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**Strategic information transmission with
sender's approval:**

the single crossing case

**Sémirat, Stéphan
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May 2021

Strategic information transmission with sender's approval: the single-crossing case

Stéphan Sémirat* and Françoise Forges^{†‡}

Abstract

We consider a sender-receiver game, in which the sender has finitely many types and the receiver's decision is a real number. We assume that utility functions are concave, single-peaked and single-crossing. After the cheap talk phase, the receiver makes a decision, which requires the sender's approval to be implemented. Otherwise, the sender "exits". At a perfect Bayesian equilibrium without exit, the receiver must maximize his expected utility subject to the participation constraints of all positive probability types. This necessary condition may not hold at the receiver's prior belief, so that a non-revealing equilibrium may fail to exist. Similarly, a fully revealing equilibrium may not exist either due to the sender's incentive compatibility conditions. We propose a constructive algorithm that always achieves a perfect Bayesian equilibrium without exit.

Keywords: Approval, cheap talk, sender-receiver game, participation constraints, single-crossing

JEL Classification: C72, D82

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1 Introduction

A standard sender-receiver game is played in two stages: first, the – informed – agent sends a message to the – uninformed – decision maker; then, the latter makes his decision. However, in many applications, there is a third stage: the agent can reject the decision in favor of an outside option.

The relevance of such a third stage can be illustrated by the relationship between a patient and a medical doctor. The patient has a lot of private information (on the diseases he had, his fear of some medications, etc.), a treatment necessitates a decision of the medical doctor. But the decision cannot be implemented without the patient’s approval.

In this paper, we consider three-stage sender-receiver games in which, given the sender’s message, the receiver proposes a decision. If the sender accepts the proposal, it is implemented. Otherwise, the sender “exits”, and his outside option is implemented. The sender’s utility (on decisions, but also on the outside option) depends on his type.

As in the most general version of Crawford and Sobel (1982), the receiver’s decisions belong to a real interval and the type-dependent utility functions, over decisions, satisfy desirable properties, namely, they are concave, single-peaked and single-crossing; the sender is upwardly biased. As in Frug (2016), the sender has finitely many types, which are originally ordered to make sense of the previous properties.

By contrast with the previous papers, the sender’s reservation utility matters in our model. We do not assume that it varies in any specific way with the sender’s type. For instance, many relationships are conceivable between the patient’s type and his utility for his outside option (to get no treatment, to see another physician, etc.).

We focus on the case where rejection of the proposal by the sender is extremely harmful to the receiver. In the above example, the medical doctor views the patient’s rejection of the treatment as a failure (e.g., due to the involved sunk cost). We show that the sender-receiver game with sender’s approval has a perfect Bayesian equilibrium (PBE) in which exit does not happen.

Without the “no exit” restriction, existence of a PBE is not an issue, because our game always has a babbling equilibrium, in which the sender possibly exits, for some of his types. A natural motivation for PBE without exit, under the assumption that the receiver’s utility, when exit occurs, is sufficiently low, is that the maximal utility the receiver can expect, over

all conceivable PBE, is achieved at a PBE without exit (provided there exists one, of course).¹ In other words, if, as Bester and Strausz (2001), we view the receiver as a principal who cannot commit to a mechanism, namely, must play a cheap talk game with the agent, but can choose among the equilibria of this game, a PBE without exit will emerge as the solution of the principal's optimization problem.²

Guided by the previous insights, we study the limit game in which the receiver's utility level in case of exit is $-\infty$. In this setup, the receiver's expected utility, at any PBE in which exit occurs with positive probability, is $-\infty$, while it is finite in every PBE without exit.

As will be illustrated in Section 4.2, existence of a PBE without exit requires specific assumptions. We provide a counterexample in which utility functions are concave and single-peaked, but do not satisfy the single-crossing condition.

The conditions that we impose on utility functions, when exit does not occur, are standard in economic applications (see, e.g., Kreps and Sobel (1994), Sobel (2013) and the references below). They cover, but go much beyond, the popular case in which the utility functions are quadratic, and the players' ideal decisions differ by a bias that is independent of the sender's type. We only restrict the sender's (type-dependent) reservation utility by assuming that, under complete information, the receiver would be able to make a decision inducing the sender's participation. Summing up, the main message of the current paper is that existence of a PBE without exit is guaranteed in a large class of economic models, provided that their underlying features are taken into account.

We establish that the limit game has a PBE in pure strategies, in which the sender accepts the receiver's proposal, whatever his type. Every such equilibrium corresponds to a partition of the sender's (finite) set of types satisfying two properties: individual rationality (IR) and incentive compatibility (IC). The idea is that the sender reveals the cell π of the partition containing his type and that, for every cell π of the partition, the receiver proposes a decision that maximizes his own utility, knowing that the sender's type belongs to π . In order to avoid exit, there must exist, for every cell π , a decision giving at least as much as the outside option to every sender's type in π . This is the IR property. If the partition is IR, let, for every cell

¹This quite intuitive result is established formally by Forges and Renault (2021) in a model that is more general than the current one.

²The same applies if instead of maximizing the receiver's expected utility, we seek to maximize a weighted sum of the players' expected utilities with a positive weight to the receiver's utility.

π of the partition, y^π be the best choice of the decision maker when the sender's type is in π . The IC property expresses that every sender's type in π prefers the decision y^π associated with π to the decision $y^{\pi'}$ associated with any other cell π' . Our existence proof takes the form of a step by step algorithm, which starts with the finest partition of the sender's types (namely, the set of singletons) and gradually modifies it – by moving types from one cell to another and possibly merging cells – so as to eventually satisfy IR and IC.

Let us go back to the principal-agent interpretation suggested above. The receiver cannot commit to a mechanism, so that the revelation principle does not hold. Hence to maximize his expected utility, the receiver can just choose the PBE to be played. Given the previous characterization of pure PBE without exit, the receiver has to select an IR and IC partition of the sender's types, with the understanding that the sender will report the partition cell containing his true type.

The paper is organized as follows: we first complete the introduction by indicating further links between this paper and related ones. Then, in Section 2, we introduce the formal model and, in Section 3, we state our main result. Section 4 contains some elementary examples (Subsection 4.1) and, as already announced, a counterexample, in Subsection 4.2. The algorithm is described in Section 5. The section starts with basic lemmas which are proved in an appendix. The different steps of the algorithm are detailed and its convergence is established. Section 6 makes some suggestions for future research.

Related papers

As pointed out above, we add an approval stage to the cheap talk game considered by Frug (2016), namely, Crawford and Sobel's (1982) model with finitely many types for the sender. In other words, Frug's (2016) model can be viewed as a particular case of ours, when the participation constraints are never binding. Among other results, in the uniform quadratic case, Frug (2016) proposes a procedure to determine a specific IR and IC partition of the types, which corresponds to an ex ante optimal equilibrium.

Matthews (1989) already considers a cheap talk game with an approval stage. He makes the same basic assumptions as Crawford and Sobel (1982). In particular, the sender has a continuum of unidimensional types. In Matthews' (1989) game, rejection of the receiver's proposal leaves the players at the status quo, which is not particularly harmful to the receiver.

By contrast, equilibria without exit are considered in Shimizu (2017). The model is again a direct extension of Crawford and Sobel (1982), with a focus on quadratic utility functions, uniformly distributed types and type-independent reservation utility levels. Shimizu’s (2017) main point is to show that the sender’s outside options can make cheap talk informative in spite of a substantial conflict of interest between the players. Shimizu (2013) illustrates the same phenomenon in the case where the receiver has a binary decision.

Finally, Forges and Renault (2021) can be viewed as a companion paper. The three-stage game and the solution concepts are the same as in this paper. But Forges and Renault (2021) only assume that the receiver’s decision set is compact and that the utility functions are continuous. In particular, the sender’s types can be multidimensional, so that they cannot be ordered in a relevant way. Forges and Renault (2021) show that a PBE without exit may fail to exist when the receiver can randomize over finitely many decisions. They propose various sufficient conditions for the existence of a pure PBE without exit. A straightforward case is when the sender has two types only. Existence of a pure PBE without exit is also guaranteed as soon as the receiver’s utility is type-independent. The proof proceeds by merging cells of the finest partition of the sender’s types, which is one of the basic ingredients of the algorithm below. Other sufficient conditions identified by Forges and Renault (2021) are that decisions belong to the real line (as in the current paper) and that, for every type, the sender’s utility function is monotonic in the receiver’s decision (in sharp contrast with the current paper).

2 Model

We consider a sender-receiver game Γ , in which the receiver’s decision cannot be implemented unless the sender approves it. The sender’s set of types Θ is a finite ordered set, p denotes a probability distribution over Θ , such that $p(\theta) > 0$ for every θ . Messages belong to a finite set M , such that $|M| \geq |\Theta|$. Finally, u_0^θ is type θ ’s reservation utility.

The timing of the game Γ is as follows:

- The sender’s type θ is chosen in Θ according to p .
- The sender learns his type and sends a message in M to the receiver.
- The receiver proposes a decision $x \in \mathbb{R}$ to the sender.

- If the sender accepts the proposal, the sender's utility is $U^\theta(x)$ and the receiver's utility is $V^\theta(x)$. Otherwise, the sender's utility is u_0^θ and the receiver's utility is $-\infty$.

At a perfect Bayesian equilibrium (PBE) of Γ , the sender accepts decision x if $U^\theta(x) > u_0^\theta$ and exits if $U^\theta(x) < u_0^\theta$. We will focus on PBE without exit and such that, at equilibrium, x is accepted if and only if $U^\theta(x) \geq u_0^\theta$. This means that the effective utility function of the sender is $\max\{U^\theta, u_0^\theta\}$. At a PBE without exit, the receiver's utility is finite.

As pointed out in the Introduction, Forges and Renault (2021) consider the more general case where the set of decisions is a compact subset of \mathbb{R}^n . They start by considering games $\Gamma(v_0)$, in which the receiver's utility, when the sender exits, is a finite number v_0 . They show that the PBE without exit of the limit game Γ are relevant to those of $\Gamma(v_0)$. Indeed, on the one hand, if Γ does not have a PBE without exit, then, the same happens in every $\Gamma(v_0)$; on the other hand, if Γ has a PBE without exit, then every $\Gamma(v_0)$ with a sufficiently low v_0 also has a PBE without exit.

In this paper, we directly concentrate on the game Γ , but we assume that the receiver's decision is a real number and that the functions $(\theta, x) \mapsto U^\theta(x)$ and $(\theta, x) \mapsto V^\theta(x)$ satisfy the following properties:

- *Strict concavity:*

For every $\theta \in \Theta$, $x \mapsto U^\theta(x)$ and $x \mapsto V^\theta(x)$ are twice continuously differentiable and for every $x \in \mathbb{R}$, $\frac{\partial^2 U^\theta(x)}{\partial x^2} < 0$ and $\frac{\partial^2 V^\theta(x)}{\partial x^2} < 0$;

(A0)

- *Single-crossing:*

For every $(\theta_1, \theta_2, x_1, x_2) \in \Theta^2 \times \mathbb{R}^2$, with $\theta_2 > \theta_1$ and $x_2 > x_1$,
if $U^{\theta_1}(x_2) - U^{\theta_1}(x_1) \geq 0$, then $U^{\theta_2}(x_2) - U^{\theta_2}(x_1) > 0$, and
if $V^{\theta_1}(x_2) - V^{\theta_1}(x_1) \geq 0$, then $V^{\theta_2}(x_2) - V^{\theta_2}(x_1) > 0$;

(A1)

- *Existence of a unique maximizing argument:*

For every $\theta \in \Theta$, there exist a unique $x^*(\theta) \in \mathbb{R}$ and a unique $y^*(\theta) \in \mathbb{R}$
such that $\frac{\partial U^\theta(x)}{\partial x} \Big|_{x=x^*(\theta)} = 0$ and $\frac{\partial V^\theta(x)}{\partial x} \Big|_{x=y^*(\theta)} = 0$;

(A2)

- *The sender is right biased:*

For every $\theta \in \Theta$, $x^*(\theta) > y^*(\theta)$;

(A3)

- For every type, the receiver has the opportunity to induce the sender's participation:

$$\text{For every } \theta \in \Theta, \text{ there exists } x \in \mathbb{R} \text{ such that } U^\theta(x) > u_0^\theta; \quad (\text{A4})$$

Note that if the functions $(x, \theta) \mapsto U^\theta(x)$ satisfy (A0)–(A3), then so do the functions $(x, \theta) \mapsto U^\theta(x) - u_0^\theta$. Therefore, we can assume w.l.o.g. that for each $\theta \in \Theta$, $u_0^\theta = 0$.³ The effective utility function of the sender becomes $\max\{U^\theta, 0\}$, which, of course, does not necessarily satisfy properties (A0) - (A3).

