



**HAL**  
open science

# The Cauchy problem for Laplace's equation via a modified conjugate gradient method and energy space approaches

Saber Amdouni, Amel Ben Abda

► **To cite this version:**

Saber Amdouni, Amel Ben Abda. The Cauchy problem for Laplace's equation via a modified conjugate gradient method and energy space approaches. 2021. hal-03232979

**HAL Id: hal-03232979**

**<https://hal.science/hal-03232979v1>**

Preprint submitted on 23 May 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The Cauchy problem for Laplace's equation via a modified conjugate gradient method and energy space approaches

S. AMDOUNI <sup>1</sup>, A. BEN ABDA <sup>2</sup>,

## Abstract

In the present work, we focus on the resolution of the Cauchy Laplace problem using an energetic variational minimization approach [18] in the framework of a finite element method. A new strategy of regularization, called a filtering procedure regularization, is developed. The advantage of using this new regularization is that it does not require a regularization parameter and is easy to implement. An optimal a priori error estimate is proven, for the first time up to our knowledge, in the context of the finite element method. Some numerical results are presented to illustrate the performance of our approach.

Keywords: Cauchy problem, Energy space approaches, Conjugate gradient method, Finite element method, Laplace problem

## Introduction

The present paper is meant to present a new numerical procedure for the resolution of the Cauchy problem. It is situated, for simplicity, within the framework of the Laplace equation which can be easily extended for any elliptic equation. The Cauchy problem has arisen from many practical contexts, such as non-destructive testing, electrocardiography, electroencephalography and hydro-geology. A tremendous body of literature has referred back to the leading work of Hadamard [7]. In his work, Hadamard highlighted the ill-posedness of such an inverse problem which enormously increases the difficulties of its numerical treatment. There are many strategies used to circumvent these difficulties. We can cite the quasi-reversibility approach (see [4, 3, 16]), the Kozlov approach [15, 14], the Fading approach [6, 8], the Steklov-Poincaré approach [1] and the variational minimization approach [12, 11, 18]. For a detailed literature review in the context of the Laplace equation we can see [?].

In this study, we adopted a variational minimization method with an energy space approach similar to that presented in [18]. The main feature is the use of a filtering procedure as a new regularization strategy in the context of the finite element method and a conjugate gradient method. This strategy does not require a regularization parameter. It is easy to implement and leads to a good accuracy similar to that of the direct problem. An optimal error estimate is shown, for the first time up to our knowledge, in the context of the finite element method.

The present contribution is organized as follows: In section 1, we define some useful space and operator needed for our study. In section 2, we model our energetic variational minimization problem. Section 3 presents a proof of the equivalence between the energetic variational minimization formulation and the Cauchy problem. Then we show the stability of our formulation according to the noise level. We also prove that our cost functional is twice Fréchet differentiable, strictly convex and coercive. In section 4, the discretization of the energetic variational minimization problem is presented in the framework of the finite element method. Then a stability

---

<sup>1</sup>University Of Tunis El Manar, National Engineering School of Tunis, Laboratory For Mathematical And Numerical Modeling In Engineering Science, 1002, Tunis, Tunisia [saber.amdouni@enit.utm.tn](mailto:saber.amdouni@enit.utm.tn)

<sup>2</sup>University Of Tunis El Manar, National Engineering School of Tunis, Laboratory For Mathematical And Numerical Modeling In Engineering Science, 1002, Tunis, Tunisia [amel.benabda@enit.utm.tn](mailto:amel.benabda@enit.utm.tn)

result of our discrete formulation is proved. An optimal and an original a priori error estimate of this formulation is shown. In section 5, a modified conjugate gradient algorithm is presented. In order to illustrate the performance of our approach, the numerical results for some synthetic examples are presented in section 6. In all cases, the numerical results are compared to the “exact solutions” of the Cauchy problem. Finally, the conclusions are given in section 7.

## 1 Preliminary

Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^2$  and  $\Gamma = \overline{\Gamma_0 \cup \Gamma_i \cup \Gamma_c}$ . We assume that  $\Gamma_i$  and  $\Gamma_c$  are separated by a non-zero measure portion  $\Gamma_0$ . Then following [17] let's define:

$$H^{\frac{1}{2}}(\Gamma_k) = \left\{ v = \tilde{v}|_{\Gamma_k} : \tilde{v} \in H^{\frac{1}{2}}(\Gamma) \right\}, \quad \text{with } k \in \{i, c\},$$

and

$$\widehat{H}^{\frac{1}{2}}(\Gamma_k) = \left\{ v = \tilde{v}|_{\Gamma_k} : \tilde{v} \in H^{\frac{1}{2}}(\Gamma), \text{supp}(\tilde{v}) \subset \Gamma_k \right\} \subset H^{\frac{1}{2}}(\Gamma_k), \quad \text{with } k \in \{i, c\}.$$

Now, in order to define a norm in the Sobolev spaces  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$  and  $\widehat{H}^{\frac{1}{2}}(\Gamma_c)$ , the following two operators are constructed:

Let  $v_k$ , with  $k \in \{i, c\}$  be the unique weak solution of the problem

$$\begin{cases} -\Delta v_k = 0 & \text{in } \Omega, \\ v_k = w & \text{on } \Gamma_k, \\ v_k = 0 & \text{on } \Gamma \setminus \Gamma_k, \end{cases} \quad w \in \widehat{H}^{\frac{1}{2}}(\Gamma_k), \quad (1)$$

then

$$\begin{aligned} A_k : \widehat{H}^{\frac{1}{2}}(\Gamma_k) &\longrightarrow H^{-\frac{1}{2}}(\Gamma_k), \\ w &\longmapsto A_k w = \partial_n v_k|_{\Gamma_k}. \end{aligned}$$

Thereby we define the norms in the Sobolev spaces  $\widehat{H}^{\frac{1}{2}}(\Gamma_k)$  with  $k \in \{i, c\}$  by:

$$\|w\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_k)}^2 = \langle A_k w, w \rangle = \|\nabla v_k\|_{L^2(\Omega)}^2. \quad (2)$$

For the numerical procedure, the Neumann to Dirichlet operator is defined by:

$$\begin{aligned} S_{ND} : H^{-\frac{1}{2}}(\Gamma_i) &\longrightarrow \widehat{H}^{\frac{1}{2}}(\Gamma_i) \\ \lambda &\longmapsto S_{ND}\lambda = \xi|_{\Gamma_i}, \end{aligned}$$

where  $\xi$  is the weak solution of:

$$\begin{cases} -\Delta \xi = 0 & \text{in } \Omega, \\ \partial_n \xi = \lambda & \text{on } \Gamma_i, \\ \xi = 0 & \text{on } \Gamma \setminus \Gamma_i. \end{cases} \quad (3)$$

In this paper we use the notation  $a \lesssim b$  to signify that there exists a constant  $C > 0$ , independent of mesh size and noisy level, such that  $a \leq C b$ .

## 2 The model problem

Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^2$ . We assume that the boundary  $\Gamma$  of  $\Omega$  is split into three open parts  $\Gamma_0$ ,  $\Gamma_i$  and  $\Gamma_c$  such that  $\Gamma = \overline{\Gamma_0 \cup \Gamma_i \cup \Gamma_c}$ . We suppose that  $\Gamma_i$  and  $\Gamma_c$  are separated by a non-zero measure portion  $\Gamma_0$ . We consider the Cauchy problem for the Laplace equation:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = T & \text{on } \Gamma_c, \\ \partial_n u = \phi & \text{on } \Gamma_c, \end{cases} \quad (4)$$

where  $\Delta$  is the Laplace operator,  $T$  and  $\phi$  are two given functions in  $\widehat{H}^{\frac{1}{2}}(\Gamma_c)$  and  $H^{-\frac{1}{2}}(\Gamma_c)$ . We assume that there exists a pair of compatible data  $(T, \phi)$ , and we denote by  $\bar{v}$  the corresponding Cauchy solution. We consider a family of unnecessary compatible data  $(T^\delta, \phi^\delta) \in \widehat{H}^{\frac{1}{2}}(\Gamma_c) \times H^{-\frac{1}{2}}(\Gamma_c)$  such that :

$$\|T - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2 + \|\phi - \phi^\delta\|_{H^{-\frac{1}{2}}(\Gamma_c)}^2 \leq \delta^2 \quad (5)$$

To solve problem (4), we define the energetic variational minimization problem:

$$\min_{\eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i)} H(\eta) \quad (6)$$

where

$$H(\eta) = J(u(\phi, \eta)), \quad \forall \eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i),$$

with

$$J(u) = \frac{1}{2} \|u - T\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2,$$

and  $u(\phi, \eta)$  is the unique weak solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \partial_n u = \phi & \text{on } \Gamma_c, \\ u = \eta & \text{on } \Gamma_i, \\ u = 0 & \text{on } \Gamma_0. \end{cases} \quad (7)$$

In the case of noisy data, we consider the noisy energetic variational minimization problem:

$$\min_{\eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i)} H^\delta(\eta), \quad (8)$$

where

$$H^\delta(\eta) = J^\delta(u(\phi^\delta, \eta)), \quad \forall \eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i),$$

with

$$J^\delta(u) = \frac{1}{2} \|u - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2,$$

and  $u(\phi^\delta, \eta)$  is the unique weak solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \partial_n u = \phi^\delta & \text{on } \Gamma_c, \\ u = \eta & \text{on } \Gamma_i, \\ u = 0 & \text{on } \Gamma_0. \end{cases} \quad (9)$$

### 3 The energetic variational minimization problem

Let  $S = \left\{ u(\phi, \eta)|_{\Gamma_c} \text{ where } u \text{ is the unique weak solution of (7) : such that } \eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i) \right\}$ .

**Proposition 1**  $S$  forms a dense set in  $\widehat{H}^{\frac{1}{2}}(\Gamma_c)$  and we have  $\inf_{\eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i)} H(\eta) = 0$ .

**Proof.** Let  $w$  in  $\widehat{H}^{\frac{1}{2}}(\Gamma_c)$  such that

$$\int_{\Gamma_c} u(\phi, \eta) w d\Gamma = 0 \quad \forall \eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i). \quad (10)$$

We shall prove that  $w = 0$ , on  $\Gamma_i$ . Let  $\psi$  be the solution of

$$\begin{cases} -\Delta\psi = 0 & \text{in } \Omega, \\ \partial_n\psi = w & \text{on } \Gamma_c, \\ \psi = 0 & \text{on } \Gamma_i, \\ \psi = 0 & \text{on } \Gamma_0. \end{cases} \quad (11)$$

Using the variational formulations of (11) and (7) and equation (10), we have :

$$\int_{\Gamma_c} \phi\psi d\Gamma = \int_{\Gamma_i} \partial_n\psi \eta d\Gamma, \quad \forall \eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i). \quad (12)$$

Replacing  $\eta$  by  $-\eta$  in the last equation, we obtain

$$\int_{\Gamma_c} \phi\psi d\Gamma = \int_{\Gamma_i} \partial_n\psi \eta d\Gamma = - \int_{\Gamma_i} \partial_n\psi \eta d\Gamma = 0, \quad \forall \eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i), \quad (13)$$

which means that

$$\partial_n\psi = 0 \text{ on } \Gamma_i.$$

Thus, the function  $\psi \in H^1(\Omega)$  satisfies the Cauchy problem

$$\begin{cases} -\Delta\psi = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma_i, \\ \partial_n\psi = 0 & \text{on } \Gamma_i \\ u = 0 & \text{on } \Gamma_0. \end{cases} \quad (14)$$

from which it follows that  $\psi \equiv 0$  and therefore  $w = \partial_n\psi = 0$  on  $\Gamma_c$ .

From the density of  $S$ , we immediately have  $\inf_{\eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i)} H(\eta) = 0$ . ■

**Proposition 2** Assume that problem (4) admits a solution (i.e. a case of compatible data), then solving the Cauchy problem (4) is equivalent to solving (6).

