\nu)$:*

$$\left| \int f_0 g d\nu - C_P(\nu) \int \Gamma(f_0, g) d\nu \right| \leq (C_P(\nu) - C_P(\mu))^{\frac{1}{2}} \left(\frac{C_P(\nu)}{C_P(\mu)} \right)^{\frac{1}{2}} \left(\int \Gamma(g) d\nu \right)^{\frac{1}{2}} \quad (16)$$

Proof. Let $t \in \mathbb{R}$ and $g \in H^1(\nu)$. Let us apply the Poincaré inequality for ν to $\alpha := f_0 + tg$. Computing that

$$\text{Var}_{\nu}(\alpha) = \int (f_0 + tg)^2 d\nu = \underbrace{\int f_0^2 d\nu}_{=1} + 2t \int f_0 g d\nu + t^2 \int g^2 d\nu,$$

and that

$$\int \Gamma(\alpha) d\nu = \underbrace{\int \Gamma(f_0) d\nu}_{=C_P(\mu)^{-1}} + 2t \int \Gamma(f_0, g) d\nu + t^2 \int \Gamma(g) d\nu,$$

we get that

$$1 + 2t \int f_0 g d\nu + t^2 \int g^2 d\nu \leq C_P(\nu) \left(C_P(\mu)^{-1} + 2t \int \Gamma(f_0, g) d\nu + t^2 \int \Gamma(g) d\nu \right).$$

Now, witting $C_P(\nu) = C_P(\nu) - C_P(\mu) + C_P(\mu) =: \Delta C_P + C_P(\mu)$, we obtain :

$$2t \int f_0 g d\nu + t^2 \int g^2 d\nu \leq \frac{\Delta C_P}{C_P(\mu)} + 2C_P(\nu)t \int \Gamma(f_0, g) d\nu + t^2 C_P(\nu) \int \Gamma(g) d\nu.$$

Hence, for all $t \in \mathbb{R}$,

$$\frac{\Delta C_P}{C_P(\mu)} + 2t \left(C_P(\nu) \int \Gamma(f_0, g) d\nu - \int f_0 g d\nu \right) + t^2 C_P(\nu) \int \Gamma(g) d\nu \geq 0.$$

This polynomial of degree 2 takes only non-negative values, hence its discriminant is non-positive:

$$\delta = b^2 - 4ac = 4 \left(C_P(\nu) \int \Gamma(f_0, g) d\nu - \int f_0 g d\nu \right)^2 - 4C_P(\nu) \int \Gamma(g) d\nu \frac{\Delta C_P}{C_P(\mu)} \leq 0,$$

which yields the result. \square

Using Theorem 9 with $g := \phi \circ f_0$ and dividing by $C_P(\nu)$, we can state now the approximate integration by part formula for the quotient process.

Corollary 10. For all smooth enough $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi \circ f_0 \in H^1(\mu)$,

$$\left| \int \left(h(x)\psi'(x) - \frac{x}{C_P(\nu)}\psi(x) \right) d\nu^* \right| \leq \left(\frac{C_P(\nu) - C_P(\mu)}{C_P(\nu)C_p(\mu)} \right)^{\frac{1}{2}} \left(\int h(x)\psi'(x)^2 d\nu^* \right)^{\frac{1}{2}}.$$

The original point of view of Courtade and Fathi is the following [7, Section 1.3]. The Poincaré constant is considered as the minimizer of the energy $\int \Gamma(f) d\mu$ for f with variance 1 in a large enough functional set. One can then write the Euler-Lagrange equation. The heuristic is that if another measure almost satisfies this equation, its Poincaré constant would not be so far from the one of μ . This is made rigorous with Stein's method. Here the integration by part formula (15) plays the role of the Euler-Lagrange equation.

4 Implementing Stein's method

The original idea of Stein's method is to control some distance between two probability measures by bounding the solution of some equation called Stein equation. For more details, see the survey [14]. We start with the approximated integration by parts formula given in Corollary 10 above. The goal is to get stability inequalities of the form

$$C_P(\nu) \geq C_P(\mu) + \alpha d(\mu^*, \nu^*)^2,$$

where $\alpha > 0$ is a multiplicative constant and d is a metric on the space of probability measures defined by

$$d(\mu, \nu) := \sup_{f \in \mathcal{F}} \left| \int f d\mu - \int f d\nu \right|$$

for some set of functions \mathcal{F} . Metrics of this form include 1-Wasserstein and total variation distances. This goal would be achieved if one could solve the equation

$$S_\nu(\psi)(x) := h(x)\psi'(x) - \frac{x}{C_P(\nu)}\psi(x) = f(x) - \mu^*(f), \quad x \in I,$$

for all $f \in \mathcal{F}$, and bound the term $\int h(x)\psi'(x)^2 d\nu^*$ independently of $f \in \mathcal{F}$. The actual Stein equation is

$$S_\mu(\psi)(x) := h(x)\psi'(x) - \frac{x}{C_P(\mu)}\psi(x) = f(x) - \mu^*(f), \quad x \in \overset{\circ}{I}.$$

Since $S_\mu(\psi') = L^*(\psi)$, for any solution ϕ of the Poisson equation $L\phi = f - \mu^*(f)$, ϕ' solves the Stein equation. Hence one can study the Stein equation via the probabilistic analysis of the Poisson equation. Observing that

$$S_\mu\psi = S_\nu\psi + \left(\frac{1}{C_P(\nu)} - \frac{1}{C_P(\mu)} \right) x\psi,$$

