

Relaxation methods and finite element schemes for the equations of visco-elastodynamics

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discussion in progress with C. Lattanzio and A.E. Tzavaras

preliminary issues of finite element relaxation schemes for elastodynamics
from physical modeling through analytical studies for numerical purposes

- relaxation approximations support *entropy inequalities* – dissipative mechanism and damping effects on oscillations
- stability/convergence of finite elements schemes
- computational performance – *adaptive meshing* yields an extra stabilization against the nonlinear response over shock regions

physical modeling : relaxation mechanisms in continuum mechanics and kinetic theory (models for elastodynamics, traffic flows, time decaying of chemical elements, biological systems with reaction time)

analytical studies : diffusive relaxation limits for hyperbolic systems by means of compensated compactness techniques, singular perturbations of first order hyperbolic (pseudo-differential) equations, BGK-type approximations towards degenerate parabolic equations

- D. Donatelli, P. Marcati, Convergence of singular limits for multi-D semilinear hyperbolic systems to parabolic systems, *Trans. Amer. Math. Soc.* (2004)

numerical purposes : relaxation approximations of hyperbolic conservation laws, reduced discrete velocity kinetic models, ...

- S. Jin, Z.P. Xin, The relaxation schemes for systems of conservation laws in arbitrary space dimensions, *Comm. Pure Appl. Math.* (1995)
- M.K. Banda, A. Klar, L. Pareschi, M. Seaïd, Compressible and incompressible limits for hyperbolic systems with relaxation, *J. Comput. Appl. Math.* (2004)

Chapman-Enskog expansions and parabolic scaling of hyperbolic systems to evidence diffusive behaviors, under a sub-characteristic condition

diffusive stress relaxation for elastodynamics in one space dimension

- deformation gradient $u^\varepsilon(t, x) \in \mathbb{R}$ and velocity $v^\varepsilon(t, x) \in \mathbb{R}$
- kinematic viscosity $\mu = \mu(\varepsilon) > 0$, relaxation parameter $\varepsilon > 0$

non-homogeneous (strictly) hyperbolic 3×3 semilinear system

$$\begin{cases} u_t^\varepsilon - v_x^\varepsilon = 0 \\ v_t^\varepsilon - Z_x^\varepsilon = 0 \\ \varepsilon Z_t^\varepsilon - \mu v_x^\varepsilon = -Z^\varepsilon + S(u^\varepsilon) \end{cases}$$

with the corresponding system for the original fields (*one-field equation*)

$$\begin{cases} u_t^\varepsilon - v_x^\varepsilon = 0 \\ v_t^\varepsilon - S(u^\varepsilon)_x - \mu v_{xx}^\varepsilon + \varepsilon v_{tt}^\varepsilon = 0 \end{cases}$$

from the 2×2 *incompletely parabolic system of visco-elastodynamics*

- supplementary relaxation variable $Z^\varepsilon(t, x) \in \mathbb{R}$ for the *Piola-Kirchhoff stress-strain tensor* $S : \mathbb{R} \rightarrow \mathbb{R}$ (gradient of a convex internal/stored energy for hyper-elastic materials with nonlinear response)

viscoelastic materials exhibit time dependent stress-strain response when undergoing deformations : viscosity/relaxation approximations describe the (isothermal) motion in Lagrangian coordinates (with diffusive stress)

$$\left(Z^\varepsilon - \frac{\mu}{\varepsilon} u^\varepsilon \right)_t = -\frac{1}{\varepsilon} \left(Z^\varepsilon - S(u^\varepsilon) \right)$$

which can be recast as

$$Z^\varepsilon = \frac{\mu}{\varepsilon} u^\varepsilon - \int_{-\infty}^t \frac{1}{\varepsilon} \exp\left(-\frac{t-s}{\varepsilon}\right) \left(\frac{\mu}{\varepsilon} u^\varepsilon - S(u^\varepsilon) \right)(s) ds$$

and the relaxation effects come from an (integral) memory-type term

putting $\varepsilon = 0$ into the system, the *equilibrium relation* $Z = S(u) + \mu v_x$ is formally recovered, that is interpreted as the **passage from viscosity of the memory-type to viscosity of the rate-type**

- B.D. Coleman, M.E. Gurtin, Thermodynamics with internal state variables, *J. Chem. Physics* (1967)
- C.M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Fundamental Principles of Mathematical Sciences **325**, Springer-Verlag, Berlin, 2000
- A.E. Tzavaras, Materials with internal variables and relaxation to conservation laws, *Arch. Ration. Mech. Anal.* (1999)

for **visco-elastodynamics**, the well-posedness in Sobolev spaces for large data is guaranteed by the results on weakly parabolic systems