**Proof.** In the case of the compatible Cauchy problem, let  $\bar{v}$  be the unique solution of (4) such that  $\bar{\eta}_C = \bar{v}|_{\Gamma_i}$ . The well-posed problem (7) has a unique solution  $u(\phi, \bar{\eta}_C)$  for  $\eta = \bar{\eta}_C$ . Let  $z = u(\phi, \bar{\eta}_C) - \bar{v}$ , then it is the unique weak solution of (4) with  $\phi = 0$  and  $\eta = 0$ . Thus  $z = 0$  on  $\Omega$ ,  $\bar{v} = u(\phi, \bar{\eta}_C)$  and  $H(\bar{\eta}_C) = 0$ . This means that  $\bar{\eta}_C$  solves the variational minimization problem (6). Conversely, let  $\bar{\eta}$  be a solution of (6) and  $u(\phi, \bar{\eta})$  the corresponding weak solution of (7), then  $H(\bar{\eta}) = 0$  ( $\inf_{\eta \in \widehat{H}^{\frac{1}{2}}(\Gamma_i)} H(\eta) = 0$ ) which follows that  $u(\phi, \bar{\eta}) = T|_{\Gamma_c}$ . In the case of compatible Cauchy

data, we deduce that  $u(\phi, \bar{\eta})$  is the solution of (4) and  $\bar{\eta} = \bar{\eta}_C$ . ■

**Proposition 3** Assume there exists a unique Cauchy solution  $\bar{v} \in H^1(\Omega)$  for a given compatible pair of data  $(T, \phi) \in \widehat{H}^{\frac{1}{2}}(\Gamma_c) \times H^{-\frac{1}{2}}(\Gamma_c)$ . Let  $(T^\delta, \phi^\delta) \in \widehat{H}^{\frac{1}{2}}(\Gamma_c) \times H^{-\frac{1}{2}}(\Gamma_c)$  be any sequence of noisy, unnecessary compatible, data such that

$$\|T - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2 + \|\phi - \phi^\delta\|_{H^{-\frac{1}{2}}(\Gamma_c)}^2 \leq \delta^2, \quad (15)$$

then the noisy energetic variational minimization problem (8) has a unique solution  $\bar{\eta}^\delta$  which strongly converges, as  $\delta \rightarrow 0$ , to the Cauchy missing data  $\bar{\eta}_C = \bar{v}|_{\Gamma_i}$ . Moreover, the solution  $u(\phi^\delta, \bar{\eta}^\delta)$  of (9) strongly converges in  $H^1(\Omega)$  to the Cauchy solution  $\bar{v}$ .

**Proof.** It is clear that, the noisy energetic variational minimization problem (8) is found to have the same property of the energetic variational minimization problem (6). Then problem (8) has a unique solution  $\bar{\eta}^\delta$  and by using a triangular inequality and the trace theorem between  $H^1(\Omega)$  et  $H^{\frac{1}{2}}(\Gamma)$  we obtain:

$$\begin{aligned} H^\delta(\bar{\eta}_C) &= J^\delta(u(\phi^\delta, \bar{\eta}_C)), \\ &= \frac{1}{2} \|u(\phi^\delta, \bar{\eta}_C) - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2, \\ &= \frac{1}{2} \|u(\phi^\delta, \bar{\eta}_C) - u(\phi, \bar{\eta}_C) + u(\phi, \bar{\eta}_C) - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2, \\ &\leq \|u(\phi^\delta, \bar{\eta}_C) - u(\phi, \bar{\eta}_C)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2 + \|u(\phi, \bar{\eta}_C) - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2, \\ &\lesssim \|u(\phi^\delta, \bar{\eta}_C) - u(\phi, \bar{\eta}_C)\|_{H^1(\Omega)}^2 + \|u(\phi, \bar{\eta}_C) - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2. \end{aligned}$$

Let  $z = u(\phi^\delta, \bar{\eta}_C) - u(\phi, \bar{\eta}_C)$ , then  $z$  is the weak solution of

$$\begin{cases} -\Delta z = 0 & \text{in } \Omega, \\ \partial_n z = \phi^\delta - \phi & \text{on } \Gamma_c, \\ z = 0 & \text{on } \Gamma_i, \\ u = 0 & \text{on } \Gamma_0. \end{cases} \quad (16)$$

then we have

$$\|z\|_{H^1(\Omega)} \lesssim \|\phi^\delta - \phi\|_{H^{-\frac{1}{2}}(\Gamma_c)}.$$

As  $u(\phi, \bar{\eta}_C) = \bar{v}$ , then we obtain

$$\begin{aligned} H^\delta(\bar{\eta}_C) &\lesssim \|\phi - \phi^\delta\|_{H^{-\frac{1}{2}}(\Gamma_c)}^2 + \|T - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2 \\ &\lesssim \delta^2. \end{aligned}$$

Now, since  $H^\delta(\bar{\eta}^\delta) \leq H^\delta(\bar{\eta}_C) \lesssim \delta^2$  therefore  $\lim_{\delta \rightarrow 0} H^\delta(\bar{\eta}^\delta) = 0$ . Given the fact that  $H^\delta$  is coercive in  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$ , we can deduce that  $\bar{\eta}^\delta$  is uniformly bonded in  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$ . Thus  $\bar{v}_i^\delta$  is uniformly bonded in  $H^1(\Omega)$  where  $\bar{v}_i^\delta$  is the unique solution of (1) with  $k = i$  and  $w = \bar{\eta}^\delta$ . Up to a subsequence, we have  $\bar{v}_i^\delta$  weakly converges to  $\bar{v}_i$  in  $H^1(\Omega)$  and strongly in  $L^2(\Omega)$  ( $\bar{v}_i$  is the unique solution of (1) with  $k = i$  and  $w = \bar{\eta}^0$ ). Using Poincarè inequality, we have the strong convergence of  $\bar{v}_i^\delta$  to  $\bar{v}_i$  in  $H^1(\Omega)$ . Consequently,  $\bar{\eta}^\delta$  strongly converges to  $\bar{\eta}^0$  in  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$ . Since the sequence  $T^\delta$  strongly converges to  $T$  in  $\widehat{H}^{\frac{1}{2}}(\Gamma_c)$  and  $\phi^\delta$  strongly converges to  $\phi$  in  $H^{-\frac{1}{2}}(\Gamma_c)$  as well, we infer that the

sequences  $u(\phi^\delta, \bar{\eta}^\delta)$  which are the solutions of (9), strongly converge in  $H^1(\Omega)$  to  $\bar{u}(\phi, \bar{\eta}^0)$  which is the unique solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \partial_n u = \phi & \text{on } \Gamma_c, \\ u = \bar{\eta}^0 & \text{on } \Gamma_i, \\ u = 0 & \text{on } \Gamma_0. \end{cases} \quad (17)$$

As  $H^\delta(\bar{\eta}^\delta) = \frac{1}{2} \|u(\phi^\delta, \bar{\eta}^\delta) - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2 \lesssim \delta^2$ , we deduce that  $u(\phi^\delta, \bar{\eta}^\delta) - T^\delta$  strongly converges to 0 in  $\widehat{H}^{\frac{1}{2}}(\Gamma_c)$ . Hence  $\bar{u}(\phi, \bar{\eta}^0) = T$  over  $\Gamma_c$ . Thanks to the uniqueness of the Cauchy problem, we conclude that  $\bar{u}(\phi, \bar{\eta}^0) = \bar{v}$  and  $\bar{\eta}^0 = \bar{\eta}_C$ . ■

**Proposition 4** *The functional  $H$  is twice Fréchet differentiable and strictly convex over  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$ .*

**Proof.** Let  $h$  be a function in  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$ , then for all  $\varepsilon \geq 0$  we have:

$$\begin{aligned} H(\eta + \varepsilon h) &= \frac{1}{2} \|u(\phi, \eta + \varepsilon h) - T\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2, \\ &= \frac{1}{2} \|u(\phi, \eta) - T + \varepsilon u(0, h)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2, \\ &= \frac{1}{2} \|u(\phi, \eta) - T\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2 + \varepsilon \int_{\Gamma_c} A_c(u(\phi, \eta) - T) u(0, h) d\Gamma + \frac{\varepsilon^2}{2} \|u(0, h)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2, \\ &= H(\eta) + \varepsilon \int_{\Gamma_c} A_c(u(\phi, \eta) - T) u(0, h) d\Gamma + \frac{\varepsilon^2}{2} \|u(0, h)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2, \end{aligned}$$

then

$$\nabla H(\eta).h = \lim_{\varepsilon \rightarrow 0} \frac{H(\eta + \varepsilon h) - H(\eta)}{\varepsilon} = \int_{\Gamma_c} A_c(u(\phi, \eta) - T) u(0, h) d\Gamma. \quad (18)$$

Let  $\psi$  be the weak solution of

$$\begin{cases} -\Delta \psi = 0 & \text{in } \Omega, \\ \partial_n \psi = A_c(u(\phi, \eta) - T) & \text{on } \Gamma_c, \\ \psi = 0 & \text{on } \Gamma_i, \\ \psi = 0 & \text{on } \Gamma_0. \end{cases} \quad (19)$$

and knowing that  $u(0, h)$  is the weak solution of (7) we infer

$$\int_{\Gamma_c} A_c(u(\phi, \eta) - T) u(0, h) d\Gamma = - \int_{\Gamma_i} \partial_n \psi u(0, h) d\Gamma = - \int_{\Gamma_i} \partial_n \psi h d\Gamma, \forall h \in \widehat{H}^{\frac{1}{2}}(\Gamma_i).$$

Consequently,

$$H'(\eta) = -\partial_n \psi|_{\Gamma_i}. \quad (20)$$

Now we prove that  $J$  is twice differentiable and strictly convex.

Consider the problem:

$$\begin{cases} -\Delta \xi = 0 & \text{in } \Omega, \\ \partial_n \xi = A_c u(0, h) & \text{on } \Gamma_c, \\ \xi = 0 & \text{on } \Gamma_i, \\ \xi = 0 & \text{on } \Gamma_0. \end{cases} \quad (21)$$

Following Green's formula, we obtain

$$\int_{\Gamma_c} A_c(u(0, h)) u(0, h) d\Gamma = - \int_{\Gamma_i} \partial_n \xi u(0, h) d\Gamma = - \int_{\Gamma_i} \partial_n \xi h d\Gamma, \forall h \in \widehat{H}^{\frac{1}{2}}(\Gamma_i),$$

Using (18), we have

$$\langle H''(\eta) h, \tau \rangle = \int_{\Gamma_c} A_c(u(0, \tau)) u(0, h) d\Gamma, \forall (h, \tau) \in \widehat{H}^{\frac{1}{2}}(\Gamma_i) \times \widehat{H}^{\frac{1}{2}}(\Gamma_i), \quad (22)$$

Thus,

$$\langle H''(\eta) h, h \rangle = \int_{\Gamma_c} A_c(u(0, h)) u(0, h) d\Gamma = - \int_{\Gamma_i} \partial_n \xi u(0, h) d\Gamma, \forall h \in \widehat{H}^{\frac{1}{2}}(\Gamma_i). \quad (23)$$

It follows that  $H$  is twice differentiable and its second gradient is given by:

$$H''(\eta) h = -\partial_n \xi$$

Now we prove that functional  $H$  is strictly convex. First, we observe that it is convex since

$$\langle H''(\eta) h, h \rangle = \int_{\Gamma_c} A_c(u(0, h)) u(0, h) d\Gamma = \|u(0, h)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2 \geq 0 \quad (24)$$

Furthermore, if  $\langle H''(\eta) h, h \rangle = 0$  then  $u(0, h)|_{\Gamma_c} = 0$ . It follows that  $u(0, h)$  satisfies the Cauchy problem with homogeneous Cauchy data:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_c, \\ \partial_n u = 0 & \text{on } \Gamma_c, \\ u = 0 & \text{on } \bar{\Gamma}_0. \end{cases} \quad (25)$$

From Holmgren's theorem [13], we have  $u(0, h) \equiv 0$  in  $\bar{\Omega}$ . Hence,  $H$  is strictly convex on  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$ .  
■

**Proposition 5** *The functional  $H$  is coercive over  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$ .*

**Proof.** Assume by contradiction that  $H$  is not coercive over  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$  then there exists a sequence  $(\psi_n)_{n \in \mathbb{N}} \subset \widehat{H}^{\frac{1}{2}}(\Gamma_i)$  such that

$$\|\psi_n\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} = 1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} H(\psi_n) = +\infty.$$

We have  $H(\psi_n) = J(u(\phi, \psi_n))$  and by using the well-posedness of the weak formulation of (7), we obtain

$$\|u(\phi, \psi_n)\|_{H^1(\Omega)} \lesssim \|\psi_n\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} + \|\phi\|_{H^{-\frac{1}{2}}(\Gamma_c)}.$$

Consequently, by using the trace theorem from  $H^1(\Omega)$  to  $H^{\frac{1}{2}}(\Gamma)$ , we get

$$\|u(\phi, \psi_n)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} \lesssim \|u(\phi, \psi_n)\|_{H^{\frac{1}{2}}(\Gamma)} \lesssim \|u(\phi, \psi_n)\|_{H^1(\Omega)} \leq \|\phi\|_{H^{-\frac{1}{2}}(\Gamma_c)} + \|\psi_n\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)},$$



and  $u(\phi, \psi_n)$  is bounded in  $\widehat{H}^{\frac{1}{2}}(\Gamma_c)$  which is in contradiction with the fact that  $J$  is coercive in  $\widehat{H}^{\frac{1}{2}}(\Gamma_c)$ .  $\blacksquare$

