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Lionel Bréhamet. Final properties of  $osp(1|2)$  Racah coefficients related to hypergeometric series and orthogonal Wilson polynomials. 2021. <hal-03232877>

**HAL Id: hal-03232877**

**<https://hal.science/hal-03232877v1>**

Preprint submitted on 22 May 2021

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# Final properties of $\text{osp}(1|2)$ Racah coefficients related to hypergeometric series and orthogonal Wilson polynomials

Lionel Bréhamet <sup>†</sup>

May 22, 2021

## Abstract

From the supersymmetric version of Biedenharn-Elliott identity, two 3-term recurrence relations satisfied by the Racah coefficients are derived. Each super Racah coefficient  $6j^S$  with parities (alpha, beta or gamma) can generally be written as linear combination of two generalized hypergeometric  $4F_3$  series. For alpha parity only one series is enough. The 144 symmetries, including those of Regge, are well recovered like in the classic case except for beta parity which only counts 48. It is realized that the polynomials associated with the  $4F_3$  series actually consist simply of two Wilson polynomials. The role of Saalschützian property is emphasized.

MSC: 33C45

Keywords: Racah coefficients, Orthogonal polynomials, Hypergeometric series,  $\text{osp}(1|2)$  super-algebra

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# 1 Introduction

Using a modern terminology one could say that the so called ‘Racah problem’ for the superalgebra  $\mathfrak{osp}(1|2)$  found a solution thanks to Daumens *et al.* [1] in 1993. Indeed, it was done by using a finite-dimensional  $R^j$ -spin irreducible representation of  $\mathfrak{osp}(1|2)$ , see [2, p. 971-972] for a clear description.

The reduction of the tensor product of three representations  $R^j$  by two intermediate ways naturally led them to define a super-Racah coefficient  $6j^S$  in a similar approach to that used for  $su(2)$ . The Casimir operator of the representations was not used.

Nowadays with a different point of view, the Racah problem is to find the overlap between respective eigenstates of some intermediate Casimir operators when adding three algebras. With the representations used, it is shown that the Bannai-Ito polynomials [3] are the Racah coefficients of the super-algebra  $\mathfrak{osp}(1|2)$  [4, Part III].

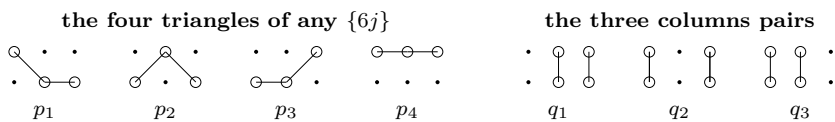
Besides, we continued studies [5, 6, 7] rather in a practical perspective. Thus any supersymmetric  $\{6j\}^S$  symbols (or supersymmetric Racah coefficients) can be computed thanks to a single summation formula over an integer  $z$  and a whole set of tools was built for  $\mathfrak{osp}(1|2)$  (tables, product formulas of tensor operators using them, Ponzano-Regge asymptotic, definition of a  $\{9j\}^S$  coefficient and so on, equivalent to those we can find in standard books about angular momentum for  $su(2)$ ). These super-Racah coefficients, called  $\{6j\}_{susy}$  by Livine and Oeckl [2] were invoked for their supersymmetric spin foam model.

Recently for deformed  $q$ -algebras like  $Uq(sl_N)$  the symmetries of quantum  $6j$ -symbols have been studied thoroughly by Alekseev *et al.* [8], with the discovering of new ones [9] in terms of  $q$ -analogues  ${}_4\Phi_3$  or  ${}_5\Phi_4$   $q$ -hypergeometric functions. In addition from a decade a lot of studies on various algebras (including  $\mathfrak{osp}(1|2)$ ) related to hypergeometric orthogonal polynomials, mainly from Montreal University, pushed us to delve deeper into unexplored properties of the  $\{6j\}^S$  coefficients. Since a long time standard  $6j$  coefficients are recognized as  ${}_4F_3$  generalized hypergeometric functions and orthogonal polynomials [10, 11], it remained today to further explore the properties of the supersymmetric  $6j^S$ , verify their symmetries and by the same token derive other interesting results like 3-term recurrence relations.

# 2 Basic recalls about $6j^S$ symbols

## 2.1 Standart $6j$ symbols

Representation of a Racah coefficient (called also  $\{6j\}$  symbol)  $\left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\}$ :



$p_i$  is the sum of the values of the three circled spins just above  $p_i$  in the diagrams. In the same way,  $q_k$  is the sum of the values of the four circled spins above  $q_j$ . The variables  $p, q$  (perimeters,

quadrangles) are not free since they satisfy the identity

$$\sum_{k=1}^3 q_k = \sum_{i=1}^4 p_i. \quad (2.1)$$

Any spin is thus determined by two  $p_i$  and one  $q_k$  according to

$$2J_1 = p_1 + p_4 - q_1, \quad 2J_2 = p_2 + p_4 - q_2, \quad 2J_3 = p_3 + p_4 - q_3, \quad (2.2)$$

$$2j_1 = p_2 + p_3 - q_1, \quad 2j_2 = p_3 + p_1 - q_2, \quad 2j_3 = p_1 + p_2 - q_3. \quad (2.3)$$

A standard 6- $j$  symbol may be written as follows [5]:

$$\left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\} = N \sum_z \frac{(-1)^z (z+1)!}{\prod_{i=1}^{i=4} (z-p_i)! \prod_{k=1}^{k=3} (q_k-z)!}, \quad \text{with} \quad N = \left[ \frac{\prod_{k=1}^{k=3} \prod_{i=1}^{i=4} (q_k-p_i)}{\prod_{i=1}^{i=4} (p_i+1)!} \right]^{\frac{1}{2}}. \quad (2.4)$$

## 2.2 Super $6j^S$ symbols

Similarly to  $su(2)$ , these coefficients arise when we recouple three representations, but here for  $\mathfrak{osp}(1|2)$ . We have obtained the following formulas:

$$\left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\}_\pi^S = (-1)^{4 \sum j_k} \sqrt{\mathbf{R}^S} \sum_z \frac{(-1)^z z! \Pi_\pi(z)}{\prod_{i=1}^{i=4} (z-[p_i+\frac{1}{2}])! \prod_{k=1}^{k=3} ([q_k+\frac{1}{2}]-z)!}, \quad (2.5)$$

where  $\mathbf{R}^S$  is a supersymmetric prefactor given by

$$\mathbf{R}^S = \frac{\prod_{k=1}^{k=3} \prod_{i=1}^{i=4} [q_k - p_i]!}{\prod_{i=1}^{i=4} [p_i + \frac{1}{2}]!}. \quad (2.6)$$

The delimiters  $[\dots]$  around an expression stands for the integer part of the expression. For any  $j$ , integer or half integer, its integer part is obtained by taking the double, then perform the Euclidean division by 2 and consider the quotient, namely

$$[j] = \{(2j)\}/2. \quad (2.7)$$

There are 3 parities ( $\pi$ ) for these Racah coefficients from the four quantum triangle perimeters:

$$\begin{cases} \pi = \alpha & \text{if } \forall i \in [1, 4] p_i \text{ integer } (\Rightarrow q_k \text{ integer } (k = 1, 2, 3)), \\ \pi = \beta & \text{if } \exists \text{ only two distinct } p_i, p_j \text{ integer } (\Rightarrow \exists! q_k \text{ integer}), \\ \pi = \gamma & \text{if } \forall i \in [1, 4] p_i \text{ half-integer } (\Rightarrow q_k \text{ integer } (k = 1, 2, 3)). \end{cases} \quad (2.8)$$

For a parity  $\beta$ , both integer triangles shall be denoted by  $p, p'$ , both other half-integer by  $\bar{p}, \bar{p}'$ . The single integer quadrangle is denoted by  $q$ , both other half-integer by  $\bar{q}, \bar{q}'$ .

In this case eq.(2.1) transforms into

$$q + \bar{q} + \bar{q}' = p + p' + \bar{p} + \bar{p}'. \quad (2.9)$$

Monomials in  $z$ , namely  $\Pi_\pi(z)$  of eq.(2.5) have been found to be [5]:

$$\Pi_\alpha(z) = 1, \quad (2.10)$$

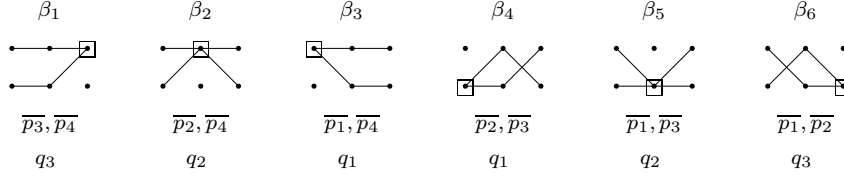
$$\Pi_\beta(z) = -z(\bar{q} + \bar{q}' - p - p' + 1) + \left(\bar{q} + \frac{1}{2}\right) \left(\bar{q}' + \frac{1}{2}\right) - pp', \quad (2.11)$$

$$\Pi_\gamma(z) = -z + 2 \sum_c j_c^\wedge j_c^\vee + \left(\sum_c (j_c^\wedge + j_c^\vee)\right) + \frac{1}{2}. \quad (2.12)$$

Note that the quantity  $(\bar{q} + \bar{q}' - p - p')$  is related to one of the six spins, let be  $\bar{J}_s^*$ , which is the single vertex common to both triangles  $\bar{p}$  and  $\bar{p}'$ .

$$\bar{J}_s^* = \frac{1}{2}(\bar{q} + \bar{q}' - p - p'). \quad (2.13)$$

Just here below for the parity  $\beta$  are shown the 6 positions of the spin  $\bar{J}_s^*$  (inserted in a small box):



$$\text{Thus } \bar{J}_s^*(\beta_3) = J_1 \parallel \bar{J}_s^*(\beta_2) = J_2 \parallel \bar{J}_s^*(\beta_1) = J_3 \parallel \bar{J}_s^*(\beta_4) = j_1 \parallel \bar{J}_s^*(\beta_5) = j_2 \parallel \bar{J}_s^*(\beta_6) = j_3. \quad (2.14)$$

The integer quadrangle  $q$  is the only one which does not contain the distinguished spin  $\bar{J}_s^*$ .  $j_c^\wedge$  stands for the three spins of the top row, and  $j_c^\vee$  for the bottom row of the same column  $c$ . It can be checked that the coefficient  $\Pi_\gamma(0)$  is an integer invariant under the tetrahedral symmetries and the five Regge transformations [6]. Indeed thanks to the expressions of any spin in terms of two  $p_i$  and one  $q_k$  [see eqs.(2.2),(2.3)], and to eq.(2.1), it can be shown that

$$4 \sum_c j_c^\wedge j_c^\vee = \sum_{k=1}^3 q_k^2 - \sum_{i=1}^4 p_i^2 \equiv \sum_{k=1}^3 q_k(q_k + 1) - \sum_{i=1}^4 p_i(p_i + 1). \quad (2.15)$$

### 3 $\mathfrak{osp}(1|2)$ superalgebra and super Racah coefficients

Taking into account the acquired experience since the historic recoupling problems of Racah, from him until Genest [18, 19] and much other, if possible we should present a study with 3- terms recurrence, orthogonality, hypergeometric functions, orthogonal polynomials. As reported in a detailed comment by Rao *et al.* [17], we know that a close relation exists between the 3-terms recurrence of Racah coefficients and the 3-terms recurrence of the associated orthogonal polynomial.

