# $(\mu, \nu)$–pseudo almost automorphic solutions of neutral type Clifford-valued high-order Hopfield neural networks with D operator

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# $(\mu, \nu)-$pseudo almost automorphic solutions of neutral type Clifford-valued high-order Hopfield neural networks with D operator

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**Abstract** The aim of this article is to study neutral type Clifford-valued high-order Hopfield neural networks with mixed delays and $D$ operator.
New criteria are established for the existence, uniqueness and global exponential stability of $(\mu, \nu)-$pseudo almost automorphic solutions of the considered model via Banach's fixed point principle and differential inequality techniques. An example is given to show the effectiveness of the main new criteria.

**Keywords** Clifford algebra · High-order Hopfield neural network · $(\mu, \nu)-$pseudo almost automorphic function · $D$ operator

## 1 Introduction

Artificial Neural Networks (NNs) are a computational technique that belongs to the field of Machine Learning (ML). Their goal is to achieve a fairly simplified model of the brain. High-Order Hopfield Neural Network (HOHNN)

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is one of the most powerful and efficient types of NNs. The key factors that affect its success are its strong approximation ability, its fast convergence rate and its high fault tolerance capability. HOHNNs have widespread applications in various fields such as associative memory, pattern recognition, signal processing, robotics, medical image edge detection, medical event detection in electronic health records, diagnosis prediction in health care and many others. The study of high-order NNs has attracted considerable multidisciplinary research. For instance, the exponential convergence of high-order cellular NNs (CNNs) with time-varying leakage delays has been obtained in [28]; the authors of [14] discussed the existence and uniqueness of pseudo almost periodic solutions of high-order type of NNs; the authors of [29] studied the existence and exponential stability of the weighted pseudo almost periodic solutions of high-order cellular NNs with mixed delays; the dynamics of the pseudo almost automorphic solutions of HOHNNs with mixed delays has been investigated in [3]; to name but a few.

One can easily see that all the NN models considered in the above-mentioned works are under time-delay effect. The investigation of delayed NNs has become an interesting world-wide focus and several types of delays such as discrete, distributed, proportional and leakage delay have been used, see for instance [4,5,8,26]. To precisely describe the dynamics of complex neural reactions, systems must contain information about the derivative of the past states. Here, we are talking about another type called "neutral-type delay". It should be mentioned that neutral type NN models can be classified into two categories: Non-Operator-Based Neutral Functional Differential Equations (NOBNFDEs) and $D$-Operator-Based Neutral Functional Differential Equations (DOBNFDEs). It is important to notice that neutral type NNs with $D$ operator have more realistic significance than non-operator-based ones in many practical applications of NN dynamics. Recently, new success stories of neutral type NNs with $D$ operator have been provided. In [16], the global exponential stability of the anti-periodic solutions for neutral type cellular NNs with $D$ operator has been studied. In [30], the anti-periodic solutions for neutral shunting inhibitory cellular NNs with time-varying delays and $D$ operator have been investigated. Reference [31] dealt with the global convergence of CNNs with neutral type delays and $D$ operator. In [32], the authors analyzed the global exponential convergence of neutral type shunting inhibitory cellular NNs with $D$ operator. Zhang studied the oscillation dynamics of almost periodic solutions for shunting inhibitory CNNs with neutral type proportional delays and $D$ operator in [33] and extended the results to the pseudo almost periodic solutions of the same model in [34].

On the one hand, to follow real phenomena in biological systems, researchers have proposed several classes of functions such as the class of Almost Automorphic (AA) functions in [27], the class of Pseudo Almost Automorphic (PAA) functions in [17] and the class of Weighted Pseudo Almost Automorphic (WPAA) functions which have extended to the class of $(\mu,\nu)-$Pseudo Almost Automorphic ($(\mu,\nu)-$PAA) functions [2]. $(\mu,\nu)-$PAA functions have been rarely used in NN theory where the main task consists of finding an an-

swer to the following problem: "what will be the nature of output when all the parameters of the NN model are $(\mu, \nu)-$PAA functions?". In [7], we found an answer to this problem by studying the dynamic oscillations of $(\mu, \nu)-$PAA solutions of bidirectional associative memory NNs. On the other hand, Clifford introduced Clifford algebra in the 19th century. It is important associative algebras within the theories of quadratic forms, orthogonal groups and theoretical physics. As an extension of real value models, Clifford-value NNs have become active research domain due to their powerful applications in many fields such as neural computing, robotic vision, image processing, control problems and other areas. Success stories of Clifford-value NNs are reported in the following. The existence and global exponential stability of the equilibrium point of Clifford-valued recurrent NNs have been studied in [37]; sufficient conditions ensuring the existence and global stability of Clifford-valued NNs with time-varying delays have been derived in [23, 21]; the globally asymptotic almost automorphic synchronization of almost automorphic solutions of Clifford-valued recurrent NNs with mixed delays is developed in [20].

In this brief, we tried to manipulate neutral type HOHNNs in Clifford algebra with $(\mu, \nu)-$PAA parameters. To our best knowledge, there are no public results considering the dynamic behavior of $(\mu, \nu)-$PAA solutions of neutral type Clifford-valued NNs.

The main aim of this work is to obtain new sufficient conditions for ensuring the existence and global exponential stability of $(\mu, \nu)-$pseudo almost automorphic solutions of neutral-type Clifford-valued HOHNNs. Most of the published articles on neutral type NNs focused on the first-order systems and analyzed real-valued, complex-valued and quaternion-valued NNs. So, this article brings several advancements listed below:

1. The study of the existence and uniqueness of the $(\mu, \nu)-$pseudo almost automorphic solutions for neutral type HOHNN.
2. The analysis of the global exponential stability of the $(\mu, \nu)-$pseudo almost automorphic solutions for the considered model without using the Lyapunov functional method.
3. The class of $(\mu, \nu)-$PAA functions covers larger classes of functions that are very sophisticated and difficult to handle. We generalize many earlier publications [3, 5, 14].
4. The parameters are $(\mu, \nu)-$PAA functions which have been considered in Clifford algebra for the first time in such context. Some previous works in the literature are significantly extended and complemented, such as [18, 19, 22].
5. Via direct method, we study the $(\mu, \nu)-$PAA solutions for Clifford-valued HOHNNs without decomposing them into real-valued systems. Compared with real-valued, complex-valued and quaternion-valued HOHNNs [24, 25], the dynamical behaviors of Clifford-valued HOHNNs are the most complicated.

The outline of this paper is arranged as follows. In Section 2, we establish useful definitions, assumptions and lemmas. Section 3 is devoted to establish

new criteria for the existence, uniqueness and global exponential stability of $(\mu,\nu)-$PAA solutions of HOHNNs. In section 4, a numerical example is given to illustrate the feasibility of the obtained results. Conclusion and meaningful remarks are drawn in Section 5.

## 2 Preliminaries

### 2.1 Real Clifford algebra

In this subsection, we recall some results about real Clifford algebra. For more details, the reader may refer to [13] and the references therein. Let us denote $\mathbb{R}^m$ the $m-$dimensional real vector space. The real Clifford algebra over $\mathbb{R}^m$ is defined as

$$\mathcal{A}=\left\{\sum_{A\subseteq\{1\cdots m\}} a^A e_A;\ a^A\in\mathbb{R}\right\},\ \text{where } e_A=e_{h_1}\cdots e_{h_\zeta} \text{ with } A=\left\{h_1\cdots h_\zeta\right\},$$
$$1\leq h_1<h_2<\cdots<h_\zeta\leq m \text{ and } 1\leq\zeta\leq m.$$

$\mathcal{A}$ equipped with $m$ generators is defined as the Clifford algebra over the real number $\mathbb{R}$ with $m$ multiplicative generators $e_1,\cdots,e_m$ such that $e_i\in\mathbb{R}^m$, $e_\emptyset=e_0=1$, $e_0^2=1$ and

$$\begin{cases} e_0e_i=e_ie_0=e_i, & i=1,2,\cdots,m,\\ e_ie_j+e_je_i=0, & i\neq j,\ i,j\in\{1,\cdots,m\},\\ e_i^2=-1, & i=1,2,\cdots,m.\end{cases}$$

When one element is the product of multiple Clifford generators, we will write its subscripts together such as $e_{h_1}e_{h_2}=e_{h_1h_2}$ and $e_{h_1}e_{h_2}e_{h_5}=e_{h_1h_2h_5}$. It is easy to see that $\dim_{\mathbb{R}}\mathcal{A}=\sum\limits_{k=0}^{m}\binom{m}{k}=2^m$. We also define the norm on $\mathcal{A}$ by

$$\|x\|_{\mathcal{A}}=\max_{1\leq i\leq n}\{|x^A|\},\ \text{for } x=\sum_A x^Ae_A\in\mathcal{A},$$

and the norm on $\mathcal{A}^n$ by

$$\|x\|_{\mathcal{A}^n}=\max_{1\leq i\leq n}\{\|x_i\|_{\mathcal{A}}\},\ \text{for } x=(x_1,x_2,\cdots,x_n)^T\in\mathcal{A}^n.$$

In the following, $\mathcal{A}^n$ denotes the $n-$dimensional real Clifford vector space.

### 2.2 Model description

In this article, we deal with the following neutral type Clifford-valued HOHNNs with mixed delays:

$$\begin{aligned}\Big[x_i(t)-q_i(t)x_i(t-r_i(t))\Big]' &= -c_i(t)x_i(t)+\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t-\tau_{ij}(t)))\\ &+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(t)g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_l(t-\nu_{ijl}(t)))\\ &+\sum_{j=1}^{n}\beta_{ij}(t)\int_0^{+\infty}G_{ij}(s)h_j(x_j(t-s))ds\\ &+\sum_{j=1}^{n}\sum_{l=1}^{n}p_{ijl}(t)\int_0^{+\infty}P_{ijl}(t-s)k_j(x_j(s))ds\\ &\times\int_0^{+\infty}Q_{ijl}(t-s)k_l(x_l(s))ds+I_i(t), \qquad (1)\end{aligned}$$

in which $n$ corresponds to the number of units in the NN, $x_i(\cdot)\in\mathcal{A}$ corresponds to the state vector of the $i^{th}$ unit, $c_i(\cdot)$ represents the rate with which the $i^{th}$ unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $q_i(\cdot)\in\mathcal{A}$ is the connection weights, $a_{ij}(\cdot),\ \beta_{ij}(\cdot)\in\mathcal{A}$ are the synaptic connection weight of the $j^{th}$ unit on the $i^{th}$ unit, $\alpha_{ijl}(\cdot),\ p_{ijl}(\cdot)\in\mathcal{A}$ represent the second-order synaptic weights of the NNs, $f_j(\cdot),\ g_j(.),\ h_j(\cdot),\ k_j(\cdot)\in\mathcal{A}$ represent the activation functions of signal transmission, $G_{ij}(\cdot),\ P_{ijl}(\cdot),\ Q_{ijl}(\cdot)\in\mathcal{A}$ are the transmission delay kernels, $r_i(\cdot),\ \tau_{ij}(\cdot),\ \sigma_{ijl}(\cdot),\ \nu_{ijl}(\cdot)\in\mathbb{R}^+$ are the transmission delays, $I_i(\cdot)\in\mathcal{A}$ denotes the external inputs. The initial conditions associated with (1) are of the form:

$$x_i(s)=\phi_i(s),\ s\in(-\infty,0], \qquad i=1,2,\ldots,n, \qquad (2)$$

where $\phi_i\in C\Big((-\infty,0],\mathcal{A}\Big)$ which is the set of continuous functions from $(-\infty,0]$ to $\mathcal{A}$.

*Remark 1* In (1), the functions $r_i(\cdot),\ \tau_{ij}(\cdot),\ \sigma_{ijl}(\cdot)$ and $\nu_{ijl}(\cdot)$ correspond to the transmission delays. In fact, time-delays exist in most NN systems because neurons cannot communicate or respond instantaneously. Sometimes they make the dynamic behaviors more complex and may destabilize the stable equilibria (see [1,6,11,12]).

### 2.3 Notations and definitions

**Definition 1 ([17])** A continuous function $f:\mathbb{R}\to\mathbb{R}^n$ is called almost automorphic if for every real sequence $(S_n)_{n\in\mathbb{N}}$, there exists a subsequence

$(s_n)_{n\in\mathbb{N}}$ such that $g(t) = \lim_{n\to\infty} f(t+s_n)$ is well defined for each $t \in \mathbb{R}$ and $\lim_{n\to\infty} g(t-s_n) = f(t)$ for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{R},\mathbb{R}^n)$ the set of all such functions.