**Remark:** Using the linearities of problems (7) and (9) we show that:

$$H(\eta) = \frac{1}{2}a(\eta, \eta) - l(\eta) + d, \quad (26)$$

$$H^\delta(\eta) = \frac{1}{2}a(\eta, \eta) - l^\delta(\eta) + d^\delta, \quad (27)$$

with  $a$  a symmetric bilinear form on  $\widehat{H}^{\frac{1}{2}}(\Gamma_i) \times \widehat{H}^{\frac{1}{2}}(\Gamma_i)$  defined by

$$a(\eta, h) = \int_{\Gamma_c} A_c(u(0, \eta)) u(0, h) d\Gamma, \quad \forall (\eta, h) \in (\widehat{H}^{\frac{1}{2}}(\Gamma_i) \times \widehat{H}^{\frac{1}{2}}(\Gamma_i)), \quad (28)$$

$l$  and  $l^\delta$  are two linear forms defined by:

$$\begin{aligned} l(h) &= - \int_{\Gamma_c} A_c(u(0, h)) (u(\phi, 0) - T) d\Gamma, \quad \forall h \in \widehat{H}^{\frac{1}{2}}(\Gamma_i), \\ l^\delta(h) &= - \int_{\Gamma_c} A_c(u(0, h)) (u(\phi^\delta, 0) - T^\delta) d\Gamma, \quad \forall h \in \widehat{H}^{\frac{1}{2}}(\Gamma_i), \end{aligned}$$

and  $d$  and  $d^\delta$  are two constant defined by:

$$\begin{aligned} d &= \int_{\Gamma_c} A_c(u(\phi, 0) - T) (u(\phi, 0) - T) d\Gamma, \\ d^\delta &= \int_{\Gamma_c} A_c(u(\phi^\delta, 0) - T^\delta) (u(\phi^\delta, 0) - T^\delta) d\Gamma. \end{aligned}$$

Using Proposition (4) and (5), we deduce that  $a$  is a symmetric bilinear form strictly convex and coercive on  $\widehat{H}^{\frac{1}{2}}(\Gamma_i) \times \widehat{H}^{\frac{1}{2}}(\Gamma_i)$ . Also,  $l$  and  $l^\delta$  are two continuous linear forms on  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$ .

**Proposition 6** *Under the assumption of Proposition (3), let  $\bar{\eta}^\delta$  (Resp.  $\bar{\eta}$ ) be the unique solution of the noisy energetic variational minimization problem (8) (Resp. the energetic variational minimization problem (6)). Then it yields*

$$\|\bar{\eta}^\delta - \bar{\eta}\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} \lesssim \delta. \quad (29)$$

**Proof.** Using the first order optimality condition of problems (8) and (6), it gives:

$$\begin{aligned} a(\bar{\eta}^\delta, h) &= l^\delta(h), \quad \forall h \in \widehat{H}^{\frac{1}{2}}(\Gamma_i), \\ a(\bar{\eta}, h) &= l(h), \quad \forall h \in \widehat{H}^{\frac{1}{2}}(\Gamma_i). \end{aligned}$$

Which lead to:

$$\begin{aligned} a(\bar{\eta} - \bar{\eta}^\delta, \bar{\eta} - \bar{\eta}^\delta) &= l(\bar{\eta} - \bar{\eta}^\delta) - l^\delta(\bar{\eta} - \bar{\eta}^\delta), \\ &= \int_{\Gamma_c} A_c(u(0, \bar{\eta} - \bar{\eta}^\delta)) \left( u(\phi^\delta - \phi, 0) + (T - T^\delta) \right) d\Gamma. \end{aligned}$$

From the  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$ -ellipticity of  $a$ , the Cauchy and the triangular inequalities we have:

$$\begin{aligned} \|\bar{\eta} - \bar{\eta}^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)}^2 &\lesssim a(\bar{\eta} - \bar{\eta}^\delta, \bar{\eta} - \bar{\eta}^\delta), \\ &= \int_{\Gamma_c} A_c(u(0, \bar{\eta} - \bar{\eta}^\delta)) \left( u(\phi^\delta - \phi, 0) + (T - T^\delta) \right) d\Gamma, \\ &\lesssim \|A_c(u(0, \bar{\eta} - \bar{\eta}^\delta))\|_{H^{-\frac{1}{2}}(\Gamma_c)} \|u(\phi^\delta - \phi, 0) + (T - T^\delta)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}, \\ &\lesssim \|A_c(u(0, \bar{\eta} - \bar{\eta}^\delta))\|_{H^{-\frac{1}{2}}(\Gamma_c)} \left( \|u(\phi^\delta - \phi, 0)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} + \|T - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} \right). \end{aligned}$$

Using the continuities of the operator  $A_c$  and the trace operator between  $H^1(\Omega)$  and  $H^{\frac{1}{2}}(\Gamma)$  we obtain

$$\begin{aligned} \|\bar{\eta} - \bar{\eta}^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)}^2 &\lesssim \|u(0, \bar{\eta} - \bar{\eta}^\delta)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} \left( \|u(\phi^\delta - \phi, 0)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} + \|T - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} \right), \\ &\lesssim \|u(0, \bar{\eta} - \bar{\eta}^\delta)\|_{H^1(\Omega)} \left( \|u(\phi^\delta - \phi, 0)\|_{H^1(\Omega)} + \|T - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} \right), \\ &\lesssim \|\bar{\eta} - \bar{\eta}^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} \left( \|\phi^\delta - \phi\|_{H^{-\frac{1}{2}}(\Gamma_c)} + \|T - T^\delta\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} \right). \end{aligned}$$

By using assumption (15), we have the a priori error estimate (29).  $\blacksquare$

## 4 Discrete energetic variational minimization problem

In this section, we present the discretized formulation of (8) using a classical  $C^0$  finite element based on polynomials of the  $k$ th degree ( $k \geq 1$ ). Let  $\tau^h$  be a regular mesh (see [5]) of the domain  $\Omega$  with a maximal size  $h$ ,  $\bar{\Omega} = \bigcup_{K \in \tau^h} \bar{K}$ . We denote by  $\mathbb{P}_k(T)$  the space of polynomials of a degree less or equal to  $k \in \mathbb{N}^*$  defined on  $T \in \tau^h$ . Then we can define the following standard finite element spaces:

$$\begin{aligned} V^h &= \left\{ v^h \in C^0(\bar{\Omega}) : v^h|_T \in \mathbb{P}_k(T), \forall T \in \tau^h \right\}, \\ W^h &= \left\{ v^h \in L^2(\bar{\Omega}) : v^h|_T \in \mathbb{P}_k(T), \forall T \in \tau^h \right\}, \\ V_\Gamma^h &= V^h|_\Gamma \subset H^{\frac{1}{2}}(\Gamma), \\ M_{\Gamma_k}^h &= W^h|_{\Gamma_k} \subset H^{-\frac{1}{2}}(\Gamma_k), \quad \text{with } k \in \{i, c\}, \\ \widehat{H}^{h, \frac{1}{2}}(\Gamma_k) &= \left\{ v^h = \tilde{v}|_{\Gamma_k} : \tilde{v}^h \in V_\Gamma^h, \text{supp}(\tilde{v}^h) \subset \Gamma_k \right\} \subset \widehat{H}^{\frac{1}{2}}(\Gamma_k), \quad \text{with } k \in \{i, c\}. \end{aligned}$$

Let  $T_h = \Pi_{\widehat{H}^{h, \frac{1}{2}}(\Gamma_c)}^h T$ ,  $\phi_h = \pi_{M_{\Gamma_c}^h}^h \phi$ ,  $T_h^\delta = \Pi_{\widehat{H}^{h, \frac{1}{2}}(\Gamma_c)}^h T^\delta$ , and  $\phi_h^\delta = \pi_{M_{\Gamma_c}^h}^h \phi^\delta$  where  $\Pi_{\widehat{H}^{h, \frac{1}{2}}(\Gamma_c)}^h$  is the standard Lagrange interpolation operators in  $\widehat{H}^{h, \frac{1}{2}}(\Gamma_c)$  and  $\pi_{M_{\Gamma_c}^h}^h$  is the  $L^2(\Gamma_c)$ -orthogonal projection operator on  $M_{\Gamma_c}^h$ . We recall that for all  $\phi \in H^s(\Gamma_c)$  and  $T \in H^{s+1}(\Gamma_c)$  with  $1 \leq s \leq k+1$  we have [5, 9]:

$$\|\phi - \pi_{M_{\Gamma_c}^h}^h \phi\|_{H^{-\frac{1}{2}}(\Gamma_c)} \lesssim h^{s+\frac{1}{2}} \|\phi\|_{H^s(\Gamma_c)}, \quad (30)$$

$$\|T - \Pi_{\widehat{H}^{h, \frac{1}{2}}(\Gamma_c)}^h T\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} \lesssim h^{s+\frac{1}{2}} \|T\|_{H^{s+1}(\Gamma_c)}. \quad (31)$$

We define the discrete energetic variational minimization problem by:

$$\min_{\eta \in \widehat{H}^{h, \frac{1}{2}}(\Gamma_i)} H_h(\eta), \quad (32)$$

where

$$H_h(\eta) = J_h(u^h(\phi_h^\delta, \eta)), \quad \forall \eta \in \widehat{H}^{h, \frac{1}{2}}(\Gamma_i),$$

with

$$J_h(u^h) = \frac{1}{2} \|u^h - T_h\|_{\widehat{H}^{h, \frac{1}{2}}(\Gamma_c)}^2,$$

and  $u^h(\phi_h, \eta)$  is the unique weak discrete solution of

$$\begin{cases} -\Delta u^h = 0 & \text{in } \Omega, \\ \partial_n u^h = \phi_h & \text{on } \Gamma_c, \\ u^h = \eta & \text{on } \Gamma_i, \\ u^h = 0 & \text{on } \Gamma_0. \end{cases} \quad (33)$$

Similarly, we define the noisy discrete energetic variational minimization problem by :

$$\min_{\eta \in \widehat{H}^{h\frac{1}{2}}(\Gamma_i)} H_h^\delta(\eta), \quad (34)$$

where

$$H_h^\delta(\eta) = J_h^\delta(u^h(\phi_h^\delta, \eta)), \quad \forall \eta \in \widehat{H}^{h\frac{1}{2}}(\Gamma_i),$$

with

$$J_h^\delta(u^h) = \frac{1}{2} \|u^h - T_h^\delta\|_{\widehat{H}^{h\frac{1}{2}}(\Gamma_c)}^2,$$

and  $u^h(\phi_h^\delta, \eta)$  is the unique weak solution of

$$\begin{cases} -\Delta u^h = 0 & \text{in } \Omega, \\ \partial_n u^h = \phi_h^\delta & \text{on } \Gamma_c, \\ u^h = \eta & \text{on } \Gamma_i, \\ u^h = 0 & \text{on } \Gamma_0. \end{cases} \quad (35)$$

**Proposition 7** *Assume there exists a unique discrete Cauchy solution  $\bar{v}^h \in V^h$  for a given discrete compatible pair of data  $(T_h, \phi_h) \in \widehat{H}^{h\frac{1}{2}}(\Gamma_c) \times M_{\Gamma_c}^h$ . Let  $(T_h^\delta, \phi_h^\delta) \in \widehat{H}^{h\frac{1}{2}}(\Gamma_c) \times M_{\Gamma_c}^h$  be any sequence of noisy, unnecessary compatible, data such that*

$$\|T_h - T_h^\delta\|_{\widehat{H}^{h\frac{1}{2}}(\Gamma_c)}^2 + \|\phi_h - \phi_h^\delta\|_{H^{-\frac{1}{2}}(\Gamma_c)}^2 \leq \delta^2 \quad (36)$$

*Then the noisy discrete energetic variational minimization problem (34) has a unique solution  $\bar{\eta}_h^\delta$  which strongly converges, as  $\delta \rightarrow 0$ , to the Cauchy missing data  $\eta_{Ch} = \bar{v}^h|_{\Gamma_i}$ . Moreover, the solution  $u(\phi_h^\delta, \bar{\eta}_h^\delta)$  of (35) strongly converges in  $V^h$  to the discrete Cauchy solution  $\bar{v}^h = u(\phi_h, \bar{\eta}_{Ch})$ .*

**Proof.** The proof is the same as that of Proposition (3) with the little difference that weak convergence implies strong convergence in a finite dimension space.