#### 3.1 3-term recurrence relation in steps of $\frac{1}{2}$

First of all, let us recall that for  $\mathfrak{osp}(1,2)$  or  $su(2)$ , orthogonality is simply a particular case of Biedenharn-Elliott identity (BE) [5, p.244]. Let's write the supersymmetric version of this identity:

$$\begin{aligned} \left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\}^S &= (-1)^{[a+b+c]} \\ &\times \sum_z (-1)^{[e+q+z]+[f+r+z]+[d+p+z]+2z} \left\{ \begin{matrix} e & q & z \\ r & f & a \end{matrix} \right\}^S \left\{ \begin{matrix} f & r & z \\ p & d & b \end{matrix} \right\}^S \left\{ \begin{matrix} d & p & z \\ q & e & c \end{matrix} \right\}^S. \end{aligned} \quad (3.1)$$

---

\*Useful relation, because if  $a$  is integer, then  $a(a+1)$  is even.

Setting  $e = a$ ,  $d = b$  and **especially**  $f = \frac{1}{2}$  leads to 3 terms because here  $f$  varies in steps of  $1/2$ .

$$\begin{aligned}
& \left\{ \begin{array}{ccc} a & b & c \\ b & a & \frac{1}{2} \end{array} \right\}^S \left\{ \begin{array}{ccc} a & b & c \\ p & q & r \end{array} \right\}^S = \\
& - (-1)^{[a+b+c]+[a+q+r-\frac{1}{2}]+[b+p+r-\frac{1}{2}]} \left\{ \begin{array}{ccc} a & q & r - \frac{1}{2} \\ r & \frac{1}{2} & a \end{array} \right\}^S \left\{ \begin{array}{ccc} \frac{1}{2} & r & r - \frac{1}{2} \\ p & b & b \end{array} \right\}^S \left\{ \begin{array}{ccc} a & b & c \\ p & q & r - \frac{1}{2} \end{array} \right\}^S \\
& + (-1)^{[a+b+c]+[a+q+r]+[b+p+r]} \left\{ \begin{array}{ccc} a & q & r \\ r & \frac{1}{2} & a \end{array} \right\}^S \left\{ \begin{array}{ccc} \frac{1}{2} & r & r \\ p & b & b \end{array} \right\}^S \left\{ \begin{array}{ccc} a & b & c \\ p & q & r \end{array} \right\}^S \\
& + (-1)^{[a+b+c]+[a+q+r+\frac{1}{2}]+[b+p+r+\frac{1}{2}]} \left\{ \begin{array}{ccc} a & q & r + \frac{1}{2} \\ r & \frac{1}{2} & a \end{array} \right\}^S \left\{ \begin{array}{ccc} \frac{1}{2} & r & r + \frac{1}{2} \\ p & b & b \end{array} \right\}^S \left\{ \begin{array}{ccc} a & b & c \\ p & q & r + \frac{1}{2} \end{array} \right\}^S. \quad (3.2)
\end{aligned}$$

• Definitions and recalls for  $\mathbf{6}\text{-}j^S$  with one spin  $1/2$ :

Directly inherited from Table II found in [5], they are replaced by a product of two factors : a first one with phase and a second that we call *reduced* (with a \* as lower index) .

$$\left\{ \begin{array}{ccc} a & b & c \\ \frac{1}{2} & c & b \end{array} \right\}^S = \frac{(-1)^{[a+b+c+\frac{1}{2}]+2a}}{(2b(2b+1)2c(2c+1))^{\frac{1}{2}}} \left| \left\{ \begin{array}{ccc} a & b & c \\ \frac{1}{2} & c & b \end{array} \right\}_*^S \right|, \quad (3.3)$$

$$\left\{ \begin{array}{ccc} a & b & c \\ \frac{1}{2} & c & b \end{array} \right\}^S = \frac{(-1)^{[a+b+c]+2b+2c}}{(2b(2b+1)2c(2c+1))^{\frac{1}{2}}} \left| \left\{ \begin{array}{ccc} a & b & c \\ \frac{1}{2} & c & b \end{array} \right\}_*^S \right|, \quad [\text{variant of eq.(3.3)}] \quad (3.4)$$

$$\left\{ \begin{array}{ccc} a & b - \frac{1}{2} & c \\ \frac{1}{2} & c & b \end{array} \right\}^S = \frac{(-1)^{[a+b+c-\frac{1}{2}]+2b-1}}{((2b)2c(2c+1))^{\frac{1}{2}}} \left| \left\{ \begin{array}{ccc} a & b - \frac{1}{2} & c \\ \frac{1}{2} & c & b \end{array} \right\}_*^S \right|, \quad (3.5)$$

$$\left\{ \begin{array}{ccc} a & b & c \\ \frac{1}{2} & c & b + \frac{1}{2} \end{array} \right\}^S = \frac{(-1)^{[a+b+c]+2b}}{((2b+1)2c(2c+1))^{\frac{1}{2}}} \left| \left\{ \begin{array}{ccc} a & b & c \\ \frac{1}{2} & c & b + \frac{1}{2} \end{array} \right\}_*^S \right|, \quad (3.6)$$

with the following *reduced* expressions

$$\left| \left\{ \begin{array}{ccc} a & b & c \\ \frac{1}{2} & c & b \end{array} \right\}_*^S \right| = \begin{cases} (-a+b+c+4bc) & \text{if } a+b+c \text{ integer} \\ (a+b+c+\frac{1}{2}+4bc) & \text{if } a+b+c \text{ half-integer} \end{cases}, \quad (3.7)$$

$$\left| \left\{ \begin{array}{ccc} a & b - \frac{1}{2} & c \\ \frac{1}{2} & c & b \end{array} \right\}_*^S \right| = \begin{cases} \{(-a+b+c)(a+b-c)\}^{\frac{1}{2}} & \text{if } a+b+c \text{ integer} \\ \{(a-b+c+\frac{1}{2})(a+b+c+\frac{1}{2})\}^{\frac{1}{2}} & \text{if } a+b+c \text{ half-integer} \end{cases}. \quad (3.8)$$

$$\left| \left\{ \begin{array}{ccc} a & b & c \\ \frac{1}{2} & c & b + \frac{1}{2} \end{array} \right\}_*^S \right| = \begin{cases} \{(a-b+c)(a+b+c+1)\}^{\frac{1}{2}} & \text{if } a+b+c \text{ integer} \\ \{(-a+b+c+\frac{1}{2})(a+b-c+\frac{1}{2})\}^{\frac{1}{2}} & \text{if } a+b+c \text{ half-integer} \end{cases}. \quad (3.9)$$

By using the integer parts identity  $[j] + [j + \frac{1}{2}] = 2j$  we obtain the following reductions:

$$\begin{aligned}
& \left\{ \begin{array}{ccc} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{array} \right\}^S \left\{ \begin{array}{ccc} p & r - \frac{1}{2} & b \\ \frac{1}{2} & b & r \end{array} \right\}^S = (-1)^{[a+q+r]+[b+p+r]+2(a+b+p+q)} \\
& \quad \times \frac{1}{2r} \left[ \frac{1}{2a(2a+1)2b(2b+1)} \right]^{\frac{1}{2}} \left| \left\{ \begin{array}{ccc} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{array} \right\}_*^S \right| \left| \left\{ \begin{array}{ccc} p & r - \frac{1}{2} & b \\ \frac{1}{2} & b & r \end{array} \right\}_*^S \right|. \quad (3.10)
\end{aligned}$$

$$\begin{aligned}
& \left\{ \begin{array}{ccc} q & r & a \\ \frac{1}{2} & a & r \end{array} \right\}^S \left\{ \begin{array}{ccc} p & r & b \\ \frac{1}{2} & b & r \end{array} \right\}^S = (-1)^{[a+q+r]+[b+p+r]+2(a+b)} \\
& \quad \times \frac{1}{2r(2r+1)} \left[ \frac{1}{2a(2a+1)2b(2b+1)} \right]^{\frac{1}{2}} \left| \left\{ \begin{array}{ccc} q & r & a \\ \frac{1}{2} & a & r \end{array} \right\}_*^S \right| \left| \left\{ \begin{array}{ccc} p & r & b \\ \frac{1}{2} & b & r \end{array} \right\}_*^S \right|. \quad (3.11)
\end{aligned}$$

$$\left\{ \begin{matrix} q & r & a \\ \frac{1}{2} & a & r + \frac{1}{2} \end{matrix} \right\}^S \left\{ \begin{matrix} p & r & b \\ \frac{1}{2} & b & r + \frac{1}{2} \end{matrix} \right\}^S = (-1)^{[q+r+a]+[p+r+b]} \\ \times \frac{1}{(2r+1)} \left[ \frac{1}{2a(2a+1)2b(2b+1)} \right]^{\frac{1}{2}} \left| \left\{ \begin{matrix} q & r & a \\ \frac{1}{2} & a & r + \frac{1}{2} \end{matrix} \right\}_*^S \right| \left| \left\{ \begin{matrix} p & r & b \\ \frac{1}{2} & b & r + \frac{1}{2} \end{matrix} \right\}_*^S \right|. \quad (3.12)$$

For avoiding square roots, we will work with the integer quantities like  $w^S \left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\}$  and super-triangles  $\Delta^S(abc)$  defined in [5].

$$\left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\}^S = (-1)^4 \sum_c j_c^\wedge j_c^\vee \Delta^S(J_1 j_2 j_3) \Delta^S(j_1 J_2 j_3) \Delta^S(j_1 j_2 J_3) \Delta^S(J_1 J_2 J_3) w^S \left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\}. \quad (3.13)$$

After some rearrangements we have

$$\left( \binom{2r(2r+1)}{\left| \left\{ \begin{matrix} c & a & b \\ \frac{1}{2} & b & a \end{matrix} \right\}_*^S \right| - \left| \left\{ \begin{matrix} q & r & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \left\{ \begin{matrix} p & r & b \\ \frac{1}{2} & b & r \end{matrix} \right\}_*^S \right|} \right) w^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\} = \\ - (-1)^{2(a+b+c)} \binom{2r+1}{\left| \left\{ \begin{matrix} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \left\{ \begin{matrix} p & r - \frac{1}{2} & b \\ \frac{1}{2} & b & r \end{matrix} \right\}_*^S \right|} w^S \left\{ \begin{matrix} a & b & c \\ p & q & r - \frac{1}{2} \end{matrix} \right\} \frac{\Delta^S(aqr - \frac{1}{2})}{\Delta^S(aqr)} \cdot \frac{\Delta^S(pbr - \frac{1}{2})}{\Delta^S(pbr)} \\ + (-1)^{2(p+q+c)} \binom{2r}{\left| \left\{ \begin{matrix} q & r & a \\ \frac{1}{2} & a & r + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} p & r & b \\ \frac{1}{2} & b & r + \frac{1}{2} \end{matrix} \right\}_*^S \right|} w^S \left\{ \begin{matrix} a & b & c \\ p & q & r + \frac{1}{2} \end{matrix} \right\} \frac{\Delta^S(aqr + \frac{1}{2})}{\Delta^S(aqr)} \cdot \frac{\Delta^S(pbr + \frac{1}{2})}{\Delta^S(pbr)}. \quad (3.14)$$