**Definition 2 ([20])** Let $f = (f_1, f_2, \cdots, f_n)^T : \mathbb{R} \mapsto \mathcal{A}^n$ where $f_i = \sum_A f_i^A e_A$. $f^A : \mathbb{R} \mapsto \mathbb{R}$ is called almost automorphic if for every $i = 1, \cdots, n$ we have $f_i^A \in AA(\mathbb{R},\mathbb{R}^n)$. Denote by $AA(\mathbb{R},\mathcal{A}^n)$ the set of all such functions.

Let $B$ the Lebesgue $\sigma$-field of $\mathbb{R}$, $\mathcal{M}$ denotes the set of all positive measures $\mu$ on $B$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$ for all $a, b \in \mathbb{R}$ $(a \le b)$.

**Definition 3** For $\mu, \nu \in \mathcal{M}$, the measures $\mu$ and $\nu$ are said to be equivalent if there exist constants $a_0$, $a_1 > 0$ and a bounded interval $\Omega \subset \mathbb{R}$ such that

$$a_0\nu(A) \le \mu(A) \le a_1\nu(A)$$

for all $A \in B$ satisfying $A \cap \Omega = \emptyset$.

Now, we introduce a new concept of ergodicity, which generalizes those previously given in the literature.

**Definition 4** Let $\mu, \nu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \mapsto \mathcal{A}^n$ is said to be $(\mu,\nu)$ ergodic if

$$\lim_{r\longrightarrow+\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} \| f(t) \|_{\mathcal{A}}\, d\mu(t) = 0.$$

We denote the collection of all such functions by $\xi(\mathbb{R},\mathcal{A}^n,\mu,\nu)$.

Let us denote $BC(\mathbb{R},\mathcal{A}^n)$ the set of bounded continued functions from $\mathbb{R}$ to $\mathcal{A}^n$, then $(BC(\mathbb{R},\mathcal{A}^n), \|\cdot\|_*)$ is a Banach space where $\|\cdot\|_*$ is the norm

$$\| f \|_* := \sup_{t\in\mathbb{R}} \max_{1\le i\le n} \{\|f_i(t)\|_{\mathcal{A}}\}.$$

**Definition 5 ([7])** Let $\mu, \nu \in \mathcal{M}$, $f \in BC(\mathbb{R},\mathcal{A}^n)$ is $(\mu,\nu)$-pseudo almost automorphic if it can be expressed as

$$f = f_1 + f_2$$

where $f_1 \in AA(\mathbb{R},\mathcal{A}^n)$ and $f_2 \in \xi(\mathbb{R},\mathcal{A}^n,\mu,\nu)$. The collection of such functions will be denoted by $PAA(\mathbb{R},\mathcal{A}^n,\mu,\nu)$.

The following assumptions are fundamental in this function space:
$(A_1)$ For all $\mu, \nu \in \mathcal{M}$, we have $\limsup_{n\to\infty} \frac{\mu([-r,r])}{\nu([-r,r])} < \infty$.
$(A_2)$ For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval $I$ such that

$$\mu(\{a+\tau : a \in A\}) \le \beta\mu(A)$$

when $A \in B$ satisfies $A \cap I = \emptyset$.

Let us state two useful theorems proved in [7] as follows.

**Theorem 1 ([7])** *Let $\mu, \nu \in \mathcal{M}$ satisfy $(A_2)$. Then the decomposition of a $(\mu,\nu)$-pseudo almost automorphic function of the form $f = f_1 + f_2$ where $f_1 \in AA(\mathbb{R}, \mathcal{A}^n)$ and $f_2 \in \xi(\mathbb{R}, \mathcal{A}^n, \mu, \nu)$ is unique.*

**Theorem 2 ([7])** *Let $\mu, \nu \in \mathcal{M}$ satisfy $(A_1)$ and $(A_2)$. Then $PAA(\mathbb{R}, \mathcal{A}^n, \mu, \nu)$ is a Banach space.*

## 2.4 Technical lemmas

For $1 \leq i, j, l \leq n$, we denote $q_i^* = \sup_{t\in\mathbb{R}} \|q_i(t)\|_{\mathcal{A}}$, $c_i^* = \sup_{t\in\mathbb{R}} \|c_i(t)\|_{\mathcal{A}}$, $c_{i*} = \inf_{t\in\mathbb{R}} \|c_i(t)\|_{\mathcal{A}}$, $a_{ij}^* = \sup_{t\in\mathbb{R}} \|a_{ij}(t)\|_{\mathcal{A}}$, $\alpha_{ijl}^* = \sup_{t\in\mathbb{R}} \|\alpha_{ijl}(t)\|_{\mathcal{A}}$, $\beta_{ij}^* = \sup_{t\in\mathbb{R}} \|\beta_{ij}(t)\|_{\mathcal{A}}$, $p_{ijl}^* = \sup_{t\in\mathbb{R}} \|p_{ijl}(t)\|_{\mathcal{A}}$, $I_{ij}^* = \sup_{t\in\mathbb{R}} \|I_{ij}(t)\|_{\mathcal{A}}$,$r_i^* = \sup_{t\in\mathbb{R}} r_i(t)$,$\tau_{ij}^* = \sup_{t\in\mathbb{R}} \tau_{ij}(t)$, $\sigma_{ijl}^* = \sup_{t\in\mathbb{R}} \sigma_{ijl}(t)$, $\nu_{ijl}^* = \sup_{t\in\mathbb{R}} \nu_{ijl}(t)$.
Moreover, assume that for all $1 \leq i, j, l \leq n$, we have

$$c_i(\cdot),\ r_i(\cdot),\ \tau_{ij}(\cdot),\ \sigma_{ijl}(\cdot),\ \nu_{ijl}(\cdot) \in AA(\mathbb{R}, \mathbb{R}^n)$$

and

$$q_i(\cdot),\ a_{ij}(\cdot),\ \alpha_{ijl}(\cdot),\ \beta_{ij}(\cdot),\ p_{ijl}(\cdot),\ I_i(\cdot) \in PAA(\mathbb{R}, \mathcal{A}^n, \mu, \nu).$$

**Assumption 1** *For all $1 \leq j \leq n$, and all $u, v \in \mathbb{R}$ there exist nonnegative constants $L_j^f$, $L_j^g$, $L_j^h$, $L_j^k$, $d_j^g$, $d_j^k$ such that*

$$\begin{array}{llll} ||f_j(u) - f_j(v)||_{\mathcal{A}} \leq L_j^f||u-v||_{\mathcal{A}}, & ||g_j(u) - g_j(v)||_{\mathcal{A}} \leq L_j^g||u-v||_{\mathcal{A}}, \\ ||g_j(u)||_{\mathcal{A}} \leq d_j^g, & ||h_j(u) - h_j(u)||_{\mathcal{A}} \leq L_j^h||u-v||_{\mathcal{A}}, \\ ||k_j(u) - k_j(u)||_{\mathcal{A}} \leq L_j^k||u-v||_{\mathcal{A}}, & ||k_j(u)||_{\mathcal{A}} \leq d_j^k. \end{array}$$

*For simplicity of calculation and without loss of generality, we assume that $f_j(0) = g_j(0) = h_j(0) = 0$.*

**Assumption 2** *$G_{ij} : [0, +\infty) \to \mathbb{R}$ is continuous and $|G_{ij}(t)|e^{\kappa_1 t}$ is integrable on $[0, +\infty)$ for a certain positive constant $\kappa_1$. $P_{ijl} : [0, +\infty) \to \mathbb{R}$ is continuous and $|P_{ijl}(t)|e^{\kappa_2 t}$ is integrable on $[0, +\infty)$ for a certain positive constant $\kappa_2$. $Q_{ijl} : [0, +\infty) \to \mathbb{R}$ is continuous and $|Q_{ijl}(t)|e^{\kappa_3 t}$ is integrable on $[0, +\infty)$ for a certain positive constant $\kappa_3$.*

The following Lemma 1 and Lemma 2 are proved in [7].

**Lemma 1 ([7])** *If $\phi(\cdot) \in PAA(\mathbb{R}, \mathcal{A}, \mu, \nu)$ then we have $\phi(\cdot - a) \in PAA(\mathbb{R}, \mathcal{A}, \mu, \nu)$.*

**Lemma 2 ([7])** *If $\varphi, \psi \in PAA(\mathbb{R}, \mathcal{A}, \mu, \nu)$ then we have $\varphi \times \psi \in PAA(\mathbb{R}, \mathcal{A}, \mu, \nu)$.*

By using Lemma 1 and Lemma 2, it is possible to prove the following lemmas.

**Lemma 3 ([7])** *If $f(\cdot) \in C(\mathcal{A}, \mathcal{A})$ satisfies the $l_f$-Lipschitz condition, $\phi(\cdot) \in PAA(\mathbb{R}, \mathcal{A}, \mu, \nu)$ and $\tau(\cdot) \in AA(\mathbb{R}, \mathcal{A}, \mu, \nu)$ then we have $f(\phi(\cdot - \tau(\cdot))) \in PAA(\mathbb{R}, \mathcal{A}, \mu, \nu)$.*

**Lemma 4 ([7])** *Assume that Assumptions 1–2 hold. For all* $1 \leq i, j \leq n$, *if* $\phi_j(\cdot) \in PAA(\mathcal{A}, \mathcal{A}, \mu, \nu)$ *then we have*

$$t \mapsto \int_0^{+\infty} G_{ij}(s) h_j(\phi_j(\cdot - s)) ds \in PAA(\mathbb{R}, \mathcal{A}, \mu, \nu).$$

**Definition 6** Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ be a $(\mu, \nu)-$pseudo almost automorphic solution of the system (1) with the initial value $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))^T \in C\Big((-\infty, 0], \mathcal{A}\Big)$ and $y(t) = (y_1(t), \ldots, y_n(t))^T$ be an arbitrary solution of the system (1) with the initial value $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))^T \in C\Big((-\infty, 0], \mathcal{A}\Big)$ respectively. If there exist positive constants $\lambda$ and $N$ such that

$$\|x(t) - y(t)\|_{\mathcal{A}^n} \leq N \|\phi\|_{\epsilon} e^{-\lambda t}, \forall t > 0,$$

where

$$\|\phi\|_{\varepsilon} = \max_{1 \leq i \leq n} \sup_{t \in (-\infty, 0]} \Big\| [\phi_i(t) - d_i(t)\phi_i(t - \tau_i(t))] - [x_i^*(t) - d_i(t) x_i^*(t - \tau_i(t))] \Big\|_{\mathcal{A}}$$

$(\mu, \nu)-$pseudo almost automorphic solution of the system (1) is said to be globally exponentially stable.

**Assumption 3** *For each* $1 \leq i \leq n$, $t \in \mathbb{R}$,

$$M[c_i] = \lim_{T \to +\infty} \frac{1}{T} \int_t^{t+T} c_i(s) ds > 0$$

*and there exist a bounded and continuous function* $\tilde{c}_i : \mathbb{R} \to (0, +\infty)$ *and a positive constant* $K_i$ *such that*

$$e^{-\int_s^t c_i(u) du} \leq K_i e^{-\int_s^t \tilde{c}_i(u) du}, \text{ for all } t, s \in \mathbb{R}, \ t - s \geq 0.$$

**Lemma 5** *For* $\phi = (\phi_1, \ldots, \phi_n)^T \in PAA(\mathbb{R}, \mathcal{A}^n, \mu, \nu)$, *we define the nonlinear operator* $\Theta = \Theta_\phi$ *as follows*

$$\Theta_\phi(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_s^t c_1(u) du} F_1(s) ds \\ \vdots \\ \int_{-\infty}^{t} e^{-\int_s^t c_n(u) du} F_n(s) ds \end{pmatrix}$$

*where*

$$\begin{aligned}F_i(s) &= c_i(s)q_i(s)\phi_i(s-r_i(s))\\ &+\sum_{j=1}^{n}a_{ij}(s)f_j(\phi_j(s-\tau_{ij}(s)))+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)g_j(\phi_j(s-\sigma_{ijl}(s)))g_l(\phi_l(s-\nu_{ijl}(s)))\\ &+\sum_{j=1}^{n}\beta_{ij}(s)\int_0^{+\infty}G_{ij}(m)h_j(\phi_j(s-m))dm\\ &+\sum_{j=1}^{n}\sum_{l=1}^{n}p_{ijl}(s)\int_0^{+\infty}P_{ijl}(m)k_j(\phi_j(s-m))dm\int_0^{+\infty}Q_{ijl}(m)k_l(\phi_l(s-m))dm+I_i(s).\end{aligned}$$