3 Existence of an equilibrium without exit

In Section 5, we establish the following proposition.

Proposition 1. *The game Γ has a pure perfect Bayesian equilibrium without exit.*

Set for every $\theta \in \Theta$,

$$X^\theta = \{x \in \mathbb{R}, U^\theta(x) \geq 0\}.$$

The set X^θ contains the decisions that are approved by type θ . By assumption (A4), $X^\theta \neq \emptyset$. We show below that it is a closed interval of \mathbb{R} . More generally, let, for every $L \subseteq \Theta$ and every $x \in \mathbb{R}$,

$$X^L = \bigcap_{\theta \in L} X^\theta,$$

$$V^L(x) = \sum_{\theta \in L} p(\theta) V^\theta(x).$$

The set X^L contains the decisions that are approved by all types in L . The function V^L is proportional to the receiver's conditional expected utility, given that the sender's type belongs to L ; it is concave and single-peaked on \mathbb{R} . If $X^L \neq \emptyset$,

$$y^L = \arg \max_{x \in X^L} V^L(x)$$

is well-defined.

³Replacing U^θ by $U^\theta - u_0^\theta$ makes reservation utilities type-independent but of course modifies the utility functions when exit does not occur. Hence, assuming a specific functional form, e.g., quadratic, for U^θ together with type-independent reservation utilities, as in Shimizu (2017), may entail a loss of generality (see Subsection 4.1).

To prove Proposition 1, we construct a partition Π of Θ such that Π induces an equilibrium without exit of Γ .

Consider the following conditions:

(IR) *Individual Rationality*:

For every cell $\pi \in \Pi$, $X^\pi \neq \emptyset$;

(IC) *Incentive Compatibility*:

If $|\Pi| > 1$, for every $\pi, \pi' \in \Pi$, $\pi \neq \pi'$ and for every $\theta \in \pi$, $U^\theta(y^\pi) \geq U^\theta(y^{\pi'})$.

Let Π be a partition of Θ satisfying (IR) and (IC).⁴ To construct strategies from Π , let us first associate a message m_π to every cell π of Π . Let the sender's strategy σ associated to Π be:

for every $\theta \in \Theta$, $\sigma(\theta) = m_{\pi(\theta)}$,

where $\pi(\theta)$ is the cell that contains θ . Let the receiver's strategy τ associated to Π be:

for every $m \in M$, if $m = m_\pi$ for some $\pi \in \Pi$, $\tau(m_\pi) = y^\pi$,
otherwise, $\tau(m) = y^{\pi_0}$,

where π_0 is an arbitrary cell of Π . It is straightforward to check that (σ, τ) defines a PBE without exit of Γ .⁵

As a first particular case, the coarsest partition $\Pi = \{\Theta\}$ induces a nonrevealing (aka pooling) equilibrium, for which (IC) is automatically satisfied. However, in our model, X^Θ may be empty, so that a nonrevealing PBE without exit does not necessarily exist. At the other extreme, the finest partition $\Pi = \{\{\theta\} : \theta \in \Theta\}$ induces a fully revealing (aka separating) equilibrium, for which (IR) is satisfied as a consequence of assumption (A4). But in this case, (IC) is quite demanding.

⁴Note that single-crossing (i.e., (A1)) implies that a partition Π satisfying (IR) and (IC) is necessarily an *interval* partition, in which every cell consists of consecutive types, i.e., if $\theta_1 < \theta_2$ belong to the same cell π of Π , then every θ such that $\theta_1 < \theta < \theta_2$ also belongs to π .

⁵Note that conversely, every pure PBE without exit (σ, τ) of Γ induces a partition Π satisfying (IR) and (IC). It is the partition whose cells contain the types that lead to the same action, for every action on the equilibrium path.

4 Examples

4.1 Elementary examples

Let us consider quadratic utility functions, with type-independent bias $b > 0$ and type-independent maximal utility c^2 for the sender⁶:

$$U^\theta(x) = c^2 - (\theta + b - x)^2$$

$$V^\theta(x) = -(\theta - x)^2$$

In this case, assuming $c > 0$,

$$X^\theta = [\theta + b - c, \theta + b + c]$$

$$x^*(\theta) = \theta + b$$

$$y^*(\theta) = \theta$$

$$y^\theta = \begin{cases} \theta + b - c & \text{if } c < b \\ \theta & \text{if } c \geq b. \end{cases}$$

Let $\Theta = \{\theta_1, \dots, \theta_n\}$, with $\theta_1 < \dots < \theta_n$. In this example, depending on whether $c < b$ or not, the sender's participation constraint is either binding for every θ or not binding for every θ . Furthermore, y^θ increases with θ . As next examples will show, these properties are not always satisfied in our model.

Let us assume $c < b$. Then for every subset $L = \{\theta_{j+1}, \dots, \theta_{j+\ell}\}$ of ℓ consecutive types such that $X^L \neq \emptyset$, the participation constraint, namely, $x \in X^L$, remains binding:

$$\sum_{\theta \in L} p(\theta)\theta \leq \theta_{j+\ell} \leq \theta_{j+\ell} + b - c = \min X^L$$

so that $y^L = \theta_{j+\ell} + b - c$.

The partition $\Pi_0 = \{\{\theta_i\}, i = 1, \dots, n\}$ satisfies IR, but every type just receives his reservation value. Regarding IC, if $\theta_i < \theta_j$, $y^{\theta_i} \notin X^{\theta_j}$ so that type θ_j cannot envy θ_i . However, θ_j envies $\theta_k > \theta_j$ as soon as $\theta_k < \theta_j + 2c$. This suggests that the effect of the parameter c , which reflects the credibility of exit (since it determines the size of X^θ) is as important as the effect of the bias b , which reflects the players' conflict of interest. This is a key insight in Shimizu (2017).

⁶We have pointed out above that w.l.o.g., we can assume that the sender's reservation utility u_0^θ equals 0. However, imposing at the same time that U^θ has a specific quadratic form is of course restrictive.

Let us show how a partition satisfying IR and IC can be gradually constructed from Π_0 . The highest type θ_n does not envy any type. Let us keep the cells as in Π_0 until we find θ_k envying θ_{k+1} ; if this happens, we move θ_k to the next cell to form $\{\theta_k, \theta_{k+1}\}$. The receiver's constrained optimal decision over the new cell is $y^{\{\theta_k, \theta_{k+1}\}} = \theta_{k+1} + b - c = y^{\theta_{k+1}}$. Hence the move is helpful to fulfill type θ_k 's IC condition but has no effect on the other types. We can go on by identifying the next type, among $\theta_1 < \dots < \theta_{k-1}$, starting with θ_{k-1} , which envies the cell on its right, and so on to end up with a partition satisfying both IR and IC. These steps are consistent with the algorithm that we propose below and all correspond to case C. of Subsection 5.7.

Even with the utility functions above, the construction gets more intricate when $c \geq b$, because the participation constraint can become binding over L when L gets larger. Note that when c is so large that the participation constraint is never binding, whatever the subset of types, we recover the standard case, in which there is no need to ask for the sender's approval.

Other examples are proposed in the sequel to illustrate the difficulties of a fully general algorithm. These examples are still generated by quadratic utility functions but both the bias and the maximum payoff of the sender depend on his type, namely

$$U^\theta(x) = c(\theta)^2 - (\theta + b(\theta) - x)^2$$

with $b(\theta) > 0$ and $\theta + b(\theta)$ increasing with θ .

4.2 Counterexample

Existence of a partition Π that satisfies (IR) and (IC) is not guaranteed in general.⁷ Indeed, conditions (IR) and (IC) tend to exert opposite effects on the players' strategies:

- Condition (IC) directly reports on the sender's Nash equilibrium conditions. When condition (IR) is irrelevant (e.g. when the sender prefers the receiver's preferred action to his outside option), condition (IC) is obtained by *pooling* adjacent types, up to the situation in which players no more conflict about the resulting pool-contingent actions.⁸

⁷For instance, Forges and Renault (2021, Section 4.2 and 4.3) exhibit settings in which there is no pure strategy equilibrium without exit (Section 4.2), or in which there is no mixed strategy equilibrium without exit (Section 4.3).

⁸The resulting set of pools possibly consists of the single set Θ . For instance, it is so in the leading example of Crawford and Sobel (1982), when the informed player's bias is sufficiently large.

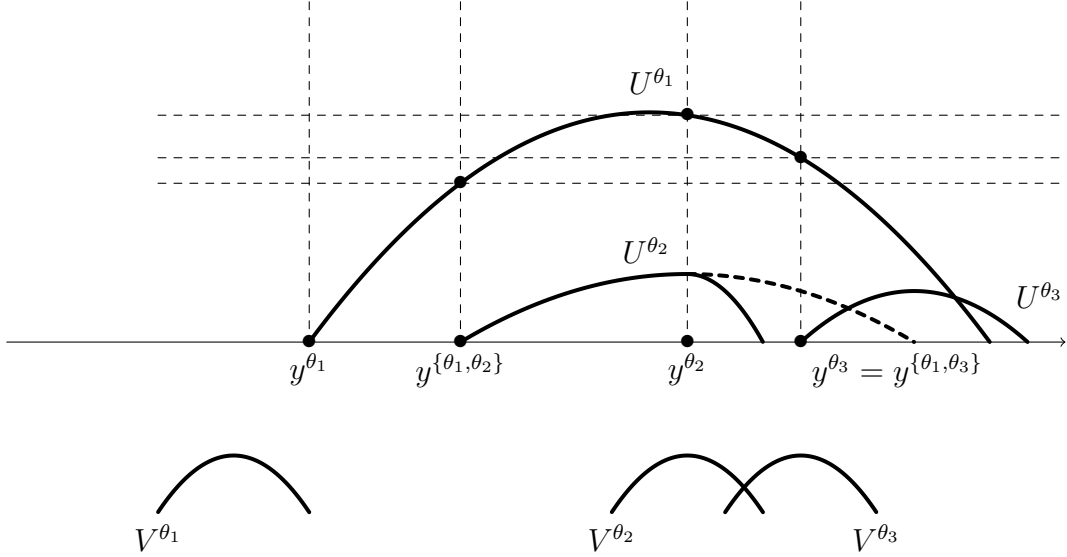


Figure 1: A situation in which (A1) does not hold, and in which (IR) and (IC) do not simultaneously hold.

- Condition (IR) requires *separating* strategies for types whose participation constraints cannot be simultaneously satisfied.

Let us illustrate, in a setting close to ours, that conditions (IC) and (IR) may not be simultaneously satisfied. In the following example, we only relax the single-crossing assumption (A1), and show that it is not possible to pool and separate the types in such a way that both conditions (IR) and (IC) hold.

Example 1. There are three types $\theta_1, \theta_2, \theta_3$, with $\theta_1 < \theta_2 < \theta_3$. The utility functions and cell-contingent optimal actions are as depicted in Figure 1. In particular, assumptions (A0), (A2), (A3) and (A4) hold. Moreover, we have $x^*(\theta_1) < x^*(\theta_2) < x^*(\theta_3)$ and $y^*(\theta_1) < y^*(\theta_2) < y^*(\theta_3)$. However, the single-crossing condition (A1) does not hold, because $\theta_2 > \theta_1$ and $y^{\theta_3} > y^{\{\theta_1, \theta_2\}}$, but

$$U^{\theta_1}(y^{\theta_3}) - U^{\theta_1}(y^{\{\theta_1, \theta_2\}}) > 0 \text{ and } U^{\theta_2}(y^{\theta_3}) - U^{\theta_2}(y^{\{\theta_1, \theta_2\}}) < 0.$$

In that context, there is no partition Π of $\Theta = \{\theta_1, \theta_2, \theta_3\}$ such that both (IR) and (IC) are satisfied:

- If $\Pi = \{\theta_1, \theta_2, \theta_3\}$ (uninformative), or if $\Pi = \{\theta_1\} \cup \{\theta_2, \theta_3\}$ (partially informative), from $X^{\{\theta_1, \theta_2, \theta_3\}} \subset X^{\{\theta_2, \theta_3\}} = X^{\theta_2} \cap X^{\theta_3} = \emptyset$, (IR) does not hold.
- If $\Pi = \{\theta_1, \theta_2\} \cup \{\theta_3\}$ (partially informative), then (IR) holds, but from $U^{\theta_1}(y^{\{\theta_1, \theta_2\}}) < U^{\theta_1}(y^{\theta_3})$, (IC) does not hold.