From eq.(3.8) we derive

$$\frac{\Delta^S(aqr - \frac{1}{2})}{\Delta^S(aqr)} = \begin{cases} \{(a-q+r)(-a+q+r)\}^{-\frac{1}{2}} & \text{if } a+q+r \text{ integer} \\ \{(a+q-r+\frac{1}{2})(a+q+r+\frac{1}{2})\}^{\frac{1}{2}} & \text{if } a+q+r \text{ half-integer} \end{cases}, \quad (3.15)$$

$$\left| \left\{ \begin{matrix} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \right| = \begin{cases} \{(-q+a+r)(q-a+r)\}^{\frac{1}{2}} & \text{if } a+q+r \text{ integer} \\ \{(a+q-r+\frac{1}{2})(a+q+r+\frac{1}{2})\}^{\frac{1}{2}} & \text{if } a+q+r \text{ half-integer} \end{cases}, \quad (3.16)$$

and are able to write the following products only with integer parts:

$$\frac{\Delta^S(aqr - \frac{1}{2})}{\Delta^S(aqr)} \times \left| \left\{ \begin{matrix} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \right| = \begin{cases} 1 & \text{if } a+q+r \text{ integer} \\ (a+q-r+\frac{1}{2})(a+q+r+\frac{1}{2}) & \text{if } a+q+r \text{ half-integer} \end{cases}. \quad (3.17)$$

$$\frac{\Delta^S(aqr - \frac{1}{2}) \Delta^S(bpr - \frac{1}{2})}{\Delta^S(aqr) \Delta^S(bpr)} \times \left| \left\{ \begin{matrix} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \left\{ \begin{matrix} p & r - \frac{1}{2} & b \\ \frac{1}{2} & b & r \end{matrix} \right\}_*^S \right| = \\ \begin{cases} 1 & \text{if } a+q+r \text{ integer} \\ (a+q-r+\frac{1}{2})(a+q+r+\frac{1}{2}) & \text{if } a+q+r \text{ half-integer} \end{cases} \times \begin{cases} 1 & \text{if } b+p+r \text{ integer} \\ (b+p-r+\frac{1}{2})(b+p+r+\frac{1}{2}) & \text{if } b+p+r \text{ half-integer} \end{cases}. \quad (3.18)$$

In the same way

$$\frac{\Delta^S(aqr + \frac{1}{2}) \Delta^S(bpr + \frac{1}{2})}{\Delta^S(aqr) \Delta^S(bpr)} \times \left| \left\{ \begin{matrix} q & r & a \\ \frac{1}{2} & a & r + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} p & r & b \\ \frac{1}{2} & b & r + \frac{1}{2} \end{matrix} \right\}_*^S \right| = \\ \begin{cases} 1 & \text{if } a+q+r \text{ integer} \\ (a-q+r+\frac{1}{2})(-a+q+r+\frac{1}{2}) & \text{if } a+q+r \text{ half-integer} \end{cases} \times \begin{cases} 1 & \text{if } b+p+r \text{ integer} \\ (b-p+r+\frac{1}{2})(-b+p+r+\frac{1}{2}) & \text{if } b+p+r \text{ half-integer} \end{cases}. \quad (3.19)$$

Equations (3.18),( 3.19) can be compacted by the notational trick

$$\frac{[m + \frac{1}{2}]!}{[m]!} = \begin{cases} 1 & \text{if } m \text{ integer} \\ (m + \frac{1}{2}) & \text{if } m \text{ half-integer} \end{cases}. \quad (3.20)$$

Thus the 3-term recurrence becomes an equation with all coefficients of the  $w^S$  integer:

$$\begin{aligned} 2r(2r+1) \left| \left\{ \begin{matrix} c & a & b \\ \frac{1}{2} & b & a \end{matrix} \right\}_*^S \right| w^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\} = \\ - (-1)^{2(a+b+c)} (2r+1) \frac{[a+q-r+\frac{1}{2}]!}{[a+q-r]!} \frac{[a+q+r+\frac{1}{2}]!}{[a+q+r]!} \frac{[b+p-r+\frac{1}{2}]!}{[b+p-r]!} \frac{[b+p+r+\frac{1}{2}]!}{[b+p+r]!} w^S \left\{ \begin{matrix} a & b & c \\ p & q & r - \frac{1}{2} \end{matrix} \right\} \\ + \left| \left\{ \begin{matrix} q & r & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \right| \left| \left\{ \begin{matrix} p & r & b \\ \frac{1}{2} & b & r \end{matrix} \right\}_*^S \right| w^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\} \\ + (-1)^{2(p+q+c)} (2r) \frac{[a-q+r+\frac{1}{2}]!}{[a-q+r]!} \frac{[-a+q+r+\frac{1}{2}]!}{[-a+q+r]!} \frac{[b-p+r+\frac{1}{2}]!}{[b-p+r]!} \frac{[-b+p+r+\frac{1}{2}]!}{[-b+p+r]!} w^S \left\{ \begin{matrix} a & b & c \\ p & q & r + \frac{1}{2} \end{matrix} \right\}. \quad (3.21) \end{aligned}$$

• Definitions:

Let us introduce three coefficients  $h_{n_2, n_3}^{n_1}, X_r, Y_r$

$$h_{n_2, n_3}^{n_1} = \left| \left\{ \begin{matrix} n_3 & n_1 & n_2 \\ \frac{1}{2} & n_2 & n_1 \end{matrix} \right\}_*^S \right| = [n_3 - n_1 + n_2 + \frac{1}{2}][n_3 + n_1 - n_2 + \frac{1}{2}] + [n_3 + n_1 + n_2 + 1][n_3 + n_1 + n_2 + \frac{1}{2}], \quad (3.22)$$

$$X_r = \frac{[a+q-r+\frac{1}{2}]!}{[a+q-r]!} \frac{[a+q+r+\frac{1}{2}]!}{[a+q+r]!} \frac{[b+p-r+\frac{1}{2}]!}{[b+p-r]!} \frac{[b+p+r+\frac{1}{2}]!}{[b+p+r]!}, \quad (3.23)$$

$$Y_r = \frac{[a-q+r+\frac{1}{2}]!}{[a-q+r]!} \frac{[-a+q+r+\frac{1}{2}]!}{[-a+q+r]!} \frac{[b-p+r+\frac{1}{2}]!}{[b-p+r]!} \frac{[-b+p+r+\frac{1}{2}]!}{[-b+p+r]!}. \quad (3.24)$$

Accordingly the 3-term recurrence in steps of  $\frac{1}{2}$  reads ultimately

$$\begin{aligned} 2r(2r+1) h_{b,c}^a w^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\} = -(-1)^{2(a+b+c)} (2r+1) X_r w^S \left\{ \begin{matrix} a & b & c \\ p & q & r - \frac{1}{2} \end{matrix} \right\} \\ + h_{a,q}^r h_{b,p}^r w^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\} + (-1)^{2(p+q+c)} (2r) Y_r w^S \left\{ \begin{matrix} a & b & c \\ p & q & r + \frac{1}{2} \end{matrix} \right\}. \quad (3.25) \end{aligned}$$

*Stricto sensu* we really have to do with 3 terms, but the drawback of this equation in steps of  $\frac{1}{2}$  lies in its dependence in the parities ( $\pi$ ) of their related  $w_{(\pi)}^S$ . At each iteration the parity of certain  $w^S$  is changed. Actually this relation shows the non-independence of the parities  $\alpha, \beta$  and  $\gamma$ . Each Racah coefficient with parity  $\alpha, \beta, \gamma$  has a different recurrence relation. Nevertheless it is possible to find a standard 3-term recurrence in steps of 1, which does not change the parity of the  $w_{\pi}^S$  by again using the supersymmetric BE identity (3.1) . Equation above was successfully checked on computer by using only integers.

### 3.2 3-term recurrence in steps of 1

Here we restart from eq.(3.14) rewritten with coefficients  $h_{n_2, n_3}^{n_1}$ , and  $r \rightarrow r - \frac{1}{2}$ , what yields

$$\begin{aligned} & \left( (2r-1)(2r) h_{b,c}^a - h_{a,q}^{r-\frac{1}{2}} h_{b,p}^{r-\frac{1}{2}} \right) \times w^S \left\{ \begin{matrix} a & b & c \\ p & q & r - \frac{1}{2} \end{matrix} \right\} = \\ & - (-1)^{2(a+b+c)} (2r) \left| \left\{ \begin{matrix} q & r - 1 & a \\ \frac{1}{2} & a & r - \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} p & r - 1 & b \\ \frac{1}{2} & b & r - \frac{1}{2} \end{matrix} \right\}_*^S \right| w^S \left\{ \begin{matrix} a & b & c \\ p & q & r - 1 \end{matrix} \right\} \frac{\Delta^S(aqr-1)}{\Delta^S(aqr-\frac{1}{2})} \cdot \frac{\Delta^S(pbr-1)}{\Delta^S(pbr-\frac{1}{2})} \\ & + (-1)^{2(p+q+c)} (2r-1) \left| \left\{ \begin{matrix} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \left\{ \begin{matrix} p & r - \frac{1}{2} & b \\ \frac{1}{2} & b & r \end{matrix} \right\}_*^S \right| w^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\} \frac{\Delta^S(aqr)}{\Delta^S(aqr-\frac{1}{2})} \cdot \frac{\Delta^S(pbr)}{\Delta^S(pbr-\frac{1}{2})}. \quad (3.26) \end{aligned}$$

and a similar equation for  $r \rightarrow r + \frac{1}{2}$ .

First we multiply eq.(3.26) by  $\frac{\Delta^S(aqr-\frac{1}{2})}{\Delta^S(aqr)} \cdot \frac{\Delta^S(pbr-\frac{1}{2})}{\Delta^S(pbr)}$   $\left| \left\{ \begin{matrix} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \left\{ \begin{matrix} p & r - \frac{1}{2} & b \\ \frac{1}{2} & b & r \end{matrix} \right\}_*^S \right|$

and then we use the following identity with integer parts:

$$\left| \left\{ \begin{matrix} q & r - 1 & a \\ \frac{1}{2} & a & r - \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \right| \frac{\Delta^S(aqr-1)}{\Delta^S(aqr)} = [q-r+a+1][q+r+a+\frac{1}{2}]. \quad (3.27)$$

It results a 3-term expression in  $r - \frac{1}{2}, r - 1, r$ :

$$\begin{aligned} & \left( (2r-1)(2r) h_{b,c}^a - h_{a,q}^{r-\frac{1}{2}} h_{b,p}^{r-\frac{1}{2}} \right) \\ & \times \left| \left\{ \begin{matrix} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \left\{ \begin{matrix} p & r - \frac{1}{2} & b \\ \frac{1}{2} & b & r \end{matrix} \right\}_*^S \right| \frac{\Delta^S(aqr-\frac{1}{2})}{\Delta^S(aqr)} \frac{\Delta^S(pbr-\frac{1}{2})}{\Delta^S(pbr)} w^S \left\{ \begin{matrix} a & b & c \\ p & q & r - \frac{1}{2} \end{matrix} \right\} \\ & = \\ & - (-1)^{2(a+b+c)} (2r) [q-r+a+1][q+r+a+\frac{1}{2}][p-r+b+1][p+r+b+\frac{1}{2}] w^S \left\{ \begin{matrix} a & b & c \\ p & q & r - 1 \end{matrix} \right\} \\ & + \\ & (-1)^{2(p+q+c)} (2r-1) \left( \left| \left\{ \begin{matrix} q & r - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \left\{ \begin{matrix} p & r - \frac{1}{2} & b \\ \frac{1}{2} & b & r \end{matrix} \right\}_*^S \right| \right)^2 \times w^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\}. \quad (3.28) \end{aligned}$$