- P. D'Ancona, S. Spagnolo The Cauchy problem for weakly parabolic systems, *Math. Ann.* (1997)
 - in one space dimension, the assumption that the nonlinear flux (stress-strain function) is globally Lipschitz, $\sup_{u \in \mathbb{R}} |S'(u)| < +\infty$, stands for the **sub-characteristic condition** (the relaxation finite element scheme yields weak solutions that dissipate all convex entropies, for the *method of symmetrizers*, and the *compensated compactness* theory holds)
 - in multi-dimensional systems, a sufficient condition to compensate for the lack of parabolicity is a poly-convexity (or growth) assumption on the internal/stored energy
 - C. Lattanzio, A.E. Tzavaras, Structural properties of stress relaxation and convergence from viscoelasticity to polyconvex elastodynamics, *Arch. Ration. Mech. Anal.* (2006)
 - these are **structural properties**, whereas it should be proven for Navier-Stokes and Euler equations
 - Y. Brenier, R. Natalini, M. Puel, On a relaxation approximation of the incompressible Navier-Stokes equations, *Proc. Amer. Math. Soc.* (2004)

features that make finite element methods computationally attractive :

- formal high-order accuracy for smooth solutions, nonlinear stability
- in curved (spatial) domains, to easily incorporate complex geometries, and support domain decomposition techniques for parallelization
- inherent implicit-explicit discretization of nonlinear terms

difficulties to solve hyperbolic and convection-dominated problems :

- to choose the correct regularization mechanism, to capture discontinuities or strong gradients without producing spurious oscillations
- for relaxation approximations, the number of unknowns is increased

RELAXATION + ADAPTIVE MESHING

provides a *dissipative mechanism against the destabilizing effect of nonlinear response*, combined with appropriate *mesh refinement* (by means of an error indicator) *to detect the location of the singularities*

- F. Coquel, B. Perthame, Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics, *SIAM J. Num. Anal.* (1998)
- L. Gosse, A.E. Tzavaras, Convergence of relaxation schemes to the equations of elastodynamics, *Math. Comp.* (2001)
- Ch. Arvanitis, Th. Katsaounis, Ch. Makridakis, Adaptive finite element relaxation schemes for hyperbolic conservation laws, *M2AN Math. Model. Numer. Anal.* (2001)

standard semi-discrete finite element schemes

- let \mathbb{T}_h be (uniformly regular) partitions, with characteristic length $h > 0$
- (conforming) finite element spaces $W_{h,k} \subset H_0^1$, $k \geq 1$, of piecewise polynomials (globally continuous) on \mathbb{T}_h with **consistency properties**

$$\inf_{w_h \in W_{h,k}} \left(\|w_h - w\|_{L^2} + h \|w_h - w\|_{H^1} \right) \leq C h^k \|w\|_{H^k}$$

- homogeneous/periodic boundary conditions
- finite element space $W_{h,k-1} \subset L^2$ of piecewise discontinuous polynomials

variational formulation : to find $u_h^\varepsilon, v_h^\varepsilon, Z_h^\varepsilon \in W_{h,k}$ solution to

$$\langle \partial_t u_h^\varepsilon, \phi^1 \rangle + \langle v_h^\varepsilon, \partial_x \phi^1 \rangle = 0$$

$$\langle \partial_t v_h^\varepsilon, \phi^2 \rangle + \langle Z_h^\varepsilon, \partial_x \phi^2 \rangle = 0$$

$$\varepsilon \langle \partial_t Z_h^\varepsilon, \phi^3 \rangle + \mu \langle v_h^\varepsilon, \partial_x \phi^3 \rangle = -\langle Z_h^\varepsilon, \phi^3 \rangle + \langle S(u_h^\varepsilon), \phi^3 \rangle$$

for all $\phi^1, \phi^2, \phi^3 \in W_{h,k}$ (for the computational issues, *semi-implicit discretizations in time* are eventually considered, for which analogous estimates could also be proven)

for $\mu > \varepsilon$, the system admits a positive definite **symmetrizer**

$$H^\varepsilon = \begin{pmatrix} \mu/\varepsilon & 0 & -1 \\ 0 & \mu/\varepsilon - 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

which defines the hessian of a *convex entropy-entropy flux pair* (with dissipative behavior for the stiff/lower order source term)