- If $\Pi = \{\theta_2\} \cup \{\theta_1, \theta_3\}$ (partially informative), then (IR) holds, but from $U^{\theta_1}(y^{\{\theta_1, \theta_3\}}) < U^{\theta_1}(y^{\theta_2})$, (IC) does not hold.
- If $\Pi = \{\theta_1\} \cup \{\theta_2\} \cup \{\theta_3\}$ (fully informative), then (IR) holds, but from $U^{\theta_1}(y^{\theta_1}) < U^{\theta_1}(y^{\theta_2})$ or $U^{\theta_1}(y^{\theta_1}) < U^{\theta_1}(y^{\theta_3})$, (IC) does not hold.

Finally, note that assumption (A1) would hold if, in Figure 1, the alternative graph for utility function U^{θ_2} (the dashed one) was considered. In that case, $X^{\{\theta_1, \theta_2, \theta_3\}} = X^{\theta_2} \cap X^{\theta_3} \neq \emptyset$, $y^{\{\theta_1, \theta_2, \theta_3\}} = y^{\theta_3}$, and the uninformative partition $\Pi = \{\theta_1, \theta_2, \theta_3\}$ does satisfy (IR) and (IC).

In the next section, we provide an algorithm showing that under assumptions (A0)–(A4), it is always possible to pool and separate the types in such a way that conditions (IR) and (IC) simultaneously hold.

5 Algorithm

We establish Proposition 1 by recursively defining a sequence of partitions $(\Pi_r)_{r \geq 0}$ that ends on a partition Π of Θ satisfying (IR) and (IC).

Let us first state some lemmas about the receiver's constrained optimum (proofs are given in the appendix).

5.1 Preliminary lemmas

The first lemma states that X^θ is a closed interval.

Lemma 1. *For every $\theta \in \Theta$, $X^\theta = [\min X^\theta, \max X^\theta]$.*

The next two lemmas deal with the receiver's constrained optimal action in the complete information case. Recall that

$$y^\Theta = \arg \max_{x \in X^\Theta} V^\Theta(x).$$

Lemma 2. *For every $\theta \in \Theta$, $y^\theta < x^*(\theta) < \max X^\theta$. In particular, U^θ is increasing at $x = y^\theta$.*

Lemma 3. *For every $\theta \in \Theta$, $y^\theta \geq y^*(\theta)$, with equality iff $y^*(\theta) \geq \min X^\theta$. In particular, V^θ is not increasing at $x = y^\theta$.*

$$\text{Summing up, } y^\theta = \arg \max_{x \in X^\theta} V^\theta(x) = \begin{cases} y^*(\theta) & \text{if } x \in X^\theta \text{ is not binding} \\ \min X^\theta & \text{if } x \in X^\theta \text{ is binding.} \end{cases}$$

Observe that, as expected from the single-crossing condition (A1), $y^*(\theta)$ increases when θ increases according to the original order over Θ , but that this monotonicity does not necessarily hold for y^θ (see Figure 2 below).

Finally, the next three lemmas identify relationships between the receiver's constrained optimal action in the partially informative case y^L , $L \subset \Theta$ and y^θ , $\theta \in L$. Note that when L is not reduced to a singleton, y^L is possibly achieved at both ends of X^L :

$$y^L = \arg \max_{x \in X^L} \sum_{\theta \in L} p(\theta) V^\theta(x) = \begin{cases} y^*(L) & \text{if } x \in X^L \text{ is not binding} \\ \min X^L \text{ or } \max X^L & \text{if } x \in X^L \text{ is binding} \end{cases}$$

where $y^*(L) = \arg \max_{x \in \mathbb{R}} \sum_{\theta \in L} p(\theta) V^\theta(x)$.

Lemma 4. *Let $L \subseteq \Theta$ be such that $X^L \neq \emptyset$. Let $\theta \in L$ be such that for every $\theta' \in L$, $y^\theta \leq y^{\theta'}$. Then $y^\theta \leq y^L$.*

Lemma 5. *Let $L \subseteq \Theta$ be such that $X^L \neq \emptyset$. Let $\theta \in L$ be such that for every $\theta' \in L$, $y^\theta \geq y^{\theta'}$. Then $y^\theta \geq y^L$.*

Lemma 6. *Let $L_1, L_2 \subseteq \Theta$ be such that $X^{L_1 \cup L_2} \neq \emptyset$. If $y^{L_1} \geq y^{L_2}$ then $y^{L_1} \geq y^{L_1 \cup L_2} \geq y^{L_2}$.*

5.2 Initialization

Let us order the types in Θ as $\theta_{i_1}, \dots, \theta_{i_n}$, so that

$$y^{\theta_{i_1}} < \dots < y^{\theta_{i_n}}.$$

We start with the following partition Π_0 :

$$\pi_0^1 = \{\theta_{i_1}\}, \dots, \pi_0^n = \{\theta_{i_n}\},$$

whose cells are ordered in the same way as the types.⁹

Partition Π_0 clearly satisfies (IR). Moreover, since for each $\theta \in \Theta$, U^θ is increasing at y^θ (see Lemma 2), partition Π_0 is such that no type in a cell strictly prefers the action induced

⁹If there exist types θ and θ' such that $y^\theta = y^{\theta'}$, then it is straightforward to see that $y^{\{\theta, \theta'\}} = y^\theta = y^{\theta'}$, so that such types can be merged into a unique cell, up to obtaining a partition $\tilde{\Pi}_0$ that does satisfy $y^{\tilde{\pi}_0^1} < y^{\tilde{\pi}_0^2} < \dots < y^{\tilde{\pi}_0^{|\tilde{\Pi}_0|}}$. In that case, the algorithm has to start from partition $\tilde{\Pi}_0$ instead of Π_0 .

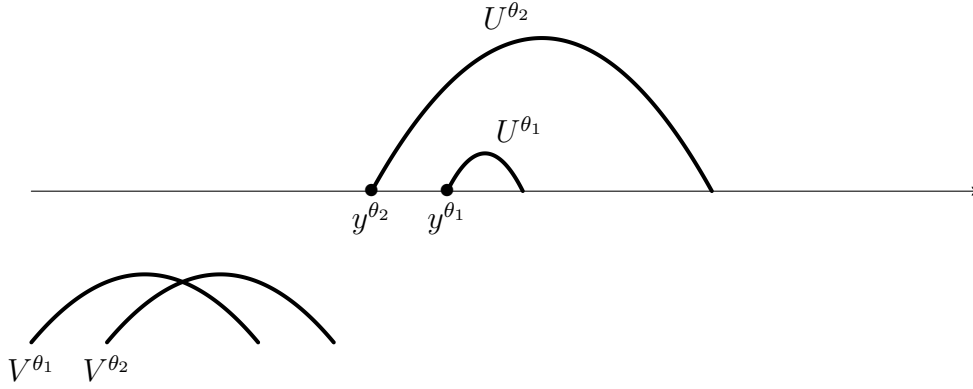


Figure 2: A situation in which $\theta_1 < \theta_2$, but $y^{\theta_1} > y^{\theta_2}$.

in a *preceding* cell. Hence partition $\Pi = \Pi_0$ also satisfies the following condition, which makes sense in any partition Π with ordered cells:

(L-IC) *Left-sided incentive compatibility*: If $|\Pi| > 1$, for every $n_2 \in \{2, \dots, |\Pi|\}$, for every $\theta \in \pi^{n_2}$, for every $n_1 \in \{1, \dots, n_2 - 1\}$, $U^\theta(y^{\pi^{n_1}}) < U^\theta(y^{\pi^{n_2}})$.

Condition (L-IC) can be seen as a half-way path to the more demanding condition (IC), which says that no type in a given cell strictly prefers the action induced in neither a preceding nor a succeeding cell. The recursive procedure detailed below will generate a sequence $(\Pi_r)_{r \geq 0}$ in which every Π_r is an ordered partition satisfying conditions (IR) and (L-IC), and such that, as r gets larger, Π_r gradually fulfills the “right-sided incentive compatibility” (R-IC) constraint too. To achieve this goal, if Π_r does not satisfy (R-IC), the idea is to construct Π_{r+1} from Π_r by merging cells of Π_r or by moving a type $\theta \in \pi^n$, with $\pi^n \in \Pi_r$, to some succeeding cell $\pi^{n'}$, $n' > n$, of Π_r . But to keep (IR) and (L-IC) all along the sequence, some care is needed in the way cells are ordered in every partition Π_r . It turns out to be useful to identify further properties (namely, (P1) and (P2) below) to be satisfied by the partitions in $(\Pi_r)_{r \geq 0}$.

5.3 Orders on cells, types and induced actions

The type ordering used to define partition Π_0 might not be the original one, for which single-crossing holds. Indeed, as illustrated in Figure 2, a lower type may have a greater outside option than a higher one. In particular, adjacent types (with respect to the original order) may not be adjacent with respect to the order of their associated type-contingent actions. The consequence is that the order of types ensuring that Π_0 satisfies (L-IC) may enter in conflict with the goal of reaching (R-IC). This point is illustrated in the following example.

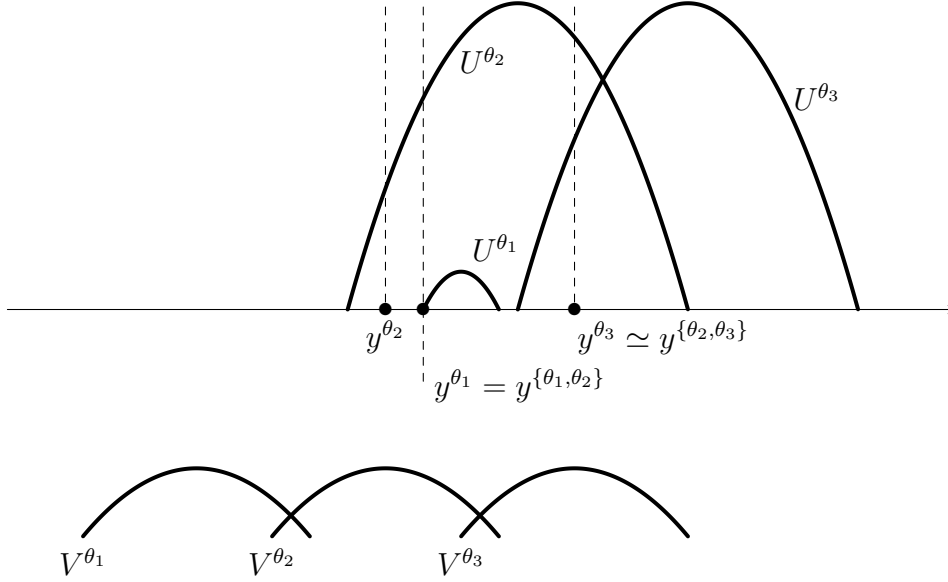


Figure 3: Non-monotonicity of the receiver's constrained optimal actions

Example 2. There are three types $\theta_1, \theta_2, \theta_3$, with $\theta_1 < \theta_2 < \theta_3$. The utility functions and cell-contingent optimal actions are as depicted in Figure 3. The initial partition is $\Pi_0 = \{\theta_2\} \cup \{\theta_1\} \cup \{\theta_3\}$. The only partition satisfying (IR) and (IC) is $\Pi = \pi^1 \cup \pi^2$ with $\pi^1 = \{\theta_1\}$, $\pi^2 = \{\theta_2, \theta_3\}$.¹⁰ This partition Π is such that $y^{\pi^1} < y^{\pi^2}$, but $\theta_1 \in \pi^1$, $\theta_2 \in \pi^2$ and $y^{\theta_1} > y^{\theta_2}$.

In order to deal with the above mentioned possible conflict, we state precise conditions (P1) and (P2), to be satisfied by the successive partitions we construct, in addition to (IR) and (L-IC):

(P1) *Increasing cell-contingent actions:* $y^{\pi^1} < y^{\pi^2} < \dots < y^{\pi^{|\Pi|}}$.

(P2) *Increasing unconstrained type-contingent actions:* If $|\Pi| > 1$, then for every $n \in \{1, \dots, |\Pi| - 1\}$, for every $\theta \in \pi^n$, if $y^\theta > \min X^\theta$, then for every $n' \in \{n+1, \dots, |\Pi|\}$, for every $\theta' \in \pi^{n'}$, $y^\theta \leq y^{\theta'}$.

Condition (P1) guarantees that the order over the cells determines the order over cell-contingent actions and condition (P2) guarantees that this order carries over for those θ 's such that $y^\theta = y^*(\theta)$, i.e., such that $y^\theta > \min X^\theta$.