We follow a similar process for  $r \rightarrow r + \frac{1}{2}$ , this time by operating with multipliers such as

$$\frac{\Delta^S(aqr+\frac{1}{2})}{\Delta^S(aqr)} \cdot \frac{\Delta^S(pbr+\frac{1}{2})}{\Delta^S(pbr)} \left| \left\{ \begin{matrix} qr & a \\ \frac{1}{2} ar + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} pr & b \\ \frac{1}{2} br + \frac{1}{2} \end{matrix} \right\}_*^S \right|. \text{ After simplification we find}$$

$$\begin{aligned} & \left( (2r+1)(2r+2) h_{b,c}^a - h_{a,q}^{r+\frac{1}{2}} h_{b,p}^{r+\frac{1}{2}} \right) \\ & \times \left| \left\{ \begin{matrix} qr & a \\ \frac{1}{2} ar + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} pr & b \\ \frac{1}{2} br + \frac{1}{2} \end{matrix} \right\}_*^S \right| \frac{\Delta^S(aqr+\frac{1}{2})}{\Delta^S(aqr)} \cdot \frac{\Delta^S(pbr+\frac{1}{2})}{\Delta^S(pbr)} w^S \left\{ \begin{matrix} ab & c \\ pq r + \frac{1}{2} \end{matrix} \right\} = \\ & - (2r+2)(-1)^{2(a+b+c)} \left( \left| \left\{ \begin{matrix} qr & a \\ \frac{1}{2} ar + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} pr & b \\ \frac{1}{2} br + \frac{1}{2} \end{matrix} \right\}_*^S \right| \right)^2 w^S \left\{ \begin{matrix} abc \\ pqr \end{matrix} \right\} \\ & + \\ & (2r+1)(-1)^{2(p+q+c)} \left| \left\{ \begin{matrix} qr & a \\ \frac{1}{2} ar + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} qr + \frac{1}{2} & a \\ \frac{1}{2} a & r + 1 \end{matrix} \right\}_*^S \right| \frac{\Delta^S(aqr+1)}{\Delta^S(aqr)} \\ & \times \left| \left\{ \begin{matrix} pr & b \\ \frac{1}{2} br + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} pr + \frac{1}{2} & b \\ \frac{1}{2} b & r + 1 \end{matrix} \right\}_*^S \right| \frac{\Delta^S(pbr+1)}{\Delta^S(pbr)} w^S \left\{ \begin{matrix} ab & c \\ pq r + 1 \end{matrix} \right\}. \quad (3.29) \end{aligned}$$

By using the identity (3.27) for integer parts we can write

$$\left| \left\{ \begin{matrix} qr & a \\ \frac{1}{2} ar + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} qr + \frac{1}{2} & a \\ \frac{1}{2} a & r + 1 \end{matrix} \right\}_*^S \right| = [q - r + a][q + r + a + \frac{3}{2}] \frac{\Delta^S(aqr+1)}{\Delta^S(aqr)}, \quad (3.30)$$

and a similar equation with  $\{prb\}$  instead of  $\{qra\}$ . Then in eq.(3.29) the coefficient of  $w^S \left\{ \begin{matrix} ab & c \\ pq r + 1 \end{matrix} \right\}$  is equal to

$$(2r+1)[q-r+a][q+r+a+\frac{3}{2}][p-r+b][p+r+b+\frac{3}{2}] \left( \frac{\Delta^S(aqr+1)}{\Delta^S(aqr)} \cdot \frac{\Delta^S(pbr+1)}{\Delta^S(pbr)} \right)^2. \quad (3.31)$$

Moreover as we have the result

$$\left( \frac{\Delta^S(aqr+1)}{\Delta^S(aqr)} \right)^2 = \frac{[-a+q+r+1][a-q+r+1]}{[a+q+r+\frac{3}{2}][a+q-r]}, \quad (3.32)$$

the coefficient of  $w^S \left\{ \begin{matrix} ab & c \\ pq r + 1 \end{matrix} \right\}$  becomes finally  $(2r+1)[-a+q+r+1][a-q+r+1][-b+p+r+1][b-p+r+1]$ .

Equation (3.29) can be rewritten as:

$$\begin{aligned} & \left( (2r+1)(2r+2) h_{b,c}^a - h_{a,q}^{r+\frac{1}{2}} h_{b,p}^{r+\frac{1}{2}} \right) \\ & \times \left| \left\{ \begin{matrix} qr & a \\ \frac{1}{2} ar + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} pr & b \\ \frac{1}{2} br + \frac{1}{2} \end{matrix} \right\}_*^S \right| \frac{\Delta^S(aqr+\frac{1}{2})}{\Delta^S(aqr)} \cdot \frac{\Delta^S(pbr+\frac{1}{2})}{\Delta^S(pbr)} w^S \left\{ \begin{matrix} ab & c \\ pq r + \frac{1}{2} \end{matrix} \right\} = \\ & - (2r+2)(-1)^{2(a+b+c)} \left( \left| \left\{ \begin{matrix} qr & a \\ \frac{1}{2} ar + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} pr & b \\ \frac{1}{2} br + \frac{1}{2} \end{matrix} \right\}_*^S \right| \right)^2 w^S \left\{ \begin{matrix} abc \\ pqr \end{matrix} \right\} \\ & + \\ & (2r+1)(-1)^{2(p+q+c)} [-a+q+r+1][a-q+r+1][-b+p+r+1][b-p+r+1] w^S \left\{ \begin{matrix} ab & c \\ pq r + 1 \end{matrix} \right\}. \quad (3.33) \end{aligned}$$

Multiplying eq.(3.14) by  $\left( (2r-1)(2r) h_{b,c}^a - h_{a,q}^{r-\frac{1}{2}} h_{b,p}^{r-\frac{1}{2}} \right) \left( (2r+1)(2r+2) h_{b,c}^a - h_{a,q}^{r+\frac{1}{2}} h_{b,p}^{r+\frac{1}{2}} \right)$  leads to

$$\begin{aligned}
& \left( (2r-1)(2r) h_{b,c}^a - h_{a,q}^{r-\frac{1}{2}} h_{b,p}^{r-\frac{1}{2}} \right) \left( (2r+1)(2r+2) h_{b,c}^a - h_{a,q}^{r+\frac{1}{2}} h_{b,p}^{r+\frac{1}{2}} \right) \\
& \quad \times w^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\} \\
& \quad = \\
& \quad - (-1)^{2(a+b+c)} (2r+1) \left( (2r+1)(2r+2) h_{b,c}^a - h_{a,q}^{r+\frac{1}{2}} h_{b,p}^{r+\frac{1}{2}} \right) \frac{\Delta^S(aqr-\frac{1}{2})}{\Delta^S(aqr)} \cdot \frac{\Delta^S(pbr-\frac{1}{2})}{\Delta^S(pbr)} \\
& \quad \times \left( (2r-1)(2r) h_{b,c}^a - h_{a,q}^{r-\frac{1}{2}} h_{b,p}^{r-\frac{1}{2}} \right) \left| \left\{ \begin{matrix} qr - \frac{1}{2} & a \\ \frac{1}{2} & a & r \end{matrix} \right\}_*^S \left\{ \begin{matrix} pr - \frac{1}{2} & b \\ \frac{1}{2} & b & r \end{matrix} \right\}_*^S \right| w^S \left\{ \begin{matrix} a & b & c \\ p & q & r - \frac{1}{2} \end{matrix} \right\} \\
& \quad + (-1)^{2(p+q+c)} (2r) \left( (2r-1)(2r) h_{b,c}^a - h_{a,q}^{r-\frac{1}{2}} h_{b,p}^{r-\frac{1}{2}} \right) \frac{\Delta^S(aqr+\frac{1}{2})}{\Delta^S(aqr)} \cdot \frac{\Delta^S(pbr+\frac{1}{2})}{\Delta^S(pbr)} \\
& \quad \times \left( (2r+1)(2r+2) h_{b,c}^a - h_{a,q}^{r+\frac{1}{2}} h_{b,p}^{r+\frac{1}{2}} \right) \left| \left\{ \begin{matrix} qr & a \\ \frac{1}{2} & a & r + \frac{1}{2} \end{matrix} \right\}_*^S \left\{ \begin{matrix} pr & b \\ \frac{1}{2} & b & r + \frac{1}{2} \end{matrix} \right\}_*^S \right| w^S \left\{ \begin{matrix} a & b & c \\ p & q & r + \frac{1}{2} \end{matrix} \right\}. \quad (3.34)
\end{aligned}$$

We are now able to replace eqs.(3.28) and (3.33) in eq.(3.34) for dropping out the  $w^S$  having  $r \pm \frac{1}{2}$ . We need again to define three new coefficients  $\eta_f$ ,  $\mathcal{N}_r$  and  $k_{n_2, n_3}^{n_1}$ .

• Definitions:

$$\eta_f = \left( 2f(2f+1)h_{b,c}^a - h_{a,q}^f h_{b,p}^f \right), \quad \mathcal{N}_r = \prod_{f=r-\frac{1}{2}}^{f=r+\frac{1}{2}} \eta_f, \quad k_{n_2, n_3}^{n_1} = \left| \left\{ \begin{matrix} n_3 & n_1 - \frac{1}{2} & n_2 \\ \frac{1}{2} & n_2 & n_1 \end{matrix} \right\}_*^S \right|. \quad (3.35)$$

Finally here is the final 3-term recurrence in steps of 1 obtained with only integer quantities:

$$\begin{aligned}
\mathcal{N}_r \cdot w^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\} &= \eta_{r+\frac{1}{2}} (2r)(2r+1) [q-r+a+1][q+r+a+\frac{1}{2}][p-r+b+1][p+r+b+\frac{1}{2}] w^S \left\{ \begin{matrix} a & b & c \\ p & q & r-1 \end{matrix} \right\} \\
&\quad - (-1)^{2(a+b+p+q)} \\
&\quad \times \left[ \eta_{r+\frac{1}{2}} (2r-1)(2r+1) \left( k_{a,q}^r k_{b,p}^r \right)^2 + \eta_{r-\frac{1}{2}} (2r)(2r+2) \left( k_{a,q}^{r+\frac{1}{2}} k_{b,p}^{r+\frac{1}{2}} \right)^2 \right] w^S \left\{ \begin{matrix} a & b & c \\ p & q & r \end{matrix} \right\} \\
&\quad + \eta_{r-\frac{1}{2}} (2r)(2r+1) [-a+q+r+1][a-q+r+1][-b+p+r+1][b-p+r+1] w^S \left\{ \begin{matrix} a & b & c \\ p & q & r+1 \end{matrix} \right\}. \quad (3.36)
\end{aligned}$$

While realizing that writing a synthetic formula that use integer parts hides subdivisions of cases with different mathematical expressions, we note the advantage of not depending formally on parities  $\alpha, \beta, \gamma$ . As for the 3-terms recurrence relation in steps of 1/2 successful tests were performed with integer numbers, but in a field more reduced because huge numbers in the coefficient  $\mathcal{N}_r$  occur and are out the range of small computers.

### 3.3 Pseudo-orthogonality relation

There are analogous properties, indeed the supersymmetric Racah coefficients  $\{6j\}_\pi^S$  have also a pseudo-orthogonality relation [1, 5] that we can rewrite here as:

$$\sum_x (-1)^{[a+e+x]+[d+b+x]+2x} \left\{ \begin{matrix} a & b & c \\ d & e & x \end{matrix} \right\}^S \left\{ \begin{matrix} a & b & c' \\ d & e & x \end{matrix} \right\}^S = (-1)^{[a+b+c]+[d+e+c]+2c} \delta_{c,c'}, \quad (3.37)$$

with the increment on  $x$  done by steps of  $1/2$ . For recall let us write the orthogonality for  $su(2)$ :

$$\sum_x \begin{Bmatrix} a & b & x \\ d & e & f \end{Bmatrix}^S \begin{Bmatrix} a & b & x \\ d & e & f' \end{Bmatrix}^S = \delta_{f,f'}, \quad (3.38)$$

with an increment in steps of 1. However, nothing really exploitable can be obtained because of parities mixing between  $\alpha, \beta_{i \in 1,6}$  and  $\gamma$ .