Corollary 10 gives

$$\left| \int S_\mu(\psi) d\nu^* \right| \leq \left(\frac{C_P(\nu) - C_P(\mu)}{C_P(\nu)C_p(\mu)} \right)^{\frac{1}{2}} \left(\int h(x)\psi'(x)^2 d\nu^* \right)^{\frac{1}{2}} + \left(\frac{C_P(\nu) - C_P(\mu)}{C_P(\nu)C_p(\mu)} \right) \left| \int x\psi d\nu^* \right|.$$

Now, using that ν^* is centered, the Cauchy-Schwarz inequality and the Poincaré inequality for ν^* , we obtain

$$\begin{aligned} \left| \int x\psi d\nu^* \right| &= \left| \int x(\psi - \int \psi d\nu^*) d\nu^* \right| \\ &\leq \left(\int (\psi - \int \psi d\nu^*)^2 d\nu^* \right)^{\frac{1}{2}} \left(\int x^2 d\nu^* \right)^{\frac{1}{2}} \\ &= \left(\int (\psi - \int \psi d\nu^*)^2 d\nu^* \right)^{\frac{1}{2}} \left(\int f_0^2 d\nu \right)^{\frac{1}{2}} \\ &= \left(\int (\psi - \int \psi d\nu^*)^2 d\nu^* \right)^{\frac{1}{2}} \\ &\leq \sqrt{C_P(\nu)} \left(\int h(x)\psi'(x)^2 d\nu^* \right)^{\frac{1}{2}}. \end{aligned}$$

Setting,

$$\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu)C_p(\mu)}$$

we have proved that

$$\left| \int S_\mu(\psi) d\nu^* \right| \leq \left(\sqrt{\delta} + \sqrt{C_P(\nu)} \delta \right) \left(\int \Gamma^*(\psi) d\nu^* \right)^{\frac{1}{2}}. \quad (17)$$

So our goal would be achieved if for all $f \in \mathcal{F}$, we can solve the Stein equation $S_\mu(\psi) = f - \mu^*(f)$, and get a bound for $\Gamma^*(\psi)$ independently of f .

4.1 Stein solution

Let $f \in L^1(\mu^*)$. We consider the Stein equation

$$h(x)\psi'(x) - \frac{x}{C_P(\mu)}\psi(x) = f(x) - \mu^*(f), \quad x \in I \quad (18)$$

In the sequel, we will call f the target function. This equation is a first order linear ODE, which can be explicitly solved.

Lemma 11. *The function*

$$\psi(x) := \exp\left(\int_0^x \frac{u du}{C_P(\mu)h(u)}\right) \int_a^x (f(y) - \mu^*(f)) \frac{1}{h(y)} \exp\left(-\int_0^y \frac{u du}{C_P(\mu)h(u)}\right) dy \quad (19)$$

is a solution of (18). Moreover, it also can be written as

$$\psi(x) = -\exp\left(\int_0^x \frac{u du}{C_P(\mu)h(u)}\right) \int_x^b (f(y) - \mu^*(f)) \frac{1}{h(y)} \exp\left(-\int_0^y \frac{u du}{C_P(\mu)h(u)}\right) dy \quad (20)$$

Proof. A solution is obtained by classical results for ODE. The second formula is obtained from Proposition 6. \square

Let us emphasize that this approach is only possible in dimension 1.

4.2 Boundedness of solutions for bounded target functions

Under the current framework one can give the following bounds for the solution of the Stein equation.

Proposition 12. *Let f be a bounded measurable function on I , and let ψ be the solution of (18) given by Lemma 11. Then*

$$\|\psi\|_\infty \leq Z \max(q(0), 1 - q(0)) \|f - \mu^*(f)\|_\infty \leq Z \|f - \mu^*(f)\|_\infty,$$

with Z the normalization constant (Cf. Proposition 6) and q the cumulative distribution of μ^* . Moreover,

$$\|x\psi(x)\|_\infty \leq C_P(\mu) \|f - \mu^*(f)\|_\infty.$$

Proof. Let us recall that $v(t) = \exp\left(-\lambda_\mu \int_0^t \frac{u du}{h(u)}\right)$. If $x \leq 0$ we use the writing (19) of ψ , and get:

$$|\psi(x)| \leq \|f - \mu^*(f)\|_\infty v(x)^{-1} Z q(x).$$

and by Lemma 7:

$$\sup_{x \in I_-} |\psi(x)| \leq Z q(0) \|f - \mu^*(f)\|_\infty.$$

If $x > 0$, we use the writing (20) of ψ in Lemma 11 together with Lemma 7:

$$\sup_{x \in I_+} |\psi(x)| \leq Z (1 - q(0)) \|f - \mu^*(f)\|_\infty.$$

To prove the second inequation, we split off again the case $x > 0$ and $x < 0$ using in each case the appropriate representation of ψ in Lemma 11 and we conclude with Lemma 7. If $x \in I_-$, then $|\psi(x)| \leq v(x)^{-1} \|f - \mu^*(f)\|_\infty Z q(x)$ but $q(x) \leq \frac{C_P(\mu)}{Z|x|} v(x)$ so $|x\psi(x)| \leq C_P(\mu) \|f - \mu^*(f)\|_\infty$. If $x \in I_+$, then $|\psi(x)| \leq v(x)^{-1} \|f - \mu^*(f)\|_\infty Z (1 - q(x))$ but $(1 - q(x)) \leq \frac{C_P(\mu)}{Zx} v(x)$ so $|x\psi(x)| \leq C_P(\mu) \|f - \mu^*(f)\|_\infty$. \square