¹⁰When considering the fully revealing partition $\Pi = \{\theta_2\} \cup \{\theta_1\} \cup \{\theta_3\}$, or the partially revealing partition $\Pi = \{\theta_2, \theta_1\} \cup \{\theta_3\}$, type θ_2 prefers y^{θ_3} to y^{θ_2} or to $y^{\{\theta_2, \theta_1\}}$ respectively, and if we consider the non revealing partition $\Pi = \{\theta_2, \theta_1, \theta_3\}$ or the partially revealing partition $\Pi = \{\theta_2\} \cup \{\theta_1, \theta_3\}$, the receiver may not satisfy the participation constraint of the sender.

The initial partition Π_0 obviously satisfies (P1) by construction. Furthermore, for every type θ , the *type*-contingent action y^θ is identical to the *cell*-contingent action of the cell of Π_0 containing θ . Hence Π_0 also satisfies (P2).¹¹ In example 2, the unique partition Π satisfying (IR) and (IC) also satisfies (P1) and (P2). According to the algorithm described below, Π will be reached from Π_0 via $\Pi_1 = \{\theta_2, \theta_1\} \cup \{\theta_3\}$, which satisfies (IR), (L-IC), (P1) and (P2) and is transformed into Π by moving θ_2 to the last cell.

5.4 Recursive procedure

Proposition 1 results from the following claim, which we establish in Section 5.7.

Claim. Let Π_r be a partition which satisfies (P1), (P2), (IR) and (L-IC). If Π_r does not satisfy (IC), then there exists a partition Π_{r+1} , obtained by moving some type $\tilde{\theta}$ from some cell $\pi^n \in \Pi_r$ to some succeeding cell $\pi^{n'}$, $n' > n$, and possibly by merging cells, such that Π_{r+1} satisfies conditions (P1), (P2), (IR) and (L-IC).

The claim ultimately guarantees the existence of a partition satisfying (IR) and (IC) because the moves considered in the claim cannot be repeated indefinitely. Indeed, there are finitely many types and finitely many cells. Furthermore, every type move is always directed to a succeeding cell. In other words, starting from Π_0 , and constructing Π_{r+1} from Π_r according to the claim, we necessarily reach a partition that satisfies (IR) and (IC).

As will be clear below, the algorithm used to establish the claim will be flexible regarding the choice of the cell of the type $\tilde{\theta}$ to be moved. Hence, different runs of the algorithm will be conceivable, possibly leading – a priori – to different (IR) and (IC) partitions. Given our primary goal, namely, to establish the existence of an equilibrium without exit, we do not constrain the algorithm unless further specification makes it simpler.

5.5 Choices for the moving type and the destination cell

According to the above procedure, we need to establish the existence of a partition Π_{r+1} that satisfies conditions (IR) and (L-IC), among the partitions that can be reached from Π_r , by using the moves prescribed in the above claim.

¹¹The same holds for partition $\tilde{\Pi}_0$.

Keeping (IR) satisfied from Π_r to Π_{r+1} is easily obtained if only appropriate cells are merged (cells π and π' such that $X^\pi \cap X^{\pi'} \neq \emptyset$), and if, as the following lemma (established in the appendix) states, type moves are restricted to moving a type to a cell that it prefers.

Lemma 7. *Let $L, L' \subset \Theta$ with $X^L \neq \emptyset$, $X^{L'} \neq \emptyset$, and suppose that there exists $\theta \in L'$, $\theta \notin L$, such that $U^\theta(y^L) > U^\theta(y^{L'})$. Then we have $X^{L \cup \{\theta\}} \neq \emptyset$.*

Keeping (L-IC) satisfied is more demanding because the considered moves from Π_r to Π_{r+1} shift some induced actions, and thereby impact condition (L-IC) for *every* types. To that end, we need to carefully monitor every cell-contingent induced actions y^{π_r} , $\pi_r \in \Pi_r$, during the process. As a first step, we will specify the way in which type $\tilde{\theta}$ is moved from its cell π^n to a succeeding cell $\pi^{n'}$. Before this specification, let us introduce further notations.

Set, for every $y \in \mathbb{R}$,

$$X^\theta(y) = \{x \in \mathbb{R}, U^\theta(x) \geq U^\theta(y)\}.$$

Since U^θ is single-peaked, we have:

$$\begin{aligned} \text{if } y < x^*(\theta), \text{ then } \min X^\theta(y) &= y; \\ \text{if } y > x^*(\theta), \text{ then } \max X^\theta(y) &= y; \\ \text{if } y = x^*(\theta), \text{ then } X^\theta(y) &= \{x^*(\theta)\}. \end{aligned}$$

Then we can write

$$X^\theta(y) = [x_-^\theta(y), x_+^\theta(y)],$$

with $y = x_-^\theta(y)$ or $y = x_+^\theta(y)$ conditional on $y \leq x^*(\theta)$ and $y \geq x^*(\theta)$ respectively. Note that from Lemma 2, for every $\theta \in \Theta$, $x_-^\theta(y^\theta) = y^\theta$.

5.5.1 Destination cell

Concerning the destination cell, if the chosen type $\tilde{\theta}$ to be moved is an element of π^n , we show that we can w.l.o.g. choose $\pi^{n'}$ with $n' = n + 1$.

Let $\tilde{\theta} \in \pi^n$, for some $n \in \{1, \dots, |\Pi| - 1\}$, that prefers the action induced in *some* succeeding cell $\pi^{n'}$, $n' > n$, *i.e.* such that $y^{\pi^{n'}} \in X^{\tilde{\theta}}(y^{\pi^n})$. Since $X^{\tilde{\theta}}(y^{\pi^n})$ is an interval, and since we also have $y^{\pi^n} \in X^{\tilde{\theta}}(y^{\pi^n})$, we obtain $[y^{\pi^n}, y^{\pi^{n'}}] \subseteq X^{\tilde{\theta}}(y^{\pi^n})$, where $y^{\pi^n} < y^{\pi^{n'}}$ results from (P1). From (P1), we also get $y^{\pi^{n+1}} \in [y^{\pi^n}, y^{\pi^{n'}}]$. Therefore, we have $y^{\pi^{n+1}} \in X^{\tilde{\theta}}(y^{\pi^n})$, *i.e.* type $\tilde{\theta}$ prefers $y^{\pi^{n+1}}$ to y^{π^n} .

5.5.2 Moving type

We choose, among the types in π^n who prefer $y^{\pi^{n+1}}$, the type θ with the greatest associated action y^θ . We set

$$\tilde{\theta} = \theta_{i_\ell},$$

where we ordered $\pi^n = \{\theta_{i_1}, \dots, \theta_{i_{|\pi^n|}}\}$ in such a way that $y^{\theta_{i_1}} \leq \dots \leq y^{\theta_{i_{|\pi^n|}}}$, and set

$$\ell = \max\{j : y^{\pi^{n+1}} \in X^{\theta_{i_j}}(y^{\pi^n})\}.$$

The next lemma allows us to characterize $\tilde{\theta}$ within π^n .

Lemma 8. *Either $\ell = |\pi^n|$ or, if $\ell < |\pi^n|$, then for every $j \in \{\ell + 1, \dots, |\pi^n|\}$, $y^{\theta_{i_j}} = \min X^{\theta_{i_j}}$. In that latter case, $y^{\pi^n} = y^{\theta_{i_{|\pi^n|}}} = \min X^{\theta_{i_{|\pi^n|}}}$.*

5.6 Action shifts

The choice of $\tilde{\theta}$ as a maximal element of π^n and the choice of moving it to the next succeeding cell π^{n+1} allow us to characterize how the induced actions y^{π^n} and $y^{\pi^{n+1}}$ shift after the type move.

Lemma 9. • *If $\pi^n \setminus \{\tilde{\theta}\} \neq \emptyset$, then*

$$y^{\pi^n \setminus \{\tilde{\theta}\}} \leq y^{\pi^n}, \tag{1}$$

and, if moreover $\ell < |\pi^n|$,

$$y^{\pi^n \setminus \{\tilde{\theta}\}} = y^{\pi^n}. \tag{1a}$$

•

$$y^{\tilde{\theta}} \leq y^{\pi^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi^{n+1}} \tag{2}$$

• *If $\ell = |\pi^n|$, then*

$$y^{\pi^n} \leq y^{\pi^{n+1} \cup \{\tilde{\theta}\}}. \tag{3}$$

5.7 Partition Π_{r+1}

Finally, we are now able to define partition Π_{r+1} from a partition $\Pi = \Pi_r$ that satisfies (IR), (L-IC), (P1) and (P2). To this end, we distinguish the following cases:

(A) $[\pi^n \setminus \{\tilde{\theta}\} \neq \emptyset \text{ and } y^{\pi^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi^n \setminus \{\tilde{\theta}\}}]$,

$$(B) [\pi^n \setminus \{\tilde{\theta}\} \neq \emptyset \text{ and } y^{\pi^{n+1} \cup \{\tilde{\theta}\}} > y^{\pi^n \setminus \{\tilde{\theta}\}}]$$

$$(C) \pi^n \setminus \{\tilde{\theta}\} = \emptyset.$$

Case C is the simplest one. It occurs when π^n is a cell of the initial partition Π_0 , in particular, if Π_0 does not satisfy (IC). We provide an illustration of the construction of Π_{r+1} in cases A and B in examples 3 and 4 respectively. The construction is greatly simplified when y^θ is increasing w.r.t. the initial order of types, a property that does not hold in examples 3 and 4 but holds when participation constraints are not binding for individual types (i.e., $y^\theta = y^*(\theta)$). Then only cases B.1 (see below) and C can arise, the initial order of types is maintained all along the procedure and at every step, properties (P1) and (P2) are satisfied in a straightforward way.

Once Π_{r+1} is constructed in the different possible cases, as described below, we have to show that it also satisfies (IR), (L-IC), (P1) and (P2). To that end, the next lemma will be used in showing that Π_{r+1} satisfies (L-IC).

Lemma 10. *Let $\Pi = \pi^1 \cup \dots \cup \pi^{|\Pi|}$ be a partition of Θ that satisfy (IR), (P1) and (L-IC). Let $n_2 \in \{2, \dots, |\Pi|\}$, $n_1 \in \{1, \dots, n_2 - 1\}$ and let $\theta \in \pi^{n_2}$. Then for every $y_1, y_2 \in \mathbb{R}$, if $y_1 < y_2$, $y_2 \leq y^{\pi^{n_2}}$ and $y_1 \leq y^{\pi^{n_1}}$, then $y_1 \notin X^\theta(y_2)$.*

(A). Suppose $\pi_r^n \setminus \{\tilde{\theta}\} \neq \emptyset$ and

$$y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi_r^n \setminus \{\tilde{\theta}\}}. \quad (4)$$

In that case, partition Π_{r+1} is obtained by merging cells π^n and π^{n+1} . The next example illustrates that possibility.

Example 3. There are three types $\theta_1, \theta_2, \theta_3$, with $\theta_1 < \theta_2 < \theta_3$. The utility functions and cell-contingent optimal actions are as depicted in Figure 4. In particular, $y^{\theta_2} < y^{\theta_1} < y^{\theta_3}$. Set $\Pi_0 = \{\theta_2\} \cup \{\theta_1\} \cup \{\theta_3\}$ accordingly. Then type θ_2 prefers $y^{\theta_1} = y^{\pi_0^2}$ to $y^{\theta_2} = y^{\pi_0^1}$. Set $\Pi_1 = \{\theta_2, \theta_1\} \cup \{\theta_3\}$. Then type θ_2 prefers $y^{\theta_3} = y^{\pi_1^2}$ to $y^{\{\theta_2, \theta_1\}} = y^{\pi_1^1}$. When $\tilde{\theta} = \theta_2$ moves from $\pi_1^1 = \{\theta_2, \theta_1\}$ to $\pi_1^2 = \{\theta_3\}$, we obtain $y^{\pi_1^2 \cup \{\tilde{\theta}\}} = y^{\{\theta_2, \theta_3\}} \leq y^{\theta_1} = y^{\pi_1^1 \setminus \{\tilde{\theta}\}}$. This is case A. In words, moving a type from a cell to a *succeeding* cell may reverse the order of the shifted cell-contingent actions. Hence the resulting cell-contingent actions do not satisfy (P1). However, as we shall show, in such a case it is always possible to merge the considered cells while keeping condition (IR) satisfied. In the present example, the next step is to set $\Pi_2 = \pi_1^1 \cup \pi_1^2 = \{\theta_2, \theta_1, \theta_3\}$ which does satisfy (IR) and (P1).