*Then $\Theta_\phi$ maps $PAA(\mathbb{R},\mathcal{A}^n,\mu,\nu)$ into itself.*

*Proof* Let $\phi\in PAA(\mathbb{R},\mathcal{A},\mu,\nu)$. Using Lemma 3, we obtain $f_j(\phi_j(s-\tau_{ij}(s)))$, $g_j(\phi_j(s-\sigma_{ijl}(s)))$ and $g_l(\phi_l(s-\nu_{ijl}(s)))$ in $PAA(\mathbb{R},\mathcal{A},\mu,\nu)$. By using Lemma 4, we have

$$\int_0^{+\infty}G_{ij}(m)h_j(x_j(s-m))dm,\ \int_0^{+\infty}P_{ijl}(m)k_j(x_j(s-m))dm \text{ and } \int_0^{+\infty}Q_{ijl}(m)k_l(x_l(s-m))dm$$

in $PAA(\mathbb{R},\mathcal{A},\mu,\nu)$. Then, $F_i$ is a $(\mu,\nu)-$PAA function. Via Theorem 1, we have $F_i=F_i^1+F_i^2$ with $F_i^1\in AA(\mathbb{R},\mathcal{A}^n)$ and $F_i^2\in\xi(\mathbb{R},\mathcal{A}^n,\mu,\nu)$.
Noting that $M[c_i]>0$ from Assumption 3 and using the theory of exponential dichotomy, we obtain that

$$\int_{-\infty}^{t}e^{-\int_s^t c_i(u)du}F_i^1(s)ds\in AA(\mathbb{R},\mathcal{A}^n) \tag{3}$$

is a solution of the following almost automorphic differential equation

$$\dot{y}(t)=-c_i(t)y(t)+F_i^1(t),\qquad 1\le i,j\le n.$$

Now, let us show that

$$\int_{-\infty}^{t}e^{-\int_s^t c_i(u)du}F_i^2(s)ds\in\xi(\mathbb{R},\mathcal{A}^n,\mu,\nu).$$

From Assumption 3, one has

$$\begin{aligned}\lim_{r\longrightarrow\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\|F_i^2(t)\|_{\mathcal{A}}d\mu(t)&=\lim_{r\longrightarrow\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\left\|\int_{-\infty}^{t}e^{-\int_s^t c_i(u)du}F_i^2(s)ds\right\|_{\mathcal{A}}d\mu(t)\\ &\le\lim_{r\longrightarrow\infty}\frac{K_i}{\nu([-r,r])}\int_{[-r,r]}\int_{-\infty}^{t}e^{-(t-s)\tilde{c}_{i*}}\|F_i^2(s)\|_{\mathcal{A}}dsd\mu(t).\end{aligned}$$

Let

$$E_1 = \lim_{r\longrightarrow\infty}\frac{K_i}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-r}^{t}\|e^{-(t-s)\tilde{c}_{i*}}F_i^2(s)\|_{\mathcal{A}}ds\right)d\mu(t),$$

$$E_2 = \lim_{r\longrightarrow\infty}\frac{K_i}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{-r}e^{-(t-s)\tilde{c}_{i*}}\|F_i^2(s)\|_{\mathcal{A}}ds\right)d\mu(t)$$

and $m=t-s$, then by Fubini's theorem we obtain

$$\begin{aligned}
E_1 &= \lim_{r\longrightarrow\infty}\frac{K_i}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-r}^{t}e^{-(t-s)\tilde{c}_{i*}}\|F_i^2(s)\|_{\mathcal{A}}ds\right)d\mu(t)\\
&\leq \lim_{r\longrightarrow\infty}\frac{K_i}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{0}^{+\infty}e^{-m\tilde{c}_{i*}}\|F_i^2(t-m)\|_{\mathcal{A}}dm\right)d\mu(t)\\
&\leq \int_{0}^{+\infty}e^{-m\tilde{c}_{i*}}\left(\lim_{r\longrightarrow\infty}\frac{K_i}{\nu([-r,r])}\int_{[-r,r]}\|F_i^2(t-m)\|_{\mathcal{A}}d\mu(t)\right)dm\\
&= \int_{0}^{+\infty}e^{-m\tilde{c}_{i*}}\left(\lim_{r\longrightarrow\infty}\frac{K_i}{\nu([-r,r])}\int_{-r-m}^{r-m}\|F_i^2(t)\|_{\mathcal{A}}d\mu_m(t)\right)dm\\
&\leq K_i\int_{0}^{+\infty}e^{-m\tilde{c}_{i*}}\left(\lim_{r\longrightarrow\infty}\frac{\nu([-r-m,r+m])}{\nu([-r,r]}\frac{\beta}{\nu([-r-m,r+m])}\right.\\
&\quad\left.\times\int_{[-r-m,r+m]}\|F_i^2(t)\|_{\mathcal{A}}d\mu(t)\right)dm.
\end{aligned}$$

On the one hand we have $F_i^2\in\xi(\mathbb{R},\mathbb{R},\mu,\nu)$ then $E_1=0$. On the other hand, we have

$$\begin{aligned}
E_2 &= \lim_{r\longrightarrow\infty}\frac{K_i}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{-r}\|e^{-(t-s)\tilde{c}_{i*}}F_i^2(s)\|_{\mathcal{A}}ds\right)d\mu(t)\\
&= \lim_{r\longrightarrow\infty}\frac{K_i}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{-r}e^{-(t-s)\tilde{c}_{i*}}\|F_i^2(s)\|_{\mathcal{A}}ds\right)d\mu(t)\\
&\leq \lim_{r\longrightarrow\infty}\frac{K_i}{\nu([-r,r])}\int_{-\infty}^{-r}e^{s\tilde{c}_{i*}}\|F_i^2(s)\|_{\mathcal{A}}ds\int_{-r}^{r}e^{-t\tilde{c}_{i*}}d\mu(t)\\
&= \lim_{r\longrightarrow\infty}K_i\frac{\|F_i^2\|_{\mathcal{A}}}{\tilde{c}_{i*}}e^{-2r\tilde{c}_{i*}}=0.
\end{aligned}$$

Combining with (3) it leads to $\Theta_\phi$ maps $PAA(\mathbb{R},\mathcal{A}^n,\mu,\nu)$ into itself. □

## 3 Main results

In this section, we establish new results for the existence, uniqueness and stability of $(\mu,\nu)-$PAA solution of the system (1). For $(\phi_1,\phi_2,\cdots,\phi_n)^T \in PAA(\mathbb{R},\mathcal{A}^n,\mu,\nu)$, we define the norm of $\phi$ as

$$\| \phi \|_*:= \max_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \Big\{ \|\phi_i(t)\|_{\mathcal{A}} \Big\}.$$

And we consider the following assumption:

**Assumption 4** *For $1 \leq i,j,l \leq n$, there exist strictly positive constants $M$, $\Lambda_i$ and $\Omega_i$ such that*

$$M = \max_{1\leq i\leq n} \left\{ \frac{K_i}{\tilde{c}_{i*}} I_i^* \right\}$$

$$\begin{aligned}
\Lambda = \max_{1\leq i\leq n} \Bigg\{ & q_i^* + \frac{K_i}{\tilde{c}_{i*}} \Bigg[ c_i^* q_i^* + \sum_{j=1}^{n} a_{ij}^* L_j^f + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}^* L_j^g L_l^g + \sum_{j=1}^{n} \beta_{ij}^* L_j^h \int_0^{+\infty} |G_{ij}(u)|du \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}^* L_j^k L_l^k \int_0^{+\infty} |P_{ijl}(u)|du \int_0^{+\infty} |Q_{ijl}(u)|du \Bigg] \Bigg\} < 1,
\end{aligned}$$

$$\begin{aligned}
\Omega = \max_{1\leq i\leq n} \Bigg\{ & q_i^* + \frac{K_i}{\tilde{c}_{i*}} \Bigg[ c_i^* q_i^* + \sum_{j=1}^{n} a_{ij}^* L_j^f + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}^* (L_j^g d_l^g + L_l^g d_j^g) \\
& + \sum_{j=1}^{n} \beta_{ij}^* L_j^h \int_0^{+\infty} |G_{ij}(u)|du + \sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}^* \int_0^{+\infty} |P_{ijl}(u)|du \int_0^{+\infty} |Q_{ijl}(u)|du (L_j^k d_l^k + L_l^k d_j^k) \Bigg] \Bigg\} \\
< 1 &
\end{aligned}$$

*and a strictly negative constant $\Upsilon_0$ such that*

$$\begin{aligned}
\Upsilon_0 = \sup_{t\in\mathbb{R}} \Bigg\{ & -\tilde{c}_i(t) + K_i \Bigg[ \frac{1}{1-q_i^*} c_i^* q_i^* + \sum_{j=1}^{n} a_{ij}^* L_j^f \frac{1}{1-q_j^*} \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}^* \left( L_j^g d_l^g \frac{1}{1-q_j^*} + L_l^g d_j^g \frac{1}{1-q_j^*} \right) + \sum_{j=1}^{n} \beta_{ij}^* L_j^h \int_0^{+\infty} |G_{ij}(u)|du \frac{1}{1-q_j^*} \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}^* \left[ L_j^k d_l^k \int_0^{+\infty} |P_{ijl}(u)|du + L_l^k d_j^k \int_0^{+\infty} |Q_{ijl}(u)|du \right] \frac{1}{1-q_j^*} \Bigg] \Bigg\}.
\end{aligned}$$

**Theorem 3** *Suppose that Assumptions 1–4 hold. Then the system (1) has only one $(\mu,\nu)-$PAA solution in the region*

$$S = \left\{ \phi \in PAA(\mathbb{R},\mathcal{A}^n,\mu,\nu) : \| \phi - \phi_0 \|_* \leq \frac{\Lambda}{1-\Lambda} M \right\},$$

*where*

$$\phi_0(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_s^t c_1(u)du} I_1(s)ds \\ \vdots \\ \int_{-\infty}^{t} e^{-\int_s^t c_n(u)du} I_n(s)ds \end{pmatrix}.$$

*which is globally exponentially stable.*

*Proof First part: existence and uniqueness of* $(\mu,\nu)-PAA$ *solutions*
Let

$$Y_i(t) = x_i(t) - q_i(t)x_i(t - r_i(t)), \text{ for all } 1 \le i \le n. \tag{4}$$

Then, we have

$$\begin{aligned} Y_i'(t) &= \Big[x_i(t) - q_i(t)x_i(t - r_i(t))\Big]' \\ &= -c_i(t)Y_i(t) - c_i(t)q_i(t)x_i(t - r_i(t)) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(t)g_j(x_j(t - \sigma_{ijl}(t)))g_l(x_l(t - \nu_{ijl}(t))) \\ &\quad + \sum_{j=1}^{n} \beta_{ij}(t) \int_0^{+\infty} G_{ij}(s)h_j(x_j(t - s))ds \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}(t) \int_0^{+\infty} P_{ijl}(s)k_j(x_j(t - s))ds \int_0^{+\infty} Q_{ijl}(s)k_l(x_l(t - s))ds \\ &\quad + I_i(t), \qquad i = 1, \dots, n. \end{aligned} \tag{5}$$

By using Lemma 5, we define an operator $\overline{\Theta}$ as follows:

$$\begin{aligned} \overline{\Theta}: \quad S \quad &\to S \\ (\phi_1 \cdots, \phi_n)^T &\mapsto (\overline{\Theta}_{\phi_1} \cdots, \overline{\Theta}_{\phi_n})^T \end{aligned}$$

such that $\overline{\Theta}_{\phi_i} = \{q_i(t)\phi_i(t - r_i(t))\} + \Theta_\phi(t)$ for all $\phi \in S$. One has

$$\begin{aligned} \|\phi_0\|_* &= \max_{1\le i\le n} \left\{ \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} I_i(s)ds \right\|_{\mathcal{A}} \right\} \\ &\le \max_{1\le i\le n} \left\{ K_i \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} e^{-\int_s^t \tilde{c}_i(u)du} I_i(s)ds \right\|_{\mathcal{A}} \right\} \\ &\le \max_{1\le i\le n} \left\{ \frac{K_i}{\tilde{c}_{i*}} I_i^* \right\} = M \end{aligned}$$

is the unique $(\mu,\nu)-$pseudo almost automorphic solution of the following differential equations:

$$Y_i'(t) = -c_i(t)Y_i(t) + I_i(t), \qquad i = 1, \dots, n.$$

The set $S = \left\{ \phi \in PAA(\mathbb{R}, \mathcal{A}^n, \mu, \nu) : \| \phi - \phi_0 \|_* \leq \frac{\Lambda}{1-\Lambda} M \right\}$ is a closed convex subset of $PAA(\mathbb{R}, \mathcal{A}^n, \mu, \nu)$. If $\phi \in S$, then

$$\begin{aligned} \| \phi \|_* &\leq \| \phi - \phi_0 \|_* + \| \phi_0 \|_* \\ &\leq \| \phi - \phi_0 \|_* + M. \end{aligned} \tag{6}$$