5 Stability in total variation under uniform ellipticity condition

Let us assume the following uniform ellipticity condition on h :

$$\kappa := \inf_{x \in \mathbb{R}} h(x) > 0 \quad (21)$$

Under this ellipticity condition, one can get better estimates on the solutions of Stein equation. The vocabulary of ellipticity comes from PDE theory, see [9, Chapter 6]. This condition extend the Gaussian case discussed at the beginning of Section 2.

Proposition 13. *Let f be a bounded function on \mathbb{R} , and let ψ be the solution of the Stein equation given by Lemma 11. Then*

$$\|\psi'\|_\infty \leq \frac{2}{\kappa} \|f - \mu^*(f)\|_\infty,$$

and

$$\|h\psi'^2\|_\infty \leq \frac{4}{\kappa} \|f - \mu^*(f)\|_\infty^2.$$

Proof. By direct calculation:

$$\psi'(x) = \frac{1}{h(x)} \left(f - \mu^*(f) + \frac{x}{C_P(\mu)} \psi(x) \right).$$

The lower bound on h and Proposition 12 yield the first claim. Similarly,

$$h(x)\psi'(x)^2 = \frac{1}{h(x)} \left(f - \mu^*(f) + \frac{x}{C_P} \psi(x) \right)^2.$$

So from the elementary inequality $(a + b)^2 \leq 2a^2 + 2b^2$, and the same reasoning as above, we get the result. \square

From these estimates, we derive a stability result.

Theorem 14. *Let (L, μ) be a Markov diffusion on a Riemannian manifold M satisfying a Poincaré inequality with sharp constant $C_P(\mu)$. Suppose that Assumptions 1, 2 and 3 are verified. If $\kappa := \inf_{x \in \mathbb{R}} h(x) > 0$, then for all measures ν on M normalized as in (6) and satisfying a Poincaré inequality with sharp constant $C_P(\nu)$,*

$$d_{TV}(\mu^*, \nu^*) \leq \frac{2}{\sqrt{\kappa}} \left(\sqrt{\delta} + \sqrt{C_P(\nu)} \delta \right), \quad (22)$$

where μ^* (resp. ν^*) is the pushforward of μ (resp. ν) by the first eigenfunction f_0 , and $\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu) C_P(\mu)}$.

Proof. Let $f : I \rightarrow \mathbb{R}$ be an indicator function. We solve the Stein equation with Lemma 11 and denote ψ the chosen solution. Then (17) and Proposition 13 give:

$$\left| \int S_\mu(\psi) d\nu^* \right| \leq \left(\sqrt{\delta} + \sqrt{C_P(\nu)} \delta \right) \left(\int \Gamma^*(\psi) d\nu^* \right)^{\frac{1}{2}} \leq \left(\sqrt{\delta} + \sqrt{C_P(\nu)} \delta \right) \frac{2}{\sqrt{\kappa}} \|f - \mu^*(f)\|_\infty.$$

But:

$$\left| \int S_\mu(\psi) d\nu^* \right| = \left| \int (f - \mu^*(f)) d\nu^* \right| = |\nu^*(f) - \mu^*(f)|.$$

Hence for any indicator function f ,

$$|\nu^*(f) - \mu^*(f)| \leq \frac{4}{\sqrt{\kappa}} \left(\sqrt{\delta} + \sqrt{C_P(\nu)} \delta \right).$$

Taking the supremum over all such f concludes the proof. \square

Let us point out that (22) is of the form (3) with $\phi(x, y) = \frac{2}{\sqrt{\kappa}} \left(\sqrt{\frac{y-x}{xy}} + \frac{y-x}{x\sqrt{y}} \right)$.

An immediate application of Theorem 22 is stability for Brascamp-Lieb inequalities. Let $M = \mathbb{R}$ and $d\mu = e^{-\phi} dx$ be a probability measure with $\phi : \mathbb{R} \rightarrow \mathbb{R}$ strictly convex and C^2 . Then μ satisfies the following Brascamp-Lieb inequality [3] :

$$\text{Var}_\mu(f) \leq \int \frac{f'^2}{\phi''} d\mu$$

for all functions f in the weighted Sobolev space $H_{\phi, \mu}^{1,2} = \{g \in L^2(\mu) \mid \frac{g'}{\sqrt{\phi''}} \in L^2(\mu)\}$. Then

$$Lf = \frac{1}{\phi''} f'' - \left(\frac{\phi'}{\phi''} + \frac{\phi^{(3)}}{\phi''^2} \right) f'$$

is a Markov diffusion generator with reversible measure μ satisfying a Poincaré inequality with sharp constant 1 and carré du champ operator $\Gamma(f) = \frac{f'^2}{\phi''}$. Moreover

$$L(\phi') = -\phi',$$

hence the spectral gap is attained with $f_0 = \phi'$ which is centered. One easily sees that ϕ' is a bijection, so $I = \mathbb{R}$ and Assumption 2 is satisfied with $h = \phi'' \circ \phi'^{-1}$. The push forward measure μ^* is then the moment measure of ϕ (see [6, 16]) and has density

$$d\mu^* = \frac{\exp(-\phi(\phi'^{-1}(t)))}{\phi''(\phi'^{-1}(t))} dt.$$

In this setting, Theorem 14 says that if $\phi'' \geq \kappa > 0$ (meanning that ϕ is uniformly convex) then μ^* satisfies a stability result in total variation distance. It is a direct generalization of the Gaussian case [18].