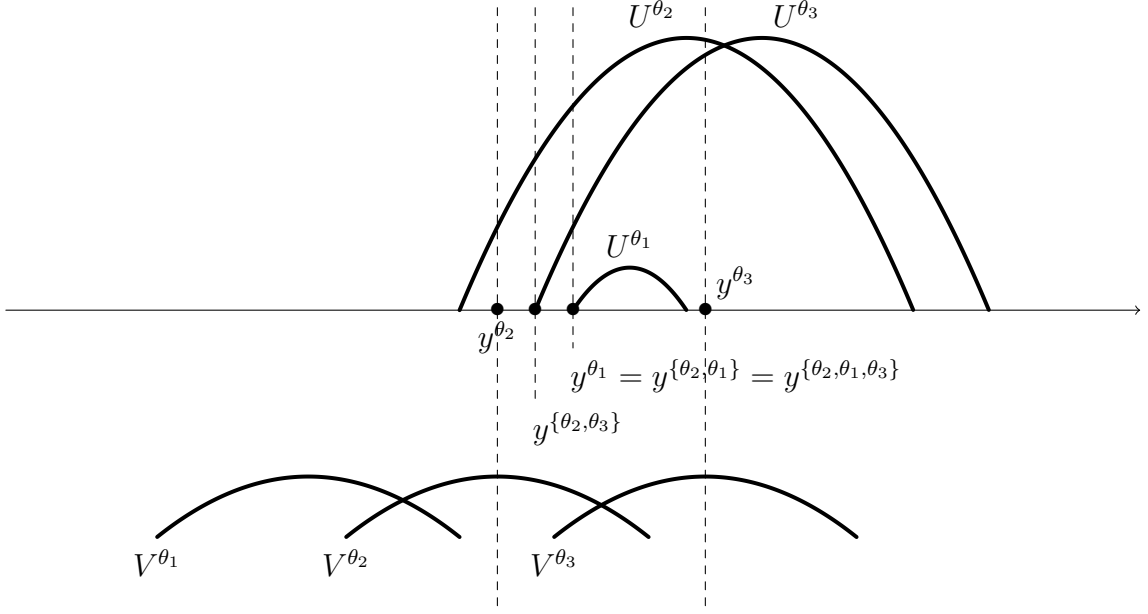


Figure 4: Illustration of Case A.

From (1), (2), (4) and $y^{\pi_r^n} < y^{\pi_r^{n+1}}$, we have

$$y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi_r^n \setminus \{\tilde{\theta}\}} \leq y^{\pi_r^n} \leq y^{\pi_r^{n+1}}. \quad (5)$$

Note that $y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \in \bigcap_{\theta \in \pi_r^{n+1} \cup \{\tilde{\theta}\}} X^\theta$ and $y^{\pi_r^{n+1}} \in \left(\bigcap_{\theta \in \pi_r^{n+1}} X^\theta \right) \cap X^{\tilde{\theta}} = \bigcap_{\theta \in \pi_r^{n+1} \cup \{\tilde{\theta}\}} X^\theta$. This means that for every $\theta \in \pi_r^{n+1} \cup \{\tilde{\theta}\}$, we have $\{y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}, y^{\pi_r^{n+1}}\} \subset X^\theta$. However, X^θ is an interval. Therefore for every $\theta \in \pi_r^{n+1} \cup \{\tilde{\theta}\}$, we have $[y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}, y^{\pi_r^{n+1}}] \subset X^\theta$. Then from (5), we have that for every $\theta \in \pi_r^{n+1} \cup \{\tilde{\theta}\}$, $y^{\pi_r^n \setminus \{\tilde{\theta}\}} \in X^\theta$. But this is also true for every $\theta \in \pi_r^n \setminus \{\tilde{\theta}\}$. Therefore, for every $\theta \in (\pi_r^n \setminus \{\tilde{\theta}\}) \cup (\pi_r^{n+1} \cup \{\tilde{\theta}\}) = \pi_r^n \cup \pi_r^{n+1}$, we have $y^{\pi_r^n \setminus \{\tilde{\theta}\}} \in X^\theta$. In particular,

$$\bigcap_{\theta \in \pi_r^n \cup \pi_r^{n+1}} X^\theta \neq \emptyset.$$

Then from Lemma 6, we have

$$y^{\pi_r^n} \leq y^{\pi_r^n \cup \pi_r^{n+1}} \leq y^{\pi_r^{n+1}}. \quad (6)$$

Then from $y^{\pi_r^{n-1}} < y^{\pi_r^n}$ and $y^{\pi_r^{n+1}} < y^{\pi_r^{n+2}}$, we obtain

$$y^{\pi_r^{n-1}} < y^{\pi_r^n \cup \pi_r^{n+1}} < y^{\pi_r^{n+2}}. \quad (7)$$

We set

$$\begin{cases} \pi_{r+1}^1 = \pi_r^1, \dots, \pi_{r+1}^{n-1} = \pi_r^{n-1}, \\ \pi_{r+1}^n = \pi_r^n \cup \pi_r^{n+1}, \\ \pi_{r+1}^{n+1} = \pi_r^{n+2}, \dots, \pi_{r+1}^{|\Pi_r|-1} = \pi_r^{|\Pi_r|}. \end{cases} \quad (8)$$

Partition Π_{r+1} satisfies (IR) by construction. From (7), it also satisfies (P1).

Let us show that it also satisfies (P2). Let $n_1 \in \{1, \dots, |\Pi_{r+1}| = |\Pi_r| - 1\}$ and let $\theta \in \pi_r^{n_1}$. Suppose that there exists $n_2 \geq n_1 + 1$ and $\theta' \in \pi_{r+1}^{n_2}$ such that $y^{\theta'} < y^\theta$. Let us show that $y^\theta = \min X^\theta$.

- If $n_1 \leq n - 1$, then from $\pi_{r+1}^{n_1} = \pi_r^{n_1}$, we have $\theta' \in \pi_r^{n_1}$. Then $n_2 \geq n$, so that $\theta' \in \pi_r^{n'_2}$ for some n'_2 such that $n'_2 = n_2$ or $n'_2 = n_2 + 1$. In particular $n'_2 \geq n_1$. Since Π_r satisfies (P2), we then get $y^\theta = \min X^\theta$.
- If $n_1 = n$, then from $n_2 \geq n_1 + 1 = n + 1$, we have $\pi_{r+1}^{n_2} = \pi_r^{n_2+1}$ and thus $\theta' \in \pi_r^{n_2+1}$ with $n_2 + 1 \geq n + 2$. We have either $\theta \in \pi_r^n$ or $\theta \in \pi_r^{n+1}$. Thus $\theta \in \pi_r^{n'_1}$ with $n'_1 = n$ or $n'_1 = n + 1$. In both cases, $n_2 + 1 \geq n + 2 \geq n'_1$. Since Π_r satisfies (P2), we get $y^\theta = \min X^\theta$.
- If $n_1 \geq n + 1$ then from $n_2 \geq n_1 + 1 \geq n + 2$ we have $\theta \in \pi_r^{n'_1}$ and $\theta' \in \pi_r^{n'_2}$, with $n'_1 = n_1 + 1$ and $n'_2 = n_2 + 1$ so that $n'_2 = n_2 + 1 \geq (n_1 + 1) + 1 = n'_1 + 1$. Since Π_r satisfies (P2), we get $y^\theta = \min X^\theta$.

Finally, we show that Π_{r+1} satisfies (L-IC). Let $n'_2 \in \{2, \dots, |\Pi_{r+1}|\}$, $n'_1 \in \{1, \dots, n'_2 - 1\}$ and $\theta \in \pi_{r+1}^{n'_2}$. We want to show that $y^{\pi_{r+1}^{n'_1}} \notin X^\theta(y^{\pi_{r+1}^{n'_2}})$. If $n'_1 \neq n$ and $n'_2 \neq n$, then $\pi_{r+1}^{n'_1} = \pi_r^{n'_1}$, and $\pi_{r+1}^{n'_2} = \pi_r^{n'_2}$, so that $y^{\pi_{r+1}^{n'_1}} \notin X^\theta(y^{\pi_{r+1}^{n'_2}})$ because $y^{\pi_r^{n'_1}} \notin X^\theta(y^{\pi_r^{n'_2}})$. Let us now consider the cases $n'_1 = n$, or $n'_2 = n$. Set

$$y_1 = y^{\pi_{r+1}^{n'_1}}$$

and

$$y_2 = y^{\pi_{r+1}^{n'_2}}.$$

Then according to Lemma 10, the result derives from $y_1 < y_2$ (guaranteed because Π_{r+1} satisfies (P1)), and the existence of $n_2 \in \{2, \dots, |\Pi_r|\}$ and $n_1 \in \{1, \dots, n_2 - 1\}$ such that $y_1 \leq y^{\pi_r^{n_2}}$ and $y_2 \leq y^{\pi_r^{n_1}}$.

- If $n'_1 = n$, then $n'_2 > n$. From (6), $y_1 \leq y^{\pi_r^{n_2+1}} = y^{\pi_r^{n_1}}$ where $n_1 = n + 1$. From $n'_2 > n$, we have $y_2 = y^{\pi_r^{n_2}}$ with $n_2 \geq n + 2$.
- If $n'_2 = n$, then $n'_1 < n$ and thus $y_1 = y^{\pi_r^{n_1}}$ with $n_1 = n'_1 < n$. Then from (6), $y_2 \leq y^{\pi_r^{n_2+1}} = y^{\pi_r^{n_2}}$ with $n_2 = n + 1 > n_1$.

(B). Suppose $\pi_r^n \setminus \{\tilde{\theta}\} \neq \emptyset$ and

$$y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} > y^{\pi_r^n \setminus \{\tilde{\theta}\}}. \quad (9)$$

In this case, we distinguish cases (B.1) [$n=1$ or $y^{\pi_r^{n-1}} < y^{\pi_r^n \setminus \{\tilde{\theta}\}}$], and (B.2) [$n > 1$ and $y^{\pi_r^{n-1}} \geq y^{\pi_r^n \setminus \{\tilde{\theta}\}}$]. The following example illustrates both cases (B.1) and (B.2).

Example 4. There are four types $\theta_1, \theta_2, \theta_3, \theta_4$, with $\theta_1 < \theta_2 < \theta_3 < \theta_4$. The utility functions and cell-contingent optimal actions are as depicted in Figure 5. In particular, $y^{\theta_2} < y^{\theta_1} < y^{\theta_3} < y^{\theta_4}$. Set $\Pi_0 = \{\theta_2\} \cup \{\theta_1\} \cup \{\theta_3\} \cup \{\theta_4\}$ accordingly. Then type θ_2 prefers $y^{\theta_1} = y^{\pi_0^2}$ to $y^{\theta_2} = y^{\pi_0^1}$. Set $\Pi_1 = \{\theta_2, \theta_1\} \cup \{\theta_3\} \cup \{\theta_4\}$. Now type θ_2 prefers $y^{\theta_3} = y^{\pi_1^2}$ to $y^{\{\theta_2, \theta_1\}} = y^{\pi_1^1}$. Set $\Pi_2 = \{\theta_1\} \cup \{\theta_2, \theta_3\} \cup \{\theta_4\}$. Then $y^{\theta_1} < y^{\{\theta_2, \theta_3\}} < y^{\theta_3}$, so that the order on the cell-contingent actions does not change from Π_1 to Π_2 . Accordingly, condition (L-IC) is satisfied. This is case B.1. Now let us go on with the algorithm in order to illustrate case B.2. At Π_2 , θ_3 prefers $y^{\theta_4} = y^{\pi_2^3}$ to $y^{\{\theta_2, \theta_3\}} = y^{\pi_2^2}$. When $\tilde{\theta} = \theta_3$ moves from $\pi_2^2 = \{\theta_2, \theta_3\}$ to $\pi_2^3 = \{\theta_4\}$, then $y^{\pi_2^1} = y^{\theta_1} \geq y^{\theta_2} = y^{\pi_2^2 \setminus \{\tilde{\theta}\}}$. This is case B.2. In words, moving a type from a cell π^n to the next succeeding cell may reverse the order of the cell-contingent actions of the n th cell (which becomes $\pi^n \setminus \{\tilde{\theta}\}$ and its *preceding* cell π^{n-1} . In particular, the resulting cell-contingent actions do not satisfy (P1). However, as we shall show, in that case it is always possible to merge $\pi^n \setminus \{\tilde{\theta}\}$ and π^{n-1} (possibly recursively, because $y^{\pi^n \setminus \{\tilde{\theta}\}}$, and $y^{\pi^n \setminus \{\tilde{\theta}\} \cup \pi^{n-1}}$ may shift further below) in such a way that the resulting partition satisfies (IR), and up to satisfy (P1). In the present example, the next step is to set $\Pi_3 = \underbrace{\left(\pi_2^1 \cup \pi_2^2 \setminus \{\tilde{\theta}\}\right)}_{\pi_3^1} \cup \underbrace{\left(\pi_2^3 \cup \{\tilde{\theta}\}\right)}_{\pi_3^2} = \{\theta_2, \theta_1\} \cup \{\theta_3, \theta_4\}$, which does satisfies (P1) and (IR).

