

# Relaxation methods and finite element schemes for the incompressible Navier-Stokes equations (and the system of visco-elastodynamics)

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- relaxation models - singular hyperbolic approximations (kinetic theory, continuum mechanics, fluid dynamics, traffic flows, ...)
- stability/convergence of finite elements schemes - *modulated energy*
- computational performance - *adaptive meshing* yields an extra stabilization against the nonlinear response over shock regions

## Numerical modeling of (viscous) incompressible flows

velocity  $u(t, x) \in \mathbb{R}^2$ , pressure  $p(t, x) \in \mathbb{R}$ , kinematic viscosity  $\mu > 0$

$$\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = 0$$

$$\nabla \cdot u = 0 \quad t \geq 0, x \in \mathbb{R}^2$$

difficulties : small viscosity - dominant nonlinear terms ;  
divergence-free constraint ; stability properties - error estimates

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- Y. Brenier, R. Natalini, M. Puel, On a relaxation approximation of the incompressible Navier-Stokes equations, *Proc. Amer. Math. Soc.* (2004)
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Relaxation approximation of the Navier-Stokes equations  
by means of a *singular perturbation* of the *nonlinear flux*

$$\begin{cases} \partial_t u^\varepsilon + \nabla \cdot U^\varepsilon + \nabla p^\varepsilon = 0 \\ \partial_t U^\varepsilon + \frac{\mu}{\varepsilon} \nabla u^\varepsilon = -\frac{1}{\varepsilon} (U^\varepsilon - u^\varepsilon \otimes u^\varepsilon) \end{cases}$$

for  $\varepsilon > 0$ , matrix valued variable  $U^\varepsilon(t, x) \in \mathbb{R}^2 \times \mathbb{R}^2$

- diffusive scaling of the relaxation system for the Euler equations  
the (incompressible) *one-field equation* with lower-order source terms

$$\begin{aligned} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \mu \Delta u^\varepsilon + \varepsilon \partial_{tt} u^\varepsilon + \nabla p^\varepsilon + \varepsilon \partial_t \nabla p^\varepsilon &= 0 \\ \nabla \cdot u^\varepsilon &= 0 \end{aligned}$$

reveals a **small dissipative correction** (regularization mechanism with finite speed of propagation as part of hyperbolic wave operators)

in one space dimension, from the  $2 \times 2$  **incompletely parabolic system of visco-elastodynamics**, to a hyperbolic  $3 \times 3$  *semilinear system*

$$\begin{cases} u_t^\varepsilon - v_x^\varepsilon = 0 \\ v_t^\varepsilon - Z_x^\varepsilon = 0 \\ \varepsilon Z_t^\varepsilon - \mu v_x^\varepsilon = -Z^\varepsilon + S(u^\varepsilon) \end{cases}$$

with the corresponding one-field equation

$$\begin{cases} u_t^\varepsilon - v_x^\varepsilon = 0 \\ v_t^\varepsilon - S(u^\varepsilon)_x - \mu v_{xx}^\varepsilon + \varepsilon v_{tt}^\varepsilon = 0 \end{cases}$$

- deformation gradient  $u^\varepsilon(t, x) \in \mathbb{R}$  and velocity  $v^\varepsilon(t, x) \in \mathbb{R}$
- supplementary relaxation variable  $Z^\varepsilon(t, x) \in \mathbb{R}$  of the *Piola-Kirchhoff stress-strain tensor*  $S : \mathbb{R} \rightarrow \mathbb{R}$  (gradient of a convex stored/internal energy for hyperelastic materials with nonlinear response)

**viscoelastic materials** exhibit time dependent stress-strain when undergoing deformations : viscosity/relaxation approximations describe the (isothermal) motion due to diffusive stress in Lagrangian coordinates,

$$\left( Z^\varepsilon - \frac{\mu}{\varepsilon} u^\varepsilon \right)_t = -\frac{1}{\varepsilon} \left( Z^\varepsilon - S(u^\varepsilon) \right)$$

which can be recast as

$$Z^\varepsilon = \frac{\mu}{\varepsilon} u^\varepsilon - \int_{-\infty}^t \frac{1}{\varepsilon} \exp\left(-\frac{t-s}{\varepsilon}\right) \left( \frac{\mu}{\varepsilon} u^\varepsilon - S(u^\varepsilon) \right)(s) ds$$

and viscous effects actually comes from an (integral) memory-type term.