Corollary 15. *Let $M = \mathbb{R}$ and $d\mu = e^{-\phi} dx$ be a C^2 uniformly log-concave probability measure. Let ν be a probability measure on \mathbb{R} verifying the following Poincaré inequality: for all functions f in the weighted Sobolev space $H_{\phi, \nu}^{1,2} = \{g \in L^2(\nu) \mid \frac{g'}{\sqrt{\phi''}} \in L^2(\nu)\}$,*

$$\text{Var}_\nu(f) \leq C_P(\nu) \int \frac{f'^2}{\phi''} d\nu,$$

and such that $\int \phi' d\nu = 0$, $\int \phi'^2 d\nu = 1$ and $\int \phi'' d\nu = \int \phi'' d\mu$. So $C_P(\nu) \geq 1$, and if $\phi'' \geq \kappa > 0$ with κ a constant, then $\delta := \frac{C_P(\nu)-1}{C_P(\nu)}$ satisfies

$$d_{TV}(\mu^*, \nu^*) \leq \frac{4}{\sqrt{\kappa}} \left(\sqrt{\delta} + \sqrt{C_P(\nu)} \delta \right),$$

where $\mu^* = \phi'_\# \mu$ and $\nu^* = \phi'_\# \nu$.

6 Stability in W_1 distance

In [4, Appendix p.37], Chen, Goldstein and Shao bound the derivatives of the solution of the Stein equation in the case where μ is the standard Gaussian measure. Here, we extend their approach to a broader class of measures. Let $f : I \rightarrow \mathbb{R}$ be absolutely continuous, and let ψ be the solution of (18) given by Lemma 11. We are looking for inequalities of the form

$$\|h\psi'^2\|_\infty \leq C \|f'\|_\infty,$$

where $C > 0$ is a constant which does not depend on f . Recall that we denote by $q(t) = \int_a^t d\mu^*$ the cumulative distribution function of the measure μ^* , and $v(t) = \exp\left(-\lambda_\mu \int_0^x \frac{u du}{h(u)}\right)$ (Cf. (13)).

Proposition 16. *Let $f : I \rightarrow \mathbb{R}$ be in $C^1(I) \cap L^1(\mu^*)$. The associated solution (20) of Lemma 11 can be written as:*

$$\psi(x) = -Z \frac{1-q(x)}{v(x)} \int_{t=a}^x f'(t)q(t) dt - Z \frac{q(x)}{v(x)} \int_{t=x}^b f'(t)(1-q(t)) dt \quad (23)$$

Proof. First, write

$$\begin{aligned} f(x) - \mu^*(f) &= \int_a^x (f(x) - f(y)) d\mu^*(y) - \int_b^x (f(x) - f(y)) d\mu^*(y) \\ &= \int_a^x \int_y^x f'(t) dt d\mu^*(y) - \int_b^x \int_y^x f'(t) dt d\mu^*(y) \\ &= \int_a^x f'(t) \int_a^t d\mu^*(y) dt - \int_x^b f'(t) \int_t^b d\mu^*(y) dt \\ &= \int_a^x f'(t)q(t) dt - \int_x^b f'(t)(1-q(t)) dt. \end{aligned} \quad (24)$$

It follows that

$$\begin{aligned}\psi(x) &= \frac{1}{v(x)} \int_a^x (f(y) - \mu^*(f)) Z d\mu^*(y) \\ &= \frac{1}{v(x)} \int_a^x \left(\int_a^y f'(t)q(t) dt - \int_y^b f'(t)(1-q(t)) dt \right) Z d\mu^*(y).\end{aligned}$$

Now, on the one hand:

$$\int_a^x \int_a^y f'(t)q(t) dt d\mu^*(y) = \int_a^x \int_t^x f'(t)q(t) d\mu^*(y) dt = \int_a^x f'(t)q(t) (q(x) - q(t)) dt,$$

on the other hand:

$$\begin{aligned}\int_a^x \int_y^b f'(t)(1-q(t)) dt d\mu^*(y) &= \int_x^b \int_a^x f'(t)(1-q(t)) d\mu^*(y) dt + \int_a^x \int_a^t f'(t)(1-q(t)) d\mu^*(y) dt \\ &= \int_x^b f'(t)(1-q(t))q(x) dt + \int_a^x f'(t)(1-q(t))q(t) dt.\end{aligned}$$

From which one gets (23). □

By definition ψ satisfies

$$h(x)\psi'(x) = \left(f(x) - \mu^*(f) + \frac{x}{C_P} \psi(x) \right).$$

Hence, using Proposition 16 and Formula (24), we are now able to give a new formula for the derivative of the solution.