**Proposition 8** *Under assumption of Proposition (7), let  $\bar{\eta}_h^\delta$  (Resp.  $\bar{\eta}_h$ ) be the unique solution of the discrete noisy energetic variational minimization problem (34) (Resp. the discrete energetic variational minimization problem (32)). Then it yields*

$$\|\bar{\eta}_h^\delta - \bar{\eta}_h\|_{\widehat{H}^{h\frac{1}{2}}(\Gamma_i)} \lesssim \delta. \quad (37)$$

**Proof.** The proof is the same as the proof of Proposition (6).

**Proposition 9** *Let  $\bar{v}$  be the exact solution of the continuous Cauchy problem for a couple of compatible data  $(T, \phi) \in \widehat{H}^{h\frac{1}{2}}(\Gamma_c) \times M_{\Gamma_c}^h$ , such that  $\bar{v} \in H^s(\Omega)$  with  $1 \leq s \leq k+1$ . Under assumption of Proposition (6) and (8) the following estimates hold:*

$$\|\bar{\eta}_h^\delta - \bar{\eta}\|_{\widehat{H}^{h\frac{1}{2}}(\Gamma_i)} \lesssim h^{s-1} \left( \|\bar{v}\|_{H^{s-\frac{1}{2}}(\Gamma_i)} + \|\phi\|_{H^{s-\frac{3}{2}}(\Gamma_c)} + \|T\|_{H^{s-\frac{1}{2}}(\Gamma_c)} \right) + \delta, \quad (38)$$

$$\|u_h(\phi_h^\delta, \bar{\eta}_h^\delta) - \bar{v}\|_{H^1(\Omega)} \lesssim h^{s-1} \|\bar{v}\|_{H^s(\Omega)} + \delta. \quad (39)$$

**Proof.** Let  $\tilde{\eta}_h$  be the unique solution of the variational formulation:

$$a(\tilde{\eta}_h, \psi_h) = l(\psi_h), \quad \forall \psi_h \in \widehat{H}^{h\frac{1}{2}}(\Gamma_i). \quad (40)$$

To prove (38), we use the triangular inequality to obtain:

$$\|\bar{\eta}_h^\delta - \bar{\eta}\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} \leq \|\bar{\eta}_h^\delta - \bar{\eta}_h\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} + \|\tilde{\eta}_h - \bar{\eta}\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} + \|\bar{\eta}_h - \tilde{\eta}_h\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} \quad (41)$$

For the second term of the right-hand side of inequality (41), we use Céa Lemma, the Galerkin orthogonality and standard finite element a priori error estimate of the Lagrange interpolation operator  $\Pi_{\widehat{H}^{h\frac{1}{2}}(\Gamma_i)}^h$  to find:

$$\|\tilde{\eta}_h - \bar{\eta}\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} \lesssim h^{s-1} \|\bar{\eta}\|_{H^{s-\frac{1}{2}}(\Gamma_i)}. \quad (42)$$

To estimate the third term of the right-hand side of inequality (41), we first write the optimality condition of problems (6) and (32):

$$\begin{aligned} a(\tilde{\eta}_h, \psi_h) &= l(\psi_h), \quad \forall \psi_h \in \widehat{H}^{h\frac{1}{2}}(\Gamma_i), \\ a(\bar{\eta}_h, \psi_h) &= l_h(\psi_h), \quad \forall \psi_h \in \widehat{H}^{h\frac{1}{2}}(\Gamma_i), \end{aligned}$$

with

$$l_h(\psi_h) = - \int_{\Gamma_c} A_c(u(0, \psi_h))(u(\phi_h, 0) - T_h) d\Gamma, \quad \forall \psi_h \in \widehat{H}^{\frac{1}{2}}(\Gamma_i).$$

Thus

$$\begin{aligned} a(\bar{\eta}_h - \tilde{\eta}_h, \bar{\eta}_h - \tilde{\eta}_h) &= l_h(\bar{\eta}_h - \tilde{\eta}_h) - l(\bar{\eta}_h - \tilde{\eta}_h), \\ &= \int_{\Gamma_c} A_c(u(0, \bar{\eta}_h - \tilde{\eta}_h)) \left( u(\phi - \phi_h, 0) + T_h - T \right) d\Gamma, \\ &\leq \|A_c(u(0, \bar{\eta}_h - \tilde{\eta}_h))\|_{H^{-\frac{1}{2}}(\Gamma_c)} \left( \|u(\phi - \phi_h, 0)\|_{H^{\frac{1}{2}}(\Gamma_c)} + \|T_h - T\|_{H^{\frac{1}{2}}(\Gamma_c)} \right). \end{aligned}$$

Then using the  $\widehat{H}^{\frac{1}{2}}(\Gamma_i)$ -ellipticity of the operator  $a$  and the continuities of the operator  $A_c$  and the trace operator between  $H^1(\Omega)$  and  $H^{\frac{1}{2}}(\Gamma)$  we deduce

$$\begin{aligned} \|\bar{\eta}_h - \tilde{\eta}_h\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)}^2 &\leq \|A_c(u(0, \bar{\eta}_h - \tilde{\eta}_h))\|_{H^{-\frac{1}{2}}(\Gamma_c)} \left( \|u(\phi - \phi_h, 0)\|_{H^{\frac{1}{2}}(\Gamma_c)} + \|T_h - T\|_{H^{\frac{1}{2}}(\Gamma_c)} \right), \\ &\lesssim \|u(0, \bar{\eta}_h - \tilde{\eta}_h)\|_{H^{\frac{1}{2}}(\Gamma_c)} \left( \|u(\phi - \phi_h, 0)\|_{H^{\frac{1}{2}}(\Gamma_c)} + \|T_h - T\|_{H^{\frac{1}{2}}(\Gamma_c)} \right), \\ &\lesssim \|u(0, \bar{\eta}_h - \tilde{\eta}_h)\|_{H^1(\Omega)} \left( \|u(\phi - \phi_h, 0)\|_{H^1(\Omega)} + \|T_h - T\|_{H^{\frac{1}{2}}(\Gamma_c)} \right), \\ &\lesssim \|\bar{\eta}_h - \tilde{\eta}_h\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} \left( \|\phi - \phi_h\|_{H^{-\frac{1}{2}}(\Gamma_c)} + \|T_h - T\|_{H^{\frac{1}{2}}(\Gamma_c)} \right). \end{aligned}$$

Due to the a priori error estimates (30) and (31) we have:

$$\|\bar{\eta}_h - \tilde{\eta}_h\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} \lesssim h^{s-1} \left( \|\phi\|_{H^{s-\frac{3}{2}}(\Gamma_c)} + \|T\|_{H^{s-\frac{1}{2}}(\Gamma_c)} \right). \quad (43)$$

From Proposition (8), inequalities (41), (42) and (43) we establish:

$$\|\bar{\eta}_h^\delta - \bar{\eta}\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} \lesssim h^{s-1} \left( \|\bar{v}\|_{H^{s-\frac{1}{2}}(\Gamma_i)} + \|\phi\|_{H^{s-\frac{3}{2}}(\Gamma_c)} + \|T\|_{H^{s-\frac{1}{2}}(\Gamma_c)} \right) + \delta. \quad (44)$$

Now to prove (39), we use a triangular inequality to have:

$$\begin{aligned} \|u_h(\phi_h^\delta, \bar{\eta}_h^\delta) - \bar{v}\|_{H^1(\Omega)} &\leq \|u_h(\phi_h^\delta, \bar{\eta}_h^\delta) - u_h(\phi_h^\delta, \bar{\eta}_h)\|_{H^1(\Omega)} + \|u_h(\phi_h^\delta, \bar{\eta}_h) - u_h(\phi_h, \bar{\eta}_h)\|_{H^1(\Omega)} \\ &\quad + \|u_h(\phi_h, \bar{\eta}_h) - \bar{v}\|_{H^1(\Omega)}. \end{aligned}$$

Thanks to the classical regularity results for elliptic Boundary Value Problems type (35), we obtain:

$$\|u_h(\phi_h^\delta, \bar{\eta}_h^\delta) - \bar{v}\|_{H^1(\Omega)} \lesssim \|\bar{\eta}_h^\delta - \bar{\eta}_h\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)} + \|\phi_h - \phi_h^\delta\|_{H^{-\frac{1}{2}}(\Gamma_c)} + \|u_h(\phi_h, \bar{\eta}_h) - \bar{v}\|_{H^1(\Omega)}.$$

As  $u_h(\phi_h, \bar{\eta}_h)$  is the weak discrete solution of (33) and  $\bar{v}$  is also the exact solution of (7), we have by using a standard finite element a priori error estimate (see [9]):

$$\|u_h(\phi_h, \bar{\eta}_h) - \bar{v}\|_{H^1(\Omega)} \lesssim h^{s-1} \|\bar{v}\|_{H^s(\Omega)}. \quad (45)$$

Finally, using Proposition (8) and inequalities (36) and (45), we deduce:

$$\|u_h(\phi_h^\delta, \bar{\eta}_h^\delta) - \bar{v}\|_{H^1(\Omega)} \lesssim \delta + h^{s-1} \|\bar{v}\|_{H^s(\Omega)}.$$

■

## 5 Numerical procedure

To solve numerically the proposed formulation (9) of the data completion problem, we apply the following modified conjugate gradient algorithm:

1. Choose  $\eta_0$  and set  $n = 0$ .
2. Solve the direct problem (7) with  $\eta = \eta_n$  and determine the residual:

$$r_n = (u(\phi^\delta, \eta_n) - T^\delta)|_{\Gamma_c}$$

3. Determine the gradient  $g_n = H'(\eta_n)$  by solving the adjoint problem (19) with

$$H'(\eta_n) = -\partial_n \psi|_{\Gamma_i},$$

then calculate

$$d_n = -g_n + \beta_{n-1} d_{n-1},$$

$$\text{with the convention } \beta_{-1} = 0, \beta_{n-1} = \frac{\|g_n\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)}^2}{\|g_{n-1}\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)}^2}, \forall n > 0.$$

4. Calculate  $z_n = S_{ND} d_n$  by solving (3).

5. Determine  $u(0, z_n)$  by solving the direct problem (7) with  $\phi = 0$  and  $\eta = z_n$

$$\alpha_n = \frac{\|g_n\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_i)}^2}{\|u(0, z_n)\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}^2},$$

$$\eta_{n+1} = \eta_n + \alpha_n z_n.$$

6. Increase  $n$  by one and go to (2) if  $\|r_{n+1}\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} \leq \|r_n\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)}$ .

The fourth step of the algorithm is a filtering procedure. It can be viewed as a regularization strategy that allows our algorithm to converge numerically with the same optimal rates of convergence proved in Proposition (9).

As a stopping criterion, we choose a classical one (see [10]), the first  $k$  such that:

$$\|r_{n+1}\|_{\widehat{H}^{\frac{1}{2}}(\Gamma_c)} < \beta * \delta,$$

where  $\beta$  is some number which is greater than unity. Similar to [10, 12] we take  $\beta = 1.1$ .

## 6 Numerical results

To illustrate the performance of our method, we consider four cases. In the first case, the numerical

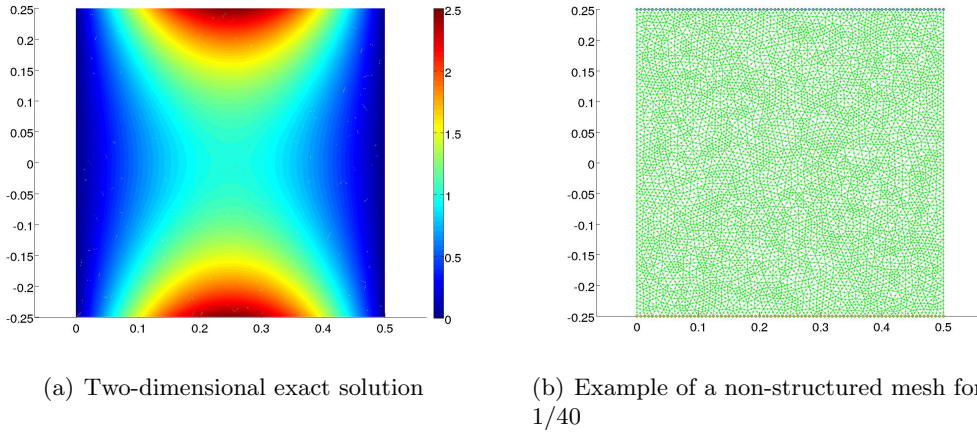


Figure 1: Exact solution and an example of non-structured mesh on the square domain

tests are made on a square domain defined by  $\Omega = ]0, 0.5[ \times ] - 0.25, 0.25[$ . The boundary of  $\Omega$  ( $\partial\Omega$ ) is divided into three parts:

$$\begin{aligned} \Gamma_c &= \{(x, -0.25) : 0 < x < 0.5\}, \\ \Gamma_i &= \{(x, 0.25) : 0 < x < 0.5\}, \\ \Gamma_0 &= \Gamma \setminus (\overline{\Gamma_c \cup \Gamma_i}). \end{aligned}$$