putting the relaxation parameter  $\varepsilon = 0$ , the *equilibrium relation*  $z = S(u) + \mu v_x$  is formally recovered, that is the **passage from viscosity of the memory-type to viscosity of the rate-type...**

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- L. Gosse, A.E. Tzavaras, Convergence of relaxation schemes to the equations of elastodynamics, *Math. Comp.* (2001)

## Semi-discrete finite element schemes

- let  $\mathbb{T}_h$  be (uniformly regular) partitions of  $\mathbb{R}^2$ , for  $h > 0$
- finite element spaces  $V_h \subset H^1$ ,  $P_h \subset L^2$  of piecewise polynomials (globally continuous) on  $\mathbb{T}_h$  with **consistency properties**

$$\inf_{v \in V_h} \left[ \|u - v\|_{L^2} + h \|u - v\|_{H^1} \right] \leq Ch^k \|u\|_{H^k}, \quad k \geq 1$$

- subspace  $Z_h \subset V_h$  of **approximately divergence-free** functions

$$Z_h := \{v \in V_h; (v, \nabla q) = 0, \forall q \in P_h\}$$

**mixed variational formulation** : to find  $u_h \in Z_h$  solution to

$$(\partial_t u_h, v) + (b(u_h; u_h), v) + \mu (\nabla u_h, \nabla v) + \varepsilon (\partial_{tt} u_h, v) = 0, \quad \forall v \in Z_h$$

$$b(u_h; u_h) = u_h \cdot \nabla u_h + \frac{1}{2} (\nabla \cdot u_h) u_h$$

with the *numerical nonlinear term* defined on  $Z_h$  to keep the *conservation property*  $(b(u_h; u_h), u_h) = 0$

**features** that make finite element methods computationally attractive :

- formal high-order accuracy, nonlinear stability, ability to capture discontinuities or strong gradients without producing spurious oscillations;
- in curved (spatial) domains, to easily incorporate complex geometries, and support domain decomposition techniques for parallelization;
- inherent implicit-explicit discretization of nonlinear terms;

**difficulties** to solve hyperbolic and convection-dominated problems :

- to choose the correct regularization mechanism, that resolve with high accuracy the smooth parts of the solution and without spurious oscillations over the shock regions;
- for relaxation approximations, the number of unknowns is increased;

**relaxation + adaptive meshing**

provides a *dissipative mechanism against the destabilizing effect of nonlinear response*, combined with appropriate *mesh refinement* (by means of an error indicator) *to detect the location of the singularities*

we consider  $v = u_h \in Z_h$  to obtain

$$\frac{1}{2} \frac{d}{dt} \|u_h\|_{L^2}^2 + \mu \|\nabla u_h\|_{L^2}^2 + \varepsilon \frac{d}{dt} (\partial_t u_h, u_h) - \varepsilon \|\partial_t u_h\|_{L^2}^2 = 0$$

we estimate the time derivative  $v = \partial_t u_h \in Z_h$  as the **correction by lower order effects**

$$\|\partial_t u_h\|_{L^2}^2 + b(u_h, u_h; \partial_t u_h) + \frac{1}{2} \mu \frac{d}{dt} \|\nabla u_h\|_{L^2}^2 + \frac{1}{2} \varepsilon \frac{d}{dt} \|\partial_t u_h\|_{L^2}^2 = 0$$

we multiply by  $2\varepsilon$  the last one before summing

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_h + \varepsilon \partial_t u_h\|_{L^2}^2 + \frac{1}{2} \varepsilon^2 \frac{d}{dt} \|\partial_t u_h\|_{L^2}^2 + \varepsilon \mu \frac{d}{dt} \|\nabla u_h\|_{L^2}^2 \\ & = -\mu \|\nabla u_h\|_{L^2}^2 - \varepsilon \|\partial_t u_h\|_{L^2}^2 - 2\varepsilon b(u_h, u_h; \partial_t u_h) \\ & = -\varepsilon \|\partial_t u_h + b(u_h; u_h)\|_{L^2}^2 - \mu \|\nabla u_h\|_{L^2}^2 + \varepsilon \|b(u_h; u_h)\|_{L^2}^2 \end{aligned}$$

to get **strong dissipative estimates** as modification of the standard energy functional suggested by the relaxation terms