Corollary 17. *We have:*

$$\begin{aligned}h(x)\psi'(x) &= \left(1 - Z(1-q(x)) \frac{x}{C_P} \exp \left(\int_0^x \frac{u du}{C_P h(u)} \right) \right) \int_a^x f'(t)q(t) dt \\ &\quad - \left(1 + Zq(x) \frac{x}{C_P} \exp \left(\int_0^x \frac{u du}{C_P h(u)} \right) \right) \int_x^b f'(t)(1-q(t)) dt.\end{aligned}$$

In order to simplify the expression above, one can see that

$$\left| 1 - Z(1-q(x)) \frac{x}{C_P} \exp \left(\int_0^x \frac{u du}{C_P h(u)} \right) \right| \frac{1}{\sqrt{h(x)}} = \sqrt{\Gamma^*(a_1)(x)} \quad (25)$$

and

$$\left| 1 + Zq(x) \frac{x}{C_P} \exp \left(\int_0^x \frac{u du}{C_P h(u)} \right) \right| \frac{1}{\sqrt{h(x)}} = \sqrt{\Gamma^*(a_2)(x)} \quad (26)$$

where

$$a_1(x) := Z(1-q(x)) \exp \left(\int_0^x \frac{u du}{C_P(\mu)h(u)} \right) \quad (27)$$

$$a_2(x) := Zq(x) \exp \left(\int_0^x \frac{u du}{C_P(\mu)h(u)} \right) \quad (28)$$

Hence the following is immediate.

Proposition 18. *Let $f : I \rightarrow \mathbb{R}$ be in $C^1(I) \cap L^1(\mu^*)$, and let ψ the associated solution (20). The following bound holds:*

$$\|\sqrt{h}\psi'\|_\infty \leq C_h \|f'\|_\infty,$$

where

$$C_h := \sup_{x \in I} \left[\sqrt{\Gamma^*(a_1)(x)} \int_a^x q(t) dt + \sqrt{\Gamma^*(a_2)(x)} \int_x^b (1-q(t)) dt \right].$$

We can now state the main theorem of this article.

Theorem 19. *Let (L, μ) be a Markov diffusion on a Riemannian manifold M satisfying a Poincaré inequality with sharp constant $C_P(\mu)$. Suppose that Assumptions 1, 2 and 3 are verified. If*

$$C_h := \sup_{x \in I} \left[\sqrt{\Gamma^*(a_1)(x)} \int_a^x q(t) dt + \sqrt{\Gamma^*(a_2)(x)} \int_x^b (1-q(t)) dt \right] < \infty,$$

where a_1 and a_2 are defined in (27) and (28), then for all measures ν on M normalized as in (6) and satisfying a Poincaré inequality with sharp constant $C_P(\nu)$, it holds:

$$W_1(\mu^*, \nu^*) \leq C_h \left(\sqrt{\delta} + \sqrt{C_P(\nu)} \delta \right) \leq C_h \left(\frac{\sqrt{C_P(\nu) - C_P(\mu)}}{C_P(\mu)} + \frac{C_P(\nu) - C_P(\mu)}{C_P(\mu)^{3/2}} \right),$$

where μ^* (resp. ν^*) is the pushforward of μ (resp. ν) by f_0 , W_1 is the 1-Wasserstein distance (Cf. Definition 5), and $\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu)C_P(\mu)}$.

Proof. The proof is the same as in Theorem 14 but we replace the indicator functions by 1-Lipschitz functions, and we use Proposition 18 instead of Proposition 13. \square

6.1 Finiteness of C_h

In this section, we give explicit conditions on h that ensure the finiteness of C_h , thus allowing to apply Theorem 19. Recall that we denote $a = \inf I < 0$ and $b = \sup I > 0$.

Theorem 20. *Assume that:*

1. There exists a non-negative function g_1 , defined on a neighborhood of a , satisfying

$$x(g_1(x) - 1) = O(1), \quad x \rightarrow a^+ \quad (29)$$

and

$$q(x) \geq -\frac{C_P(\mu)}{Z}v(x)\frac{g_1(x)}{x}, \quad x \rightarrow a^+ \quad (30)$$

2. There exists a non-negative function g_2 , defined on a neighborhood of b , satisfying

$$x(g_2(x) - 1) = O(1), \quad x \rightarrow b^-$$

and

$$1 - q(x) \geq \frac{C_P(\mu)}{Z}v(x)\frac{g_2(x)}{x}, \quad x \rightarrow b^- \quad (31)$$

Then the constant C_h defined in Theorem 19 is finite.

