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# Thin layer approximations in mechanical structures : the Dirichlet boundary condition case

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## Abstract

We consider the solution of a transmission problem at a thin layer interface of thickness  $\varepsilon > 0$  in a mechanical structure. We build a multi-scale expansion for that solution as  $\varepsilon \rightarrow 0$  that enables to replace the thin layer with an improved boundary condition and leads to optimal estimates for the remainders. This short note presents new results when a Dirichlet condition is imposed on the internal boundary of the thin layer and is the counterpart of [2] where the Neumann case was considered.

*Keywords:* Asymptotic analysis, Generalized impedance boundary conditions, Linear elasticity, Dirichlet boundary condition

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## Résumé

Cette note concerne le problème de transmission dans une structure mécanique contenant une couche d'épaisseur mince  $\varepsilon > 0$ . Nous construisons un développement asymptotique de la solution lorsque  $\varepsilon \rightarrow 0$  qui permet de remplacer la couche mince par une condition aux limites approchées et nous en déduisons des estimations d'erreurs optimales. Nous présentons de nouveaux résultats lorsqu'une condition de Dirichlet est imposée sur la frontière interne de la couche mince tandis que le cas d'une condition de Neumann est étudié dans [2].

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## 1. Introduction and problem settings

Let  $\Omega$  be a Lipschitz bounded open set of  $\mathbb{R}^d$ , where  $d \geq 2$  is an integer representing the dimension. We assume that the solid  $\Omega$  consists of an isotropic material with a linear behavior. The boundary of  $\Omega$  is such that  $\partial\Omega =: \Gamma_D \cup \Gamma_N$  where  $\Gamma_D$  and  $\Gamma_N$  are two non-empty open sets of  $\partial\Omega$  and  $|\Gamma_D| > 0$ . We consider a nonempty inclusion  $\omega \subset\subset \Omega$  with analytic boundary  $\partial\omega =: \Gamma$ . We denote by  $\mathbf{n}$  the unit normal vector to  $\partial\Omega$  and  $\Gamma$  directed outward to  $\Omega \setminus \bar{\omega}$ .

Let  $\varepsilon > 0$  be small enough. We consider that  $\Gamma$  has an interior thin layer with thickness  $\varepsilon$  bordering  $\omega$  defined by

$$\omega_i^\varepsilon := \{x + s\mathbf{n}(x) \mid x \in \Gamma \text{ and } 0 < s < \varepsilon\}.$$

We recall that the normal vector  $\mathbf{n}$  is directed inward the inclusion  $\omega$ . We set  $\omega^\varepsilon := \omega \setminus \bar{\omega}_i^\varepsilon$  and we denote its boundary by  $\Gamma^\varepsilon$ . In the sequel, we use the lower index  $e$  for all quantities related to  $\Omega \setminus \bar{\omega}$  and the lower index  $i$  for all quantities related to  $\omega_i^\varepsilon$ . These notations are illustrated in [2, Fig. 1].

We denote by  $A_e$  the Hooke's law defined, for any symmetric matrix  $\xi$ , by

$$A_e \xi := 2\mu_e \xi + \lambda_e \text{Tr}(\xi) \mathbf{I}_d,$$

where  $\mu_e > 0$  and  $\lambda_e > 0$  are two positive constants which represent the Lamé coefficients in  $\Omega \setminus \bar{\omega}$ . The Hooke's law associated to  $\omega_i^\varepsilon$  is denoted  $A_i$  with Lamé coefficients  $\mu_i > 0$  and  $\lambda_i > 0$ . Moreover the stress vector relative

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to the material properties  $A_i$  on  $\Gamma$  is defined by

$$\mathbf{T}_i(\mathbf{u}) := A_i e(\mathbf{u}) \mathbf{n}, \quad \text{where} \quad e(\mathbf{u}) := \frac{1}{2} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}),$$

and we define similarly  $\mathbf{T}_e$  the stress vector relative to the material properties  $A_e$  on either  $\Gamma$  or  $\Gamma_N$ .