## 4 Expansion on hypergeometric ${}_4F_3$ functions

### 4.1 Rao's calculations as model

From a series like eq.(2.4) which define a standard 6- $j$ , use of function  $\Gamma$  and Pochhammer symbol  $(x)_n$  leads to an hypergeometric formulation. For example by making the change  $z \rightarrow q_1 - n$  in eq.(2.4) we have to handle expressions like

$$N(-1)^{q_1} \sum_n \frac{(-1)^n (q_1+1-n)!}{\prod_{i=1}^{i=4} (q_1-p_i-n)! \prod_{k=1}^{k=3} (n+q_k-q_1)!}. \quad (4.1)$$

$$(-1)^{(n+q_1)} (q_1+1-n)! = (-1)^{q_1} [(-q_1-1)_n]^{-1} \Gamma(q_1+2), \quad (4.2)$$

$$\frac{1}{(q_1-p_1-n)!} = (-1)^n (p_1-q_1)_n [\Gamma(1+q_1-p_1)]^{-1}, \quad \frac{1}{(q_1-p_2-n)!} = (-1)^n (p_2-q_1)_n [\Gamma(1+q_1-p_2)]^{-1}, \quad (4.3)$$

$$\frac{1}{(q_1-p_3-n)!} = (-1)^n (p_3-q_1)_n [\Gamma(1+q_1-p_3)]^{-1}, \quad \frac{1}{(q_1-p_4-n)!} = (-1)^n (p_4-q_1)_n [\Gamma(1+q_1-p_4)]^{-1}, \quad (4.4)$$

$$\frac{1}{(n+q_2-q_1)!} = \frac{1}{(q_2-q_1+1)_n} [\Gamma(q_2-q_1+1)]^{-1}, \quad \frac{1}{(n+q_3-q_1)!} = \frac{1}{(q_3-q_1+1)_n} [\Gamma(q_3-q_1+1)]^{-1}. \quad (4.5)$$

Then, a  $\{6j\}$  symbol is written as a generalized hypergeometric function  ${}_4F_3$  of argument 1. That yields a generalized hypergeometric function  ${}_4F_3$  of argument 1 [12, 13]:

$$\begin{aligned} \left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\} &= (-1)^{q_k} N \Gamma(q_k+2) [\prod_{i=1}^{i=4} \Gamma(1+q_k-p_i) \prod_{l \neq k} \Gamma(q_l-q_k+1)]^{-1} \\ &\times {}_4F_3 \left( \begin{matrix} (p_1-q_k), (p_2-q_k), (p_3-q_k), (p_4-q_k) \\ (-q_k-1), (q_{k_2}-q_k+1), (q_{k_3}-q_k+1) \end{matrix} ; 1 \right). \end{aligned} \quad (4.6)$$

with  $(k, k_2, k_3)$  cyclic  $= (1, 2, 3)$  and  $k \in [1, 3]$ .

Let us remind that  ${}_rF_s \left( \begin{matrix} a_1, a_2, \dots \\ b_1, b_2, \dots \end{matrix} ; x \right)$  is said Saalschützian [14] if

$$\sum b_k = \sum a_i + 1. \quad (4.7)$$

Thereafter we compute the "parametric excess" of balance condition of Slater [14, p.49], that we will denote in this paper by  $\tau_s$ :

$$\tau_s = \sum b_k - \sum a_i. \quad (4.8)$$

As  $\tau_s = 1$  in the present case, we recover the set I of the three Saalschützian  ${}_4F_3$  functions of Rao [13]. The total number of permutations for  $p_i - q_k$  and  $(q_{k_2} - q_k + 1)$  is  $3 \times 4! \times 2 = 144$ .

A second change of summation variable is given by  $z \rightarrow n + p_1$  yields

$$N(-1)^{p_1} \sum_n \frac{(-1)^n (n+p_1+1)!}{\prod_{i=1}^{i=4} (n+p_1-p_i)! \prod_{k=1}^{k=3} (q_k-n-p_1)!}. \quad (4.9)$$

Hence

$$(-1)^{(n+p_1)}(n+p_1+1)! = (-1)^{n+p_1} (p_1+2)_n \Gamma(p_1+2), \quad (4.10)$$

$$\frac{1}{(q_1-p_1-n)!} = (-1)^n (p_1-q_1)_n [\Gamma(1+q_1-p_1)]^{-1}, \quad \frac{1}{(q_2-p_1-n)!} = (-1)^n (p_1-q_2)_n [\Gamma(1+q_2-p_1)]^{-1}, \quad (4.11)$$

$$\frac{1}{(q_3-p_1-n)!} = (-1)^n (p_1-q_3)_n [\Gamma(1+q_3-p_1)]^{-1}, \quad \frac{1}{(n+p_1-p_2)!} = \frac{1}{(p_1-p_2+1)_n} [\Gamma(p_1-p_2+1)]^{-1}, \quad (4.12)$$

$$\frac{1}{(n+p_1-p_3)!} = \frac{1}{(p_1-p_3+1)_n} [\Gamma(p_1-p_3+1)]^{-1}, \quad \frac{1}{(n+p_1-p_4)!} = \frac{1}{(p_1-p_4+1)_n} [\Gamma(p_1-p_4+1)]^{-1}. \quad (4.13)$$

Then the set II of the four  ${}_4F_3$  Saalschützian [thanks to eq.(2.1)] functions of Rao [13] is obtained.

$$\left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\} = (-1)^{p_i} N \Gamma(p_i+2) [\prod_{k=1}^{k=3} \Gamma(1+q_k-p_i) \prod_{j \neq i} \Gamma(p_i-p_j+1)]^{-1} \times {}_4F_3 \left( \begin{matrix} (p_i+2), (p_i-q_1), (p_i-q_2), (p_i-q_3) \\ (p_i-p_{i_2}+1), (p_i-p_{i_3}+1), (p_i-p_{i_4}+1) \end{matrix}; 1 \right), \quad (4.14)$$

with  $(i, i_2, i_3, i_4)$  cyclic = (1, 2, 3, 4) and  $i \in [1, 3]$ . Number of permutations =  $4 \times 3! \times 3! = 144$ .

Thus the 144 symmetries of a Racah coefficient, including the Regge symmetries [15]. Set I and set II are equivalent. Hereafter we will follow this method and generalize it to the supersymmetric case.

#### 4.1.1 Remark for simply obtaining Regge symmetries

Here is our way of managing Regge symmetries by simple permutations: suppose we represent a  $\{6j\}$  as a function of its  $p_i$  and  $q_k$  by  $(p_1, p_2, p_3, p_4; q_1, q_2, q_3)$ . Then the 5 Regge transformations may be written as

$$\mathcal{R}_1(p_1, p_2, p_3, p_4; q_1, q_2, q_3) = (p_1, p_3, p_2, p_4; q_1, q_2, q_3), \quad (4.15)$$

$$\mathbf{p}_2 \leftrightarrow \mathbf{p}_3$$

$$\mathcal{R}_2(p_1, p_2, p_3, p_4; q_1, q_2, q_3) = (p_3, p_2, p_1, p_4; q_1, q_2, q_3), \quad (4.16)$$

$$\mathbf{p}_1 \leftrightarrow \mathbf{p}_3$$

$$\mathcal{R}_3(p_1, p_2, p_3, p_4; q_1, q_2, q_3) = (p_2, p_1, p_3, p_4; q_1, q_2, q_3), \quad (4.17)$$

$$\mathbf{p}_1 \leftrightarrow \mathbf{p}_2$$

$$\mathcal{R}_4(p_1, p_2, p_3, p_4; q_1, q_2, q_3) = (p_3, p_1, p_2, p_4; q_2, q_3, q_1), \quad (4.18)$$

$$(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \rightarrow (\mathbf{p}_3, \mathbf{p}_1, \mathbf{p}_2) \text{ and } (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \rightarrow (\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_1)$$

$$\mathcal{R}_5(p_1, p_2, p_3, p_4; q_1, q_2, q_3) = (p_2, p_3, p_1, p_4; q_3, q_1, q_2). \quad (4.19)$$

$$(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \rightarrow (\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_1) \text{ and } (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \rightarrow (\mathbf{q}_3, \mathbf{q}_1, \mathbf{q}_2)$$

Now just use the expressions calculating the spins, eqs.(2.2),(2.3), after replacing the  $p_i$  or  $q_k$  given by the equations just above (4.15)-(4.19) to recover the well-known Regge formulas.

## 5 Expansion of super Racah coefficients

The triplet of equations (2.10)-(2.11)-(2.12) is the key for displaying the coefficients under a hypergeometric form  ${}_4F_3$ . The numerator factors  $(-1)^z z! \Pi_\pi(z)$  containing the monomials  $\Pi_\pi(z)$ , can be

used for splitting each supersymmetric symbols into two  ${}_3F_4$  at most:

$$\mathcal{N}_\alpha(z) = (-\mathbf{1})^{\mathbf{z}}(\mathbf{z})!, \quad (5.1)$$

$$\begin{aligned} \mathcal{N}_\beta(z) &= (-\mathbf{1})^{\mathbf{z}+\mathbf{1}}(\bar{p} + \bar{p}' - q + 1)(\mathbf{z} + \mathbf{1})! \\ &\quad + (-\mathbf{1})^{\mathbf{z}}[(\bar{p} + \bar{p}' - q + 1) + ((\bar{q} + \frac{1}{2})(\bar{q}' + \frac{1}{2}) - pp')](\mathbf{z})!, \end{aligned} \quad (5.2)$$

$$\mathcal{N}_\gamma(z) = (-\mathbf{1})^{\mathbf{z}+\mathbf{1}}(\mathbf{z} + \mathbf{1})! + (-\mathbf{1})^{\mathbf{z}} \left( \frac{1}{2} \left( \sum_k q_k(q_k + 1) - \sum_i p_i^2 \right) + \frac{3}{2} \right) (\mathbf{z})!. \quad (5.3)$$

For the parity  $\gamma$  one may also note  $p_i$  as  $\bar{p}_i$  since they are all half-integer.

Then eqs.(2.5,2.6) compared to eq.(2.4) allows us to quickly guess how to obtain hypergeometric  ${}_4F_3$  functions for the supersymmetric  $\{6j\}_\pi^S$  symbols from the standard formulas eqs.(4.6),(4.14): Under the sum over  $z$ , then *inside the  ${}_4F_3$ , do the change  $p_i \rightarrow [p_i + \frac{1}{2}]$  and  $q_k \rightarrow [q_k + \frac{1}{2}]$ . Also for the change of variable which need to be integer, we have to make changes as  $z \rightarrow [q_k + \frac{1}{2}] - n$  and  $z \rightarrow [p_i + \frac{1}{2}] + n$ .*

As the identity between perimeters and quadrangles (2.1) is fundamental for checking if the new hypergeometric  ${}_4F_3$  functions are Saalschützian or not (by means of the indicator  $\tau_s$ ), it is useful to pre-calculate some sums of integer parts like  $\sum_{i=1}^4 [p_i + \frac{1}{2}]$  and  $\sum_{k=1}^3 [q_k + \frac{1}{2}]$ :

$$\sum_{i=1}^4 [p_i + \frac{1}{2}] = p_1 + p_2 + p_3 + p_4, \quad \sum_{k=1}^3 [q_k + \frac{1}{2}] = q_1 + q_2 + q_3. \quad \text{parity } \alpha \quad (5.4)$$

$$\sum_{i=1}^4 [p_i + \frac{1}{2}] = \bar{p} + \bar{p}' + p + p' + 1, \quad \sum_{k=1}^3 [q_k + \frac{1}{2}] = q + \bar{q} + \bar{q}' + 1. \quad \text{parity } \beta \quad (5.5)$$

$$\sum_{i=1}^4 [\bar{p}_i + \frac{1}{2}] = \bar{p}_1 + \bar{p}_2 + \bar{p}_3 + \bar{p}_4 + 2, \quad \sum_{k=1}^3 [q_k + \frac{1}{2}] = q_1 + q_2 + q_3. \quad \text{parity } \gamma \quad (5.6)$$

The supersymmetric form of identity (2.1) remains unchanged for the parities  $\alpha$  and  $\beta$  but differs for the third case  $\gamma$ . This can be visualized as follows:

$$\sum_{k=1}^3 [q_k + \frac{1}{2}] = \sum_{i=1}^4 [p_i + \frac{1}{2}] \text{ for } (\alpha, \beta) \quad \text{and} \quad \sum_{k=1}^3 [q_k + \frac{1}{2}] = \sum_{i=1}^4 [p_i + \frac{1}{2}] - 2 \text{ for } (\gamma). \quad (5.7)$$

For the frontal phase as function of the parities  $\pi = \alpha, \beta, \gamma$ , it can be checked that

$$(-1)^{4 \sum j_i J_i} = \begin{cases} (-1)^{\phi_\alpha} = 1 & \text{if } \pi = \alpha, \\ (-1)^{\phi_\beta} = (-1)^{p+p'-q} & \text{if } \pi = \beta, \\ (-1)^{\phi_\gamma} = (-1)^{1+\sum_k q_k} & \text{if } \pi = \gamma. \end{cases} \quad (5.8)$$

It just remains to apply the Rao-process given as example above for the classic  $\{6j\}$  symbols.