Proof. We only study boundedness of $\sqrt{\Gamma^*(a_1)(x)} \int_{-\infty}^x q(t) dt$ at a and b and conclude by continuity. The case of $\sqrt{\Gamma^*(a_2)(x)} \int_x^{+\infty} (1 - q(t)) dt$ is similar. We only deal with the case of a , the one of b is similar. Since $v'(x) = -\lambda_\mu h(x)v(x)$ with an integration by parts, we get

$$\int_a^x q(t) dt = [tq(t)]_a^x - \int_a^x \frac{1}{Z}t \frac{v(t)}{h(t)} dt \leq xq(x) + \frac{C_P(\mu)}{Z}v(x).$$

The last inequality comes from Lemma 7 which gives $\lim_{t \rightarrow a} (-tq(t) - \frac{C_P(\mu)}{Z}v(t)) \leq 0$. Moreover, it is obvious that $1 - Z\lambda_\mu(1 - q(x))x \exp\left(\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) \geq 0$, for $x < 0$. Hence for $x < 0$, using (25),

$$\begin{aligned} \sqrt{\Gamma^*(a_1)(x)} \int_a^x q(t) dt &= \left(1 - Z\lambda_\mu(1 - q(x))x \exp\left(\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) \right) \int_a^x q(t) dt \\ &\leq \left(xq(x) + \frac{C_P(\mu)}{Z}v(x) \right) - Z\lambda_\mu(1 - q(x))\frac{x^2}{v(x)}q(x) - (1 - q(x))x. \end{aligned}$$

On the one hand, using first the assumption on g_1 , and next Lemma 7, one gets that for $x \rightarrow a^+$,

$$xq(x) + \frac{C_P(\mu)}{Z}v(x) \leq -\frac{C_P(\mu)}{Z}v(x)g_1(x) + \frac{C_P(\mu)}{Z}v(x) \leq (g_1(x) - 1)xq(x) \xrightarrow{x \rightarrow a^+} 0.$$

On the other hand, the existence of g_1 gives

$$-Z\lambda_\mu(1 - q(x))\frac{x^2}{v(x)}q(x) - (1 - q(x))x \leq xg_1(x)(1 - q(x)) - (1 - q(x))x = x(g_1(x) - 1)(1 - q(x)),$$

which is bounded by assumption.

The boundedness at b is obtained in the same way. By Lemma 7, one gets that $1 - Z\lambda_\mu(1 - q(x))x \exp\left(\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) \geq 0$, for $x > 0$. By the assumption on g_2 , one gets

$$1 - Z\lambda_\mu(1 - q(x))x \exp\left(\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) \leq 1 - g_2(x).$$

Hence using the same integration by parts as before,

$$\sqrt{\Gamma^*(a_1)(x)} \int_a^x q(t) dt \leq \left(xq(x) + \frac{C_P(\mu)}{Z}v(x) \right) (1 - g_2(x)).$$

Now $(1 - g_2(x))v(x) = x(1 - g_2(x))\frac{v(x)}{x}$ goes to zero when x goes to b since (31) gives $\frac{v(x)}{x} \xrightarrow{x \rightarrow b} 0$. This concludes the proof. \square

The standard Gaussian satisfies the requirements of Theorem 20 with $g_1(x) = g_2(x) = \frac{x^2}{1+x^2}$. Let us now give sufficient conditions on h to ensure that such g_1, g_2 functions exist. We prove the case of g_1 at a , the case of g_2 at b is similar.

Lemma 21. *Let g_1 be a bounded non-negative C^1 function defined on a neighborhood of a such that for $t \rightarrow a^+$, $g_1(t) > tg_1'(t)$ and*

$$h(t) \leq \lambda_\mu \frac{t^2(1 - g_1(t))}{g_1(t) - tg_1'(t)}, \quad t \rightarrow a^+ \quad (32)$$

If $\lim_{t \rightarrow a} v(t) = 0$, then

$$q(t) \geq -\frac{C_P}{Z}v(t)\frac{g_1(t)}{t}, \quad t \rightarrow a^+.$$

Proof. Let $f(t) := q(t) + \frac{C_P}{Z}\frac{g_1(t)}{t}v(t)$. Compute

$$f'(t) = \frac{v(t)}{Z} \left(\frac{C_P g_1'(t)}{t} - \frac{C_P g_1(t)}{t^2} + \frac{1}{h(t)} - \frac{g_1(t)}{h(t)} \right)$$

The assumption implies then that $f' \geq 0$. Hence $f(t) \geq \lim_{x \rightarrow a} f(x) = 0$. \square

Lemma 22. *Let g_2 be a bounded non-negative C^1 function defined on a neighborhood of b such that for $t \rightarrow b^-$, $g_2(t) > tg_2'(t)$ and*

$$h(t) \leq \lambda_\mu \frac{t^2(1 - g_2(t))}{g_2(t) - tg_2'(t)}, \quad t \rightarrow b^- \quad (33)$$

If $\lim_{t \rightarrow b} v(t) = 0$, then

$$1 - q(t) \geq \frac{C_P(\mu)}{Z}v(t)\frac{g_2(t)}{t}, \quad t \rightarrow b^-.$$

Under a growth control on h , the condition $v(t) \xrightarrow{x \rightarrow a} 0$ can easily be verified. One needs however to distinguish the case where a is finite from the one where it is infinite. For $a = -\infty$: if $h(t) \leq c|t|^\alpha$ for $t \rightarrow -\infty$ with $c > 0$ and $\alpha \leq 2$, then for $x \rightarrow -\infty$,

$$-\lambda_\mu \int_0^x \frac{u}{h(u)} du \leq -\frac{\lambda_\mu}{c} \int_x^0 \frac{1}{|u|^{\alpha-1}} du \xrightarrow{x \rightarrow -\infty} -\infty,$$

and hence $v(t) \xrightarrow{x \rightarrow -\infty} 0$. For $a > -\infty$: if $h(t) \leq c(t - a)^\alpha$ for $t \sim a$ with $c > 0$ and $\alpha \geq 1$, then for $x \in]a, 0[$:

$$-\lambda_\mu \int_0^x \frac{u}{h(u)} du \leq \lambda_\mu \int_x^0 \frac{u}{(u - a)^\alpha} du = \lambda_\mu \int_x^0 \frac{du}{(u - a)^{\alpha-1}} + a\lambda_\mu \int_x^0 \frac{du}{(u - a)^\alpha} \xrightarrow{x \rightarrow a} -\infty,$$

and hence $v(t) \xrightarrow{x \rightarrow a} 0$. The same results are valid at b , only replacing $(t - a)^\alpha$ by $(b - t)^\alpha$ when $b < +\infty$.