We claim that for any $\phi \in S$ the mapping $\overline{\Theta}_\phi \in S$. Note that

$$\begin{aligned} \left\| \overline{\Theta}_\varphi - \phi_0 \right\|_* = \max_{1\leq i\leq n} \Bigg\{ \sup_{t\in\mathbb{R}} \Bigg\| & q_i(t)\phi_i(t-r_i(t)) + \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} \Big[ - c_i(s)q_i(s)\phi_i(s-r_i(s)) \\ & + \sum_{j=1}^{n} a_{ij}(s) f_j(\phi_j(s-\tau_{ij}(s))) \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(s) g_j(\phi_j(s-\sigma_{ijl}(s))) g_l(\phi_l(s-\nu_{ijl}(s))) \\ & + \sum_{j=1}^{n} \beta_{ij}(s) \int_0^{+\infty} G_{ij}(u) h_j(\phi_j(s-u))du \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}(s) \int_0^{+\infty} P_{ijl}(u) k_j(\phi_j(s-u))du \int_0^{+\infty} Q_{ijl}(u) k_l(\phi_l(s-u))du \Big] ds \Bigg\|_{\mathcal{A}} \Bigg\} \end{aligned}$$

and then we have

$$\begin{aligned} \left\| \overline{\Theta}_\varphi - \phi_0 \right\|_* \leq \max_{1\leq i\leq n} \Bigg\{ & \sup_{t\in\mathbb{R}} \|q_i(t)\phi_i(t-r_i(t))\|_{\mathcal{A}} \\ & + \int_{-\infty}^{t} e^{-\int_s^t \tilde{c}_i(u)du} K_i \Big[ \sup_{t\in\mathbb{R}} \| - c_i(s)q_i(s)\|_{\mathcal{A}} \sup_{t\in\mathbb{R}} \|\phi_i(s-r_i(s))\|_{\mathcal{A}} \\ & + \sum_{j=1}^{n} \sup_{t\in\mathbb{R}} \|a_{ij}(s)\|_{\mathcal{A}} \sup_{t\in\mathbb{R}} \|f_j(\phi_j(s-\tau_{ij}(s)))\|_{\mathcal{A}} \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \sup_{t\in\mathbb{R}} \|\alpha_{ijl}(s)\|_{\mathcal{A}} \sup_{t\in\mathbb{R}} \|g_j(\phi_j(s-\sigma_{ijl}(s)))\|_{\mathcal{A}} \sup_{t\in\mathbb{R}} \|g_l(\phi_l(s-\nu_{ijl}(s)))\|_{\mathcal{A}} \\ & + \sum_{j=1}^{n} \sup_{t\in\mathbb{R}} \|\beta_{ij}(s)\|_{\mathcal{A}} \int_0^{+\infty} |G_{ij}(u)| \sup_{t\in\mathbb{R}} \|h_j(\phi_j(s-u))\|_{\mathcal{A}} du \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \sup_{t\in\mathbb{R}} \|p_{ijl}(s)\|_{\mathcal{A}} \int_0^{+\infty} |P_{ijl}(u)| \sup_{t\in\mathbb{R}} \|k_j(\phi_j(s-u))\|_{\mathcal{A}} du \\ & \times \int_0^{+\infty} |Q_{ijl}(u)| \sup_{t\in\mathbb{R}} \|k_l(\phi_l(s-u))\|_{\mathcal{A}} du \Big] ds \Bigg\}. \end{aligned}$$

It leads to

$$\begin{aligned}\left\|\overline{\Theta}_{\varphi}-\phi_0\right\|_* &\leq \max_{1\leq i\leq n}\Big\{q_i^*+\int_{-\infty}^{t} e^{-\int_s^t \tilde{c}_i(u)du}K_i\Big[c_i^*q_i^*+\sum_{j=1}^{n}a_{ij}^*L_j^f\\ &\quad+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}^*L_j^gL_l^g+\sum_{j=1}^{n}\beta_{ij}^*L_j^h\int_0^{+\infty}|G_{ij}(u)|du\\ &\quad+\sum_{j=1}^{n}\sum_{l=1}^{n}p_{ijl}^*L_j^kL_l^k\int_0^{+\infty}|P_{ijl}(u)|du\int_0^{+\infty}|Q_{ijl}(u)|du\Big]ds\Big\}\|\phi\|_*\\ &=\Lambda\|\phi\|_*.\end{aligned}$$

As $\Lambda<1$, it implies that $\overline{\Theta}_{\varphi}\in S$. Next, we prove that the mapping $\overline{\Theta}$ is a contraction mapping of $S$. For $\phi,\psi\in S$, we have

$$\begin{aligned}\left\|\overline{\Theta}_{\phi}-\overline{\Theta}_{\psi}\right\|_* &\leq \max_{1\leq i\leq n}\Big\{\sup_{t\in\mathbb{R}}\Big\|q_i(t)\Big[\phi_i(t-r_i(t))-\psi_i(t-r_i(t))\Big]\\ &\quad+\int_{-\infty}^{t}e^{-\int_s^t c_i(u)du}\Big[-c_i(s)q_i(s)\Big(\phi_i(s-r_i(s))+\psi_i(s-r_i(s))\Big)\\ &\quad+\sum_{j=1}^{n}a_{ij}(s)\Big(f_j(\phi_j(s-\tau_{ij}(s)))-f_j(\psi_j(s-\tau_{ij}(s)))\Big)\\ &\quad+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)\Big(g_j(\phi_j(s-\sigma_{ijl}(s)))g_l(\phi_l(s-\nu_{ijl}(s)))\\ &\quad-g_j(\psi_j(s-\sigma_{ijl}(s)))g_l(\psi_l(s-\nu_{ijl}(s)))\Big)\\ &\quad+\sum_{j=1}^{n}\beta_{ij}(s)\int_0^{+\infty}G_{ij}(u)\Big(h_j(\phi_j(s-u))-h_j(\psi_j(s-u)\Big)du\\ &\quad+\sum_{j=1}^{n}\sum_{l=1}^{n}p_{ijl}(s)\Big(\int_0^{+\infty}P_{ijl}(u)h_j(\phi(s-u))du\int_0^{+\infty}Q_{ijl}(u)k_l(\phi(s-u))du\\ &\quad-\int_0^{+\infty}P_{ijl}(u)h_j(\psi(s-u))du\int_0^{+\infty}Q_{ijl}(u)k_l(\phi(s-u))du\\ &\quad+\int_0^{+\infty}P_{ijl}(u)h_j(\psi(s-u))du\int_0^{+\infty}Q_{ijl}(u)k_l(\phi(s-u))du\\ &\quad-\int_0^{+\infty}P_{ijl}(u)h_j(\psi(s-u))du\int_0^{+\infty}Q_{ijl}(u)k_l(\psi(s-u))du\Big)\Big]ds\Big\|_{\mathcal{A}}\Big\}.\end{aligned}$$

It leads to

$$\begin{aligned}\left\|\overline{\Theta}_{\phi}-\overline{\Theta}_{\psi}\right\|_{*} \leq \max_{1\leq i\leq n}\Bigg\{q_i^{*}+\int_{-\infty}^{t} e^{-\int_{s}^{t}\tilde{c}_i(u)du}K_i\Bigg[c_i^{*}q_i^{*}+\sum_{j=1}^{n}a_{ij}^{*}L_j^{f}+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}^{*}(L_j^{g}d_l^{g}+L_l^{g}d_j^{g})\\ +\sum_{j=1}^{n}\beta_{ij}^{*}L_j^{h}\int_{0}^{+\infty}|G_{ij}(u)|du+\sum_{j=1}^{n}\sum_{l=1}^{n}p_{ijl}^{*}\int_{0}^{+\infty}|P_{ijl}(u)|du\\ \times\int_{0}^{+\infty}|Q_{ijl}(u)|du(L_j^{k}d_l^{k}+L_l^{k}d_j^{k})\Bigg]ds\Bigg\}\|\phi-\psi\|_{*}\end{aligned}$$

and then

$$\begin{aligned}\left\|\overline{\Theta}_{\phi}-\overline{\Theta}_{\psi}\right\|_{*} &\leq \max_{1\leq i\leq n}\Bigg\{q_i^{*}+\frac{K_i}{\tilde{c}_{i*}}\Bigg[c_i^{*}q_i^{*}+\sum_{j=1}^{n}a_{ij}^{*}L_j^{f}+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}^{*}\left(L_j^{g}d_l^{g}+L_l^{g}d_j^{g}\right)\\ &\quad+\sum_{j=1}^{n}\beta_{ij}^{*}L_j^{h}\int_{0}^{+\infty}|G_{ij}(u)|\,du+\sum_{j=1}^{n}\sum_{l=1}^{n}p_{ijl}^{*}\int_{0}^{+\infty}|P_{ijl}(u)|\,du\\ &\quad\times\int_{0}^{+\infty}|Q_{ijl}(u)|\,du(L_j^{k}d_l^{k}+L_l^{k}d_j^{k})\Bigg]\Bigg\}\|\phi-\psi\|_{*}\\ &=\Omega\|\phi-\psi\|_{*}.\end{aligned}$$

As $\Omega<1$, then we obtain that $\overline{\Theta}$ is a contraction. Via Theorem 2, $PAA(\mathbb{R},\mathcal{A}^n,\mu,\nu)$ is a Banach space. Then due to Banach's fixed point principle, $\overline{\Theta}$ possesses one and only one fixed point $x^{*}=\{x_i^{*}(t)\}\in S$, such that

$$\begin{aligned}x_i^{*}(t)&=\{q_i(t)x_i^{*}(t-r_i(t))\}+\Theta_{x^{*}}(t)\\ &=q_i(t)x_i^{*}(t-r_i(t))+\int_{-\infty}^{t}e^{-\int_{s}^{t}c_i(u)du}\Bigg[-c_i(s)q_i(s)x_i^{*}(s-r_i(s))\\ &\quad+\sum_{j=1}^{n}a_{ij}(s)f_j(x_j^{*}(s-\tau_{ij}(s)))+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)g_j(s-x_j^{*}(\sigma_{ijl}(s)))g_l(x_l^{*}(s-\nu_{ijl}(s)))\\ &\quad+\sum_{j=1}^{n}\beta_{ij}(s)\int_{0}^{+\infty}G_{ij}(u)h_j(x_j^{*}(s-u))du\\ &\quad+\sum_{j=1}^{n}\sum_{l=1}^{n}p_{ijl}(s)\int_{0}^{+\infty}P_{ijl}(u)k_j(x_j(s-u))du\int_{0}^{+\infty}Q_{ijl}(u)k_l(x_l(s-u))du+I_i(s)\Bigg]ds.\end{aligned}$$

Then it leads to

$$\begin{aligned}\Big[x_i^*(t)-q_i(t)x_i^*(t-r_i(t))\Big]' &= -c_i(t)x_i^*(t)+\sum_{j=1}^{n}a_{ij}(s)f_j(x_j^*(t-\tau_{ij}(t)))\\ &+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(t)g_j(x_j^*(t-\sigma_{ijl}(t)))g_l(x_l^*(t-\nu_{ijl}(t)))\\ &+\sum_{j=1}^{n}\beta_{ij}(t)\int_0^{+\infty}G_{ij}(u)h_j(x_j^*(t-u))du\\ &+\sum_{j=1}^{n}\sum_{l=1}^{n}p_{ijl}(t)\int_0^{+\infty}P_{ijl}(s)k_j(x_j^*(t-s))ds\\ &\times\int_0^{+\infty}Q_{ijl}(s)k_l(x_l^*(t-s))ds+I_i(t).\end{aligned}$$

Thus the system (1) has an unique $(\mu,\nu)-$PAA solution $x^*(t)$.