### 5.1 One hypergeometric function ${}_4F_3$ for parity $\alpha$

$$\left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\}_\alpha^S = \sqrt{R^S} \sum_z \frac{(-1)^z z!}{\prod_{i=1}^4 (z - [p_i + \frac{1}{2}])! \prod_{k=1}^3 ([q_k + \frac{1}{2}] - z)!}. \quad (5.9)$$

The calculation is simple because we handle only one sum over  $z$ . Since all the  $p_i$  and  $q_k$  are integers, then we can replace  $[p_i + \frac{1}{2}]$  and  $[q_k + \frac{1}{2}]$  by  $p_i$  and  $q_k$  respectively. If necessary, use will be made of the usual notation [14, p. 41]:

$$\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\cdots\Gamma(a_A)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\cdots\Gamma(b_B)} = \Gamma \left[ \begin{matrix} a_1 & a_2 & a_3 & \cdots & a_A \\ b_1 & b_2 & b_3 & \cdots & b_B \end{matrix} \right]. \quad (5.10)$$

After rearranging some products of  $\Gamma$  as  $\Gamma([q_k + \frac{1}{2}] + 1) [\prod_{i=1}^{i=4} \Gamma(1 + [q_k + \frac{1}{2}] - [p_i + \frac{1}{2}]) \prod_{l \neq k} \Gamma([q_l + \frac{1}{2}] - [q_k + \frac{1}{2}] + 1)]^{-1}$  we are led to a first result.

**Set  $\mathbf{I}_{(\alpha)}$**  for the  $\{6j\}_{\alpha}^S$ :

$$\begin{aligned} \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\}_{\alpha}^S &= \sqrt{\mathbb{R}^S} (-1)^{q_k} \Gamma(q_k + 1) [\prod_{i=1}^{i=4} \Gamma(1 + q_k - p_i) \prod_{l \neq k} \Gamma(q_l - q_k + 1)]^{-1} \\ &\times {}_4F_3 \left( \begin{array}{c} -(q_k - p_1), -(q_k - p_2), -(q_k - p_3), -(q_k - p_4) \\ -(q_k), (q_{k_2} - q_k + 1), (q_{k_3} - q_k + 1) \end{array} ; 1 \right), \end{aligned} \quad (5.11)$$

with  $(k, k_2, k_3)$  cyclic = (1, 2, 3). Indicator  $\tau_s = 2$ . In the same way we derive the second set.

**Set  $\mathbf{II}_{(\alpha)}$**  for the  $\{6j\}_{\alpha}^S$ :

$$\begin{aligned} \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\}_{\alpha}^S &= \sqrt{\mathbb{R}^S} (-1)^{p_i} \Gamma(p_i + 1) [\prod_{k=1}^3 \Gamma(1 + q_k - p_i) \prod_{m \neq i} \Gamma(p_i - p_m + 1)]^{-1} \\ &\times {}_4F_3 \left( \begin{array}{c} (p_i + 1), (p_i - q_1), (p_i - q_2), (p_i - q_3) \\ (p_i - p_{i_2} + 1), (p_i - p_{i_3} + 1), (p_i - p_{i_4} + 1) \end{array} ; 1 \right), \end{aligned} \quad (5.12)$$

with  $(i, i_2, i_3, i_4)$  cyclic = (1, 2, 3, 4). Indicator  $\tau_s = 2$ . Set  $\mathbf{I}_{(\alpha)}$  and set  $\mathbf{II}_{(\alpha)}$  are not Saalschützian. Note that the  ${}_4F_3$  obtained above have not the feature of an eventual *physical* parameter, like a standard symbol  $\{6j\}$  for example. Especially the Saalschützian property is not satisfied. By looking at each of the two  ${}_4F_3$  related to the sets  $\mathbf{I}_{(\alpha)}$  and  $\mathbf{II}_{(\alpha)}$ , and by taking up Rao's argumentation [13], we can see that the 144 symmetries of a standard  $\{6j\}$  symbol are well recovered. Both **Set  $\mathbf{I}_{(\alpha)}$**  and **Set  $\mathbf{II}_{(\alpha)}$**  are equivalent.

## 5.2 A sum of two hypergeometric functions ${}_4F_3$ for parity $\beta$

$$\begin{aligned} \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\}_{\beta}^S &= (-1)^{p+p'-q} \sqrt{\mathbb{R}^S} \\ &\times \left[ \begin{aligned} &(\overline{p} + \overline{p}' - q + 1) \sum_z \frac{(-1)^{z+1} (z+1)!}{\prod_{i=1}^{i=4} (z - [p_i + \frac{1}{2}])! \prod_{k=1}^{k=3} ([q_k + \frac{1}{2}] - z)!} \\ &+ \\ &[(\overline{p} + \overline{p}' - q + 1) + ((\overline{q} + \frac{1}{2})(\overline{q}' + \frac{1}{2}) - p p')] \sum_z \frac{(-1)^z (z)!}{\prod_{i=1}^{i=4} (z - [p_i + \frac{1}{2}])! \prod_{k=1}^{k=3} ([q_k + \frac{1}{2}] - z)!} \end{aligned} \right], \end{aligned} \quad (5.13)$$

with  $p_i \in [\overline{p}, \overline{p}', p, p']$  and  $q_k \in [\overline{q}, \overline{q}', q]$ . With the change  $z \rightarrow q - n$ , a first set is obtained. Note that here  $q$  is integer, if not we have to do 2 other changes like  $z \rightarrow [\overline{q} + \frac{1}{2}] - n$  or  $z \rightarrow [\overline{q}' + \frac{1}{2}] - n$ .

**Set  $\mathbf{I}_{(\beta)}$**  for the  $\{6j\}_{\beta}^S$ :

$$\begin{aligned} \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\}_{\beta}^S &= (-1)^{p+p'} \sqrt{\mathbb{R}^S} \\ &\times \Gamma(q+1) [\prod_{i=1}^{i=4} \Gamma(1 + q - [p_i + \frac{1}{2}]) \Gamma(\overline{q} + \frac{1}{2} - q + 1), (\overline{q}' + \frac{1}{2} - q + 1)]^{-1} \\ &\times \left[ \begin{aligned} &-(\overline{p} + \overline{p}' - q + 1) ([q_k + \frac{1}{2}] + 1) \\ &\times {}_4F_3 \left( \begin{array}{c} ([p_1 + \frac{1}{2}] - [q_k + \frac{1}{2}]), ([p_2 + \frac{1}{2}] - [q_k + \frac{1}{2}]), ([p_3 + \frac{1}{2}] - [q_k + \frac{1}{2}]), ([p_4 + \frac{1}{2}] - [q_k + \frac{1}{2}]) \\ (-[q_k + \frac{1}{2}] - 1), ([q_{k_2} + \frac{1}{2}] - [q_k + \frac{1}{2}] + 1), ([q_{k_3} + \frac{1}{2}] - [q_k + \frac{1}{2}] + 1) \end{array} ; 1 \right) \\ &+ \\ &[(\overline{p} + \overline{p}' - q + 1) + ((\overline{q} + \frac{1}{2})(\overline{q}' + \frac{1}{2}) - p p')] \\ &\times {}_4F_3 \left( \begin{array}{c} ([p_1 + \frac{1}{2}] - [q_k + \frac{1}{2}]), ([p_2 + \frac{1}{2}] - [q_k + \frac{1}{2}]), ([p_3 + \frac{1}{2}] - [q_k + \frac{1}{2}]), ([p_4 + \frac{1}{2}] - [q_k + \frac{1}{2}]) \\ (-[q_k + \frac{1}{2}]), ([q_{k_2} + \frac{1}{2}] - [q_k + \frac{1}{2}] + 1), ([q_{k_3} + \frac{1}{2}] - [q_k + \frac{1}{2}] + 1) \end{array} ; 1 \right) \end{aligned} \right], \end{aligned} \quad (5.14)$$

with  $(k, k_2, k_3)$  cyclic = (1, 2, 3). Indicator  $\tau_s = 1$  for the first hypergeometric  ${}_4F_3$  and  $\tau_s = 2$  for the second. The whole cannot to be said Saalschützian.

Similarly with the change  $z \rightarrow n + p$ ,  $z \rightarrow n + p'$  or  $z \rightarrow n + [\bar{p} + \frac{1}{2}]$  and  $z \rightarrow n + [\bar{p}' + \frac{1}{2}]$  we obtain the second set.

**Set II**<sub>( $\beta$ )</sub> for the  $\{6j\}_\beta^S$ :

$$\left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\}_\beta^S = (-1)^{[p_i + \frac{1}{2}] + p + p' - q} \sqrt{\mathbb{R}^S} \times \Gamma([p_i + \frac{1}{2}] + 1) \left[ \prod_{k=1}^3 \Gamma(1 + [q_k + \frac{1}{2}] - [p_i + \frac{1}{2}]) \prod_{m \neq i} \Gamma([p_i + \frac{1}{2}] - [p_m + \frac{1}{2}] + 1) \right]^{-1} \\ - (\bar{p} + \bar{p}' - q + 1) ([p_i + \frac{1}{2}] + 1) \\ \times {}_4F_3 \left( \begin{array}{c} ([p_i + \frac{1}{2}] + 2), ([p_i + \frac{1}{2}] - [q_1 + \frac{1}{2}]), ([p_i + \frac{1}{2}] - [q_2 + \frac{1}{2}]), ([p_i + \frac{1}{2}] - [q_3 + \frac{1}{2}]) \\ ([p_i + \frac{1}{2}] - [p_{i_2} + \frac{1}{2}] + 1), ([p_i + \frac{1}{2}] - [p_{i_3} + \frac{1}{2}] + 1), ([p_i + \frac{1}{2}] - [p_{i_4} + \frac{1}{2}] + 1) \end{array} ; 1 \right) \\ + \\ [(\bar{p} + \bar{p}' - q + 1) + ((\bar{q} + \frac{1}{2})(\bar{q}' + \frac{1}{2}) - pp')] \\ \times {}_4F_3 \left( \begin{array}{c} ([p_i + \frac{1}{2}] + 1), ([p_i + \frac{1}{2}] - [q_1 + \frac{1}{2}]), ([p_i + \frac{1}{2}] - [q_2 + \frac{1}{2}]), ([p_i + \frac{1}{2}] - [q_3 + \frac{1}{2}]) \\ ([p_i + \frac{1}{2}] - [p_{i_2} + \frac{1}{2}] + 1), ([p_i + \frac{1}{2}] - [p_{i_3} + \frac{1}{2}] + 1), ([p_i + \frac{1}{2}] - [p_{i_4} + \frac{1}{2}] + 1) \end{array} ; 1 \right) \quad , \quad (5.15)$$

with  $(i, i_2, i_3, i_4)$  cyclic = (1, 2, 3, 4). Indicator  $\tau_s = 1$  for the first hypergeometric  ${}_4F_3$  and  $\tau_s = 2$  for the second. The whole is not Saalschützian.