Proposition 23. *If $a = -\infty$ and $h(t) \leq c|t|^\alpha$ for $t \rightarrow -\infty$ with $c > 0$ and $\alpha \leq 2$, then q satisfies (30) with $g_1(t) = 1 - \frac{c}{\lambda_\mu}|t|^{\alpha-2}$. Moreover, (29) is satisfied if $\alpha \leq 1$.*

If $a > -\infty$ and $h(t) \leq c(t - a)^\alpha$ for $t \rightarrow a^+$ with $c > 0$ and $\alpha \geq 1$, then q satisfies (30) with $g_1(t) = 1 - \frac{4c}{\lambda_\mu a^2}(t - a)^\alpha$.

Proof. Let us compute, if $a = -\infty$ and $t < 0$:

$$\lambda_\mu \frac{t^2(1 - g_1(t))}{g_1(t) - tg_1'(t)} = \frac{ct^2|t|^{\alpha-2}}{1 - \frac{c}{\lambda_\mu}(3 - \alpha)|t|^{\alpha-2}} \geq c|t|^\alpha,$$

for $t \sim -\infty$. If $a > -\infty$ and $t \in]a, \frac{a}{2}[$,

$$\lambda_\mu \frac{t^2(1 - g_1(t))}{g_1(t) - tg_1'(t)} = \frac{4c}{a^2} \frac{t^2(t - a)^\alpha}{1 + \frac{4c}{\lambda_\mu a^2}(t - a)^{\alpha-1}[\alpha t - (t - a)]} \geq c(t - a)^\alpha,$$

since $\alpha t - (t - a) < 0$ and $t^2 \geq \frac{a^2}{4}$.

Hence Lemma 21 and the above remark give us the result. \square

The same result is valid at b , using Lemma 33. We sum it up in the following Proposition:

Proposition 24. *Assume that one of these two conditions is verified at a :*

- *either $a = -\infty$ and $h(t) \leq c|t|$ for $t \rightarrow -\infty$ with $c > 0$,*
- *either $a > -\infty$ and $h(t) \leq c(t - a)$ for $t \rightarrow a^+$ with $c > 0$,*

and one of these two conditions is satisfied at b :

- *either $b = +\infty$ and $h(t) \leq ct$ for $t \rightarrow +\infty$ with $c > 0$,*
- *either $b < +\infty$ and $h(t) \leq c(b - t)$ for $t \rightarrow b^-$ with $c > 0$.*

Then the constant C_h defined in Theorem 19 is finite.

From Proposition 24 and Theorem 19 one can infer the following more practical theorem.

Theorem 25. *Let (L, μ) be a Markov diffusion on a Riemannian manifold M satisfying a Poincaré inequality with sharp constant $C_P(\mu)$. Suppose that Assumptions 1, 2 and 3 are verified. Assume that one of these two conditions is verified at a :*

- *either $a = -\infty$ and $h(t) \leq c|t|$ for $t \rightarrow -\infty$ with $c > 0$,*
- *either $a > -\infty$ and $h(t) \leq c(t - a)$ for $t \rightarrow a^+$ with $c > 0$,*

and one of these two conditions is satisfied at b :

- *either $b = +\infty$ and $h(t) \leq ct$ for $t \rightarrow +\infty$ with $c > 0$,*
- *either $b < +\infty$ and $h(t) \leq c(b - t)$ for $t \rightarrow b^-$ with $c > 0$.*

Then for all measures ν on M normalized as in (6) and satisfying a Poincaré inequality with sharp constant $C_P(\nu)$, it holds:

$$W_1(\mu^*, \nu^*) \leq C_h \left(\sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right),$$

where μ^ (resp. ν^*) is the pushforward of μ (resp. ν) by f_0 , W_1 is the 1-Wasserstein distance (Cf. Definition 5), and $\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu) C_P(\mu)}$.*