With regard to the second case, the numerical tests are performed on an upper annular half disc centred on the point  $(0,0)$  with an external radius  $R_1 = 0.5$  and an internal radius  $R_2 = 0.75 R_1$ . The boundary of this domain is divided into three parts:

$$\begin{aligned} \Gamma_0 &= \{(x, 0) : R_2 < |x| < R_1\}, \\ \Gamma_i &= \{(x, y) : x^2 + y^2 = R_2, \quad y > 0\}, \\ \Gamma_c &= \{(x, y) : x^2 + y^2 = R_1, \quad y > 0\}. \end{aligned}$$

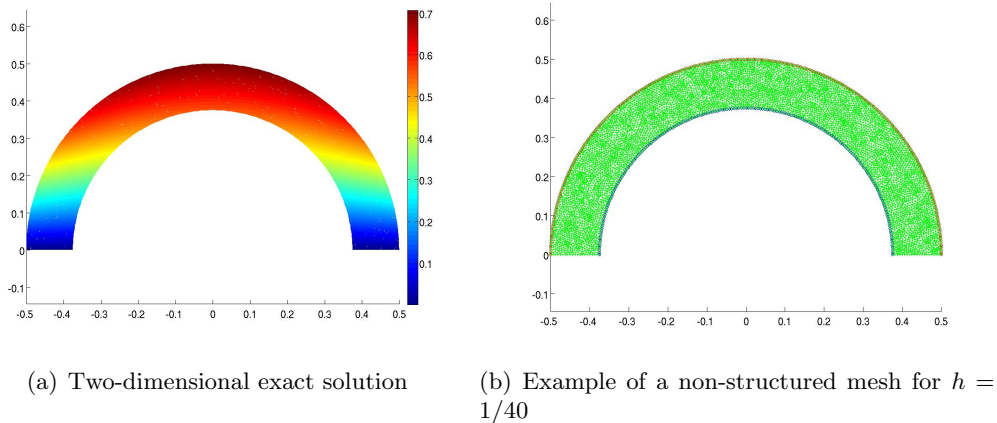


Figure 2: Exact solution and an example of a non-structured mesh on an upper annular half disc with  $mes(\Gamma_c) > mes(\Gamma_i)$

The third case is the same as the second. We just switch between  $\Gamma_c$  and  $\Gamma_i$  to illustrate the

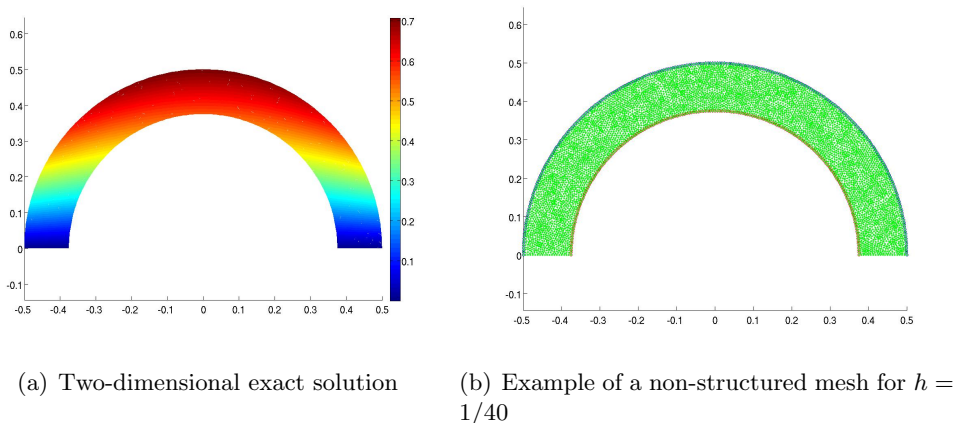


Figure 3: Exact solution and an example of a non-structured mesh on an upper annular half disc with  $mes(\Gamma_c) < mes(\Gamma_i)$

influence of the measure of the boundary  $\Gamma_c$  (when we have overabundant data) on the quality of the recovered data on  $\Gamma_i$ . As for the fourth case, the numerical tests are done on an L-shaped domain  $\Omega = ]0, 0.5[ \times ]-0.25, 0.25[ \setminus ]0, 0.35[ \times ]-0.25, 0.1[$ . The boundary of  $\Omega$  ( $\partial\Omega$ ) is also divided in three parts:

$$\begin{aligned} \Gamma_0 &= \{(0, y) : 0.1 < y < 0.25\} \cup \{(0.5, y) : -0.25 < y < 0.25\}, \\ \Gamma_i &= \{(x, 0.25) : 0 < x < 0.5\}, \\ \Gamma_c &= \Gamma \setminus \overline{\Gamma_0 \cup \Gamma_i}. \end{aligned}$$

We note that in the last case the boundary  $\Gamma_c$  presents two corner points at  $(0.35, 0.1)$  and  $(0.35, 0)$  which decreases the regularity of the prescribed data on  $\Gamma_c$ . For the four cases, the over-specified Cauchy data in  $\Gamma_c$  are extracted from the analytic solution:

$$u(x, y) = \sin(k \pi x) \cosh(k \pi y),$$

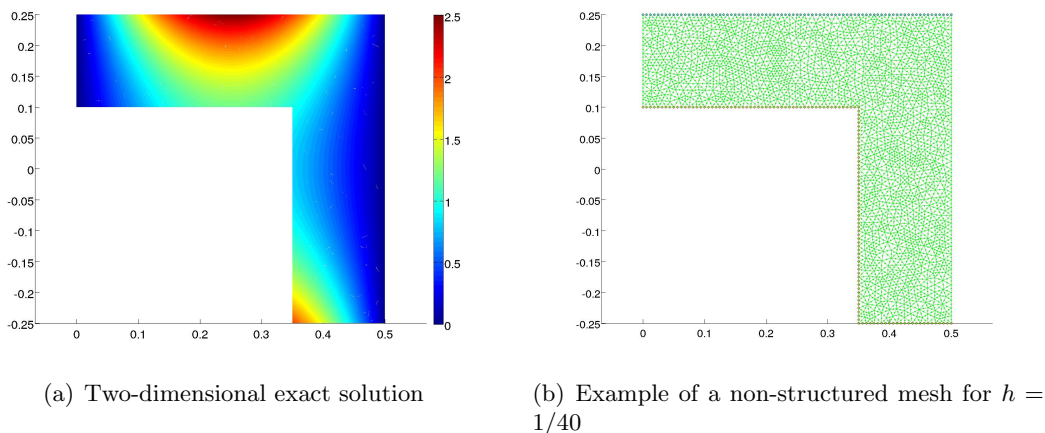


Figure 4: Exact solution and an example of a non-structured mesh on the L-shaped domain

with  $k = 0.5$  for the first as well as the last cases, and  $k = 2$  for the two remaining cases. The perturbed Cauchy data  $(T^\delta, \phi^\delta)$  are calculated by adding a multiplicative white noise to the Cauchy data  $(T, \phi)$  :

$$T^\delta = (1 + \delta \omega) T; \quad \phi^\delta = (1 + \delta \omega) \phi,$$

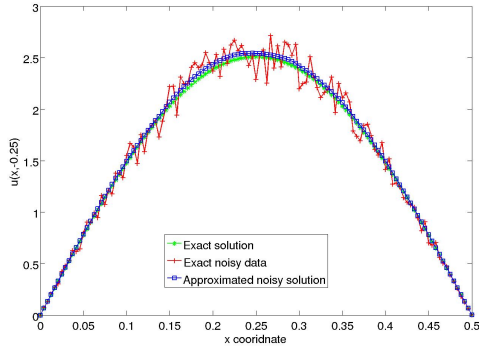
where  $\delta$  denotes the noise level and  $\omega$  is a normally distributed random function (with values ranging between  $-1$  and  $1$ ). For the first case (Resp. the second, third and fourth case), the analytic solution is shown in Fig. 1(a) (Resp. in Fig. 2(a), Fig. 3(a) and Fig. 4(a)) on a non-structured mesh with  $h = 1/40$  (see Fig. 1(b), Fig. 2(b), Fig. 3(b) and Fig. 4(b)). In the last four figures, the nodes of  $\Gamma_c$  (Resp. of  $\Gamma_i$ ) are plotted with a red circle (Resp. with a blue circle).

Note that we have chosen to use the Lagrange multiplier method to impose Dirichlet condition in all the direct solved problems. This choice plays an important role in smoothing noisy data in  $\Gamma_c$ . In all the performed tests, we used a  $\mathbb{P}_2$  finite element method to approximate the primal variable  $u$  and a  $\mathbb{P}_1$  finite element method to approximate the multiplier (used to impose Dirichlet condition).

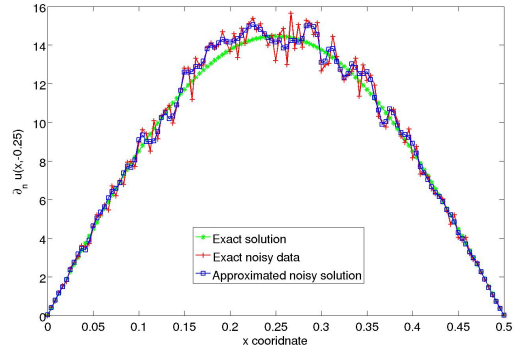
## 6.1 First case: square domain

As illustrated in Fig. (5(a)), we can observe that the trace of the noisy approximated solution on  $\Gamma_c$  is smoother than the exact noisy data imposed on  $\Gamma_c$ . This is due to the use of the Lagrange multiplier. The regularity of the trace of the noisy approximated solution obtained on  $\Gamma_c$  led to an accurate recovered missing data on  $\Gamma_i$  (see Fig. 6). In Fig. 7 (resp. Fig. 8), we plot the variation of the trace of  $u$  and the normal derivative of  $u$  ( $\partial_n u$ ) in  $\Gamma_c$  (resp.  $\Gamma_i$ ) according to the variation of  $\delta$  with  $h = 1/80$ . Despite the high noise level on the over-specified data in  $\Gamma_c$ , our variational minimization method gives an accurate reconstructed fields on  $\Gamma_i$  ( $u$  and  $\partial_n u$ ). As shown in Fig. 9 (Resp. Fig. 10 and Fig. 11), the variation of the relative error (norm  $L^2(\Omega)$ ,  $H^1(\Omega)$ ,  $L^2(\Gamma_c)$  and  $L^2(\Gamma_i)$  of  $u$  and  $L^2(\Gamma_c)$  and  $L^2(\Gamma_i)$  of  $\partial_n u$ ) is depicted according to the mesh size  $h$  for  $\delta = 0$  (Resp.  $\delta = h$  and  $\delta = h^2$ ). First, we see that for a low level of noise ( $\delta \leq h^2$ ) the obtained relative errors are very close to the relative error obtained when we solve direct problem with exact data. This means that for a low level of noise our strategy to solve Cauchy problem allows to obtain the same precision of the direct problem which is in concordance with the theoretical result proved in Proposition (9). Then we can observe that optimal rates of convergence are obtained, which



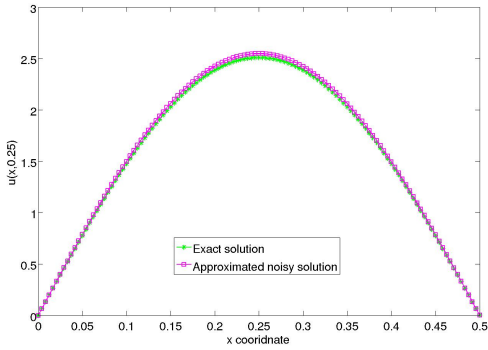


(a) Trace of  $u$  in  $\Gamma_c$

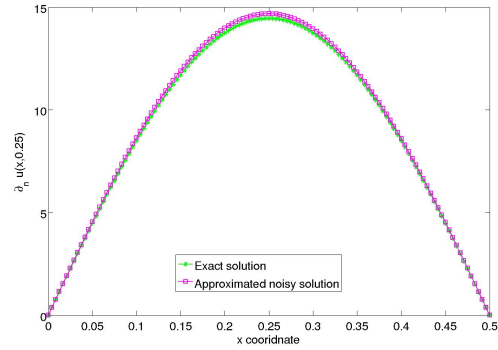


(b) Normal derivative of  $u$  ( $\partial_n u$ ) in  $\Gamma_c$

Figure 5: Trace and normal derivative of  $u$  on  $\Gamma_c$  for  $h = 1/60$  and  $\delta = 0.1$