*Second part: global exponential stability of the $(\mu,\nu)-$PAA solution*
In view of the first part, the system (1) has only one $(\mu,\nu)-$pseudo almost automorphic solution denoted by $x^*(t)=(x_1^*(t),\ldots,x_n^*(t))^T$ and satisfying (2). Let $x(t)=(x_1(t),\cdots,x_n(t))^T$ be an arbitrary solution of the system (1) with initial value $\phi(t)=(\phi_1(t),\cdots,\phi_n(t))^T$ satisfying (2). Let

$$\begin{aligned}z_i(t)&=x_i(t)-x_i^*(t),\\ Z_i(t)&=\Big[x_i(t)-q_i(t)x_i(t-r_i(t))\Big]-\Big[x_i^*(t)-q_i(t)x_i^*(t-r_i(t))\Big].\end{aligned}$$

We have

$$
\begin{aligned}
&Z_i'(t) = -c_i(t)Z_i(t) - c_i(t)q_i(t)z_i(t-r_i(t)) \\
&+\sum_{j=1}^{n} a_{ij}(t)\Big[f_j(x_j(t-\tau_{ij}(t)) - f_j(x_j^*(t-\tau_{ij}(t))\Big] \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(t)\Big[g_j(x_j(t-\sigma_{ijl}(t))g_l(x_l(t-\nu_{ijl}(t)))) - g_j(x_j^*(t-\sigma_{ijl}(t)))g_l(x_l^*(t-\nu_{ijl}(t)))\Big] \\
&+\sum_{j=1}^{n} \beta_{ij}(t)\int_0^{+\infty} G_{ij}(u)\Big[h_j(x_j(t-u)) - h_j(x_j^*(t-u))\Big]du \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}(t)\Big[\int_0^{+\infty} P_{ijl}(s)k_j(x_j(t-s))ds \int_0^{+\infty} Q_{ijl}(s)k_l(x_l(t-s))ds \\
&-\int_0^{+\infty} P_{ijl}(s)k_j(x_j^*(t-s))ds \int_0^{+\infty} Q_{ijl}(s)k_l(x_l^*(t-s))ds\Big]. \qquad (7)
\end{aligned}
$$

Set

$$
\begin{aligned}
\Upsilon(\gamma) = \sup_{t\in\mathbb{R}}\Bigg\{&\gamma - \tilde{c}_i(t) + K_i\Bigg[\frac{e^{\gamma r_i^*}}{1-q_i^* e^{\gamma r_i^*}} c_i^* q_i^* + \sum_{j=1}^{n} a_{ij}^* L_j^f \frac{e^{\gamma\tau_{ij}^*}}{1-q_j^* e^{\gamma r_j^*}} \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}^*\Bigg(L_j^g d_l^g \frac{e^{\gamma\sigma_{ijl}^*}}{1-q_j^* e^{\gamma r_j^*}} + L_l^g d_j^g \frac{e^{\gamma\nu_{ijl}^*}}{1-q_j^* e^{\gamma r_j^*}}\Bigg) \\
&+\sum_{j=1}^{n}\beta_{ij}^* L_j^h \int_0^{+\infty}|G_{ij}(u)|e^{\gamma u}du \frac{1}{1-q_j^* e^{\gamma r_j^*}} + \sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}^*\Bigg[L_j^k d_l^k \int_0^{+\infty}|P_{ijl}(u)|e^{\gamma u}du \\
&+L_l^k d_j^k \int_0^{+\infty}|Q_{ijl}(u)|e^{\gamma u}du\Bigg]\frac{1}{1-q_j^* e^{\gamma r_j^*}}\Bigg]\Bigg\}, \qquad \gamma\in[0, \min_{1\le\iota\le 3}\kappa_\iota]. \qquad (8)
\end{aligned}
$$

From Assumption 4 and by continuity of $\Upsilon(\gamma)$ we have

$$\begin{aligned}\Upsilon(0) = \sup_{t\in\mathbb{R}}\Bigg\{ &- \tilde{c}_i(t) + K_i\Bigg[\frac{1}{1-q_i^*}c_i^*q_i^* + \sum_{j=1}^{n} a_{ij}^* L_j^f \frac{1}{1-q_j^*} \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}^*\left(L_j^g d_l^g \frac{1}{1-q_j^*} + L_l^g d_j^g \frac{1}{1-q_j^*}\right) \\ &+ \sum_{j=1}^{n}\beta_{ij}^* L_j^h \int_0^{+\infty}|G_{ij}(u)|du\frac{1}{1-q_j^*} + \sum_{j=1}^{n}\sum_{l=1}^{n}p_{ijl}^*\Bigg[L_j^k d_l^k \int_0^{+\infty}|P_{ijl}(u)|du \\ &+ L_l^k d_j^k \int_0^{+\infty}|Q_{ijl}(u)|du\Bigg]\frac{1}{1-q_j^*}\Bigg]\Bigg\} < 0.\end{aligned} \tag{9}$$

We can choose a positive constant $\lambda$ such that $0 < \lambda < \min\{\kappa_1, \kappa_2, \kappa_3, , \tilde{c}_{1*}, \ldots, \tilde{c}_{n*}\}$ satisfying $1 - q_i^* e^{\lambda r_j^*} > 0$, and

$$\begin{aligned}\Upsilon(\lambda) = \sup_{t\in\mathbb{R}}\Bigg\{ &\lambda - \tilde{c}_i(t) + K_i\Bigg[\frac{e^{\lambda r_i^*}}{1-q_i^* e^{\lambda r_i^*}}c_i^*q_i^* + \sum_{j=1}^{n} a_{ij}^* L_j^f \frac{e^{\lambda\tau_{ij}^*}}{1-q_j^* e^{\lambda r_j^*}} \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}^*\left(L_j^g d_l^g \frac{e^{\lambda\sigma_{ijl}^*}}{1-q_j^* e^{\lambda r_j^*}} + L_l^g d_j^g \frac{e^{\lambda\nu_{ijl}^*}}{1-q_j^* e^{\lambda r_j^*}}\right) \\ &+ \sum_{j=1}^{n}\beta_{ij}^* L_j^h \int_0^{+\infty}|G_{ij}(u)|e^{\lambda u}du\frac{1}{1-q_j^* e^{\lambda r_j^*}} + \sum_{j=1}^{n}\sum_{l=1}^{n}p_{ijl}^*\Bigg[L_j^k d_l^k \int_0^{+\infty}|P_{ijl}(u)|e^{\lambda u}du \\ &+ L_l^k d_j^k \int_0^{+\infty}|Q_{ijl}(u)|e^{\lambda u}du\Bigg]\frac{1}{1-q_j^* e^{\lambda r_j^*}}\Bigg]\Bigg\} < 0.\end{aligned} \tag{10}$$

Let

$$\|\phi\|_\varepsilon = \max_{1\le i\le n}\sup_{t\in(-\infty,0]}\Big\|[\phi_i(t) - d_i(t)\phi_i(t-\tau_i(t))] - [x_i^*(t) - d_i(t)x_i^*(t-\tau_i(t))]\Big\|_{\mathcal{A}} \tag{11}$$

For any $\epsilon > 0$, we obtain

$$\|Z(0)\|_{\mathcal{A}} < (\|\phi\|_\varepsilon + \epsilon). \tag{12}$$

For all $t \in (-\infty, 0]$, we have

$$\begin{aligned}\|Z(t)\|_{\mathcal{A}} &\le (\|\phi\|_\varepsilon + \epsilon)e^{-\lambda t} \\ &< N(\|\phi\|_\varepsilon + \epsilon)e^{-\lambda t}\end{aligned} \tag{13}$$

where $N$ is a constant satisfying $N > \max_{1\le i\le n} K_i + 1$. Let us now prove that

$$\| Z(t) \|_{\mathcal{A}} < N(\|\phi\|_\varepsilon + \epsilon)e^{-\lambda t}, \qquad \forall t > 0. \tag{14}$$

If it is not the case, there exist $i \in \{1, \ldots, n\}$ and $t_1 > 0$ such that

$$\| Z(t_1) \|_{\mathcal{A}} = N(\|\phi\|_{\varepsilon} + \epsilon)e^{-\lambda t_1} \tag{15}$$

and

$$\| Z(t) \|_{\mathcal{A}} < N(\|\phi\|_{\varepsilon} + \epsilon)e^{-\lambda t} \qquad \forall t \in (-\infty, t_1). \tag{16}$$

Moreover, we have

$$\begin{aligned} e^{\lambda}\|z_j(\eta)\|_{\mathcal{A}} &\leq e^{\lambda\eta}\|z_j(\eta) - q_j(\eta)z_j(\eta - r_j(\eta))\|_{\mathcal{A}} + e^{\lambda\eta}\|q_j(\eta)z_j(\eta - r_j(\eta))\|_{\mathcal{A}} \\ &\leq e^{\lambda\eta}\|Z_j(\eta)\|_{\mathcal{A}} + e^{\lambda r_i^*} q_j^* e^{\lambda(\eta - r_j(\eta))}\|z_j(\eta - r_j(\eta))\|_{\mathcal{A}} \\ &\leq (\|\phi\|_{\varepsilon} + \epsilon)N + e^{\lambda r_j^*} q_j^* \sup_{s\in(-\infty,t]} e^{\lambda s}\|z_j(s)\|_{\mathcal{A}} . \end{aligned} \tag{17}$$

For all $\eta \in (-\infty, t]$, $t \in (-\infty, t_1)$, $i \in \{1, \cdots, n\}$, it leads to

$$\begin{aligned} e^{\lambda t}\|z_j(t)\|_{\mathcal{A}} &\leq \sup_{s\in(-\infty,t]} e^{\lambda s}\|z_j(s)\|_{\mathcal{A}} \\ &\leq \frac{N(\|\phi\|_{\varepsilon} + \epsilon)}{1 - e^{\lambda q_j^* r_j^*}}. \end{aligned} \tag{18}$$

Multiplying Equation (7) by $e^{-\int_0^s c_i(u)du}$ and integrating on $[0, t]$, we obtain

$$\begin{aligned} Z_i(t) = Z_i(0)e^{-\int_0^t c_i(u)du} + \int_0^t e^{-\int_s^t c_i(u)du} \Big\{ & - c_i(s)q_i(s)z_i(s - r_i(s)) \\ & + \sum_{j=1}^{n} a_{ij}(s)\left[f_j(x_j(s - \tau_{ij}(s)) - f_j(x_j^*(s - \tau_{ij}(s))\right] \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(s)[g_j(x_j(s - \sigma_{ijl}(s))g_l(x_l(s - \nu_{ijl}(s)))) \\ & - g_j(x_j^*(s - \sigma_{ijl}(s)))g_l(x_l^*(s - \nu_{ijl}(s)))] \\ & + \sum_{j=1}^{n} \beta_{ij}(s) \int_0^{+\infty} G_{ij}(u)\left[h_j(x_j(s - u)) - h_j(x_j^*(s - u))\right] du \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}(s)\Big[ \int_0^{+\infty} P_{ijl}(u)k_j(x_j(s - u))du \\ & \times \int_0^{+\infty} Q_{ijl}(u)k_l(x_l(s - u))du - \int_0^{+\infty} P_{ijl}(u)k_j(x_j^*(s - u))du \\ & \times \int_0^{+\infty} Q_{ijl}(u)k_l(x_l^*(s - u))du\Big]\Big\}ds, \qquad t \in [0, t_1]. \end{aligned} \tag{19}$$

Using equations (8), (10)-(19), we obtain

$$\begin{aligned}\|Z_i(t_1)\|_{\mathcal{A}} = \Big\| & Z_i(0)e^{-\int_0^{t_1} c_i(u)du} + \int_0^{t_1} e^{-\int_s^{t_1} c_i(u)du}\Big\{ - c_i(s)q_i(s)z_i(s-r_i(s)) \\ & + \sum_{j=1}^{n} a_{ij}(s)(f_j(x_j(s-\tau_{ij}(s))) - f_j(x_j^*(s-\tau_{ij}(s)))) \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(s)(g_j(x_j(s-\sigma_{ijl}(s)))g_l(x_l(s-\sigma_{ijl}(s))) \\ & - g_j(x_j^*(s-\sigma_{ijl}(s)))g_l(x_l^*(s-\sigma_{ijl}(s)))) \\ & + \sum_{j=1}^{n} \beta_{ij}(s) \int_0^{+\infty} G_{ij}(u)[h_j(x_j(s-u)) - h_j(x_j^*(s-u))]du \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}(s)[\int_0^{+\infty} P_{ijl}(u)k_j(x_j(s-u))du \\ & \times \int_0^{+\infty} Q_{ijl}(u)k_l(x_l(s-u))du - \int_0^{+\infty} P_{ijl}(u)k_j(x_j^*(s-u))du \\ & \times \int_0^{+\infty} Q_{ijl}(u)k_l(x_l^*(s-u))du\Big\}ds\Big\|_{\mathcal{A}}.\end{aligned}$$