(B.1). Suppose that $n = 1$, or

$$y^{\pi_r^{n-1}} < y^{\pi_r^n \setminus \{\tilde{\theta}\}}. \quad (10)$$

In that case, the move of $\tilde{\theta}$ from π_r^n to π_r^{n+1} keeps the order of the corresponding cell-contingent actions y^π .

We set

$$\left\{ \begin{array}{l} \pi_{r+1}^1 = \pi_r^1, \dots, \pi_{r+1}^{n-1} = \pi_r^{n-1} \text{ if } n > 1, \\ \pi_{r+1}^n = \pi_r^n \setminus \{\tilde{\theta}\}, \\ \pi_{r+1}^{n+1} = \pi_r^{n+1} \cup \{\tilde{\theta}\}, \\ \pi_{r+1}^{n+2} = \pi_r^{n+2}, \dots, \pi_{r+1}^{|\Pi_r|} = \pi_r^{|\Pi_r|}. \end{array} \right. \quad (11)$$

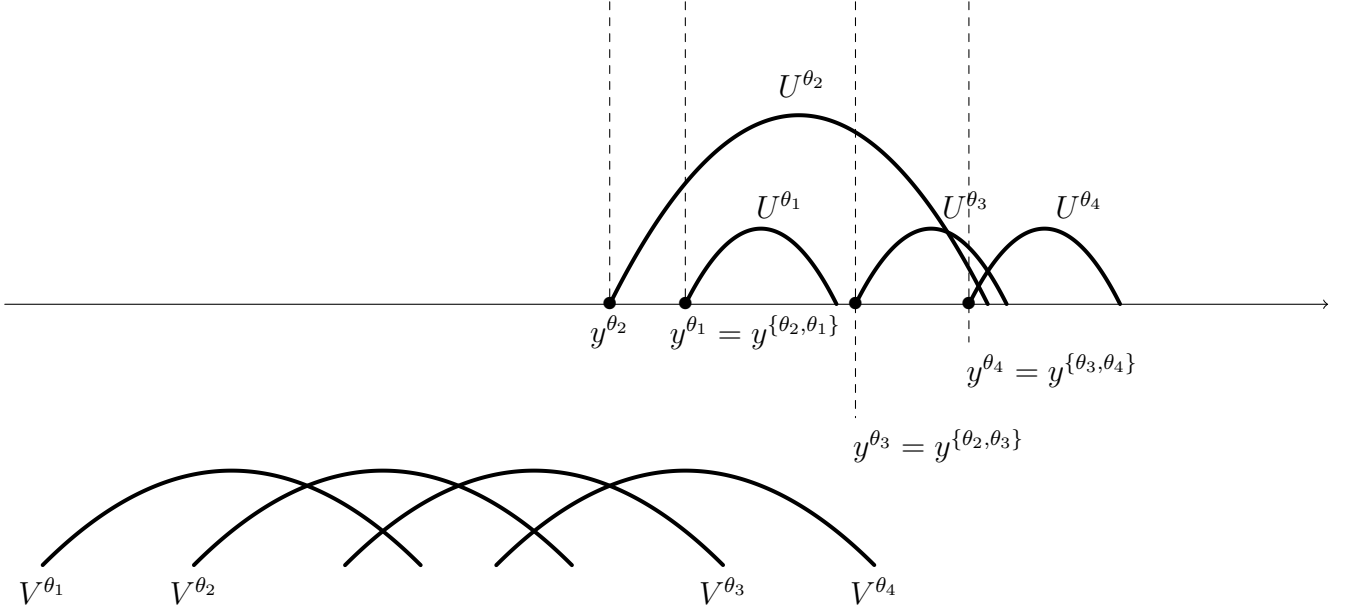


Figure 5: Illustration of Case B

Then Π_{r+1} satisfies (IR) by construction. Concerning (P1), we clearly have $y^{\pi_{r+1}^1} < \dots < y^{\pi_{r+1}^{n-1}}$ (if $n > 1$) and $y^{\pi_{r+1}^{n+2}} < \dots < y^{\pi_{r+1}^{|\Pi_{r+1}|}}$. From (9), we also have $y^{\pi_{r+1}^n} = y^{\pi^n \setminus \{\tilde{\theta}\}} < y^{\pi^{n+1} \cup \{\tilde{\theta}\}} = y^{\pi_{r+1}^{n+1}}$. Moreover, from (2) we have $y^{\pi_{r+1}^{n+1}} = y^{\pi^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi_r^{n+1}} < y^{\pi_r^{n+2}} = y^{\pi_{r+1}^{n+2}}$. Then Π_{r+1} satisfies (P1) whenever $y^{\pi_{r+1}^{n-1}} < y^{\pi_{r+1}^n}$, *i.e.* whenever $y^{\pi_r^{n-1}} < y^{\pi_r^n \setminus \{\tilde{\theta}\}}$, *i.e.* (10).

Now we show that it satisfies (P2). Let $n_1 \in \{1, \dots, |\Pi_{r+1}| - 1 = |\Pi_r| - 1\}$ and let $\theta \in \pi_{r+1}^{n_1}$. Suppose that there exists $n_2 \geq n_1 + 1$ and $\theta' \in \pi_{r+1}^{n_2}$ such that $y^{\theta'} < y^\theta$. We show that necessarily $y^\theta = \min X^\theta$.

- If $\theta \neq \tilde{\theta}$ and $\theta' \neq \tilde{\theta}$, then $\theta \in \pi_r^{n_1}$ and $\theta' \in \pi_r^{n_2}$ with $n_2 \geq n_1 + 1$, so that we have $y^\theta = \min X^\theta$ because Π_r satisfies (P2).
- If $\theta = \tilde{\theta}$, then $\theta \in \pi_r^n$. Moreover, $n_1 = n + 1$, and from $n_2 \geq n_1 + 1 = n + 2$, we have $\theta' \in \pi_{r+1}^{n_2} = \pi_r^{n_2}$. Since $n_2 \geq n + 2 > n$, and since Π_r satisfies (P2), we obtain $y^{\theta'} = \min X^{\theta'}$.
- If $\theta' = \tilde{\theta}$, then $\theta' \in \pi_r^n$. Moreover, $n_2 = n + 1$, and from $n_2 \geq n_1 + 1$, we obtain $n + 1 \geq n_1 + 1$, *i.e.* $n_1 \leq n$. From $\theta \neq \theta'$, we derive $\theta \in \pi_r^{n_1}$.
 - If $n_1 < n$, since Π_r satisfies (P2), we obtain $y^\theta = \min X^\theta$.
 - If $n_1 = n$, from $y^{\theta'} = y^{\tilde{\theta}} < y^\theta$ we have $\theta = \theta_{ij}$ for some $j > \ell$. In that case, Lemma 8 gives $y^\theta = \min X^\theta$.

Finally, we show that Π_{r+1} satisfies (L-IC). Let $n'_2 \in \{2, \dots, |\Pi_{r+1}|\}$, $n'_1 \in \{1, \dots, n'_2 - 1\}$

and $\theta \in \pi_{r+1}^{n'_2}$. We want to show that $y^{\pi_{r+1}^{n'_1}} \notin X^\theta(y^{\pi_{r+1}^{n'_2}})$. Set

$$y_1 = y^{\pi_{r+1}^{n'_1}}$$

and

$$y_2 = y^{\pi_{r+1}^{n'_2}}.$$

Then according to Lemma 10, the result derives from $y_1 < y_2$ (guaranteed because π_{r+1} satisfies (P1)), and the existence of $n_2 \in \{2, \dots, |\pi_r|\}$ and $n_1 \in \{1, \dots, n_2 - 1\}$ such that $y_1 \leq y^{\pi_r^{n_2}}$ and $y_2 \leq y^{\pi_r^{n_1}}$.

- If $n'_2 \leq n - 1$, then $\pi_{r+1}^{n'_2} = \pi_r^{n'_2}$, so that $y_2 = y^{\pi_r^{n'_2}}$, and from $n_1 \leq n'_2 - 1 \leq n - 2$, we also have $y_1 = y^{\pi_r^{n'_1}}$. This gives the result.
- If $n'_2 = n$, then $\pi_{r+1}^{n'_2} = \pi_r^{n'_2} \setminus \{\tilde{\theta}\}$, and according to (1), $y_2 \leq y^{\pi_r^{n'_2}}$. From $n_1 \leq n'_2 - 1 = n - 1$, we have $y_1 = y^{\pi_r^{n'_1}}$. This gives the result.
- If $n'_2 = n + 1$, then $\pi_{r+1}^{n'_2} = \pi_r^{n'_2} \cup \{\tilde{\theta}\}$, and according to (2), $y_2 \leq y^{\pi_r^{n'_2}}$.
 - If $n'_1 = n'_2 - 1 = n$, then $\pi_{r+1}^{n'_1} = \pi_r^{n'_1} \setminus \{\tilde{\theta}\}$, and according to (1), $y_1 \leq y^{\pi_r^{n'_1}}$. This gives the result.
 - If $n'_1 < n'_2 - 1 = n$, then $\pi_{r+1}^{n'_1} = \pi_r^{n'_1}$ and thus $y_1 = y^{\pi_r^{n'_1}}$, which also gives the result.
- If $n'_2 \geq n + 2$, then $\pi_{r+1}^{n'_2} = \pi_r^{n'_2}$ and thus $y_2 = y^{\pi_r^{n'_2}}$.
 - If $n'_1 \geq n + 2$ or $n'_1 \leq n - 1$, then $\pi_{r+1}^{n'_1} = \pi_r^{n'_1}$ and thus $y_1 = y^{\pi_r^{n'_1}}$, which gives the result.
 - If $n \leq n'_1 \leq n + 1$, inequality (1) (if $n'_1 = n$) or inequality (2) (if $n'_1 = n + 1$) gives $y_1 \leq y^{\pi_r^{n'_1}}$ and the result follows.

(B.2). Suppose

$$y^{\pi_r^{n-1}} \geq y^{\pi_r^n \setminus \{\tilde{\theta}\}}. \quad (12)$$

In that case, when type $\tilde{\theta}$ is removed from π_r^n , the n th action $y^{\pi_r^n}$ in partition Π_r moves to a rank which is possibly strictly lower than n . In that case, we merge $\pi_r^n \setminus \{\tilde{\theta}\}$ with π_r^n .

We first show that necessarily $\ell = |\pi_r^n|$. Indeed, if $\ell < |\pi_r^n|$, then from Lemma 8, $y^{\pi_r^n} = y^{\theta_i}_{|\pi_r^n|} = \min X^{\theta_i}_{|\pi_r^n|}$. Then from (1a), we have $y^{\pi_r^n \setminus \{\tilde{\theta}\}} = y^{\pi_r^n}$. Since $y^{\pi_r^{n-1}} < y^{\pi_r^n}$, inequality (12) cannot hold.

Next we show that $X^{\pi_r^{n-1}} \cap X^{\pi_r^n \setminus \{\tilde{\theta}\}} \neq \emptyset$. From (12) and $y^{\pi_r^n} > y^{\pi_r^{n-1}}$ we get

$$y^{\pi_r^n} > y^{\pi_r^{n-1}} \geq y^{\pi_r^n \setminus \{\tilde{\theta}\}}.$$

Since $X^{\pi_r^n \setminus \{\tilde{\theta}\}}$ is an interval, from $y^{\pi_r^n} \in X^{\pi_r^n} \subseteq X^{\pi_r^n \setminus \{\tilde{\theta}\}}$ and $y^{\pi_r^n \setminus \{\tilde{\theta}\}} \in X^{\pi_r^n \setminus \{\tilde{\theta}\}}$, we get $[y^{\pi^{[0]}}, y^{\pi_r^n}] \subseteq X^{\pi_r^n \setminus \{\tilde{\theta}\}}$. We obtain $y^{\pi_r^{n-1}} \in X^{\pi_r^n \setminus \{\tilde{\theta}\}}$. Since moreover $y^{\pi_r^{n-1}} \in X^{\pi_r^{n-1}}$, we obtain

$$y^{\pi_r^{n-1}} \in X^{\pi_r^{n-1}} \cap X^{\pi_r^n \setminus \{\tilde{\theta}\}} \neq \emptyset.$$

This allows us to set

$$\pi = \pi_r^{n-1} \cup \left(\pi_r^n \setminus \{\tilde{\theta}\} \right),$$

with π such that $X^\pi \neq \emptyset$.