## Inverse inequalities and uniform bound

the main argument for the  $L^2$ -stability estimates is that

$$-\mu \|\nabla u_h\|_{L^2}^2 + \varepsilon \|b(u_h; u_h)\|_{L^2}^2 \leq -\left(\mu - \frac{5}{2}\varepsilon \|u_h\|_{L^\infty}^2\right) \|\nabla u_h\|_{L^2}^2 \leq 0$$

provided that  $\|u_h\|_{L^\infty} \leq \sqrt{\frac{2\mu}{5\varepsilon}}$ ,  $\forall t \geq 0$

which is interpreted as the sub-characteristic condition

by using an inverse inequality for the finite element spaces,

$$\|u_h\|_{L^\infty} \leq C h^{-1} \|u_h\|_{L^2},$$

the numerical solution satisfies  $\|u_h\|_{L^2} \leq C \sqrt{\frac{2\mu}{5\varepsilon}} h$ ,  $\forall t \geq 0$

then (at least) we choose  $\varepsilon = \mathcal{O}(h^2)$  as  $h \rightarrow 0$ ,  $\varepsilon \rightarrow 0$

Euler equations : for  $\mu = \mathcal{O}(\varepsilon)$ , we need stability properties uniformly with respect to the diffusion parameters...

## Error estimate for the velocity

convergence toward the (smooth) solutions to the incompressible Navier-Stokes equations

$$u - u_h = (u - \mathbb{P}_h u) + (\mathbb{P}_h u - u_h) =: \eta(u) + \xi(u; u_h)$$

with  $\mathbb{P}_h u \in Z_h$  the elliptic projection onto the finite element space

- the **consistency error** is bounded as  $\|\eta\|_{L^2} \leq Ch^k \|u\|_{H^k}$ ,  $k \geq 1$
- the **stability error** satisfies the variational equation

$$\begin{aligned} & (\partial_t \xi, v) + (u \cdot \nabla u - b(u_h; u_h), v) + \mu (\nabla \xi, \nabla v) + \varepsilon (\partial_{tt} \xi, v) \\ & = -(\partial_t \eta, v) + \varepsilon (\partial_{tt} \mathbb{P}_h u, v) - (\nabla p, v), \quad \forall v \in Z_h \end{aligned}$$

with  $b(u_h; u_h) = u_h \cdot \nabla u_h + \frac{1}{2} (\nabla \cdot u_h) u_h$

so that the error estimate follows, as for the classical case!!

$$\|u - u_h\|_{L^2} \leq C e^{\frac{1}{\mu} h^k}, \quad \forall t \geq 0$$

for **visco-elastodynamics**, the well-posedness in Sobolev spaces for large data is guaranteed by the results on weakly parabolic systems

- P. D'Ancona, S. Spagnolo The Cauchy problem for weakly parabolic systems, *Math. Ann.* (1997)
- in one space dimension, the assumption that the nonlinear flux (stress-strain function) is globally Lipschitz,  $\sup_{u \in \mathbb{R}} |S'(u)| < +\infty$ , stands for the **sub-characteristic condition** (the relaxation finite element scheme yields weak solutions that dissipate all convex entropies, for the method of *symmetrizers* and a theory of *compensated compactness* holds)
- in multi-dimensional systems, a sufficient condition to compensate for the lack of full-parabolicity is a convexity (and growth) assumption on the internal/stored energy, that is not always (physically) satisfied...
- these are **structural properties**, whereas it should be proven for Navier-Stokes and Euler equations!!
- C. Lattanzio, A.E. Tzavaras, Structural properties of stress relaxation and convergence from viscoelasticity to polyconvex elastodynamics, *Arch. Ration. Mech. Anal.* (2006)

## Work in progress...

- 1) implementation by the **FreeFem++ software** for multi-dimensional problems (including complex geometries and boundary conditions)
- 2) to employ **discontinuous finite elements** to effectively handle shock waves arising in nonlinear elastic materials, for real applications
- 3) for the Euler equations, to devise an (incomplete) artificial/numerical viscosity with a (mesh dependent) higher order parameter, in view of an alternative approach to the resolution of contact discontinuities
- 4) generalization to physical models with **viscosity of memory-type** leading to **integro-differential operators** to be treated numerically
- 5) kinetic formulation of elastodynamics and related topics, as the relaxation method corresponds to some discrete kinetic approximation (Euler equations of gas dynamics, system of magneto-hydrodynamics, shallow water equations, compressible Navier-Stokes equations with high Reynolds numbers, hydrodynamic models for semiconductor devices, ...)