For a smooth bounded open set  $\omega$  of  $\mathbb{R}^d$  ( $d \geq 2$ ) with a boundary  $\Gamma$ , we denote by  $H^s(\omega)$  and  $H^s(\Gamma)$  the standard complex valued, Hilbert-Sobolev spaces of order  $s \in \mathbb{R}$  defined on  $\omega$  and  $\Gamma$  respectively (with the convention  $H^0 = L^2$ ). Spaces of vector functions will be denoted by boldface letters, thus  $\mathbf{H}^s = (H^s)^d$ . We introduce the following Sobolev space

$$\mathbf{H}_{\Gamma_D}^1(\Omega \setminus \bar{\omega}) := \{\mathbf{v} \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) ; \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}.$$

The dual space of  $\mathbf{H}_{\Gamma_D}^1(\Omega \setminus \bar{\omega})$  is denoted by  $\widetilde{\mathbf{H}}_{\Gamma_D}^{-1}(\Omega \setminus \bar{\omega})$ . Let  $\mathbf{f} \in \widetilde{\mathbf{H}}_{\Gamma_D}^{-1}(\Omega \setminus \bar{\omega})$  be some exterior forces and a load  $\mathbf{g} \in \mathbf{H}^{-1/2}(\Gamma_N)$ . We are concerned with the following transmission problem

$$\left\{ \begin{array}{ll} -\operatorname{div}(A_e e(\mathbf{u}_e^\varepsilon)) = \mathbf{f} & \text{in } \Omega \setminus \bar{\omega} \\ -\operatorname{div}(A_i e(\mathbf{u}_i^\varepsilon)) = \mathbf{0} & \text{in } \omega_i^\varepsilon \\ \mathbf{u}_e^\varepsilon = \mathbf{0} & \text{on } \Gamma_D \\ \mathbf{T}_e(\mathbf{u}_e^\varepsilon) = \mathbf{g} & \text{on } \Gamma_N \\ \mathbf{T}_i(\mathbf{u}_i^\varepsilon) = \mathbf{T}_e(\mathbf{u}_e^\varepsilon) & \text{on } \Gamma \\ \mathbf{u}_i^\varepsilon = \mathbf{u}_e^\varepsilon & \text{on } \Gamma \\ \mathbf{u}_i^\varepsilon = \mathbf{0} & \text{on } \Gamma^\varepsilon. \end{array} \right. \quad (1.1)$$

The solution of such a problem exists, is unique and belongs to  $\mathbf{H}_{\Gamma_D \cup \Gamma^\varepsilon}^1(\Omega \setminus \bar{\omega}^\varepsilon)$  thanks to the Lax-Milgram theorem and Korn's inequality (see, e.g., [3, Theorem 6.3-4]).

To avoid instabilities in the numerical treatment of the transmission problem (1.1) we approximate the solution  $\mathbf{u}_e^\varepsilon \in \mathbf{H}_{\Gamma_D}^1(\Omega \setminus \bar{\omega})$  by the solution  $\mathbf{v}_{[N]}^\varepsilon$  to some boundary value problems of the form

$$\left\{ \begin{array}{ll} -\operatorname{div}(A_e e(\mathbf{v}_{[N]}^\varepsilon)) = \mathbf{f} & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{v}_{[N]}^\varepsilon = \mathbf{0} & \text{on } \Gamma_D \\ \mathbf{T}_e(\mathbf{v}_{[N]}^\varepsilon) = \mathbf{g} & \text{on } \Gamma_N \\ \mathbf{B}_N^\varepsilon(\varepsilon, \mathbf{v}_{[N]}^\varepsilon, \mathbf{T}_e(\mathbf{v}_{[N]}^\varepsilon)) = \mathbf{0} & \text{on } \Gamma, \end{array} \right. \quad (1.2)$$