Next we show $y^\pi = y^{\pi_r^{n-1}}$. Let $\underline{\theta} \in \pi_r^n \setminus \{\tilde{\theta}\}$ be such that for every $\theta \in \pi_r^n \setminus \{\tilde{\theta}\}$, $y^\theta \leq y^{\underline{\theta}}$.

Then from Lemma 4 and from (12),

$$y^{\underline{\theta}} \leq y^{\pi_r^n \setminus \{\tilde{\theta}\}} < y^{\pi_r^{n-1}}.$$

Now let $\bar{\theta} \in \pi_r^{n-1}$ be such that for every $\theta \in \pi_r^{n-1}$, $y^\theta \geq y^{\bar{\theta}}$. Then from Lemma 5,

$$y^{\bar{\theta}} \geq y^{\pi_r^{n-1}}.$$

Therefore $y^{\bar{\theta}} \geq y^{\underline{\theta}}$, where $\underline{\theta} \in \pi_r^n$ and $\bar{\theta} \in \pi_r^{n-1}$. Since partition Π_r satisfies (P2), we get

$$y^{\bar{\theta}} = \min X^{\bar{\theta}}.$$

Consequently, from $y^{\pi_r^{n-1}} \geq \min X^{\pi_r^{n-1}} \geq \min X^{\bar{\theta}}$, we have $y^{\pi_r^{n-1}} \geq y^{\bar{\theta}}$. Hence $y^{\bar{\theta}} \geq y^{\pi_r^{n-1}}$ and $y^{\pi_r^{n-1}} \geq y^{\bar{\theta}}$, and thus

$$y^{\pi_r^{n-1}} = y^{\bar{\theta}} = \min X^{\bar{\theta}}.$$

From Lemma 6 and $y^{\pi_r^{n-1}} \geq y^{\pi_r^n \setminus \{\tilde{\theta}\}}$, we have $y^{\pi_r^{n-1}} \geq y^\pi \geq y^{\pi_r^n \setminus \{\tilde{\theta}\}}$, and in particular $y^\pi \leq \min X^{\bar{\theta}}$. But $\bar{\theta} \in \pi$, and necessarily $y^\pi \geq X^{\bar{\theta}}$. Thus $y^\pi = \min X^{\bar{\theta}} = y^{\pi_r^{n-1}}$.

We set:

$$\left\{ \begin{array}{l} \pi_{r+1}^1 = \pi_r^1, \dots, \pi_{r+1}^{n-2} = \pi_r^{n-2}, \\ \pi_{r+1}^{n-1} = \pi_r^{n-1} \cup \left(\pi_r^n \setminus \{\tilde{\theta}\} \right), \\ \pi_{r+1}^n = \pi_r^{n+1} \cup \{\tilde{\theta}\}, \\ \pi_{r+1}^{n+1} = \pi_r^{n+2}, \dots, \pi_{r+1}^{|\Pi_r|-1} = \pi_r^{|\Pi_r|}. \end{array} \right. \quad (13)$$

By construction, Π_{r+1} satisfies (IR). It satisfies (P1) if $y^{\pi_r^1} < \dots < y^{\pi_r^{n-2}}$ and $\pi_r^{n+2} < \dots < \pi_r^{|\Pi_r|}$, which hold by assumption (P1) on Π_r , if moreover $y^{\pi_r^{n-2}} < y^{\pi_r^{n-1} \cup (\pi_r^n \setminus \{\tilde{\theta}\})} = y^\pi = y^{\pi_r^{n-1}}$, which also holds by assumption (P1) on Π_r , and if moreover

$$y^{\pi_r^{n-1} \cup (\pi_r^n \setminus \{\tilde{\theta}\})} = y^{\pi_r^{n-1}} < y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}, \quad (14)$$

and

$$y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} < y^{\pi_r^{n+2}}. \quad (15)$$

Inequality (14) results from $y^{\pi_r^{n-1}} < y^{\pi_r^n}$, and, from $\ell = |\pi_r^n|$, inequality (3), that is: $y^{\pi_r^n} < y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}$. Inequality (15) results from $y^{\pi_r^{n+1}} < y^{\pi_r^{n+2}}$ and from the second inequality in (2), that is: $y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} < y^{\pi_r^{n+1}}$.

Now we show that Π_{r+1} satisfies (P2). Note that $|\Pi_{r+1}| = |\Pi_r| - m$. Let $n_1 \in \{1, \dots, |\Pi_r| - m - 1\}$ and let $\theta \in \Pi_{r+1}^{n_1}$. Suppose that there exists $n_2 \geq n_1 + 1$ and $\theta' \in \pi_{r+1}^{n_2}$ such that $y^{\theta'} < y^\theta$. We show that necessarily $y^\theta = \min X^\theta$.

- If $n_2 < n - m$ (which occurs only in case (i)), then $n_1 \leq n_2 - 1 < n - m - 1$ and then $\theta \in \pi_r^{n_1}$ and $\theta' \in \pi_r^{n_2}$. Since Π_r satisfies (P2), $y^{\theta'} < y^\theta$ implies $y^\theta = \min X^\theta$.
- If $n_2 = n - m$, then $n_1 \leq n_2 - 1 = n - m - 1$ (which occurs only in case (i)), and then the same argument holds.
- If $n_2 = n - (m - 1)$, then $\theta' \in \pi_r^{n+1}$ or $\theta' = \tilde{\theta} \in \pi_r^n$. Moreover, either $n_1 \leq n_2 - 2 = n - (m + 1)$ (which occurs only in case (i)), or $n_1 = n_2 - 1 = n - m$.
 - If $n_1 \leq n - (m + 1)$ then $\theta \in \pi_r^{n_1}$ with $n_1 < n + 1$ and $n_1 < n$, and the same argument as above holds.
 - If $n_1 = n - m$, then

$$\theta \in \pi[m] = \pi[m - 1] \cup \pi_r^{n-m} = \dots = \pi_r^{n-m} \cup \dots \cup \pi_r^{n-1} \cup \left(\pi_r^n \setminus \{\tilde{\theta}\} \right).$$

In particular, either $\theta \in \pi_r^{n'_1}$ with $n'_1 \leq n - 1 < n_2$, or $\theta \in \pi_r^n \setminus \{\tilde{\theta}\}$. If $\theta \in \pi_r^{n'_1}$ with $n'_1 \leq n - 1 < n_2$ then we are done. If $\theta \in \pi_r^n \setminus \{\tilde{\theta}\}$ and $\theta' \in \pi_r^{n+1}$ then we are done too. It remains the case in which $\theta \in \pi_r^n \setminus \{\tilde{\theta}\}$ and $\theta' = \tilde{\theta} \in \pi_r^n$. In that case, $y^{\theta'} < y^\theta$ guarantees that $\theta = \theta_{i_j}$ for some $j > \ell$, and Lemma 8 gives the result.

Finally, let us show that Π_{r+1} satisfies (L-IC). Let $n'_2 \in \{2, \dots, |\Pi_{r+1}|\}$, $n'_1 \in \{1, \dots, n'_2 - 1\}$ and $\theta \in \pi_{r+1}^{n'_2}$. We want to show that $y^{\pi_{r+1}^{n'_1}} \notin X^\theta(y^{\pi_{r+1}^{n'_2}})$. As above, we set $y_1 = y^{\pi_{r+1}^{n'_1}}$ and

$y_2 = y^{\pi_{r+1}^{n'_2}}$. Then according to Lemma 10, the result derives from $y_1 < y_2$ (guaranteed because Π_{r+1} satisfies (P1)), and the existence of $n_2 \in \{2, \dots, |\Pi_r|\}$ and $n_1 \in \{1, \dots, n_2 - 1\}$ such that $y_1 \leq y^{\pi_r^{n_2}}$ and $y_2 \leq y^{\pi_r^{n_1}}$.

- If $n'_2 \leq n - (m + 1)$ (which occurs only in case (i)), then $\pi_{r+1}^{n'_2} = \pi_r^{n'_2}$, so that $y_2 = y^{\pi_r^{n'_2}}$, and from $n_1 \leq n'_2 - 1 \leq n - (m + 2)$, we also have $y_1 = y^{\pi_r^{n'_1}}$. This gives the result.
- If $n'_2 = n - m$, then $\pi_{r+1}^{n'_2} = \pi[m]$ and property $Q[m]$ gives $y^{\pi[m]} \leq y^{\pi_r^{n-m}}$. From $n'_1 \leq n'_2 - 1 = n - m - 1$ (which occurs only in case (i)), we have $y_1 = y^{\pi_r^{n'_1}}$. This gives the result.
- If $n'_2 = n - (m - 1)$, then $\pi_{r+1}^{n'_2} = \pi_r^{n+1} \cup \{\tilde{\theta}\}$, and according to (2), $y_2 \leq y^{\pi_r^{n+1}}$.
 - If $n'_1 = n'_2 - 1 = n - m$, then $\pi_{r+1}^{n'_1} = \pi[m]$, and from $Q[m]$, $y_1 \leq y^{\pi_r^{n-m}}$. This gives the result.
 - If $n_1 < n'_2 - 1 = n - m$, then $\pi_{r+1}^{n'_1} = \pi_r^{n'_1}$ and thus $y_1 = y^{\pi_r^{n'_1}}$, which also gives the result.
- If $n'_2 \geq n - (m - 2)$, then $\pi_{r+1}^{n'_2} = \pi_r^{n'_2+m}$ and thus $y_2 = y^{\pi_r^{n'_2+m}}$.
 - If $n'_1 \geq n - (m - 2)$ or $n'_1 \leq n - (m + 1)$, then $\pi_{r+1}^{n'_1} = \pi_r^{n'_1}$ or $\pi_{r+1}^{n'_1} = \pi_r^{n'_1+m}$ respectively, and the result follows.
 - If $n - m \leq n'_1 \leq n - (m - 1)$, $Q[m]$ (if $n'_1 = n - m$) or inequality (2) (if $n'_1 = n - (m - 1)$) gives the result.

(C). Suppose $\pi_r^n \setminus \{\tilde{\theta}\} = \emptyset$. In that case, $\pi_r^n = \{\tilde{\theta}\}$. This case occurs for instance at step $r = 0$ to $r = 1$ (but it may also occurs later). Set

$$\begin{cases} \pi_{r+1}^1 = \pi_r^1, \dots, \pi_{r+1}^{n-1} = \pi_r^{n-1}, \\ \pi_{r+1}^n = \pi_r^{n+1} \cup \{\tilde{\theta}\}, \\ \pi_{r+1}^{n+1} = \pi_r^{n+2}, \dots, \pi_{r+1}^{|\pi_r|-1} = \pi_r^{|\pi_r|}. \end{cases} \quad (16)$$

By construction, Π_{r+1} satisfies (IR). From $\pi_r^n = \{\tilde{\theta}\}$, $y^{\pi_r^n} = y^{\tilde{\theta}}$. From $y^{\pi_r^n} < y^{\pi_r^{n+1}}$, we get $y^{\tilde{\theta}} < y^{\pi_r^{n+1}}$ and thus, from Lemma 6, $y^{\tilde{\theta}} \leq y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi_r^{n+1}}$, that is:

$$y^{\pi_r^n} \leq y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} < y^{\pi_r^{n+1}}.$$

Then $y^{\pi_r^{n-1}} < y^{\pi_r^n}$ and $y^{\pi_r^{n+1}} < y^{\pi_r^{n+2}}$ imply

$$y^{\pi_r^{n-1}} < y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} < y^{\pi_r^{n+2}}.$$

Consequently, Π_{r+1} satisfies (P1).

Now let us show that Π_{r+1} satisfies (P2). Let $n_1 \in \{1, \dots, |\pi_{r+1}| = |\pi_r| - 1\}$ and let $\theta \in \pi_{r+1}^{n_1}$. Suppose that there exists $n_2 \geq n_1 + 1$ and $\theta' \in \pi_{r+1}^{n_2}$ such that $y^{\theta'} < y^\theta$. We show that necessarily $y^\theta = \min X^\theta$.