6.2 Application: stability of the $\Gamma(s, 1)$ distributions, $s > 0$

Let us consider the $\Gamma(s, 1)$ distribution $d\mu(x) := \frac{x^{s-1} e^{-x}}{\Gamma(s)} \mathbf{1}_{\mathbb{R}_+}(x) dx$, where Γ denotes the Euler's Gamma function. This probability measure is the reversible measure of the Laguerre process with following generator on $[0, \infty)$:

$$Lf(x) = xf''(x) - (s - x)f'(x).$$

We have $\Gamma(f) = x(f')^2$, $C_P(\mu) = 1$ and the first Laguerre polynomial $f_0(x) = \frac{s-x}{\sqrt{s}}$ is a normalized eigenfunction attaining the spectral gap. Hence $I =]-\infty, \sqrt{s}]$, so we get $h(x) = s - \sqrt{s}x$. So Assumptions 1, 2 and 3 are satisfied. Moreover, one can see that we are also under the conditions of Proposition 24, with at the infimum, $a = -\infty$ and $h(t) \leq c|t|$ for $c = 2\sqrt{s}$, and at the supremum, $b = \sqrt{s} < +\infty$, $h(t) \leq c(b - t)$ for $c = \sqrt{s}$. In this case the three conditions (6) are reduced to only two:

$$\int x d\nu = s \quad \text{and} \quad \int x^2 d\nu = s(s + 1) \tag{34}$$

meaning that we only ask ν to have the same first and second moments as μ . So we can apply Theorem 25 for the gamma distributions $\Gamma(s, 1)$, with $s > 0$: for all measures ν on \mathbb{R}_+ normalized as in (34) and satisfying the following Poincaré inequality with sharp constant $C_P(\nu)$:

$$\text{Var}_\nu(f) \leq C_P(\nu) \int_0^{+\infty} xf'(x)^2 d\nu(x), \tag{35}$$

it holds for a finite constant C_h :

$$W_1(\mu^*, \nu^*) \leq C_h \left(\sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right), \tag{36}$$

where W_1 is the 1-Wasserstein distance (Cf. Definition 5), $\delta := \frac{C_P(\nu) - 1}{C_P(\nu)}$, $\nu^* := f_0^\# \nu$, and

$$d\mu^*(x) = \frac{e^{-s}}{\Gamma(s)} (s - x)^{s-1} e^x \mathbf{1}_{]-\infty, s]}(x) dx.$$

6.3 Application : stability for Brascamp-Lieb inequalities

Let $M = \mathbb{R}$ and $d\mu = e^{-\phi} dx$ be a probability measure with $\phi : \mathbb{R} \rightarrow \mathbb{R}$ strictly convex and C^2 , as in Section 5. Recall that in this case $h = \phi'' \circ \phi'^{-1}$ and $I = \mathbb{R}$. A straightforward application of Theorem 25 yields the following.

Corollary 26. *Let $M = \mathbb{R}$ and $d\mu = e^{-\phi} dx$ be a C^2 strictly log-concave probability measure. Let ν be a probability measure on \mathbb{R} verifying the following Poincaré inequality: for all functions f in the weighted Sobolev space $H_\phi^{1,2} = \{g \in L^2(\nu) \mid \frac{g'}{\sqrt{\phi''}} \in L^2(\nu)\}$,*

$$\text{Var}_\nu(f) \leq C_P(\nu) \int \frac{f'^2}{\phi''} d\nu,$$

and such that $\int \phi' d\nu = 0$, $\int \phi'^2 d\nu = 1$ and $\int \phi'' d\nu = \int \phi'' d\mu$. So $C_P(\nu) \geq 1$, and if

$$\phi''(t) \leq c|t|, \quad t \rightarrow \pm\infty,$$

then $\delta := \frac{C_P(\nu)-1}{C_P(\nu)}$ satisfies

$$W_1(\mu^*, \nu^*) \leq C_h \left(\sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right),$$

where $\mu^* = \phi'_{\#} \mu$ and $\nu^* = \phi'_{\#} \nu$.

7 Comparison

In this section we get stability results involving the Kolmogorov distance from one involving the Wasserstein-1 distance. Let us begin recalling:

Proposition 27. *e.g.[14, Prop. 1.2] If μ^* has bounded density ρ with respect to the Lebesgue measure then*

$$d_K(\nu^*, \mu^*) \leq 2\sqrt{C W_1(\nu^*, \mu^*)}$$

where $\rho \leq C$ and W_1 is the 1-Wasserstein distance.

Let (L, μ) and h satisfy the requirements of Theorem 25. Then μ satisfies the stability result (36) with respect to the W_1 distance. If moreover $\frac{v(x)}{h(x)}$ is bounded, then it also satisfies a stability result with respect to the Kolmogorov distance. For $s \geq 1$, it is the case of the $\Gamma(s, 1)$ distribution since $\frac{e^{-s}}{\Gamma(s)}(s-x)^{s-1}e^x \mathbf{1}_{]-\infty, \sqrt{s}]}(x)$ is bounded by $C = \frac{(s-1)^{s-1}e^{1-s}}{\Gamma(s)}$. Hence, if $\mu = \Gamma(s, 1)$ with $s \geq 1$, one gets that for all measures ν on \mathbb{R}_+ normalized as in (6) and satisfying the Poincaré inequality (35) with sharp constant $C_P(\nu)$, it holds:

$$d_K(\mu^*, \nu^*) \leq 2\sqrt{\frac{(s-1)^{s-1}e^{1-s}}{\Gamma(s)}} \sqrt{C_h \left(\sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right)},$$

with $\delta := \frac{C_P(\nu)-1}{C_P(\nu)}$, $\nu^* := f_0^{\#} \nu$, and

$$d\mu^*(x) = \frac{e^{-s}}{\Gamma(s)}(s-x)^{s-1}e^x \mathbf{1}_{]-\infty, \sqrt{s}]}(x)dx.$$

Acknowledgments

I would like to thank my PhD advisors Franck Barthe and Max Fathi for their availability and all their fruitful advice.

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