Then it leads to

$$\begin{aligned}\|Z_i(t_1)\|_{\mathcal{A}} \leq & (\|\phi\|_{\varepsilon} + \epsilon)K_i e^{-\int_0^{t_1} \tilde{c}_i(u)du} + \int_0^{t_1} e^{-\int_s^{t_1} \tilde{c}_i(u)du} K_i\Big\{ c_i^* q_i^* \|x_i(s-r_i(s)) - x_i^*(s-r_i(s))\|_{\mathcal{A}} \\ & + \sum_{j=1}^{n} a_{ij}^* \left\|f_j(x_j(s-\tau_{ij}(s)) - f_j(x_j^*(s-\tau_{ij}(s))\right\|_{\mathcal{A}} \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}^* \|g_j(x_j(s-\sigma_{ijl}(s))g_l(x_l(s-\nu_{ijl}(s))) \\ & - g_j(x_j^*(s-\sigma_{ijl}(s)))g_l(x_l^*(s-\nu_{ijl}(s)))\|_{\mathcal{A}} \\ & + \sum_{j=1}^{n} \beta_{ij}^* \int_0^{+\infty} |G_{ij}(u)| \left\|h_j(x_j(s-u)) - h_j(x_j^*(s-u))\right\|_{\mathcal{A}} du \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}^* \sup_{t\in\mathbb{R}} \Big\| \int_0^{+\infty} P_{ijl}(u)k_j(x_j(s-u))du \int_0^{+\infty} Q_{ijl}(u)k_l(x_l(s-u))du \\ & - \int_0^{+\infty} P_{ijl}(u)k_j(x_j^*(s-u))du \int_0^{+\infty} Q_{ijl}(u)k_l(x_l^*(s-u))du\Big\|_{\mathcal{A}} \Big\}ds\end{aligned}$$

and finally it yields

$$\begin{aligned}\|Z_i(t_1)\|_{\mathcal{A}} \leq (\|\phi\|_{\varepsilon}+\epsilon)e^{-\lambda t_1}K_i e^{-\int_0^{t_1}[\tilde{c}_i(u)-\lambda]du} + \int_0^{t_1} e^{-\int_s^{t_1}[\tilde{c}_i(s)-\lambda]du}K_i\Bigg\{\frac{e^{\lambda r_i^*}}{1-q_i^* e^{\lambda r_i^*}}c_i^* q_i^* \\ + \sum_{j=1}^{n} a_{ij}^* L_j^f \frac{e^{\lambda\tau_{ij}^*}}{1-q_j^* e^{\lambda r_j^*}} + \sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}^*\left(L_j^g d_l^g \frac{e^{\lambda\sigma_{ijl}^*}}{1-q_j^* e^{\lambda r_j^*}} + L_l^g d_j^g \frac{e^{\lambda\nu_{ijl}^*}}{1-q_j^* e^{\lambda r_j^*}}\right) \\ + \sum_{j=1}^{n}\beta_{ij}^* L_j^h \int_0^{+\infty}|G_{ij}(u)|e^{\lambda u}du \frac{1}{1-q_j^* e^{\lambda r_j^*}} + \sum_{j=1}^{n}\sum_{l=1}^{n} p_{ijl}^*\Bigg[L_j^k d_l^k \int_0^{+\infty}|P_{ijl}(u)|e^{\lambda u}du \\ + L_l^k d_j^k \int_0^{+\infty}|Q_{ijl}(u)|e^{\lambda u}du\Bigg]\frac{1}{1-q_j^* e^{\lambda r_j^*}}\Bigg\}dsN(\|\phi\|_{\varepsilon}+\epsilon)e^{-\lambda t_1}\end{aligned}$$

and then

$$\begin{aligned}\|Z_i(t_1)\|_{\mathcal{A}} &\leq (\|\phi\|_{\varepsilon}+\epsilon)e^{-\lambda t_1}K_i e^{-\int_0^{t_1}[\tilde{c}_i(u)-\lambda]du} \\ &\quad + \int_0^{t_1} e^{-\int_s^{t_1}[\tilde{c}_i(s)-\lambda]du}\Big\{\tilde{c}_i(u)-\lambda\Big\}dsN(\|\phi\|_{\varepsilon}+\epsilon)e^{-\lambda t_1} \\ &\leq N(\|\phi\|_{\varepsilon}+\epsilon)e^{-\lambda t_1}\left\{(\frac{K_i}{N}-1)e^{-\int_0^{t_1}(\tilde{c}_i(u)-\lambda)du}+1\right\} \\ &< N(\|\phi\|_{\varepsilon}+\epsilon)e^{-\lambda t_1}\end{aligned}$$

which contradicts equality (15), so (14) holds. Letting $\epsilon \to 0^+$, then for all $t>0$, we have $\|Z(t)\|_{\mathcal{A}} \leq N\|\phi\|_{\varepsilon}e^{-\lambda t}$. Similarly, we have

$$\begin{aligned}e^{\lambda t}\|z_j(t)\|_{\mathcal{A}} &\leq \sup_{s\in(-\infty,t]} e^{\lambda s}\|z_j(s)\|_{\mathcal{A}} \\ &\leq \frac{N\|\phi\|_{\varepsilon}}{1-q_j^* e^{\lambda r_j^*}}\end{aligned}$$

and

$$\|z_j(t)\|_{\mathcal{A}} \leq \frac{N\|\phi\|_{\varepsilon}}{1-q_j^* e^{\lambda r_j^*}}e^{-\lambda t} \qquad \forall t>0,\ j\in\{1,\cdots,n\}.$$

The $(\mu,\nu)-$PAA solution of the system (1) is global exponentially stable. □

*Remark 2* In this article, we not only consider the effects of the first-order terms $a_{ij}(\cdot)$, $\beta_{ij}(\cdot)$ on NNs but also the influences of the second-order terms $\alpha_{ijl}(\cdot)$ and $p_{ijl}(\cdot)$. If $\alpha_{ijl}(\cdot)=p_{ijl}(\cdot)=0$ then the following classical (first order) neutral type Hopfiled NNs

$$\begin{aligned}\Big[x_i(t)-q_i(t)x_i(t-r_i(t))\Big]' &= -c_i(t)x_i(t)+\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \\ &\quad +\sum_{j=1}^{n}\beta_{ij}(t)\int_0^{+\infty}G_{ij}(s)h_j(x_j(t-s))ds \\ &\quad +I_i(t), \qquad (20)\end{aligned}$$

has a unique $(\mu,\nu)$-pseudo almost automorphic solution which is global exponential stable. Hence, our results generalize the results in [10].

*Remark 3* If $q_i(t)=0$ then the following Hopfiled NN

$$\dot{x}_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} \beta_{ij}(t) \int_0^{+\infty} G_{ij}(s)h_j(x_j(t-s))ds + I_i(t), \tag{21}$$

has an unique $(\mu,\nu)$-pseudo almost automorphic solution which is global exponential stable. It is worth pointing out that our main results are also valid for the case of the non-neutral systems. Our results not only improve but also complement earlier results in [3,5,7,8,14,29].

*Remark 4* In [21], the authors considered a class of Clifford-valued neutral HOHNN with leakage delays. They studied the existence and global exponential stability of pseudo-almost periodic solutions for this class of NNs. In [22], the authors dealt with a class of inertial quaternion-valued HOHNNs with state-dependent delays. They analyzed the existence of anti-periodic solutions of the NNs. In our article, we investigate the $(\mu,\nu)-$ pseudo almost automorphic solution for neutral-type HOHNNs. This class of functions covers larger classes of functions such as almost periodic, pseudo almost periodic, almost automorphic and pseudo almost automorphic functions. Hence, our results are not only new but also most general.

*Remark 5* Although the multiplication of Clifford numbers does not satisfy the commutativity, which brings great difficulties to the study of Clifford-valued systems, we have found a method that does not decompose Clifford-valued systems into real-valued systems.

*Remark 6* Pseudo almost automorphic functions play an important role in describing the dynamics of differential equations. In [3], the authors investigated the dynamics behavior of the pseudo almost automorphic solutions of a class of HOHNNs with mixed delays. Those results cannot be applicable for the systems studied in this article. Consequently, our analysis of dynamics behavior of neutral type Clifford-valued HOHNNs model with $(\mu,\nu)-$PAA functions as coefficients and mixed delays improve the previous study in [3]. If $\mu=\nu=1$, then the system (1) has an unique PAA solution in Clifford algebra.

*Remark 7* In [7], the authors studied a class of delayed high-order Hopfield bidirectional associative memory NNs. They used fixed delays because time-varying delays are difficult to handle when dealing with $(\mu,\nu)$-pseudo almost automorphic parameters. Hence, it is not obvious to prove the composition theorem of $(\mu,\nu)$-pseudo almost automorphic functions i.e. $f(\phi(\cdot-\tau(\cdot)))\in PAA(\mathbb{R},\mathbb{R},\mu,\nu)$. In 2020, this problem has been solved in article [10].

*Remark 8* In [9], the pseudo almost periodic solutions of Clifford-valued inertial neutral NNs with time-varying delays and infinite distributed delay are investigated. Then, by using the same approach we can extend the results to the space of pseudo almost automorphic functions. Our results can complement the results in [9].

## 4 Simulation results

In this section, we apply our main results to a specific system and demonstrate the efficiency of our new stability criteria. Consider the following neutral type Clifford-valued HOHNNs model

$$\begin{aligned}\Big[x_i(t)-q_i(t)x_i(t-r_i(t))\Big]' &= -c_i(t)x_i(t)+\sum_{j=1}^{2}a_{ij}(t)f_j(x_j(t-\tau_{ij}(t)))\\ &+\sum_{j=1}^{2}\sum_{l=1}^{2}\alpha_{ijl}(t)g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_l(t-\nu_{ijl}(t)))\\ &+\sum_{j=1}^{2}\beta_{ij}(t)\int_0^{+\infty}G_{ij}(s)h_j(x_j(t-s))ds\\ &+\sum_{j=1}^{2}\sum_{l=1}^{2}p_{ijl}(t)\int_0^{+\infty}P_{ijl}(t-s)k_j(x_j(s))ds\\ &\times\int_0^{+\infty}Q_{ijl}(t-s)k_l(x_l(s))ds+I_i(t), \qquad (22)\end{aligned}$$

where $x_i(t)=x_i^0(t)e_0+x_i^1(t)e_1+x_i^2(t)e_2+x_i^{12}(t)e_{12}$ and $i=1,2$.
For $1\le i\le 2$, let $e_0=\begin{pmatrix}1&0\\0&1\end{pmatrix}, e_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, e_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $e_{12}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$.
For $\mu,\nu\in\mathcal{M}$, satisfying $(A_1)$ and $(A_2)$, we consider the measure $\mu(t)$ where its Radon-Nikodym derivative is $\rho_1(t)=e^{\sin t}$, and the measure $\nu(t)$ where its Radon-Nikodym derivative is

$$\rho_2(t)=\begin{cases}e^t & \text{if } t\le 0,\\ 1 & \text{if } t>0.\end{cases}$$

We take the following parameters:

$$\begin{aligned} f(x_i) &= g(x_i) = h(x_i) = k(x_i),\ i = 1, 2, \\ f(x_i) &= \frac{|x_i^0 + 1| - |x_i^0 + 1|}{22} e_0 + \frac{1}{20} \cos x_i^1 e_1 + \frac{1}{27} \cos x_i^2 e_2 + \frac{1}{21} \cos x_i^{12} e_{12} \\ &\Rightarrow L_j^f = L_j^g = L_j^h = L_j^k = 1, d_j^g = d_j^k = 0.1, \\ r_i(t) &= 0.5,\ \tau_{ij}(t) = \sigma_{ijl}(t) = \nu_{ijl}(t) = 0.5|\cos t|. \end{aligned}$$

Let

$$\begin{aligned} G_{ij}(t) = P_{ijl}(t) = Q_{ijl}(t) = e^{-t},\ K_i = e^{0.1} \text{ and } \kappa_\iota = 0.5 \text{ for } \iota = 1, 2, 3. \\ c_1(t) = (4 + \cos^2 t)e_0,\ c_2(t) = (4 + \sin^2 t)e_0, \end{aligned}$$

and

$$\begin{aligned} q_1(t) &= 0.12 \sin(t) e_0, \\ q_2(t) &= 0.12 \cos(t) e_0, \end{aligned}$$

and

$$\begin{aligned} a_{11}(t) &= 0.1 \sin\left(\frac{2\pi}{2 + \sin t + \sin\left(\sqrt{3}t\right)}\right) e_0 + 0.3 \sin\left(\sqrt{3}t\right) e_1, \\ a_{12}(t) &= 0.1 e_0 + 0.1 e^{-t^2} e_2, \\ a_{21}(t) &= 0,\ a_{22}(t) = 0.1 e_0 + 0.1 e^{-t} e_{12}, \end{aligned}$$

and

$$\begin{aligned} \alpha_{111}(t) &= \alpha_{121}(t) = \alpha_{122}(t) = 0, \\ \alpha_{112}(t) &= 0.3 \cos\left(\frac{1}{2 + \sin t + \sin\sqrt{2}t}\right) e_0 + \frac{0.1}{1 + t^2} e_{12}, \\ \alpha_{211}(t) &= 0.6 \sin(\sqrt{5}t) e_1 + 0.3 \sin\left(\frac{1}{2 + \cos t + \sin\sqrt{5}t}\right) e_2 + \frac{0.1}{1 + t} e_{12}, \\ \alpha_{212}(t) &= \alpha_{221}(t) = \alpha_{222}(t) = 0, \end{aligned}$$

and

$$\begin{aligned} \beta_{11}(t) &= 0.4 \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right) e_0 + 0.4 \sin\left(\sqrt{2}t\right) e_1, \\ \beta_{12}(t) &= \beta_{21}(t) = 0, \\ \beta_{22}(t) &= 0.2 e_0 + 0.2 \cos\left(\sqrt{7}t\right) e_1 + \frac{0.1}{1 + t^2} e_{12}, \end{aligned}$$

and

$$\begin{aligned} p_{111}(t) &= p_{121}(t) = p_{122}(t) = 0, \\ p_{112}(t) &= 0.2 \sin\left(\frac{1}{2 + \sin t + \sin\sqrt{2}t}\right) e_2 + \frac{0.1}{1 + t} e_{12}, \\ p_{211}(t) &= p_{212}(t) = p_{222}(t) = 0, \\ p_{221}(t) &= 0.4 \sin\left(\frac{1}{2 + \sin t + \sin\sqrt{2}t}\right) e_1 + \frac{0.1}{1 + t} e_2, \end{aligned}$$

and

$$I_1(t) = 0.7\sin\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right)e_0 + 0.7e_1 + 0.7\sin te_2 + 0.7\cos\left(\sqrt{2}t\right)e_{12},$$

$$I_2(t) = 0.7\cos\left(\frac{1}{2+\cos t+\cos\sqrt{5}t}\right)e_0 + 0.7\left(\sin t + \frac{0.3}{1+t^2}\right)e_1 + 0.7\cos t\ e_2 + 0.7\sin\left(\sqrt{7}t\right) + e^{-t^2}e_{12}.$$