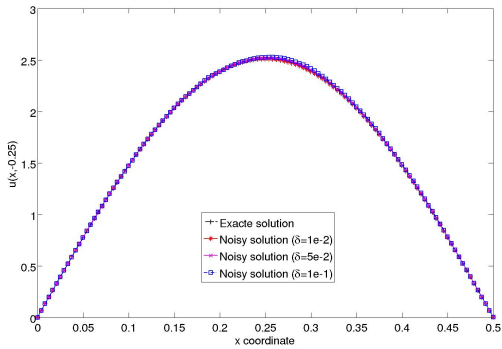


(a) Trace of  $u$  in  $\Gamma_i$

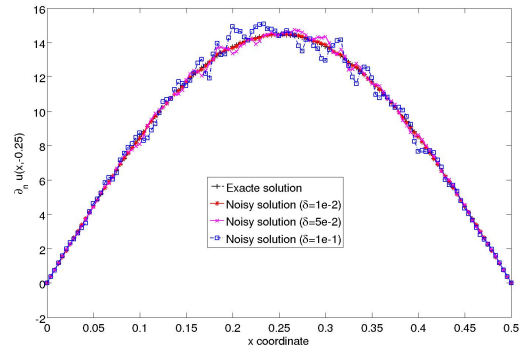


(b) Normal derivative of  $u$  ( $\partial_n u$ ) on  $\Gamma_i$

Figure 6: Trace and normal derivative of  $u$  on  $\Gamma_i$  for  $h = 1/60$  and  $\delta = 0.1$



(a) Trace of  $u$  in  $\Gamma_c$



(b) Normal derivative of  $u$  in  $\Gamma_c$

Figure 7: Variation of the trace of  $u$  and the normal derivative of  $u$  ( $\partial_n u$ ) on  $\Gamma_c$  according to the variation of  $\delta$ .

approve the shown a priori errors estimates in Proposition (9). The same optimality of the rates

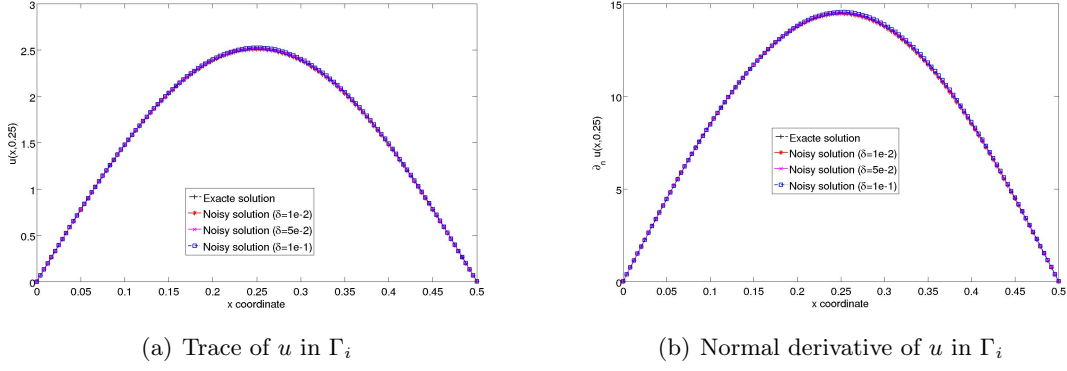


Figure 8: Variation of the trace of  $u$  and the normal derivative of  $u$  ( $\partial_n u$ ) on  $\Gamma_i$  according to the variation of  $\delta$ .

of convergence is obtained with a little difference for the remained cases. We note here that the use of a filtering procedure plays a crucial role in obtaining rates of convergence. Now in

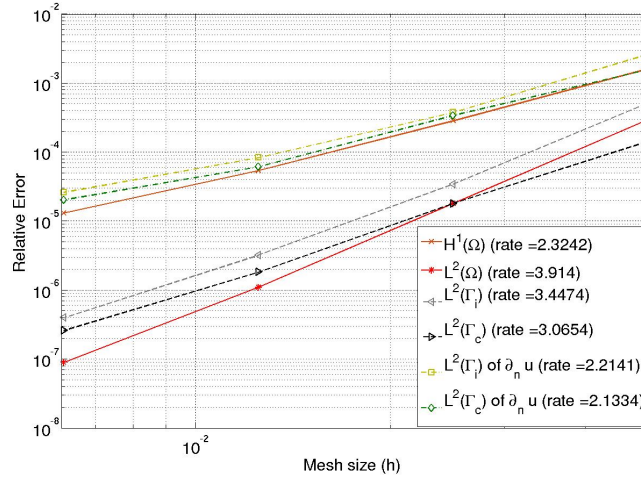


Figure 9: Variation of the relative error of  $u$  according to the variation of the mesh size ( $\delta = 0$ ).

order to illustrate the importance of using the filtering procedure, we list in Table 1 the relative error found (norm  $L^2(\Omega)$ ,  $H^1(\Omega)$ , and  $L^2(\Gamma_i)$  of  $u$  and  $L^2(\Gamma_i)$  of  $\partial_n u$ ) as well as the number of iteration needed to have convergence. Moreover, in Fig. 11 we trace the variation of the relative error (norm  $L^2(\Omega)$ ,  $H^1(\Omega)$ ,  $L^2(\Gamma_c)$  and  $L^2(\Gamma_i)$  of  $u$  and  $L^2(\Gamma_c)$  and  $L^2(\Gamma_i)$  of  $\partial_n u$ ) according to the mesh size  $h$  for  $\delta = 0$  and without a filtering procedure. From Table 1 we can observe that the relative errors obtained by the filtering procedure are less than the relative errors reached without the filtering procedure. Which means that the filtering procedure improves enormously the qualities of the recovered data. The same remark can be observed when comparing Fig. 12 and Fig. 9. Indeed in Fig. 12 the obtained rates of convergence of our method without the filtering procedure are not optimal, compared to the theoretical result presented in Proposition (9). It is limited to 2.2, 1.6 and 2 for the  $L^2(\Omega)$ ,  $H^1(\Omega)$ , and  $L^2(\Gamma_i)$  relative error of  $u$  and 1.2 for  $L^2(\Gamma_i)$  relative error of  $\partial_n u$ . Conversely the obtained rates of convergence of our method with filtering procedure, as show in Fig. 9, are super-optimal. It's limited to 3.9, 2.3 and 3.4 for the  $L^2(\Omega)$ ,  $H^1(\Omega)$ , and  $L^2(\Gamma_i)$  relative error of  $u$  and 2.2 for  $L^2(\Gamma_i)$  relative error of  $\partial_n u$ .

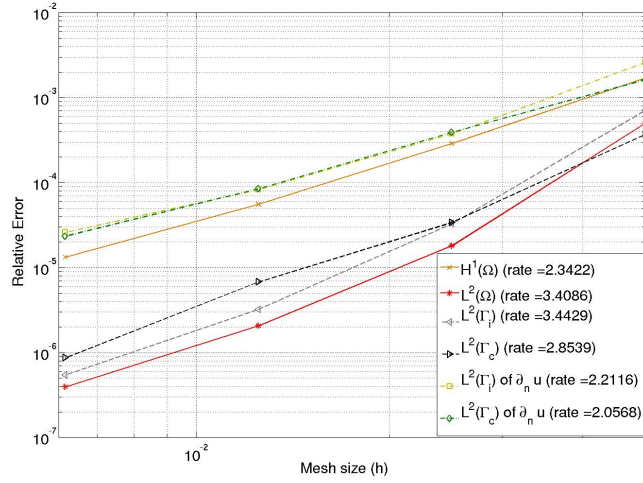


Figure 10: Variation of the relative error of  $u$  according to the variation of the mesh size ( $\delta = h^2$ ).

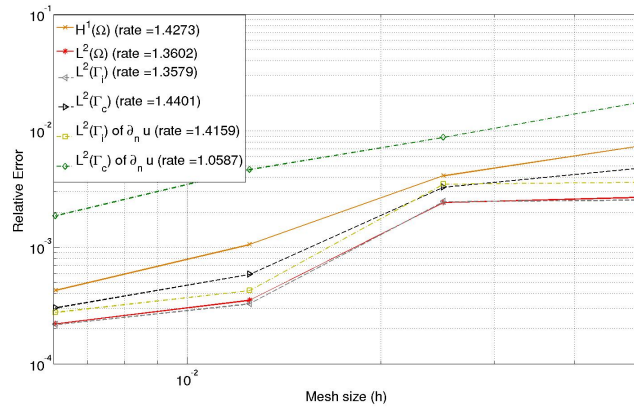


Figure 11: Variation of the relative error of  $u$  according to the variation of the mesh size ( $\delta = h$ ).

Table 1: Variation of the relative error of recovered data and the number of iteration required to find optimal convergence according to the use or not of the filtering procedure ( $\delta = 0$  and  $h = 0.5/60$ )

Strategy of resolution	$\ u\ _{L^2(\Gamma_i)}$	$\ \partial_n u\ _{L^2(\Gamma_i)}$	$\ u\ _{L^2(\Omega)}$	$\ u\ _{H^1(\Omega)}$	Number of iteration
Without a filtering	$6.5e - 5$	$1.4e - 3$	$1.27e - 5$	$2.5e - 4$	3
With a filtering	$1e - 6$	$4.64e - 5$	$3e - 7$	$2.4e - 5$	83

## 6.2 Second case: upper annular half disc with $mes(\Gamma_c) > mes(\Gamma_i)$

In Fig. 13 (resp. Fig. 14), we represent the variation of the trace of  $u$  and the normal derivative of  $u$  ( $\partial_n u$ ) on  $\Gamma_c$  (resp.  $\Gamma_i$ ) with  $h = 1/60$  and  $\delta = 0.05$ . We can see that a good approximated solutions are obtained but they are less than the one obtained in the first case. It can be explained by the fact that the ratio between the measure of  $\Gamma_c$  plus the measure of  $\Gamma_0$  over the measure of  $\Gamma_i$  is more than the one obtained for the first case. In Fig. 15 (resp. Fig. 16), we plot the variation of the trace of  $u$  and the normal derivative of  $u$  ( $\partial_n u$ ) in  $\Gamma_c$  (resp.  $\Gamma_i$ ) according to the

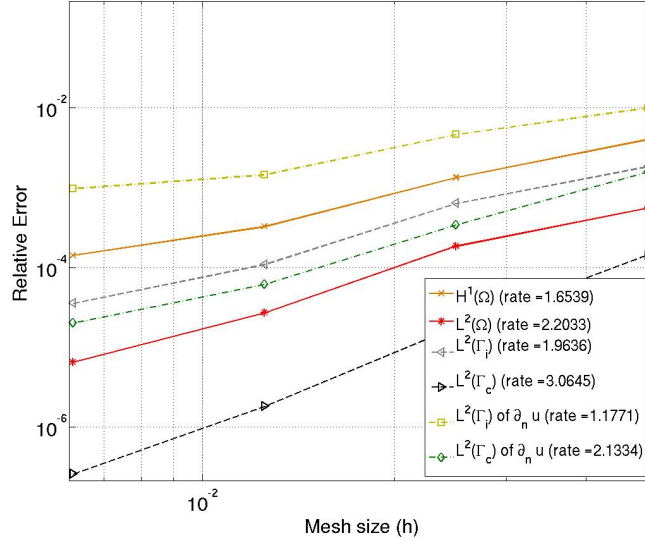


Figure 12: Variation of the relative error of  $u$  according to the variation of the mesh size ( $\delta = h$ ) without filtering procedure.

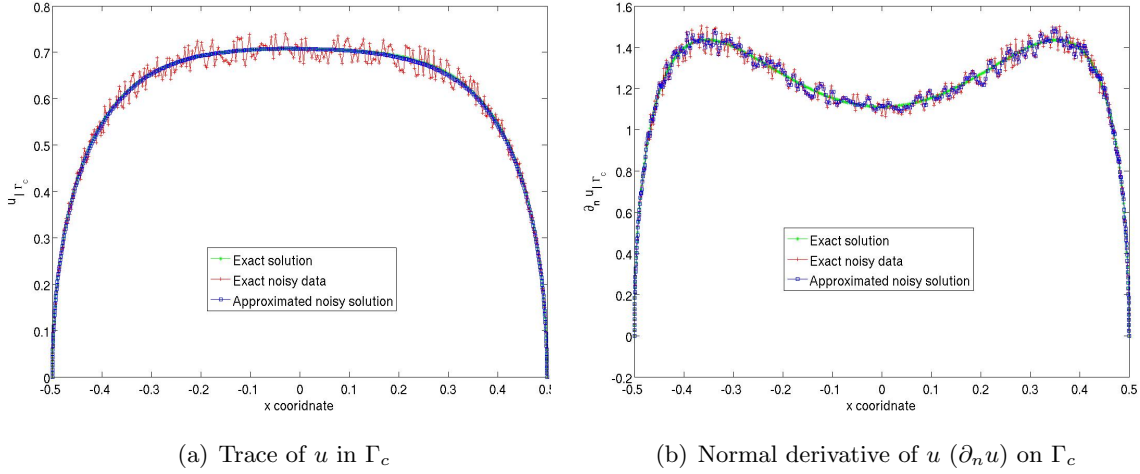


Figure 13: Trace and normal derivative of  $u$  in  $\Gamma_c$  for  $h = 1/60$  and  $\delta = 0.05$

variation of  $\delta$  with  $h = 1/80$ . We can note that for lower level of noisy parameter ( $\delta$ ) we have a good approximated solution on  $u$  and  $\partial_n u$ . When  $\delta$  increases, the quality of the approximated normal derivative of  $u$  decreases more than the trace of  $u$ . For all the values of the noisy level  $\delta$ , the obtained relative errors of our approximation solution (for all study norms) are less than the noisy level which confirm the given a priori error estimates shown in (38) and (39).