where  $\|\mathbf{u}_e^\varepsilon - \mathbf{v}_{[N]}^\varepsilon\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} = O(\varepsilon^{N+1})$ , for any  $N \in \mathbb{N}$ , and the last equation of (1.2) is a so-called *Generalized Impedance Boundary Condition* (GIBC). This approximate condition is obtained by expanding the Navier equation in  $\omega_i^\varepsilon$  in terms of  $\varepsilon$  and surface derivatives on  $\Gamma$ . The transmission problem is then split into a sequence of coupled boundary value problems in  $\omega_i^\varepsilon$  (rescaled through a nondimensional variable) and in  $\Omega \setminus \bar{\omega}$  (that will constitute the GIBCs). Both the exterior and interior solutions are expanded as power series of the thickness  $\varepsilon$  whose coefficients functions are obtained iteratively. The GIBC of order  $N \in \mathbb{N}$  is deduced from the boundary condition satisfied by the truncated series of the exterior field up to the index  $N$ . The method originates from [10] for  $d = 2$  and leads to optimal error estimates. The results are now established for the Laplace, Helmholtz and Maxwell equations and more recently for the Lamé system when a Neumann condition is imposed on  $\Gamma^\varepsilon$  (see [2] and references therein for a bibliographical overview). It is the purpose of this short note to address the case when a Dirichlet condition is considered on  $\Gamma^\varepsilon$ . The coefficient functions and the GIBCs, for  $N = 0, 1, 2$ , are given in proposition 2.1 and its proof. The expected optimal error estimates are stated in Theorem 3.1 and its proof for both the interior and exterior solutions.

Mechanical engineering applications of the proposed results include the modeling of delaminated elastic area with thin opening by Dirichlet and Neumann crack jumps [1, 4] in layered media followed by the mathematical analysis of the associated inverse problem of delamination detection [1, 8, 9] for both the static and dynamic framework.

## 2. Generalized impedance boundary conditions

The operator  $\mathbf{B}_N^\varepsilon$  is composed of curvature operators and/or surface differential operators and depends on the interior Lamé parameters. Thus we use the classical surface differential operators: the tangential gradient  $\nabla_\Gamma$  defined in [7, pages 68-75] and the surface divergence  $\operatorname{div}_\Gamma$  defined as the trace of  $\nabla_\Gamma$  applied to vector functions. Moreover,  $\mathcal{R}$  and  $\mathcal{H}$  represent the curvature operator of  $\Gamma$  and its trace, respectively.

To determine the approximate boundary condition, we follow the procedure described in [5]. For any  $x \in \Gamma$  and  $s \geq 0$ , we set  $\mathbf{u}(x + s\mathbf{n}(x)) =: \bar{\mathbf{u}}(x, s)$  and we use the change of variables  $y = x + s\mathbf{n}(x) = x + \varepsilon S\mathbf{n}(x)$  with  $S \in [0, 1]$ . We set  $\bar{\mathbf{u}}(x, s) = \bar{\mathbf{u}}(x, \varepsilon S) =: \mathbf{U}^\varepsilon(x, S)$ . Firstly we obtain the following asymptotic expansion when  $\varepsilon \rightarrow 0$ :

$$\operatorname{div}(A_i e(\mathbf{u}))(x + \varepsilon S\mathbf{n}(x)) = \frac{1}{\varepsilon^2} \left( \Lambda_0 \partial_S^2 + \sum_{n \geq 1} \varepsilon^n \Lambda_n \right) \mathbf{U}^\varepsilon(x, S), \quad (2.1)$$

where

$$\Lambda_0 := (\lambda_i + 2\mu_i)\mathbf{n} \otimes \mathbf{n} + \mu_i(\mathbf{I}_d - \mathbf{n} \otimes \mathbf{n}), \quad (2.2)$$

and

$$\Lambda_1 \mathbf{U}^\varepsilon := \Lambda_{1,1} \partial_S \mathbf{U}^\varepsilon, \quad \text{with } \Lambda_{1,1} \mathbf{U}^\varepsilon := \mu_i \mathcal{H} \mathbf{U}^\varepsilon + (\lambda_i + \mu_i)(\mathbf{n} \operatorname{div}_\Gamma \mathbf{U}^\varepsilon + \nabla_\Gamma(\mathbf{U}^\varepsilon \cdot \mathbf{n})). \quad (2.3)$$

Moreover the traction trace operator is defined on  $\Gamma$ , i.e. for  $S = 0$ , by

$$\mathbf{T}_i \mathbf{U}^\varepsilon := \frac{1}{\varepsilon} \Lambda_0 \partial_S \mathbf{U}^\varepsilon + \lambda_i \mathbf{n} \operatorname{div}_\Gamma \mathbf{U}^\varepsilon + \mu_i [\nabla_\Gamma \mathbf{U}^\varepsilon] \mathbf{n} = \frac{1}{\varepsilon} \Lambda_0 \partial_S \mathbf{U}^\varepsilon + \mathcal{B}_i^0 \mathbf{U}^\varepsilon.$$