We have

$$\begin{aligned}\Lambda = \max_{1\le i\le 2}\Bigg\{q_i^* + \frac{K_i}{\tilde{c}_{i*}}\Bigg[c_i^*q_i^* + \sum_{j=1}^{2}a_{ij}^*L_j^f + \sum_{j=1}^{2}\sum_{l=1}^{2}\alpha_{ijl}^*L_j^gL_l^g + \sum_{j=1}^{2}\beta_{ij}^*L_j^h\int_0^{+\infty}|G_{ij}(u)|du \\ + \sum_{j=1}^{2}\sum_{l=1}^{2}p_{ijl}^*L_j^kL_l^k\int_0^{+\infty}|P_{ijl}(u)|du\int_0^{+\infty}|Q_{ijl}(u)|du\Bigg]\Bigg\} = 0.9509 < 1,\end{aligned}$$

$$\begin{aligned}\Omega = \max_{1\le i\le 2}\Bigg\{q_i^* + \frac{K_i}{\tilde{c}_{i*}}\Bigg[c_i^*q_i^* + \sum_{j=1}^{2}a_{ij}^*L_j^f + \sum_{j=1}^{2}\sum_{l=1}^{2}\alpha_{ijl}^*(L_j^gd_l^g + L_l^gd_j^g) \\ + \sum_{j=1}^{2}\beta_{ij}^*L_j^h\int_0^{+\infty}|G_{ij}(u)|du + \sum_{j=1}^{2}\sum_{l=1}^{2}p_{ijl}^*\int_0^{+\infty}|P_{ijl}(u)|du\int_0^{+\infty}|Q_{ijl}(u)|du(L_j^kd_l^k + L_l^kd_j^k)\Bigg]\Bigg\} \\ = 0.8825 < 1\end{aligned}$$

and

$$\begin{aligned}\Upsilon_0 = \sup_{t\in\mathbb{R}}\Bigg\{-\tilde{c}_i(t) + K_i\Bigg[\frac{1}{1-q_i^*}c_i^*q_i^* + \sum_{j=1}^{2}a_{ij}^*L_j^f\frac{1}{1-q_j^*} \\ + \sum_{j=1}^{2}\sum_{l=1}^{2}\alpha_{ijl}^*\left(L_j^gd_l^g\frac{1}{1-q_j^*} + L_l^gd_j^g\frac{1}{1-q_j^*}\right) \\ + \sum_{j=1}^{2}\beta_{ij}^*L_j^h\int_0^{+\infty}|G_{ij}(u)|du\frac{1}{1-q_j^*} + \sum_{j=1}^{2}\sum_{l=1}^{2}p_{ijl}^*\Bigg[L_j^kd_l^k\int_0^{+\infty}|P_{ijl}(u)|du \\ + L_l^kd_j^k\int_0^{+\infty}|Q_{ijl}(u)|du\Bigg]\frac{1}{1-q_j^*}\Bigg]\Bigg\} = -1.1779 < 0.\end{aligned}$$

By a direct computation, we can check that all the conditions of Theorem 3 are satisfied. Therefore, the system (22) has an unique $(\mu,\nu)-$PAA solution which is represented in Figures 1, 2, 3 and 4. Besides, the unique $(\mu,\nu)-$PAA solution of the system (22) is global exponential stable. Almost automorphy is not always as easy to identify visually. In the above example $x_i^0$, $x_i^1$, $x_i^2$ and $x_i^{12}$ with $i = 1, 2$ never exactly repeat themselves. They are not periodic. Figures 5,6,7 and 8 confirm the global exponential stability of the $(\mu,\nu)-$PAA

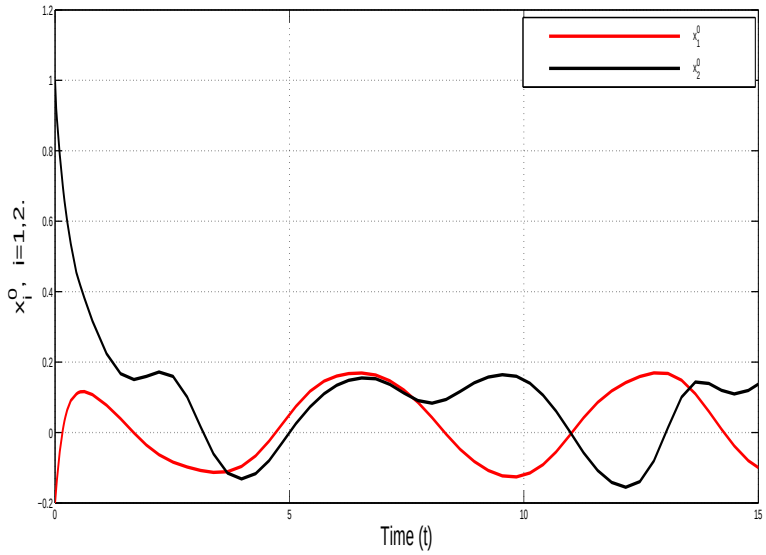


**Fig. 1** Curves of $x_i^0,\ i = 1, 2$, of the system (22).

solution for the system (22). Figures 1 to 8 confirm that the proposed conditions in our theoretical results are effective for the above example.

*Remark 9* In the above example, $r_i(\cdot)$, $\tau_{ij}(\cdot)$, $\sigma_{ijl}(\cdot)$, $\nu_{ijl}(\cdot)$ represent the time-delay functions. The time-delay as an inherent feature of signal transmission between different neurons is one of the main sources for causing dynamic properties of the system (22). It should be mentioned that the presence of time-delay is a particularly harmful source of potential instability. In this example, the established criteria are straightforward to test and independent of delays, that is, the stability of the considered NN models is insensitive to the presence of the delays.

## 5 Conclusion

In this manuscript, neutral type Clifford-valued HOHNNs with mixed delays and D operator have been studied. By employing the fixed point theorem and differential inequalities, new sufficient conditions for the existence, uniqueness and global exponential stability of the $(\mu, \nu)$-pseudo almost automorphic solutions have been established. To our best knowledge, this is the first paper studying the $(\mu, \nu)$-pseudo almost automorphic solutions in Clifford algebra for such kind of NNs. As future research, there are some paths in this article that can be explored further. For instance:

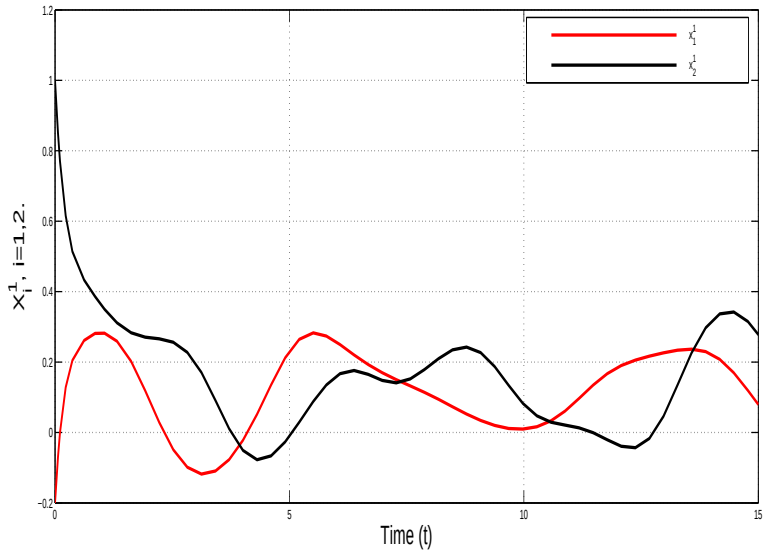


**Fig. 2** Curves of $x_1^1$ and $x_2^1$ of the system (22).

1. In the model (1), the activation functions of signal transmission $f_j(\cdot)$, $g_j(.)$, $h_j(\cdot)$ and $k_j(\cdot)$ are continuous functions. They can be considered as discontinuous functions due to the impulse behavior of firing neurons.
2. The concept of Stepanov like pseudo weighted almost automorphy ($WPAAS^p$) is quite sophisticated. However, the dynamic oscillations of delayed systems with $WPAAS^p$ parameters are still relatively new. Soon, we will try to investigate dynamic oscillations of HOHNNs with $WPAAS^p$ parameters in Clifford algebra.
3. The study of second-order such systems [35,36].

## References

1. Achouri, H., Aouiti, C., & Hamed, B. B. (2020). Bogdanov-Takens Bifurcation in a Neutral Delayed Hopfield Neural Network with Bidirectional Connection. International Journal of Biomathematics, 13(06), 2050049.
2. Ait Dads, E. H., Ezzinbi, K., Miraoui, M. (2015). $(\mu, \nu)$-Pseudo almost automorphic solutions for some non-autonomous differential equations. International Journal of Mathematics, doi.org/10.1142/S0129167X15500901
3. Alimi, A. M., Aouiti, C., Chérif, F., Dridi, F., & M'hamdi, M. S. (2018). Dynamics and oscillations of generalized high-order Hopfield neural networks with mixed delays. Neurocomputing, 321, 274-295.
4. Aouiti, C., & Dridi, F. (2019). Piecewise asymptotically almost automorphic solutions for impulsive non-autonomous high-order Hopfield neural networks with mixed delays. Neural Computing and Applications, 31(9), 5527-5545.
5. Aouiti, C., M'hamdi, M. S., Touati, A. (2017). Pseudo Almost Automorphic Solutions of Recurrent Neural Networks with Time-Varying Coefficients and Mixed Delays. Neural Processing Letters, 45(1): 121-140.

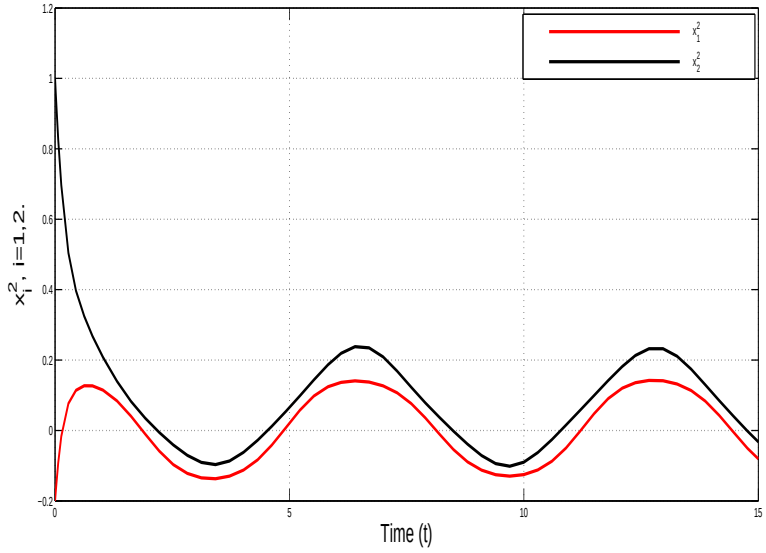


**Fig. 3** Curves of $x_i^2$ for $i = 1, 2$, of the system (22).


6. Aouiti, C., & Dridi, F. (2019). New results on impulsive Cohen-Grossberg neural networks. Neural Processing Letters, 49(3), 1459-1483.
7. Aouiti, C., Dridi, F. $(\mu, \nu)$-Pseudo-almost automorphic solutions for high-order Hopfield bidirectional associative memory neural networks. Neural Computing and Applications.doi.org/10.1007/s00521-018-3651-6
8. Aouiti, C., Gharbia, I. B., Cao, J., M'hamdi, M. S., Alsaedi, A. (2018). Existence and global exponential stability of pseudo almost periodic solution for neutral delay BAM neural networks with time-varying delay in leakage terms. Chaos, Solitons & Fractals, 107, 111-127.
9. Aouiti, C., & Ben Gharbia, I. (2020). Dynamics behavior for second-order neutral Clifford differential equations: inertial neural networks with mixed delays. Computational and Applied Mathematics, 39, doi.org/10.1007/s40314-020-01148-0
10. Aouiti, C., Ghanmi, B., & Miraoui, M. (2020). On the differential equations of recurrent neural networks. International Journal of Computer Mathematics, doi.org/10.1080/00207160.2020.1820493
11. Aouiti, C., Sakthivel, R., & Touati, F. (2020). Global dissipativity of fuzzy cellular neural networks with inertial term and proportional delays. International Journal of Systems Science, (51), 1392-1405.
12. Aouiti, C., Sakthivel, R., & Touati, F. (2020). Global dissipativity of fuzzy bidirectional associative memory neural networks with proportional delays. Iranian Journal of Fuzzy Systems, doi.org/ 10.22111/IJFS.2020.5709
13. Delanghe, R., Sommen, F., & Souček, V. (1992). Clifford algebra and spinor-valued functions: a function theory for the Dirac operator. (Vol. 53). Springer Science & Business Media.
14. Brahmi, H., Ammar, B., Chérif, F., Alimi, A. M. (2014). On the dynamics of the high-order type of neural networks with time varying coefficients and mixed delay. Neural Networks 2063-2070.
15. Brahmi, H., Ammar, B., Alimi, A. M., Chérif, F. (2016). Pseudo almost periodic solutions of impulsive recurrent neural networks with mixed delays. In Neural Networks (IJCNN), 2016 International Joint Conference on 464-470.