### 6.3 Third case: upper annular half disc with $mes(\Gamma_c) < mes(\Gamma_i)$

Now we switch between  $\Gamma_c$  and  $\Gamma_i$  in order to illustrate the influence of the ratio  $\rho = \frac{mes(\Gamma_c)}{mes(\Gamma_i)}$  on the quality of the recovered data and the sensitivity to noise levels. Fig. 18 shows the

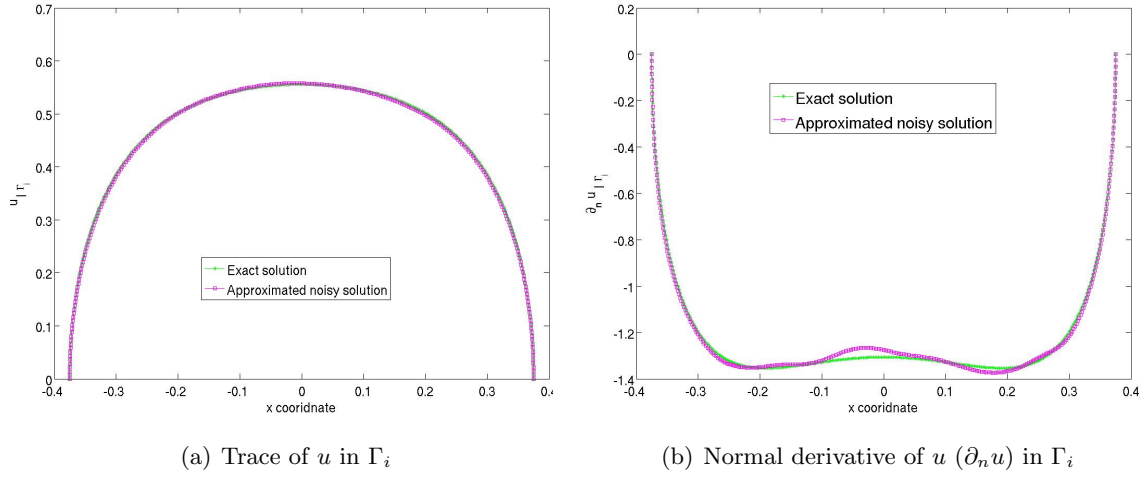


Figure 14: Trace and normal derivative of  $u$  on  $\Gamma_i$  for  $h = 1/60$  and  $\delta = 0.05$

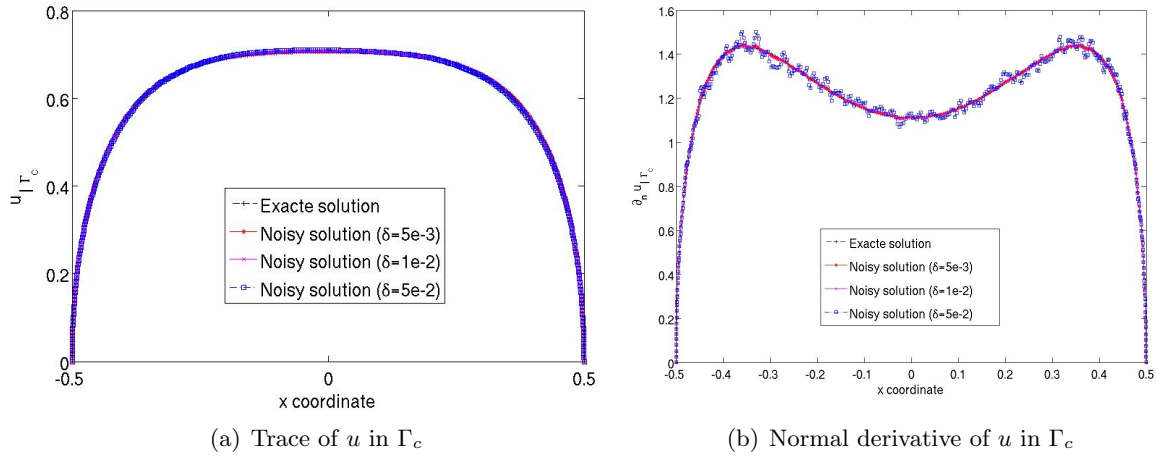


Figure 15: Variation of the trace of  $u$  and the normal derivative of  $u$  ( $\partial_n u$ ) on  $\Gamma_c$  according to the variation of  $\delta$

reconstructed data  $u$  and  $\partial_n u$  for  $h = 1/60$  and  $\delta = 0.05$ . Compared to the result obtained in the second case (Fig. 6), we deduce that the quality of reconstructed data decrease when  $\rho$  decreases. It's can be explained by the fact that when  $\rho$  decreases the quality of subscribed data on  $\Gamma_c$  decrease (as shown in Fig.13 and Fig.17), which affect the quality of the recovered data on  $\Gamma_i$ .

Fig. 19 (Resp. Fig. 20) represents the reconstruction data  $u$  and  $\partial_n u$  on  $\Gamma_c$  (Resp.  $\Gamma_i$ ) with different noise levels ( $\delta = 5e - 3$ ,  $\delta = 1e - 2$  and  $\delta = 5e - 2$ ). Compared to the result obtained in the second case (Fig. 15 and Fig. 16), we can see that the quality of the recovered data in this case is less than the one obtained in the second case. Which means that when the ratio  $\rho$  decreases the sensitivity of the noisy level increases.

#### 6.4 Fourth case: L-shape domain

In this case we like to test the influence of the presence of two corner points, at  $(0.35, 0.1)$  and  $(0.35, 0)$ , on the recovered data on  $\Gamma_i$ . For a lower level of noisy ( $\delta \leq 5e - 4$ ), our algorithm

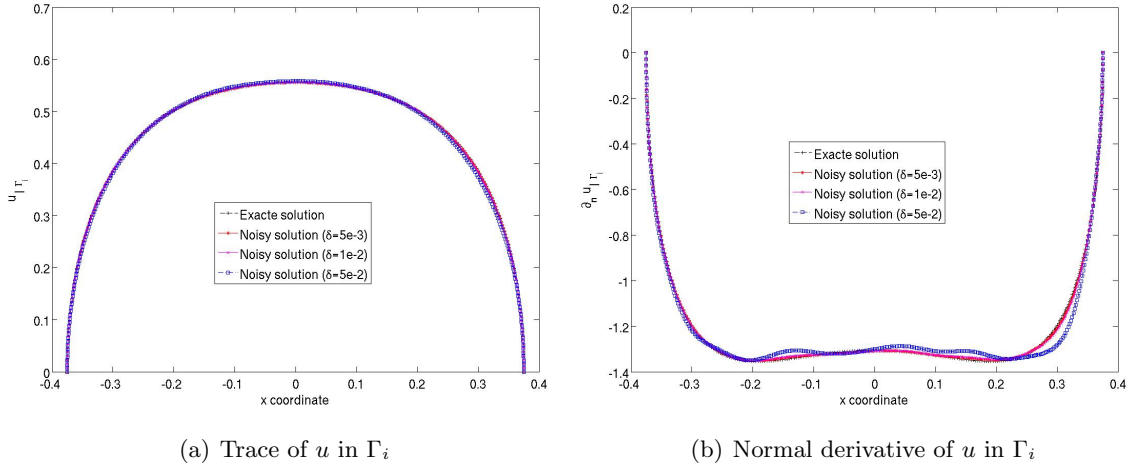


Figure 16: Variation of the trace of  $u$  and the normal derivative of  $u$  ( $\partial_n u$ ) on  $\Gamma_i$  according to the variation of  $\delta$

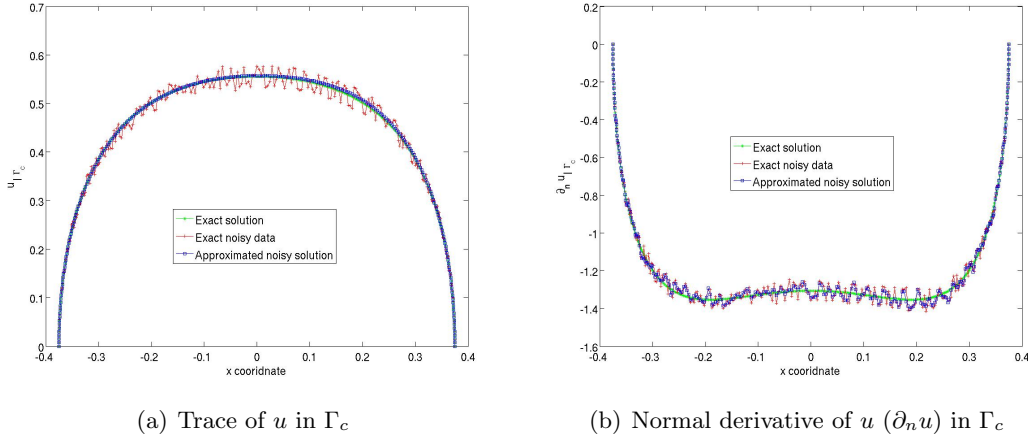


Figure 17: Trace and normal derivative of  $u$  in  $\Gamma_c$  for  $h = 1/60$  and  $\delta = 0.05$

converges slowly but gives a good quality of solution. For a high level of noisy ( $\geq 5e - 3$ ), our algorithm converges quickly but it requires more iteration than the other cases. Also, the quality of the recovered data on  $\Gamma_i$  (see Fig. 22) are less than the other cases. It can be explained by the fact that the presence of the two corner points which decrease the regularity of the prescribed data on  $\Gamma_c$ . Fig 23 shows the recovered data  $u$  and  $\partial_n u$  on  $\Gamma_i$  with different noisy levels ( $\delta = 5e - 3$ ,  $\delta = 1e - 2$  and  $\delta = 5e - 2$ ). The quality of the recovered data in this case is less than the ones obtained in the other cases. For all tested noisy levels, the quality of the recovered trace of  $u$  on  $\Gamma_i$  is similar to that obtained in the other cases 23(a). And the quality of the recovered trace of the normal derivative of  $u$  on  $\Gamma_i$  is less than the ones obtained for the other cases (see Fig. 23(b)). This means that the presence of two corner points on  $\Gamma_c$  increases the sensitivity of our algorithm to noisy data.

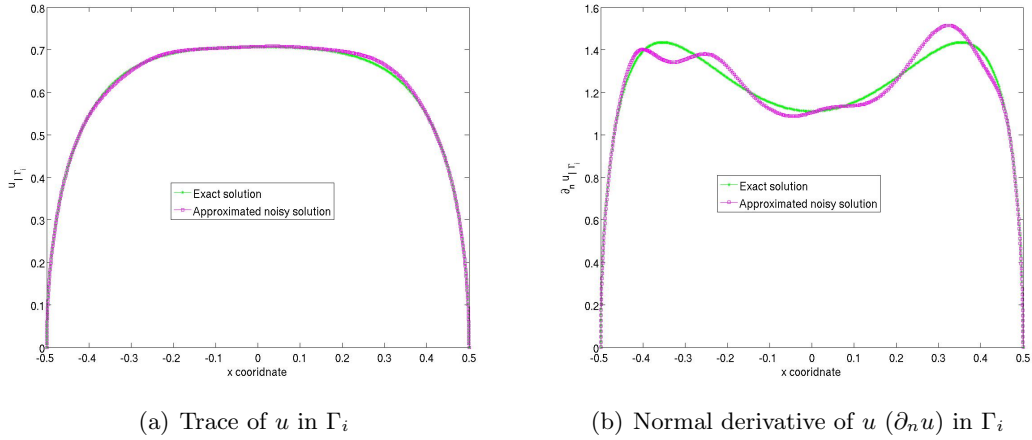


Figure 18: Trace and normal derivative of  $u$  in  $\Gamma_i$  for  $h = 1/60$  and  $\delta = 0.05$

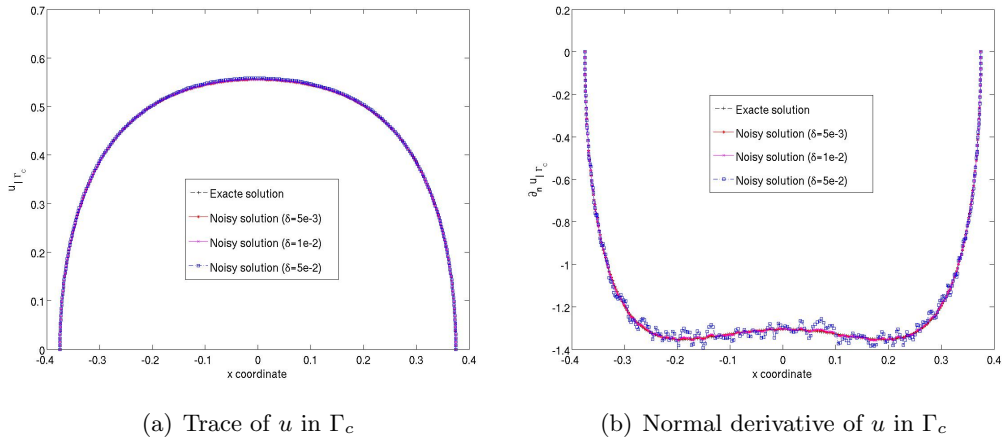


Figure 19: Variation of the trace of  $u$  and the normal derivative of  $u$  ( $\partial_n u$ ) in  $\Gamma_c$  according to the variation of  $\delta$

## 7 Concluding remarks

In this paper, we proposed an energetic variational minimization method to solve numerically the Cauchy Laplace problem in the framework of a finite element method. A filtering procedure has been used as a new regularization procedure to the suggested method. The shown a priori error estimates are optimal and take into account the level of noise. The numerical results confirm the theoretical estimates: The numerical solution converge, with optimal rate of convergence, when decreasing the mesh size and the noise level.