Secondly we set  $\mathbf{u}_e^\varepsilon := \sum_{n \geq 0} \varepsilon^n \mathbf{u}_e^n$  in  $\Omega \setminus \bar{\omega}$  and  $\bar{\mathbf{u}}_i^\varepsilon(x, s) := \mathbf{U}_i^\varepsilon(x, S) = \sum_{n \geq 0} \varepsilon^n \mathbf{U}_i^n(x, S)$  in  $\Gamma \times [0, 1]$ , with the convention  $\mathbf{U}_i^\ell = \mathbf{u}_e^\ell = \mathbf{0}$  for any integer  $\ell < 0$ . In the case of a Dirichlet interior boundary condition, the original transmission problem (1.1) can then be rewritten as a couple of two boundary value problem for every coefficient functions  $(\mathbf{U}_i^\ell, \mathbf{u}_e^\ell)$ :

$$\begin{cases} \partial_S^2 \Lambda_0 \mathbf{U}_i^\ell &= - \sum_{k=1}^{\ell} \Lambda_k \mathbf{U}_i^{\ell-k} & \text{in } \Gamma \times (0, 1) \\ \partial_S \Lambda_0 \mathbf{U}_i^\ell &= \mathbf{T}_e(\mathbf{u}_e^{\ell-1}) - \mathcal{B}_i^0 \mathbf{U}_i^{\ell-1} & \text{on } \Gamma \times \{0\} \\ \Lambda_0 \mathbf{U}_i^\ell &= \mathbf{0} & \text{on } \Gamma \times \{1\}, \end{cases}$$

and

$$\begin{cases} \operatorname{div}(A_e e(\mathbf{u}_e^\ell)) &= \delta_\ell^0 \mathbf{f} & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u}_e^\ell &= \mathbf{0} & \text{on } \Gamma_D \\ \mathbf{T}_e(\mathbf{u}_e^\ell) &= \delta_\ell^0 \mathbf{g} & \text{on } \Gamma_N \\ \Lambda_0 \mathbf{u}_e^\ell &= \Lambda_0 \mathbf{U}_i^\ell(\cdot, 0) & \text{on } \Gamma. \end{cases} \quad (2.4)$$

where  $\delta_\ell^j$  is kronecker delta. For  $\ell = 0, 1, 2$ , we solve iteratively the new systems to compute first  $\mathbf{U}_i^\ell$  and then recover the boundary condition satisfied by  $\mathbf{u}_e^\ell$ . The truncated fields are denoted by  $\mathbf{u}_{e,[N]}^\varepsilon := \sum_{k=0}^N \varepsilon^k \mathbf{u}_e^k$  and

$\mathbf{U}_{i,[N]}^\varepsilon := \sum_{k=0}^N \varepsilon^k \mathbf{U}_i^k$ . From these results we deduce the GIBC satisfied by  $\mathbf{v}_{[N]}^\varepsilon$ , which is an approximation of  $\mathbf{u}_{e,[N]}^\varepsilon$  up to  $O(\varepsilon^{N+1})$ . The results are stated in the following proposition.

**Proposition 2.1.** *The GIBC, defined on  $\Gamma$ , modeling thin layer effects for  $N = 0$ , corresponds to the homogeneous Dirichlet condition. For  $N = 1, 2$  it can be written in the form*

$$\varepsilon \mathbf{T}_e \mathbf{v}_{[N]}^\varepsilon + \mathbf{C}_N^\varepsilon(\mathbf{v}_{[N]}^\varepsilon) = \mathbf{0},$$

with

$$\begin{aligned} \mathbf{C}_1^\varepsilon(\mathbf{w}) &:= \Lambda_0 \mathbf{w}|_\Gamma, \\ \mathbf{C}_2^\varepsilon(\mathbf{w}) &:= \Lambda_0 \mathbf{w}|_\Gamma + \frac{1}{2} \varepsilon (\mu_i \mathcal{H} + (\lambda_i + \mu_i) \mathcal{R} + (\lambda_i - \mu_i) \mathcal{M}) \mathbf{w}|_\Gamma. \end{aligned}$$

where  $\mathcal{M}$  represents the tangential Gunter derivative defined by [6, Chapter V, §1]  $\mathcal{M}\mathbf{w} = [\nabla_\Gamma \mathbf{w}] \mathbf{n} - \mathbf{n} \operatorname{div}_\Gamma \mathbf{w}$ .