$$\begin{aligned} \eta(w^\varepsilon) &= \frac{1}{2} \left(\frac{\mu}{\varepsilon} - 1 \right) \left(|u^\varepsilon|^2 + |v^\varepsilon|^2 \right) + \frac{1}{2} |u^\varepsilon - Z^\varepsilon|^2 \\ q(w^\varepsilon) &= - \left(\frac{\mu}{\varepsilon} - 1 \right) v^\varepsilon Z^\varepsilon \end{aligned}$$

with $w^\varepsilon = (u^\varepsilon, v^\varepsilon, Z^\varepsilon)^T$, in the sense that $H^\varepsilon w^\varepsilon = \nabla_{w^\varepsilon} \eta(w^\varepsilon)$

because of the symmetry, the *mechanical energy* is given by

$$E^\varepsilon(w^\varepsilon) = \int_{\mathbb{R}} \eta(w^\varepsilon) dx = \frac{1}{2} \langle H^\varepsilon w^\varepsilon, w^\varepsilon \rangle = \frac{1}{2} \langle w^\varepsilon, H^\varepsilon w^\varepsilon \rangle$$

and it satisfies

$$\frac{d}{dt} E^\varepsilon(w^\varepsilon(t)) = \langle w_t^\varepsilon, \nabla_{w^\varepsilon} \eta(w^\varepsilon) \rangle$$

moreover, it bounds from above (it is equivalent to) the Euclidean norm

$$E^\varepsilon(w^\varepsilon) \geq \frac{1}{4} \int_{\mathbb{R}} \left(\frac{\mu}{\varepsilon} |u^\varepsilon|^2 + \frac{\mu}{\varepsilon} |v^\varepsilon|^2 + |Z^\varepsilon|^2 \right) dx$$

under the (admissible) hypothesis that $\frac{\mu}{\varepsilon} \geq 4$

- the system is equipped with a positive definite energy functional, namely $\varepsilon E^\varepsilon(w^\varepsilon)$, which is in fact equivalent to the (square of the) L^2 -norm of the triple $(u^\varepsilon, v^\varepsilon, \sqrt{\varepsilon} Z^\varepsilon)$ for at least $\varepsilon \ll 1$
- thanks to the semi-linearity and the constant symmetrizer,

$$E_\gamma^\varepsilon(w^\varepsilon) = \int_{\mathbb{R}} \eta(\partial_\gamma w^\varepsilon) dx = \frac{1}{2} \langle \partial_\gamma w^\varepsilon, H^\varepsilon \partial_\gamma w^\varepsilon \rangle$$

defines a high-order energy, which is equivalent to higher order Sobolev norms $H^\gamma(\mathbb{R})$, $\gamma \geq 1$

- it holds also for multi-dimensional systems, and for the equations of elastodynamics with a (strictly) polyconvex internal/stored energy

we use $\phi^1 = \frac{\mu}{\varepsilon} u_h^\varepsilon - Z_h^\varepsilon$, $\phi^2 = (\frac{\mu}{\varepsilon} - 1) v_h^\varepsilon$, $\phi^3 = Z_h^\varepsilon - u_h^\varepsilon \in W_{h,k}$ to obtain

$$\frac{d}{dt} E^\varepsilon(w_h^\varepsilon) = \frac{1}{\varepsilon} \langle S(u_h^\varepsilon) - Z_h^\varepsilon, Z_h^\varepsilon - u_h^\varepsilon \rangle$$

revealing that the symmetrized form of the system is actually introduced to keep the **conservation property** for the finite element framework

$$\begin{aligned} & \frac{1}{\varepsilon} \int_{\mathbb{R}} \left(S(u_h^\varepsilon) Z_h^\varepsilon - |Z_h^\varepsilon|^2 - S(u_h^\varepsilon) u_h^\varepsilon + Z_h^\varepsilon u_h^\varepsilon \right) dx \\ & \leq \frac{1}{\varepsilon} \left(-\frac{1}{2} \int_{\mathbb{R}} |Z_h^\varepsilon|^2 dx + \frac{3}{2} \int_{\mathbb{R}} |S(u_h^\varepsilon)|^2 dx + \frac{3}{2} \int_{\mathbb{R}} |u_h^\varepsilon|^2 dx \right) \\ & \leq -\frac{1}{2\varepsilon} \|Z_h^\varepsilon\|_{L^2}^2 + \frac{3}{2\varepsilon} \left(1 + \sup_{\xi \in W_h} |S'(\xi(u_h^\varepsilon))|^2 \right) \|u_h^\varepsilon(t)\|_{L^2}^2 \end{aligned}$$

under the generic assumption $S(0)=0$, and finally

$$E_T^\varepsilon(w_h^\varepsilon) + \frac{1}{2\varepsilon} \int_0^T \|Z_h^\varepsilon(t)\|_{L^2}^2 dt \leq E_0^\varepsilon(w_h^\varepsilon) + \frac{C_0}{\varepsilon} \int_0^T \|u_h^\varepsilon(t)\|_{L^2}^2 dt$$

for any $T > 0$, with C_0 only depending upon the (global) Lipschitz condition for the nonlinear stress-strain function