We have seen that the first hypergeometric  ${}_4F_3$  function in **Set I**<sub>( $\beta$ )</sub> was Saalschützian as well as the first one of **Set II**<sub>( $\beta$ )</sub>. It may happen that non-physical spins may appear with values like  $\frac{1}{4}$  from the division of a half-integer by two. Then it is clear that the total number of symmetries for the case  $\beta$  is reduced from 144 to 48 since, here, only one Regge symmetry yields valid spins, integer or half-integer.

### 5.3 A sum of two hypergeometric functions ${}_4F_3$ for parity $\gamma$

$$\left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\}_\gamma^S = (-1)^{1 + \sum_k q_k} \sqrt{\mathbb{R}^S} \times \left[ \sum_z \frac{(-1)^{z+1} (z+1)!}{\prod_{i=1}^{i=4} (z - [\bar{p}_i + \frac{1}{2}])! \prod_{k=1}^{k=3} ([q_k + \frac{1}{2}] - z)!} + \left( \frac{1}{2} (\sum_k q_k (q_k + 1) - \sum_i \bar{p}_i^2) + \frac{3}{2} \right) \sum_z \frac{(-1)^z (z)!}{\prod_{i=1}^{i=4} (z - [\bar{p}_i + \frac{1}{2}])! \prod_{k=1}^{k=3} ([q_k + \frac{1}{2}] - z)!} \right] \quad . \quad (5.16)$$

Similarly to previous sections, we can write right now the results.

**Set I**<sub>( $\gamma$ )</sub> for the  $\{6j\}_\gamma^S$ :

$$\left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\}_\gamma^S = (-1)^{q_k + 1 + \sum_l q_l} \sqrt{\mathbb{R}^S} \Gamma(q_k + 1) \left[ \prod_{i=1}^{i=4} \Gamma(1 + q_k - (\bar{p}_i + \frac{1}{2})) \prod_{l \neq k} \Gamma(q_l - q_k + 1) \right]^{-1} \\ - (q_k + 1) \\ \times {}_4F_3 \left( \begin{array}{c} (\bar{p}_1 + \frac{1}{2} - q_k), (\bar{p}_2 + \frac{1}{2} - q_k), (\bar{p}_3 + \frac{1}{2} - q_k), (\bar{p}_4 + \frac{1}{2} - q_k) \\ (-q_k - 1), (q_{k_2} - q_k + 1), (q_{k_3} - q_k + 1) \end{array} ; 1 \right) \\ + \\ \left( \frac{1}{2} (\sum_k q_k (q_k + 1) - \sum_i \bar{p}_i^2) + \frac{3}{2} \right) \\ \times {}_4F_3 \left( \begin{array}{c} (\bar{p}_1 + \frac{1}{2} - q_k), (\bar{p}_2 + \frac{1}{2} - q_k), (\bar{p}_3 + \frac{1}{2} - q_k), (\bar{p}_4 + \frac{1}{2} - q_k) \\ (-q_k), (q_{k_2} - q_k + 1), (q_{k_3} - q_k + 1) \end{array} ; 1 \right) \quad , \quad (5.17)$$

with  $(k, k_2, k_3)$  cyclic = (1, 2, 3). Indicator (first  ${}_4F_3$ )  $\tau_s = -1$ , second function  $\tau_s = 0$ .

Set I<sub>( $\gamma$ )</sub> is non-Saalschützian.

Set  $\mathbf{II}_{(\gamma)}$  for the  $\{6j\}_\gamma^S$ :

$$\left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\}_\gamma^S = (-1)^{\overline{p_i} + \frac{1}{2} + 1 + \sum_k q_k} \sqrt{\mathbf{R}^S} \Gamma(\overline{p_i} + \frac{1}{2} + 1) \left[ \prod_{k=1}^3 \Gamma(1 + q_k - (\overline{p_i} + \frac{1}{2})) \prod_{m \neq i} \Gamma(\overline{p_i} - \overline{p_m} + 1) \right]^{-1}$$

$$\times \left[ \begin{aligned} & -(\overline{p_i} + \frac{1}{2} + 1) \\ & \times {}_4F_3 \left( \begin{matrix} (\overline{p_i} + \frac{1}{2} + 2), (\overline{p_i} + \frac{1}{2} - q_1), (\overline{p_i} + \frac{1}{2} - q_2), (\overline{p_i} + \frac{1}{2} - q_3) \\ (\overline{p_i} - \overline{p_{i_2}} + 1), (\overline{p_i} - \overline{p_{i_3}} + 1), (\overline{p_i} - \overline{p_{i_4}} + 1) \end{matrix} ; 1 \right) \\ & + \\ & \left( \frac{1}{2} (\sum_k q_k (q_k + 1) - \sum_i \overline{p_i}^2) + \frac{3}{2} \right) \\ & \times {}_4F_3 \left( \begin{matrix} (\overline{p_i} + \frac{1}{2} + 1), (\overline{p_i} + \frac{1}{2} - q_1), (\overline{p_i} + \frac{1}{2} - q_2), (\overline{p_i} + \frac{1}{2} - q_3) \\ (\overline{p_i} - \overline{p_{i_2}} + 1), (\overline{p_i} - \overline{p_{i_3}} + 1), (\overline{p_i} - \overline{p_{i_4}} + 1) \end{matrix} ; 1 \right) \end{aligned} \right], \quad (5.18)$$

with  $(i, i_2, i_3, i_4)$  cyclic  $= (1, 2, 3, 4)$ . Indicator (first  ${}_4F_3$ )  $\tau_s = -1$ , second function  $\tau_s = 0$ . Set  $\mathbf{II}_{(\gamma)}$  is non-Saalschützian. As for the symbols of parity  $\alpha$  the 144 symmetries are recovered.

## 6 Super Racah coefficients and $\mathcal{P}$ Wilson polynomials

Recall that Wilson orthogonal polynomials of degree  $n$  [11] are given by:

$$\mathcal{P}_n(t^2; a', b', c', d') = (a' + b')_n (a' + c')_n (a' + d')_n {}_4F_3 \left( \begin{matrix} -n, (a' + b' + c' + d' + n - 1), (a' + t), (a' - t) \\ a' + b', a' + c', a' + d' \end{matrix} ; 1 \right). \quad (6.1)$$

As found in a comment by Rao *et. al.* [17] the relation between a standard 6- $j$  coefficient and the polynomial  $\mathcal{P}$  can be written as:

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\} =$$

$$\begin{aligned} & (-1)^{a+e+f} \Delta(b, d, f) \Delta(a, e, f) \Delta(a, b, c) \Delta(d, e, c) \Gamma(a + e + f + 2) \\ & \times [\Gamma(1 + b + d - f, 1 + a + e - f, 1 + d + e - c, 1 + c + d - e, 1 + b + c - a, \\ & 1 + a + b - c)]^{-1} \mathcal{P}_{b+d-f} \left( (c + \frac{1}{2})^2; -d - e - \frac{1}{2}, -a - b - \frac{1}{2}, a - b + \frac{1}{2}, -d + e + \frac{1}{2} \right), \end{aligned} \quad (6.2)$$

Now we turn back to the original formulas which led us to expand super Racah coefficients as a single summation over an integer  $z$  [5].

According to our method of calculation, see [5, pp. 261-266], it is clear that the three super Racah coefficients can be written as the linear combination of two standard 6- $j$  coefficients. Indeed if we let  $N^S = \Delta^S(J_1 j_2 j_3) \Delta^S(j_1 J_2 j_3) \Delta^S(j_1 j_2 J_3) \Delta^S(J_1 J_2 J_3)$ , we are able to write the following formulas below.

### 6.1 The three doublets of 6- $j$ symbols used for the super Racah coefficients

• Parity  $\alpha$  :

$$\left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\}_\alpha^S = (-1)^{\phi_\alpha} \left[ x_\alpha \frac{N^S}{N^\alpha} \left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 - \frac{1}{2} \end{matrix} \right\} + y_\alpha \frac{N^S}{N^\alpha} \left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\} \right], \quad (6.3)$$

We will denote  $d_n^\alpha$  for the degrees of the Wilson polynomials used for the calculation.

$$d_1^\alpha = j_1 + J_2 - j_3, \quad d_2^\alpha = j_1 + J_2 - j_3, \quad (6.4)$$

$$x_\alpha = (J_1 + J_2 + J_3 + 1)^{-1} = (p_4 + 1)^{-1}, \quad y_\alpha = (J_1 + J_2 + J_3 + 1)^{-1} = (p_4 + 1)^{-1}, \quad (6.5)$$

$$(-1)^{\phi_\alpha} = (-1)^{4 \sum j_i J_i} = 1 \text{ for the case } \alpha. \quad (6.6)$$

with, in this example the normalization coefficients  $N_1$  and  $N_2$  are given by

$$N_1^\alpha = \Delta(J_1 j_2 - \frac{1}{2} j_3 - \frac{1}{2}) \Delta(j_1 - \frac{1}{2} J_2 j_3 - \frac{1}{2}) \Delta(j_1 - \frac{1}{2} j_2 - \frac{1}{2} J_3) \Delta(J_1 J_2 J_3), \quad (6.7)$$

$$N_2^\alpha = \Delta(J_1 j_2 j_3) \Delta(j_1 J_2 j_3) \Delta(j_1 j_2 J_3) \Delta(J_1 J_2 J_3). \quad (6.8)$$

• **Parity  $\beta$**  : (1/6 possible cases [ $p_3$  and  $p_4$  integer])

$$\left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\}_\beta^S = (-1)^{\phi_\beta} \left[ x_\beta \frac{N^S}{N_1^\beta} \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 \end{array} \right\} + y_\beta \frac{N^S}{N_2^\beta} \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 - \frac{1}{2} \end{array} \right\} \right], \quad (6.9)$$

where

$$d_1^o = j_1 + J_2 - j_3 - \frac{1}{2}, \quad d_2^o = j_1 + J_2 - j_3 + \frac{1}{2}, \quad (6.10)$$

$$x_\beta = (J_1 + J_2 + J_3 + 1)^{-1} (J_1 + j_2 + j_3 + \frac{1}{2}) (-j_1 + J_2 + j_3 + \frac{1}{2}), \quad (6.11)$$

$$y_\beta = (J_1 + J_2 + J_3 + 1)^{-1} (J_1 + j_2 - j_3 + \frac{1}{2}) (j_1 + J_2 - j_3 + \frac{1}{2}), \quad (6.12)$$

and

$$N_1^\beta = \Delta(J_1 j_2 - \frac{1}{2} j_3) \Delta(j_1 - \frac{1}{2} J_2 j_3) \Delta(j_1 - \frac{1}{2} j_2 - \frac{1}{2} J_3) \Delta(J_1 J_2 J_3), \quad (6.13)$$

$$N_1^\beta = \Delta(J_1 j_2 j_3 - \frac{1}{2}) \Delta(j_1 J_2 - \frac{1}{2} j_3) \Delta(j_1 j_2 J_3) \Delta(J_1 J_2 J_3). \quad (6.14)$$