PROOF. First, we obtain

$$\Lambda_0 \mathbf{U}_i^0 = \mathbf{0}, \quad \Lambda_0 \mathbf{U}_i^1(\cdot, S) = (S-1) \mathbf{T}_e \mathbf{u}_e^0.$$

Using  $\mathbf{U}_i^1(\cdot, 0) = -\Lambda_0^{-1} \mathbf{T}_e \mathbf{u}_e^0 = \mathbf{u}_{e|\Gamma}^1$ , we rewrite  $\mathbf{U}_i^1(\cdot, S) = -(S-1) \mathbf{u}_{e|\Gamma}^1$  and we get

$$\Lambda_0 \mathbf{U}_i^2(\cdot, S) = \frac{S^2-1}{2} \Lambda_{1,1} \mathbf{u}_{e|\Gamma}^1 + (S-1) \left( \mathbf{T}_e \mathbf{u}_e^1 - \mathcal{B}_t^0 \mathbf{u}_{e|\Gamma}^1 \right).$$

When  $S = 0$ , we simplifies  $\Lambda_0 \mathbf{u}_{e|\Gamma}^2 = \Lambda_0 \mathbf{U}_i^2(\cdot, 0) = -\mathbf{T}_e \mathbf{u}_e^1 - \frac{1}{2} \Lambda_{1,1} \mathbf{u}_{e|\Gamma}^1 + \mathcal{B}_t^0 \mathbf{u}_{e|\Gamma}^1$  and

$$\frac{1}{2} \Lambda_{1,1} - \mathcal{B}_t^0 = \frac{1}{2} \left( \mu_i \mathcal{H} + (\lambda_i + \mu_i) \mathcal{R} + (\lambda_i - \mu_i) ([\nabla_\Gamma \cdot] \mathbf{n} - \mathbf{n} \operatorname{div}_\Gamma) \right).$$

Then, substituing these results in (2.4), we get the following boundary conditions for  $\mathbf{u}_{e[N]}^\varepsilon$  on  $\Gamma$ .

$$\begin{aligned} \Lambda_0 \mathbf{u}_{e[0]}^\varepsilon &= \Lambda_0 \mathbf{u}_e^0 = \mathbf{0} \\ \Lambda_0 \mathbf{u}_{e[1]}^\varepsilon &= \Lambda_0 (\mathbf{u}_e^0 + \varepsilon \mathbf{u}_e^1) = -\varepsilon \mathbf{T}_e \mathbf{u}_e^0 = -\varepsilon \mathbf{T}_e \mathbf{u}_{e[1]}^\varepsilon + \varepsilon^2 \Lambda_0 \mathbf{u}_e^1 \\ \Lambda_0 \mathbf{u}_{e[2]}^\varepsilon &= \Lambda_0 (\mathbf{u}_e^0 + \varepsilon \mathbf{u}_e^1 + \varepsilon^2 \mathbf{u}_e^2) \\ &= -\varepsilon \mathbf{T}_e (\mathbf{u}_e^0 + \varepsilon \mathbf{u}_e^1) - \frac{1}{2} \varepsilon^2 \left( \mu_i \mathcal{H} + (\lambda_i + \mu_i) \mathcal{R} + (\lambda_i - \mu_i) \mathcal{M} \right) \mathbf{u}_e^1 \\ &= -\varepsilon \mathbf{T}_e \mathbf{u}_{e[2]}^\varepsilon - \frac{1}{2} \varepsilon^2 \left( \mu_i \mathcal{H} + (\lambda_i + \mu_i) \mathcal{R} + (\lambda_i - \mu_i) \mathcal{M} \right) \mathbf{u}_{e[2]}^\varepsilon \\ &\quad + \varepsilon^3 \mathbf{T}_e \mathbf{u}_e^2 + \frac{1}{2} \varepsilon^3 \left( \mu_i \mathcal{H} + (\lambda_i + \mu_i) \mathcal{R} + (\lambda_i - \mu_i) \mathcal{M} \right) \mathbf{u}_e^2. \end{aligned}$$