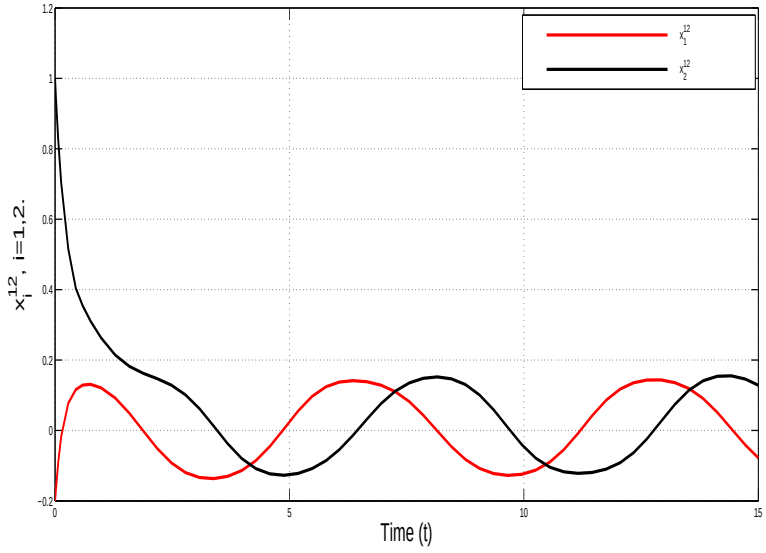


**Fig. 4** Curves of $x_i^{12}$ for $i = 1, 2$, of the system (22).

16. Chen, Z. (2018). Global exponential stability of anti-periodic solutions for neutral type CNNs with D operator. International Journal of Machine Learning and Cybernetics, 9(7), 1109-1115.
17. Diagana, T. (2013). Almost automorphic type and almost periodic type functions in abstract spaces. New York: Springer.
18. Huo, N., Li, B., & Li, Y. (2020). AntiPeriodic Solutions for CliffordValued HighOrder Hopfield Neural Networks with StateDependent and Leakage Delays. International Journal of Applied Mathematics and Computer Science, 30(1), 83-98.
19. Li, Y., & Xiang, J. (2019). Existence and global exponential stability of anti-periodic solution for Clifford-valued inertial Cohen-Grossberg neural networks with delays. Neurocomputing, 332, 259-269.
20. Li, Y., Xiang, J., & Li, B. (2019). Globally asymptotic almost automorphic synchronization of Clifford-valued RNNs With Delays. IEEE Access, 7, 54946-54957.
21. Li, B., & Li, Y. (2019). Existence and global exponential stability of pseudo almost periodic solution for Clifford-valued neutral high-order Hopfield neural networks with leakage delays. IEEE Access, 7, 150213-150225.
22. Li, Y., & Xiang, J. (2020). Existence and global exponential stability of anti-periodic solutions for quaternion-valued cellular neural networks with time-varying delays. Advances in Difference Equations, 2020(1), 47.
23. Liu, Y., Xu, P., Lu, J., & Liang, J. (2016). Global stability of Clifford-valued recurrent neural networks with time delays. Nonlinear Dynamics, 84(2), 767-777.
24. Liu, Y., Zhang, D., Lu, J., & Cao, J. (2016). Global $\mu-$stability criteria for quaternion-valued neural networks with unbounded time-varying delays. Information Sciences, 360, 273-288.
25. Liu, Y., Zhang, D., & Lu, J. (2017). Global exponential stability for quaternion-valued recurrent neural networks with time-varying delays. Nonlinear Dynamics, 87(1), 553-565.
26. Ren, F., Cao, J. (2007). Periodic oscillation of higher-order bidirectional associative memory neural networks with periodic coefficients and delays. Nonlinearity, 20(3): 605-629.

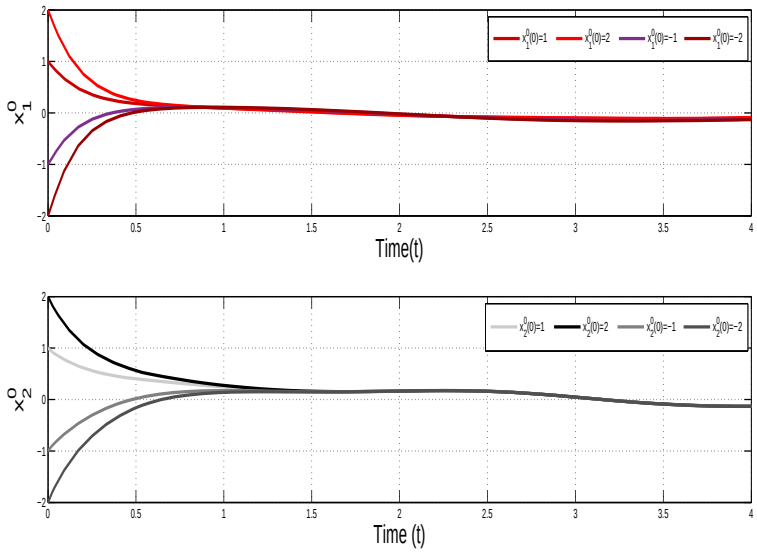


**Fig. 5** Stability of $x_i^0,\ i = 1, 2$, of the system (22).


27. Veech, W. A. (1965). Almost automorphic functions on groups. American Journal of Mathematics, 87(3), 719-751.
28. Xu, C. J., Li, P. L. (2017). New Stability Criteria for High-Order Neural Networks with Proportional Delays. Communications in Theoretical Physics, 67(3), 235.
29. Xu, Y. (2017). Exponential stability of weighted pseudo almost periodic solutions for HCNNs with mixed delays. Neural Processing Letters, 46(2), 507-519.
30. Xu, C., Li, P. (2018). On anti-periodic solutions for neutral shunting inhibitory cellular neural networks with time-varying delays and D operator. Neurocomputing, 275, 377-382.
31. Yao, L. (2018). Global convergence of CNNs with neutral type delays and D operator. Neural Computing and Applications, 29(1), 105-109.
32. Yao, L. (2017). Global exponential convergence of neutral type shunting inhibitory cellular neural networks with $D$ operator. Neural Processing Letters, 45(2), 401-409.
33. Zhang, A. (2018). Almost periodic solutions for SICNNs with neutral type proportional delays and D operators. Neural Processing Letters, 47(1), 57-70.
34. Zhang, A. (2017). Pseudo almost periodic solutions for neutral type SICNNs with $D$ operator. Journal of Experimental Theoretical Artificial Intelligence, 29(4), 795-807.
35. Zhang, W., Zuo, Z., Wang, Y., & Zhang, Z. (2019). Double-integrator dynamics for multiagent systems with antagonistic reciprocity. IEEE Transactions on Cybernetics. 50(9), 4110 - 4120.
36. Zhang, Y., Liu, Y., Yang, X., & Qiu, J. (2020). Velocity Constraint on Double-Integrator Dynamics Subject to Antagonistic Information. IEEE Transactions on Circuits and Systems II: doi 10.1109/TCSII.2020.2999375
37. Zhu, J., & Sun, J. (2016). Global exponential stability of Clifford-valued recurrent neural networks. Neurocomputing, 173, 685-689.

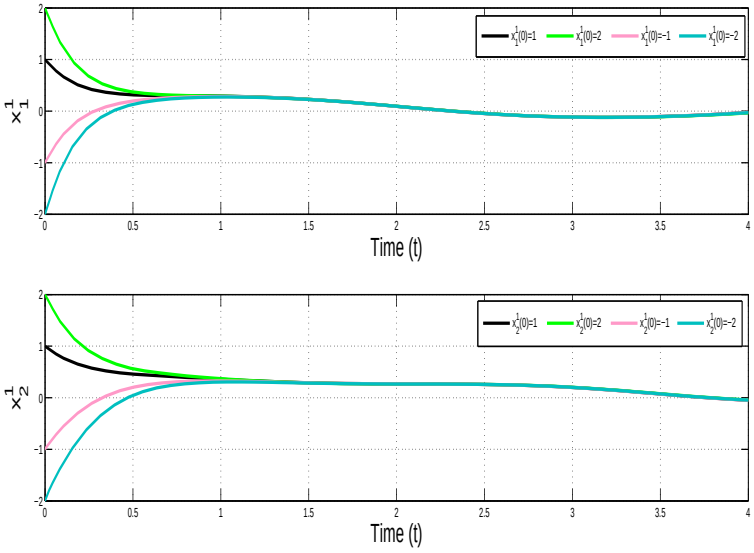


**Fig. 6** Stability of $x_i^1$ for $i = 1, 2$, of the system (22).

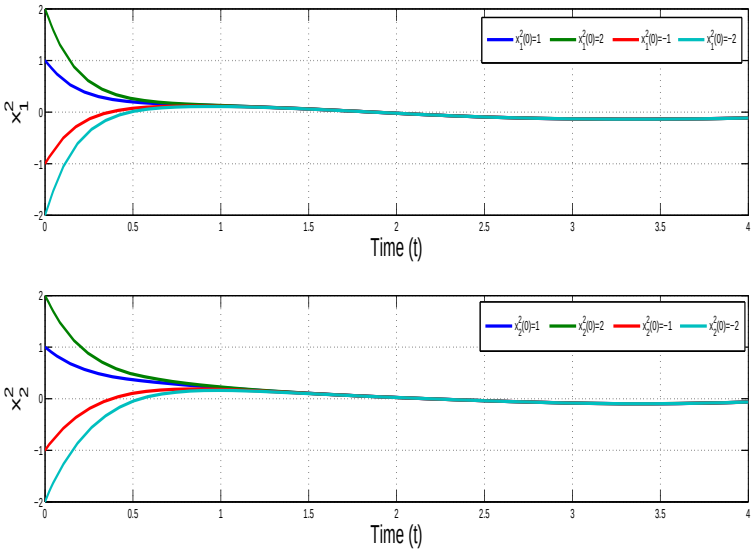


**Fig. 7** Stability of $x_i^2$ for $i = 1, 2$, of the system (22).

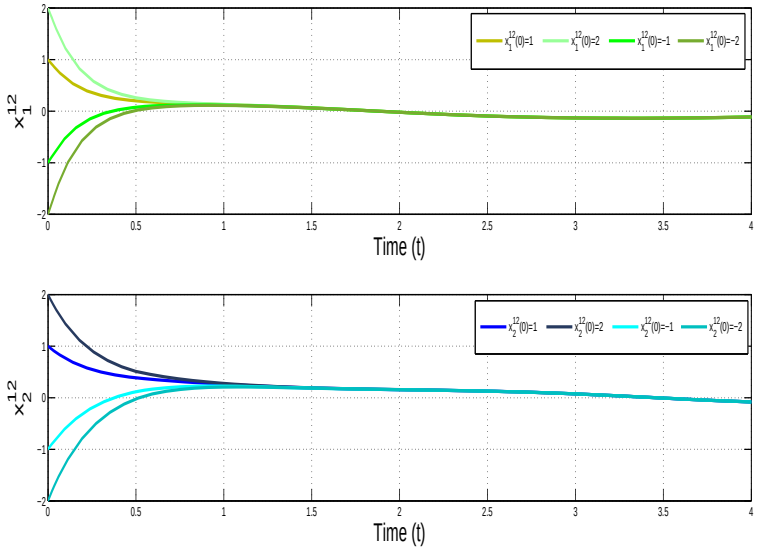


**Fig. 8** Stability of $x_i^{12}$ for $i = 1, 2$, of the system (22).