- by testing *weak derivatives* of the finite element solution, we also get H^1 -stability estimates = **strong dissipative estimates**
uniformly with respect to the relaxation and numerical parameters

• corresponding L^2 -error estimates hold for the *numerical errors*
indeed, for some *projection operator* \mathbb{P}_h into the finite element space,
and the *auxiliary relaxation variable* given by $Z = S(u) + \mu \partial_x v$, we have

$$W = w_h^\varepsilon - w = (w_h^\varepsilon - \mathbb{P}_h w) + (\mathbb{P}_h w - w) = \mathcal{S}(w_h^\varepsilon, \mathbb{P}_h w) + \mathcal{C}(\mathbb{P}_h w, w)$$

which satisfies an analogous semilinear system : in particular, the last equation can be rewritten as follows, for any $\phi^3 \in W_{h,k}$

$$\langle \partial_t Z, \phi^3 \rangle + \frac{\mu}{\varepsilon} \langle V, \partial_x \phi^3 \rangle = -\frac{1}{\varepsilon} \langle Z, \phi^3 \rangle + \frac{1}{\varepsilon} \langle S(u_h^\varepsilon) - S(u), \phi^3 \rangle - \langle \partial_t z, \phi^3 \rangle$$

- for *well-prepared initial data*, this implies convergence with the optimal (polynomial) rate, without specific relationship between the parameters

for the **method of modulated energy**, the approximation of the relaxation variables is sought in lower-order finite element spaces

we select $\phi^3 = \partial_x \phi^2 \in W_{h,k-1}$ to obtain

$$\varepsilon \langle \partial_t Z_h^\varepsilon, \partial_x \phi^2 \rangle - \mu \langle \partial_x v_h^\varepsilon, \partial_x \phi^2 \rangle = -\langle Z_h^\varepsilon, \partial_x \phi^2 \rangle + \langle S(u_h^\varepsilon), \partial_x \phi^2 \rangle$$

by differentiating the second equation with respect to time, we get

$$\langle \partial_{tt} v_h^\varepsilon, \phi^2 \rangle + \langle \partial_t Z_h^\varepsilon, \partial_x \phi^2 \rangle = 0$$

and finally the following *one-field equation*, for any $\phi^2 \in W_{h,k}$

$$\langle \partial_t v_h^\varepsilon, \phi^2 \rangle + \langle S(u_h^\varepsilon), \partial_x \phi^2 \rangle + \mu \langle \partial_x v_h^\varepsilon, \partial_x \phi^2 \rangle + \varepsilon \langle \partial_{tt} v_h^\varepsilon, \phi^2 \rangle = 0$$

together with, for any $\phi^1 \in W_{h,k}$

$$\langle \partial_t u_h^\varepsilon, \phi^1 \rangle + \langle v_h^\varepsilon, \partial_x \phi^1 \rangle = 0$$

so that **the numerical system admits an equivalent (mixed) formulation**, that is a standard finite element scheme perturbed by a wave operator (similar computation for fully-conforming finite element spaces)

we choose $\phi^1 = u_h^\varepsilon$, $\phi^2 = v_h^\varepsilon \in W_{h,k}$ to obtain

$$\frac{1}{2} \frac{d}{dt} \|u_h^\varepsilon\|_{L^2}^2 + \langle v_h^\varepsilon, \partial_x u_h^\varepsilon \rangle = 0,$$

$$\frac{1}{2} \frac{d}{dt} \|v_h^\varepsilon\|_{L^2}^2 + \langle S(u_h^\varepsilon), \partial_x v_h^\varepsilon \rangle + \mu \|\partial_x v_h^\varepsilon\|_{L^2}^2 + \varepsilon \frac{d}{dt} \langle \partial_t v_h^\varepsilon, v_h^\varepsilon \rangle_{L^2} - \varepsilon \|\partial_t v_h^\varepsilon\|_{L^2}^2 = 0,$$