We observe that the truncated series up to the index  $N$  satisfy  $\varepsilon \mathbf{T}_e \mathbf{u}_{e[N]}^\varepsilon + \mathbf{C}_N^\varepsilon (\mathbf{u}_{e[N]}^\varepsilon) = O(\varepsilon^{N+1})$ . Then we choose to approach the total exterior field  $\mathbf{u}_e^\varepsilon$  by a new field  $\mathbf{v}_{[N]}^\varepsilon$  that satisfies  $\varepsilon \mathbf{T}_e \mathbf{v}_{[N]}^\varepsilon + \mathbf{C}_N^\varepsilon (\mathbf{v}_{[N]}^\varepsilon) = \mathbf{0}$ . ■

**Remark 2.2.** *Even if we provide the formula for  $N = 0, 1, 2$  only, we guess that, for any  $N \in \mathbb{N}^*$ , the impedance operator  $\mathbf{C}_N^\varepsilon$  is a surface differential operator of order  $(N-1)$  for the three components of the state. Although, it differs a bit from the Laplace equation case, the procedure to prove optimal error estimates in  $\mathbf{H}^1$ -norm presented in [10, Chapter 1], extends to the elastic case at any order  $N \in \mathbb{N}$  and is sketched in Section 3 for the sake of completeness.*

For  $N = 1, 2$ , the associated weak variational formulations of the GIBC problems (1.2) with non vanishing right hand side  $\mathbf{T}_e \mathbf{v} + \mathbf{C}_N^\varepsilon (\mathbf{v}) = \mathbf{h}$  when  $\mathbf{h} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$  read : Find  $\mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega \setminus \bar{\omega})$  satisfying

$$\mathbf{a}_N^\varepsilon (\mathbf{v}, \mathbf{w}) = \boldsymbol{\ell}(\mathbf{w}), \quad \forall \mathbf{w} \in \mathbf{H}_{\Gamma_D}^1(\Omega \setminus \bar{\omega}) \quad (2.5)$$

where

$$\begin{aligned} \mathbf{a}_1^\varepsilon (\mathbf{v}, \mathbf{w}) &:= \lambda_e \int_{\Omega \setminus \bar{\omega}} (\operatorname{div} \mathbf{v}) (\operatorname{div} \mathbf{w}) + 2\mu_e \int_{\Omega \setminus \bar{\omega}} e(\mathbf{v}) : e(\mathbf{w}) + \varepsilon^{-1} \int_\Gamma \Lambda_0 \mathbf{v} \cdot \mathbf{w}, \\ \mathbf{a}_2^\varepsilon (\mathbf{v}, \mathbf{w}) &:= \mathbf{a}_1^\varepsilon (\mathbf{v}, \mathbf{w}) + \frac{1}{2} \int_\Gamma (\mu_i \mathcal{H} + (\lambda_i + \mu_i) \mathcal{R} + (\lambda_i - \mu_i) \mathcal{M}) \mathbf{v} \cdot \mathbf{w}, \end{aligned}$$

and

$$\boldsymbol{\ell}(\mathbf{w}) := \int_{\Omega \setminus \bar{\omega}} \mathbf{f} \cdot \mathbf{w} + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{w} + \varepsilon^{-1} \int_\Gamma \mathbf{h} \cdot \mathbf{w}.$$

The bilinear forms  $\mathbf{a}_N^\varepsilon$  are symmetric and continuous on  $\mathbf{H}_{\Gamma_D}^1(\Omega \setminus \bar{\omega}) \times \mathbf{H}_{\Gamma_D}^1(\Omega \setminus \bar{\omega})$  (see [6, Chapter V, §1] for the properties of  $\mathcal{M}$ ). The coercivity of  $\mathbf{a}_1^\varepsilon$  is obvious since  $\Lambda_0$  is a positive definite matrix.