## References

- [1] Stéphane Andrieux, TN Baranger, and A Ben Abda. Solving cauchy problems by minimizing an energy-like functional. *Inverse problems*, 22(1):115, 2006.

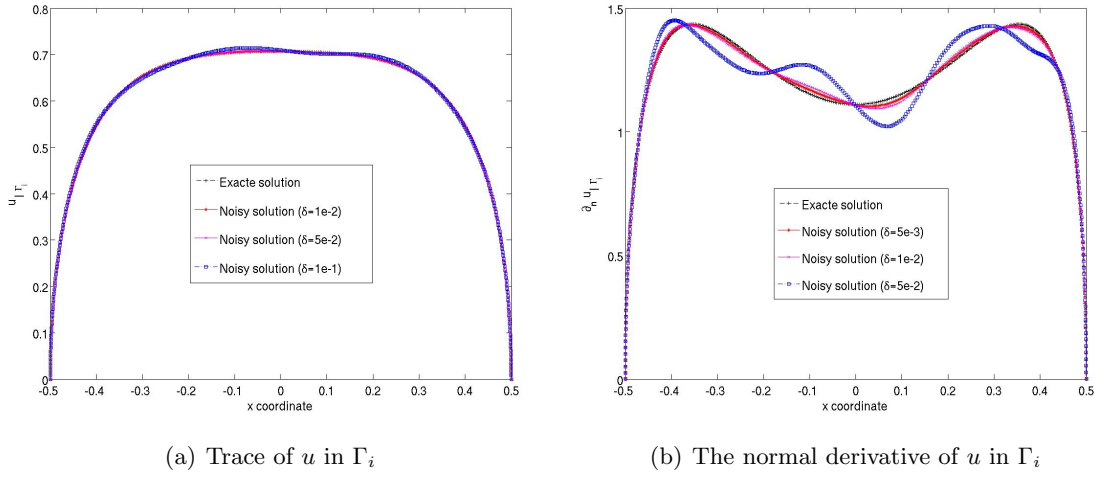


Figure 20: Variation of the trace of  $u$  and the normal derivative of  $u$  ( $\partial_n u$ ) in  $\Gamma_i$  according to the variation of  $\delta$

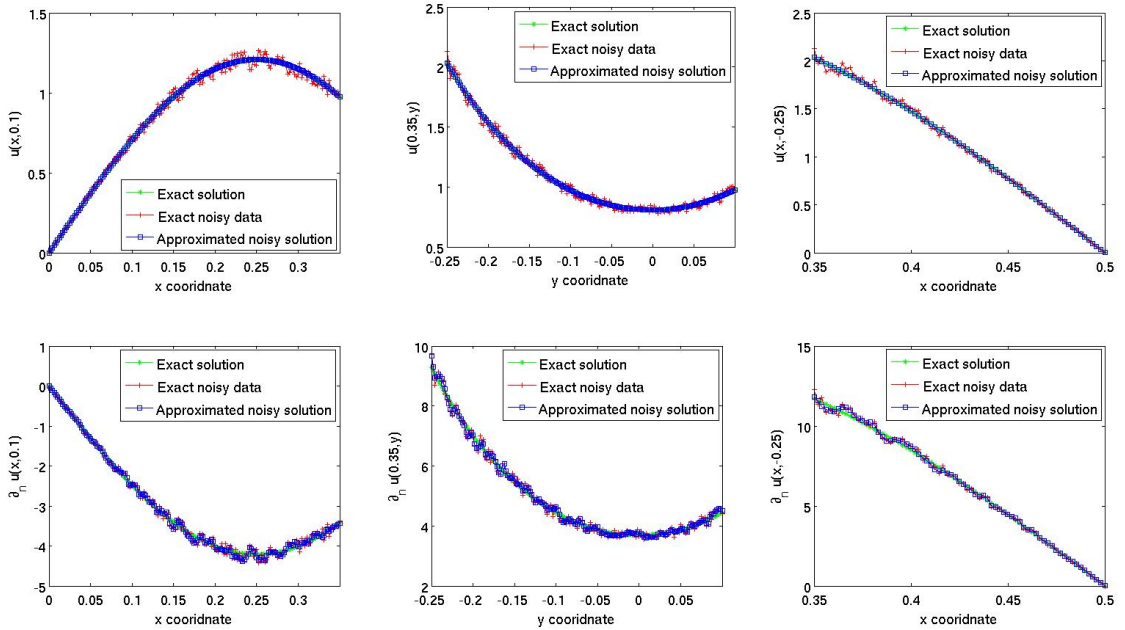


Figure 21: Trace and normal derivative of  $u$  in  $\Gamma_c$  for  $h = 1/60$  and  $\delta = 0.05$

- [2] Faker Ben Belgacem. Why is the cauchy problem severely ill-posed? *Inverse problems*, 23(2):823, 2007.
- [3] Laurent Bourgeois. A mixed formulation of quasi-reversibility to solve the cauchy problem for laplace's equation. *Inverse problems*, 21(3):1087, 2005.
- [4] Laurent Bourgeois and Arnaud Recoquilly. A mixed formulation of the tikhonov regularization and its application to inverse pde problems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 52(1):123–145, 2018.



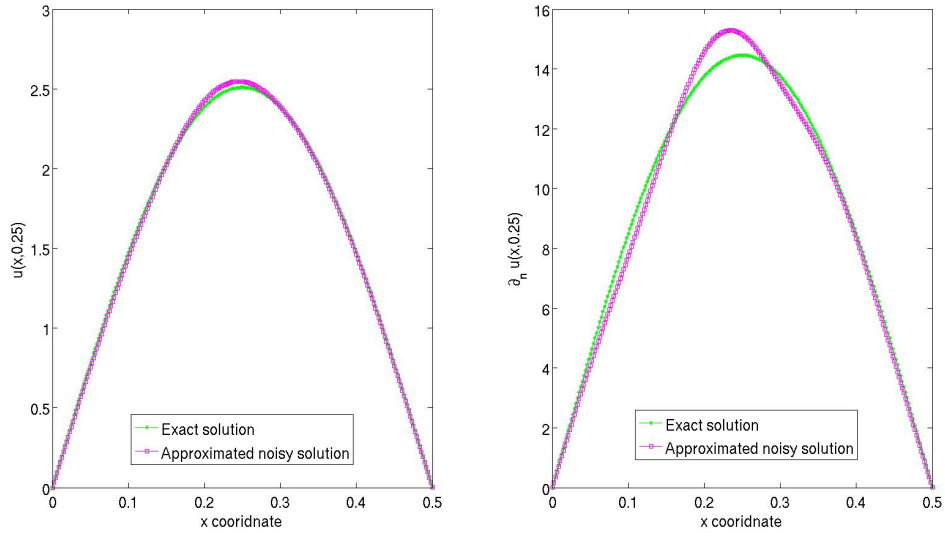
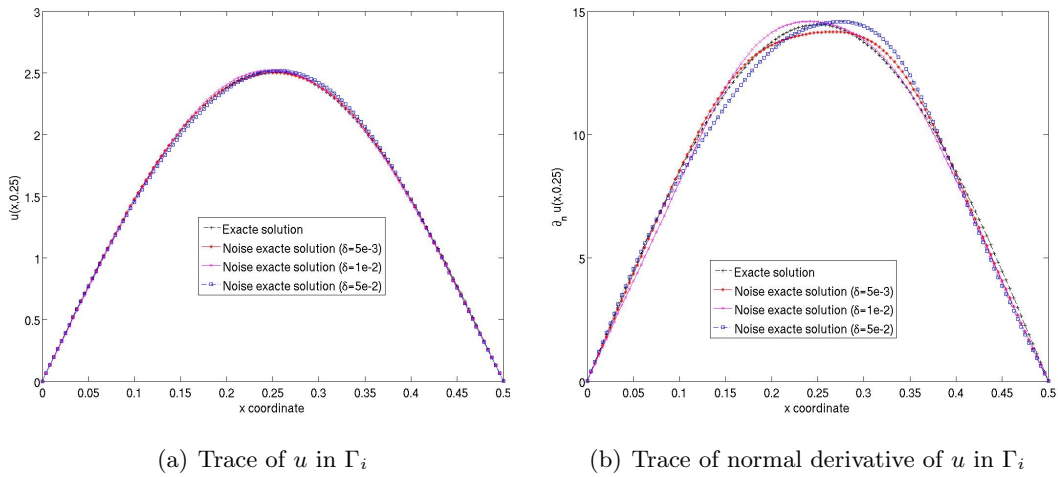


Figure 22: Trace and normal derivative of  $u$  in  $\Gamma_i$  for  $h = 1/60$  and  $\delta = 0.05$



(a) Trace of  $u$  in  $\Gamma_i$

(b) Trace of normal derivative of  $u$  in  $\Gamma_i$

Figure 23: Variation of the trace of  $u$  and the normal derivative of  $u$  ( $\partial_n u$ ) in  $\Gamma_i$  according to the variation of  $\delta$

- [5] Philippe G Ciarlet. *The finite element method for elliptic problems*. SIAM, 2002.
- [6] A Cimetiere, F Delvare, M Jaoua, and F Pons. Solution of the cauchy problem using iterated tikhonov regularization. *Inverse problems*, 17(3):553, 2001.
- [7] PJ Daniell. Lectures on cauchy's problem in linear partial differential equations. by j. hadamard. pp. viii+ 316. 15s. net. 1923.(per oxford university press.). *The Mathematical Gazette*, 12(171):173–174, 1924.
- [8] F Delvare, A Cimetière, and F Pons. An iterative boundary element method for cauchy inverse problems. *Computational mechanics*, 28(3-4):291–302, 2002.

- [9] Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements*, volume 159. Springer Science & Business Media, 2013.
- [10] M Hanke and PC Hansen. *Regularization methods for large-scale problems: Surveys on mathematics for industry*, 1993.
- [11] Dinh Nho Hao, BT Johansson, D Lesnic, and Pham Minh Hien. A variational method and approximations of a cauchy problem for elliptic equations. *Journal of Algorithms & Computational Technology*, 4(1):89–119, 2010.
- [12] Dinh Nho Hao and D Lesnic. The cauchy problem for laplace’s equation via the conjugate gradient method. *IMA Journal of Applied Mathematics*, 65(2):199–217, 2000.
- [13] Erik Albert Holmgren. *Über Systeme von linearen partiellen Differentialgleichungen*. 1901.
- [14] M Jourhmane and Abdeljalil Nachaoui. Convergence of an alternating method to solve the cauchy problem for poisson’s equation. *Applicable analysis*, 81(5):1065–1083, 2002.
- [15] VA Kozlov, VG Maz’ya, and AV Fomin. An iterative method for solving the cauchy problem for elliptic equmâions£ 9. 1991.
- [16] Robert Lattès. *Méthode de quasi-réversibilité et applications*, volume 15. Dunod, 1967.
- [17] Olaf Steinbach. *Numerical approximation methods for elliptic boundary value problems: finite and boundary elements*. Springer Science & Business Media, 2007.
- [18] Phan Xuan Thanh and Olaf Steinbach. Energy space approaches to the cauchy problem for poisson’s equation. *Acta Mathematica Vietnamica*, pages 1–15, 2019.