• **Parity  $\gamma$**  :

$$\left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\}_\gamma^S = (-1)^{\phi_\gamma} \left[ x_\gamma \frac{N^S}{N_1^\gamma} \left\{ \begin{array}{ccc} J_1 - \frac{1}{2} & J_2 - \frac{1}{2} & J_3 - \frac{1}{2} \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 - \frac{1}{2} \end{array} \right\} + y_\gamma \frac{N^S}{N_2^\gamma} \left\{ \begin{array}{ccc} J_1 - \frac{1}{2} & J_2 - \frac{1}{2} & J_3 - \frac{1}{2} \\ j_1 & j_2 & j_3 \end{array} \right\} \right], \quad (6.15)$$

where

$$d_1^o = j_1 + J_2 - j_3 - \frac{1}{2}, \quad d_2^o = j_1 + J_2 - j_3 - \frac{1}{2}, \quad (6.16)$$

$$x_\gamma = (J_1 + j_2 + j_3 + \frac{1}{2}) (j_1 + J_2 + j_3 + \frac{1}{2}) (j_1 + j_2 + J_3 + \frac{1}{2}), \quad (6.17)$$

$$y_\gamma = (-J_1 + j_2 + j_3 + \frac{1}{2}) (j_1 - J_2 + j_3 + \frac{1}{2}) (j_1 + j_2 - J_3 - \frac{1}{2}), \quad (6.18)$$

and

$$N_1^\gamma = \Delta(J_1 - \frac{1}{2} j_2 - \frac{1}{2} j_3 - \frac{1}{2}) \Delta(j_1 - \frac{1}{2} J_2 - \frac{1}{2} j_3 - \frac{1}{2}) \Delta(j_1 - \frac{1}{2} j_2 - \frac{1}{2} J_3 - \frac{1}{2}) \Delta(J_1 - \frac{1}{2} J_2 - \frac{1}{2} J_3 - \frac{1}{2}), \quad (6.19)$$

$$N_2^\gamma = \Delta(J_1 - \frac{1}{2} j_2 j_3) \Delta(j_1 J_2 - \frac{1}{2} j_3) \Delta(j_1 j_2 J_3 - \frac{1}{2}) \Delta(J_1 - \frac{1}{2} J_2 - \frac{1}{2} J_3 - \frac{1}{2}). \quad (6.20)$$

## 6.2 An example of equating two different series

Recalls:

$$\begin{aligned} \frac{1}{N_2^\alpha} \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{array} \right\} &= (-1)^{q_k} \Gamma(q_k + 2) \left[ \prod_{i=1}^{i=4} \Gamma(1 + q_k - p_i) \prod_{l \neq k} \Gamma(q_l - q_k + 1) \right]^{-1} \\ &\times {}_4\mathbf{F}_3 \left( \begin{array}{c} (p_1 - q_k), (p_2 - q_k), (p_3 - q_k), (p_4 - q_k) \\ (-q_k - 1), (q_{k_2} - q_k + 1), (q_{k_3} - q_k + 1) \end{array} ; 1 \right). \end{aligned} \quad (6.21)$$

For the first 6- $j$  symbol we change all  $q_k \rightarrow q_k - 1$  and  $p_1 \rightarrow p_1 - 1$ ,  $p_2 \rightarrow p_2 - 1$  and  $p_3 \rightarrow p_3 - 1$ :

$$\begin{aligned} \frac{1}{N_1^\alpha} \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 - \frac{1}{2} \end{array} \right\} &= (-1)^{q_k - 1} \Gamma(q_k + 1) \left[ \frac{\Gamma(q_k - p_4)}{\Gamma(1 + q_k - p_4)} \times \prod_{i=1}^{i=4} \Gamma(1 + q_k - p_i) \prod_{l \neq k} \Gamma(q_l - q_k + 1) \right]^{-1} \\ &\times {}_4\mathbf{F}_3 \left( \begin{array}{c} (p_1 - q_k), (p_2 - q_k), (p_3 - q_k), (p_4 - q_k + 1) \\ (-q_k), (q_{k_2} - q_k + 1), (q_{k_3} - q_k + 1) \end{array} ; 1 \right), \end{aligned} \quad (6.22)$$

or more

$$\frac{1}{N_1^\alpha} \left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 - \frac{1}{2} \end{matrix} \right\} = -(-1)^{q_k} \Gamma(q_k+1) \left[ \frac{\Gamma(q_k-p_4)}{\Gamma(1+q_k-p_4)} \times \prod_{i=1}^{i=4} \Gamma(1+q_k-p_i) \prod_{l \neq k} \Gamma(q_l-q_k+1) \right]^{-1} \times {}_4F_3 \left( \begin{matrix} (p_1-q_k), (p_2-q_k), (p_3-q_k), (p_4-q_k+1) \\ (-q_k), (q_{k_2}-q_k+1), (q_{k_3}-q_k+1) \end{matrix}; 1 \right). \quad (6.23)$$

We have  $\left[ \frac{\Gamma(q_k-p_4)}{\Gamma(1+q_k-p_4)} \right]^{-1} = \frac{\Gamma(1+q_k-p_4)}{\Gamma(q_k-p_4)} = \frac{(q_k-p_4)!}{(q_k-p_4-1)!} = (q_k-p_4)$ .

Then we can write

$$\frac{1}{N_1^\alpha} \left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 - \frac{1}{2} \end{matrix} \right\} = -(q_k-p_4)(-1)^{q_k} \Gamma(q_k+1) \left[ \prod_{i=1}^{i=4} \Gamma(1+q_k-p_i) \prod_{l \neq k} \Gamma(q_l-q_k+1) \right]^{-1} \times {}_4F_3 \left( \begin{matrix} (p_1-q_k), (p_2-q_k), (p_3-q_k), (p_4-q_k+1) \\ (-q_k), (q_{k_2}-q_k+1), (q_{k_3}-q_k+1) \end{matrix}; 1 \right), \quad (6.24)$$

We have  $N^S = \sqrt{R^S}$  from equation (2.6) and after use of  $\Gamma(q_k+2) = \Gamma(q_k+1) \times (q_k+1)$  we can perform the sum of the equation (6.3):

$$\left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\}_\alpha^S = \sqrt{R^S} (-1)^{q_k} \Gamma(q_k+1) \left[ \prod_{i=1}^{i=4} \Gamma(1+q_k-p_i) \prod_{l \neq k} \Gamma(q_l-q_k+1) \right]^{-1} \frac{1}{(p_4+1)} \times \left[ -(q_k-p_4) {}_4F_3 \left( \begin{matrix} (p_1-q_k), (p_2-q_k), (p_3-q_k), (p_4-q_k+1) \\ (-q_k), (q_{k_2}-q_k+1), (q_{k_3}-q_k+1) \end{matrix}; 1 \right) + (q_k+1) {}_4F_3 \left( \begin{matrix} (p_1-q_k), (p_2-q_k), (p_3-q_k), (p_4-q_k) \\ (-q_k-1), (q_{k_2}-q_k+1), (q_{k_3}-q_k+1) \end{matrix}; 1 \right) \right]. \quad (6.25)$$

$$\left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{matrix} \right\}_\alpha^S = \sqrt{R^S} (-1)^{q_k} \Gamma(q_k+1) \left[ \prod_{i=1}^{i=4} \Gamma(1+q_k-p_i) \prod_{l \neq k} \Gamma(q_l-q_k+1) \right]^{-1} \times {}_4F_3 \left( \begin{matrix} (p_1-q_k), (p_2-q_k), (p_3-q_k), (p_4-q_k) \\ (-q_k), (q_{k_2}-q_k+1), (q_{k_3}-q_k+1) \end{matrix}; 1 \right). \quad (6.26)$$

That yields

$$\begin{aligned} & (p_4+1) \times {}_4F_3 \left( \begin{matrix} (p_1-q_k), (p_2-q_k), (p_3-q_k), (p_4-q_k) \\ (-q_k), (q_{k_2}-q_k+1), (q_{k_3}-q_k+1) \end{matrix}; 1 \right) \equiv \\ & \left[ -(q_k-p_4) {}_4F_3 \left( \begin{matrix} (p_1-q_k), (p_2-q_k), (p_3-q_k), (p_4-q_k+1) \\ (-q_k), (q_{k_2}-q_k+1), (q_{k_3}-q_k+1) \end{matrix}; 1 \right) + (q_k+1) {}_4F_3 \left( \begin{matrix} (p_1-q_k), (p_2-q_k), (p_3-q_k), (p_4-q_k) \\ (-q_k-1), (q_{k_2}-q_k+1), (q_{k_3}-q_k+1) \end{matrix}; 1 \right) \right] \end{aligned} \quad (6.27)$$

Indeed the right hand side can be transformed into

$$\sum_n \left[ (p_4-q_k+n) \frac{(p_1-q_k)_n (p_2-q_k)_n (p_3-q_k)_n (p_4-q_k)_n}{(-q_k)(q_{k_2}-q_k+1)(q_{k_3}-q_k+1)} + (q_k+1-n) \frac{(p_1-q_k)_n (p_2-q_k)_n (p_3-q_k)_n (p_4-q_k)_n}{(-q_k)(q_{k_2}-q_k+1)(q_{k_3}-q_k+1)} \right].$$

Equation (6.27) is indeed an identity (rather trivial), like those we could derive from the other Set  $\text{II}_{(\alpha)}$ , Set  $\text{I-II}_{(\beta)}$  and Set  $\text{I-II}_{(\gamma)}$ . However this shows that the following examples with non Saalschützian ( $\tau_s \neq 1$ ) series  ${}_4F_3$  can be gathered into both Saalschützian series ( $\tau_s = 1$ ). Symbolically: Set  $\text{I-II}_{(\alpha)}$  :  $\tau_s = 2$  for Set  $\text{I}_{(\alpha)}$ ,  $\tau_s = 2$  for Set  $\text{II}_{(\alpha)}$   $\rightsquigarrow (\tau_s = 1) + (\tau_s = 1)$ , Set  $\text{I-II}_{(\beta)}$  : ( $\tau_s = 1$  and  $\tau_s = 2$ ) for Set  $\text{I}_{(\beta)}$ , ( $\tau_s = 1$  and  $\tau_s = 2$ ) for Set  $\text{II}_{(\beta)}$   $\rightsquigarrow (\tau_s = 1) + (\tau_s = 1)$ , Set  $\text{I-II}_{(\gamma)}$  : ( $\tau_s = -1$  and  $\tau_s = 0$ ) for Set  $\text{I}_{(\gamma)}$ , ( $\tau_s = -1$  and  $\tau_s = 0$ ) for Set  $\text{II}_{(\gamma)}$   $\rightsquigarrow (\tau_s = 1) + (\tau_s = 1)$ . This property could be useful in particular cases.

## 7 Conclusion and prospective

Each of the three super Racah coefficients ( $6j^S$  symbols) with parities  $\alpha, \beta, \gamma$  is a linear combination of two standard  $6-j$  coefficients, *i.e.*  $\left\{ \begin{matrix} L_1 & L_2 & L_3 \\ j_1 - \frac{1}{2} & l_2 & l_3 \end{matrix} \right\}$  and  $\left\{ \begin{matrix} L_1 & L_2 & L_3 \\ j_1 & l'_2 & l'_3 \end{matrix} \right\}$ , thus nothing more than a linear combination of two Wilson (W) polynomials differing by their first argument  $a'-1$  and  $a'$ . For parities  $\alpha, \gamma$  both polynomials have the same degree. For parity  $\beta$  the degrees are  $n$  and  $n+1$ . Let us recall [4, p. 401] that the Bannai-Ito (B-I) polynomials were found to be the Racah coefficients of  $\mathfrak{osp}(1|2)$ . This is an approach which differs of our own because of the use of other generators and representation bases. Precisely because of this we should be able to formulate a correlation between the  $6j^S$  symbols and the super Racah coefficients expressed in terms of (B-I)'s, as well another possible link between (B-I)'s and (W)'s. While knowing that a generalization has been already proposed [4, p. 402] for the superalgebra  $\mathfrak{osp}(1|2)_q$  with  $q$ -deformed polynomials (B-I) $_q$ 's, we think that the results with  $6j^S$  symbols should be able to generalize to some deformed  $6j^S_q$  symbols and algebras  $U_q(\mathfrak{osp}(1|2))$  [20].

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