### 3. Convergence analysis

Assuming the existence and uniqueness of the solution  $\mathbf{v}_{[N]}^\varepsilon$  to GIBC problems at any order  $N \in \mathbb{N}$ , one can establish optimal error estimates between  $\mathbf{u}_e^\varepsilon$  and its approximate field  $\mathbf{v}_{[N]}^\varepsilon$  as done below.

**Theorem 3.1.** *Let  $N \in \mathbb{N}$  and  $\mathbf{f} \in \mathcal{C}^\infty(\overline{\Omega \setminus \bar{\omega}})$ . Then there exists a constant  $C_{\Omega \setminus \bar{\omega}}$  independent of  $\varepsilon$  such that*

$$\|\mathbf{v}_{[N]}^\varepsilon - \mathbf{u}_e^\varepsilon\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} \leq C_{\Omega \setminus \bar{\omega}} \varepsilon^{N+1}.$$

PROOF. For any  $N \in \mathbb{N}$ , the assumption  $\mathbf{f} \in \mathcal{C}^\infty(\overline{\Omega \setminus \bar{\omega}})$  allows us to ensure higher order local regularity for  $\mathbf{u}_e^\ell$ , with  $\ell = 0, \dots, N-1$  in the neighborhood of  $\Gamma$  so that the right hand side to problems (2.4) belong to  $\mathbf{H}^{\frac{1}{2}}(\Gamma)$  and the solution satisfies  $u_e^N \in \mathbf{H}_{\Gamma_D}^1(\Omega \setminus \bar{\omega})$ .

• We decompose the remainder as  $\mathbf{v}_{[N]}^\varepsilon - \mathbf{u}_e^\varepsilon := (\mathbf{v}_{[N]}^\varepsilon - \mathbf{u}_{e,[N]}^\varepsilon) + (\mathbf{u}_{e,[N]}^\varepsilon - \mathbf{u}_e^\varepsilon)$ .

• Firstly, we set  $\mathbf{r}_{[N]}^\varepsilon := \mathbf{u}^\varepsilon - \sum_{n=0}^N \varepsilon^n \mathbf{u}^n$  and we denote by  $\mathbf{r}_{e,[N]}^\varepsilon$  and  $\mathbf{r}_{i,[N]}^\varepsilon$  the restriction of  $\mathbf{r}_{[N]}^\varepsilon$  respectively to  $\Omega \setminus \bar{\omega}$  and to  $\omega_i^\varepsilon$ . The remainders  $(\mathbf{r}_{e,[N]}^\varepsilon, \mathbf{R}_{i,[N]}^\varepsilon)$  solve the thin layer transmission problem with right-hand sides up to  $O(\varepsilon^{N-1})$ . Using a judicious rewriting of  $(\mathbf{r}_{e,[N]}^\varepsilon, \mathbf{R}_{i,[N]}^\varepsilon)$  (see [2, subsection 2.2] or [10, Subsection 1.3.5]) and change of variable formula for  $s = \varepsilon S$  to evaluate  $\mathbf{H}^1$ -norms, we get the estimates

$$\|\mathbf{r}_{e,[N]}^\varepsilon\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} \leq C \varepsilon^{N+1} \quad \text{and} \quad \sqrt{\varepsilon} \|\mathbf{r}_{i,[N]}^\varepsilon\|_{\mathbf{H}^1(\omega_i^\varepsilon)} \leq C \varepsilon^{N+1}, \quad (3.1)$$

with  $C$  depending on  $N$  but independent of  $\varepsilon$  (thanks to the uniform coercivity of the bilinear form associated to the transmission problem).

• Secondly, we focus on  $\mathbf{d}_{[N]}^\varepsilon := \mathbf{u}_{e,[N]}^\varepsilon - \mathbf{v}_{[N]}^\varepsilon$ . For  $N = 0$ , we have  $\mathbf{u}_{e,[0]}^\varepsilon = \mathbf{u}_e^0 = \mathbf{v}_{[0]}^\varepsilon$ , i.e.  $\mathbf{d}_{[0]}^\varepsilon = \mathbf{0}$ . Then the estimate is deduced from the previous result. To get optimal results, for  $N \geq 1$ , the trick is different than in the Neumann case [2, subsection 2.2] and is well explained in [10, page 36]. Indeed, using the bounds provided by the Lax-Milgram theorem applied to the variational formulations (2.5) do not lead to optimal estimates due to the factor  $\varepsilon^{-1}$  in the right hand side. Instead, we use an asymptotic expansion of the solution  $\mathbf{v}_{[N]}^\varepsilon$  to the GIBC problems (1.2). We write  $\mathbf{v}_{[N]}^\varepsilon = \sum_{\ell=0}^N \varepsilon^\ell \mathbf{v}_N^\ell$ , where the coefficient functions  $\mathbf{v}_N^\ell$  are defined iteratively as