• taking advantage from the relaxation term, we deduce an estimate for the time derivative, and we modulate with a lower-order energy coming from the wave equation

we fix $\lambda > 1$ and we choose $\phi^2 = \varepsilon \lambda \partial_t v_h^\varepsilon \in W_{h,k}$ to obtain

$$\varepsilon \lambda \|\partial_t v_h^\varepsilon\|_{L^2}^2 + \varepsilon \lambda \langle S(u_h^\varepsilon), \partial_x \partial_t v_h^\varepsilon \rangle + \mu \varepsilon \lambda \frac{1}{2} \frac{d}{dt} \|\partial_x v_h^\varepsilon\|_{L^2}^2 + \varepsilon^2 \lambda \frac{1}{2} \frac{d}{dt} \|\partial_t v_h^\varepsilon\|_{L^2}^2 = 0$$

we sum together, rearranging the equalities as

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|u_h^\varepsilon\|_{L^2}^2 + \|v_h^\varepsilon + \varepsilon \partial_t v_h^\varepsilon\|_{L^2}^2 + \varepsilon^2 (\lambda - 1) \|\partial_t v_h^\varepsilon\|_{L^2}^2 + \mu \varepsilon \lambda \|\partial_x v_h^\varepsilon\|_{L^2}^2 \right) \\ & + \varepsilon (\lambda - 1) \|\partial_t v_h^\varepsilon\|_{L^2}^2 + \mu \|\partial_x v_h^\varepsilon\|_{L^2}^2 + \langle S(u_h^\varepsilon) - u_h^\varepsilon, \partial_x v_h^\varepsilon \rangle \\ & + \varepsilon \lambda \frac{d}{dt} \langle S(u_h^\varepsilon), \partial_x v_h^\varepsilon \rangle - \varepsilon \lambda \langle \partial_t S(u_h^\varepsilon), \partial_x v_h^\varepsilon \rangle = 0 \end{aligned}$$

– for the generic assumption $S(0)=0$, it holds

$$\left| \langle S(u_h^\varepsilon) - u_h^\varepsilon, \partial_x v_h^\varepsilon \rangle \right| \leq \frac{(1 + L_S)^2}{\mu} \|u_h^\varepsilon\|_{L^2}^2 + \frac{1}{4} \mu \|\partial_x v_h^\varepsilon\|_{L^2}^2$$

with $L_S > 0$ the *Lipschitz constant* of the stress-strain function

– using $\phi^1 = S'(u_h^\varepsilon) \partial_x v_h^\varepsilon \in W_{h,k-1}$ (with abuse of notation), we get

$$\langle \partial_t u_h^\varepsilon, S'(u_h^\varepsilon) \partial_x v_h^\varepsilon \rangle = \langle \partial_x v_h^\varepsilon, S'(u_h^\varepsilon) \partial_x v_h^\varepsilon \rangle$$

that implies

$$\mu \|\partial_x v_h^\varepsilon\|_{L^2}^2 - \varepsilon \lambda \langle S'(u_h^\varepsilon) \partial_t u_h^\varepsilon, \partial_x v_h^\varepsilon \rangle \geq (\mu - \varepsilon \lambda L_S) \|\partial_x v_h^\varepsilon\|_{L^2}^2 \geq 0$$

under the sub-characteristic condition given by $\mu \geq \varepsilon \lambda L_S$, which is satisfied for ε small enough in diffusive relaxation limits, but also essentially for a hyperbolic scaling

thus, we conclude **strong dissipative estimates** for the *modulated energy* functional suggested by the relaxation terms (uniformly with respect to the relaxation and numerical parameters)

??? ??? ???

- 1) implementation for multi-dimensional problems (including boundary conditions) and **inverse inequalities** for weak finite element spaces
- 2) to employ **discontinuous finite elements** to effectively handle shock waves arising in nonlinear elastic materials, for real applications
- 3) generalization to physical models with **viscosity of memory-type** leading to **integro-differential operators** (non-local with general kernels)
- 4) kinetic formulation of elastodynamics and related topics, as the relaxation method corresponds to some discrete kinetic approximation (equations of gas dynamics and magneto-hydrodynamics, shallow water equations, compressible Navier-Stokes equations with high Reynolds numbers, hydrodynamic models for semiconductor devices, ...)
- 5) for the **Euler equations**, to devise an (incomplete) artificial/numerical viscosity with a (mesh dependent) higher order parameter, in view of an alternative approach to the resolution of **contact discontinuities**