the solution to the following mixed boundary value problems with Dirichlet type condition on  $\Gamma$ . For example, when  $N = 1, 2$  we have

$$\begin{cases} \operatorname{div}(A_e e(\mathbf{v}_1^\ell)) = \delta_\ell^0 \mathbf{f} & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{v}_1^\ell = \mathbf{0} & \text{on } \Gamma_D \\ \mathbf{T}_e(\mathbf{v}_1^\ell) = \delta_\ell^0 \mathbf{g} & \text{on } \Gamma_N \\ \mathbf{v}_1^\ell = -\Lambda_0^{-1} \mathbf{T}_e \mathbf{v}_1^{\ell-1} & \text{on } \Gamma. \end{cases} \quad (3.2)$$

and

$$\begin{cases} \operatorname{div}(A_e e(\mathbf{v}_2^\ell)) = \delta_\ell^0 \mathbf{f} & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{v}_2^\ell = \mathbf{0} & \text{on } \Gamma_D \\ \mathbf{T}_e(\mathbf{v}_1^\ell) = \delta_\ell^0 \mathbf{g} & \text{on } \Gamma_N \\ \mathbf{v}_2^\ell = -\Lambda_0^{-1} \left( \mathbf{T}_e \mathbf{v}_2^{\ell-1} + \frac{1}{2} (\mu_i \mathcal{H} + (\lambda_i + \mu_i) \mathcal{R} + (\lambda_i - \mu_i) \mathcal{M}) \mathbf{v}_2^{\ell-1} \right) & \text{on } \Gamma. \end{cases} \quad (3.3)$$

We observe that

$$\forall \ell = 0, \dots, N, \quad \mathbf{v}_N^\ell = \mathbf{u}_e^\ell.$$

Higher order coefficient function  $\mathbf{v}_N^\ell$ , with  $\ell \geq N+1$ , surely differs from  $\mathbf{u}_e^\ell$ . Thus, by applying Lax-Milgram theorem to the mixed boundary value problem with a Dirichlet-type boundary condition on  $\Gamma$ , we get the estimates

$$\|\mathbf{v}_{[1]}^\varepsilon - \mathbf{u}_{e,[1]}^\varepsilon\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} = \|\mathbf{v}_{[1]}^\varepsilon - \mathbf{v}_1^0 - \varepsilon \mathbf{v}_1^1\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} \leq C_1 \varepsilon^2,$$

and

$$\|\mathbf{v}_{[2]}^\varepsilon - \mathbf{u}_{e,[2]}^\varepsilon\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} = \|\mathbf{v}_{[2]}^\varepsilon - \mathbf{v}_2^0 - \varepsilon \mathbf{v}_2^1 - \varepsilon^2 \mathbf{v}_2^2\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} \leq C_2 \varepsilon^3.$$

And so on, we get for any  $N \in \mathbb{N}$ ,  $\|\mathbf{v}_{[N]}^\varepsilon - \mathbf{u}_{\varepsilon,[N]}^\varepsilon\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} \leq C_N \varepsilon^{N+1}$ , with  $C_N$  independant of  $\varepsilon$ . Using the triangular inequality and (3.1), we get the announced estimates. ■

**Remark 3.2.** *The convergence estimates at any order  $N \in \mathbb{N}$  are obtained assuming analyticity of the boundary  $\Gamma$  and smoothness of the data around the thin layer. However, a boundary  $\Gamma$  of class  $\mathcal{C}^{1,1}$  and  $\mathbf{f} \in \mathbf{L}_{\Gamma_D}^2(\Omega \setminus \bar{\omega})$  are sufficient to get the previous estimates for  $N = 1$ .*

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