- If $n_2 \leq n - 1$, then $n_1 \leq n - 2$ and $\theta \in \pi_r^{n_1}$, $\theta' \in \pi_r^{n_2}$ with $n_2 \geq n_1 + 1$, so that we have $y^\theta = \min X^\theta$ because Π_r satisfies (P2).
- If $n_2 = n$, then either $\theta' \in \pi_r^{n+1}$ or $\theta' = \tilde{\theta} \in \pi_r^n$. Moreover, $n_1 \leq n_2 - 1 = n - 1$ implies $\theta \in \pi_r^{n_1}$ with $n_1 < n$. This gives the result.
- If $n_2 \geq n + 1$, then $\theta' \in \pi_r^{n_2+1}$.
 - If $n_1 \geq n + 1$ too, then $\theta \in \pi_r^{n_1+1}$ and the result follows.
 - If $n_1 \leq n$, then either $\theta \in \pi_r^{n+1}$, or $\theta = \tilde{\theta} \in \pi_r^n$ or $\theta \in \pi_r^{n_1}$. The result is obtained in all three cases.

Finally, we show that Π_{r+1} satisfies (L-IC). Let $n'_2 \in \{2, \dots, |\pi_{r+1}|\}$, $n'_1 \in \{1, \dots, n'_2 - 1\}$ and $\theta \in \pi_{r+1}^{n'_2}$. We want to show that $y^{\pi_{r+1}^{n'_1}} \notin X^\theta(y^{\pi_{r+1}^{n'_2}})$. Set $y_1 = y^{\pi_{r+1}^{n'_1}}$ and $y_2 = y^{\pi_{r+1}^{n'_2}}$ so that according to Lemma 10, the result derives from $y_1 < y_2$ (guaranteed because π_{r+1} satisfies (P1)), and the existence of $n_2 \in \{2, \dots, |\pi_r|\}$ and $n_1 \in \{1, \dots, n_2 - 1\}$ such that $y_1 \leq y^{\pi_r^{n_1}}$ and $y_2 \leq y^{\pi_r^{n_2}}$.

- If $n'_2 \leq n - 1$, then $\pi_{r+1}^{n'_2} = \pi_r^{n'_2}$, so that $y_2 = y^{\pi_r^{n'_2}}$, and from $n_1 \leq n'_2 - 1 \leq n - 2$, we also have $y_1 = y^{\pi_r^{n'_1}}$. This gives the result.
- If $n'_2 = n$, then $\pi_{r+1}^{n'_2} = \pi_r^{n'_2} \cup \{\tilde{\theta}\}$, and according to (2), $y_2 \leq y^{\pi_r^{n'_2}}$. From $n'_1 \leq n'_2 - 1 = n - 1$, we have $y_1 = y^{\pi_r^{n'_1}}$, which gives the result.
- If $n'_2 \geq n + 1$, then $\pi_{r+1}^{n'_2} = \pi_r^{n'_2+1}$ and thus $y_2 = y^{\pi_r^{n'_2+1}}$ with $n'_2 + 1 \geq n + 2$.
 - If $n'_1 \geq n + 1$ or $n'_1 \leq n - 1$, then $\pi_{r+1}^{n'_1} = \pi_r^{n'_1}$ and thus $y_1 = y^{\pi_r^{n'_1}}$, which gives the result.
 - If $n'_1 = n$, inequality (2) gives $y_1 \leq y^{\pi_r^{n+1}}$ and the result follows.

6 Concluding remarks

We have established the existence of a perfect Bayesian equilibrium without exit in a class of games of information transmission with sender’s approval. To do so, we have proposed a step by step algorithm which achieves an IR and IC partition of the sender’s type. In the equilibrium associated with the partition, the sender reveals the cell containing his type and the receiver makes the optimal decision subject to participation, given the cell that is reported.

The algorithm starts by ordering the types so that no type envies a lower one (L-IC). The initial partition of singletons, which is trivially IR, is gradually modified into a not finer partition satisfying L-IC, as long as some type envies a higher one. Our construction is applicable in every game of the class under consideration. As noted above, our algorithm allows for some flexibility, at every step r , to pick the cell $\pi^n \in \Pi_r$ containing an envying type; then, a specific envying type $\tilde{\theta}$ is selected, namely, the maximal element of π^n and $\tilde{\theta}$ is moved to the next cell π^{n+1} . As a consequence, it can happen that the algorithm has several distinct runs.

In any case, our algorithm guarantees that the sender-receiver game with sender’s exit option has a partitional equilibrium without exit. Since, in our model, exit gives an infinitely low payoff to the receiver, our existence result implies that the problem of maximizing the equilibrium payoff of the receiver (interpreted as a principal) has a finite solution. A natural topic for future research would be to identify conditions guaranteeing that the receiver’s best equilibrium is partitional, as may happen in models like the one we consider, with finitely many types for the sender, when participation constraints do not matter (see Frug (2016)). Under such conditions, one might hope that (variants of) our algorithm converge to this best partitional equilibrium.

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A Appendix : proof of the lemmas

Lemma 1. For every $\theta \in \Theta$, $X^\theta = [\min X^\theta, \max X^\theta]$.

Proof. The proof is elementary analysis. Let us give the arguments. Function U^θ is concave with respect to x , with a maximum value at $x = x^*(\theta)$, where it is strictly positive. Since $U^\theta(x^*(\theta)) > 0$, there exists $\tilde{x} < x^*(\theta)$ such that $U^\theta(\tilde{x}) > 0$. By concavity, on $(-\infty, \tilde{x}]$, the graph of $x \mapsto U^\theta(x)$ is below its tangent line at \tilde{x} . Since $x \mapsto U^\theta(x)$ is increasing on $(-\infty, x^*(\theta))$, the tangent line has a negative slope. Therefore $U^\theta(x) < 0$ if x is sufficiently small. Since U^θ is strictly positive at \tilde{x} , continuous and strictly increasing on $(-\infty, \tilde{x}]$, the Intermediate Value Theorem ensures the existence of a unique $\underline{x}(\theta) < \tilde{x}$ at which $U^\theta(x)$ vanishes. Then $U^\theta(x) < 0$ on $(-\infty, \underline{x}(\theta))$, and $U^\theta(x) > 0$ on $(\underline{x}(\theta), x^*(\theta))$. Similarly, we can define $\bar{x}(\theta)$ that annihilates U^θ and that is such that $U^\theta(x) > 0$ on $(x^*(\theta), \bar{x}(\theta))$, and $U^\theta(x) < 0$ on $(\bar{x}(\theta), +\infty)$. Then $\min X^\theta = \underline{x}(\theta)$, $\max X^\theta = \bar{x}(\theta)$ and $X^\theta = [\min X^\theta, \max X^\theta]$. \square

Lemma 2. For every $\theta \in \Theta$, $y^\theta < x^*(\theta) < \max X^\theta$. In particular, U^θ is increasing at $x = y^\theta$.

Proof. Let $\theta \in \Theta$. From $y^*(\theta) < x^*(\theta)$, function $x \mapsto V^\theta(x)$ decreases on $[x^*(\theta) - \varepsilon, \max X^\theta] \subset X^\theta$ for some $\varepsilon > 0$. Therefore its maximal argument y^θ on X^θ is lower than $x^*(\theta)$. \square

Lemma 3. For every $\theta \in \Theta$, $y^*(\theta) \leq y^\theta$, with equality iff $y^*(\theta) \geq \min X^\theta$. In particular, V^θ is not increasing at $x = y^\theta$.

Proof. Let $\theta \in \Theta$. Function $x \mapsto V^\theta(x)$ increases on $(-\infty, y^*(\theta))$ and decreases on $(y^*(\theta), +\infty)$. From Lemma 2, V^θ is not increasing at $x = y^\theta$. Therefore $y^*(\theta) \leq y^\theta$.

Now we show that $y^*(\theta) = y^\theta$ iff $y^*(\theta) \geq \min X^\theta$. If $y^*(\theta) < \min X^\theta$, then V^θ decreases on X^θ , so that its maximal argument on X^θ , that is y^θ , is $\min X^\theta$, and thus $y^*(\theta) < y^\theta$. This shows that $y^*(\theta) \geq y^\theta$ (and in particular, $y^*(\theta) = y^\theta$) implies $y^*(\theta) \geq \min X^\theta$. Conversely, if $y^*(\theta) \geq \min X^\theta$, then from $x^*(\theta) < \max X^\theta$ and $y^*(\theta) < x^*(\theta)$ we obtain $y^*(\theta) \in X^\theta$. Since $y^*(\theta)$ maximizes V^θ on \mathbb{R} , it maximizes V^θ on X^θ and therefore $y^*(\theta) = y^\theta$. \square

Lemma 4. Let $L \subseteq \Theta$ be such that $X^L \neq \emptyset$. Let $\theta \in L$ be such that for every $\theta' \in L$, $y^\theta \leq y^{\theta'}$. Then $y^\theta \leq y^L$.

Proof. We distinguish the four cases: (i) $y^\theta = \min X^\theta$; (ii) $y^\theta > \min X^\theta$ and $y^L = \min X^L$; (iii) $y^\theta > \min X^\theta$ and $y^L = \max X^L$; (iv) $y^\theta > \min X^\theta$ and $\min X^L < y^L < \max X^L$.

(i) $y^\theta = \min X^\theta$. Then $y^L \geq \min X^L = \max\{\min X^{\theta'}, \theta' \in L\} \geq \min X^\theta = y^\theta$.

(ii) $y^\theta > \min X^\theta$ and $y^L = \min X^L$. Let θ' be such that $\min X^L = \min X^{\theta'}$. Then $y^{\theta'} = \min X^{\theta'}$. Thus $y^L = y^{\theta'} \geq y^\theta$.

(iii) $y^\theta > \min X^\theta$ and $y^L = \max X^L$. Let θ' be such that $\max X^L = \max X^{\theta'}$. Then $y^L = \max X^L = \max X^{\theta'} > y^{\theta'} \geq y^\theta$,

(iv) $y^\theta > \min X^\theta$ and $\min X^L < y^L < \max X^L$. If the derivative of V^L is positive at y^θ , then the result follows. Otherwise, there is some $\theta' \in L$ such that $V^{\theta'}$ is decreasing at $x = y^\theta$. Then $V^{\theta'}$ is also decreasing at $x = y^{\theta'} \geq y^\theta$. This implies $y^{\theta'} = \min X^{\theta'}$. Then $y^L > \min X^L \geq \min X^{\theta'} = y^{\theta'} \geq y^\theta$. \square

Lemma 5. Let $L \subseteq \Theta$ be such that $X^L \neq \emptyset$. Let $\theta \in L$ be such that for every $\theta' \in L$, $y^\theta \geq y^{\theta'}$. Then $y^\theta \geq y^L$.

Proof. We distinguish the two cases: (i) $y^L = \min X^L$; and (ii) $y^L > \min X^L$.

(i) $y^L = \min X^L$. Let θ' be such that $\min X^L = \min X^{\theta'}$. Then $y^{\theta'} \geq \min X^{\theta'} = \min X^L = y^L$. The result follows from $y^\theta > y^{\theta'}$.

(ii) $y^L > \min X^L$. Function V^L vanishes at y^L . For every $\theta' \in L$, $V^{\theta'}$ is decreasing on the right side of $x = y^{\theta'}$. Thus it decreases at $x = y^{\theta'}$. Therefore V^L decreases at $x = y^{\theta'}$. Hence $y^{\theta'} > y^L$. \square

Lemma 6. *Let $L_1, L_2 \subseteq \Theta$ be such that $X^{L_1 \cup L_2} \neq \emptyset$. If $y^{L_1} \geq y^{L_2}$ then $y^{L_1} \geq y^{L_1 \cup L_2} \geq y^{L_2}$.*

Proof. First we show $y^{L_1 \cup L_2} \geq y^{L_2}$ by distinguishing the cases (i) $\min X^{L_1} \geq y^{L_2}$; (ii) $\min X^{L_1} < y^{L_2}$ and $y^{L_2} = \min X^{L_2}$; and (iii) $\min X^{L_1} < y^{L_2}$ and $y^{L_2} > \min X^{L_2}$.

(i) $\min X^{L_1} \geq y^{L_2}$. From $\min X^{L_1} \geq y^{L_2}$ and $y^{L_2} \geq \min X^{L_2} \geq \min X^{L_1 \cup L_2}$, we obtain $\min X^{L_1} \geq \min X^{L_1 \cup L_2}$. But since $X^{L_1 \cup L_2} \subseteq X^{L_1}$, we have $\min X^{L_1} \leq \min X^{L_1 \cup L_2}$. Therefore $\min X^{L_1 \cup L_2} = \min X^{L_1}$, and then $\min X^{L_1 \cup L_2} \geq y^{L_2}$. The result follows from $y^{L_1 \cup L_2} \geq \min X^{L_1 \cup L_2}$.

(ii) $\min X^{L_1} < y^{L_2}$ and $y^{L_2} = \min X^{L_2}$. From $\min X^{L_1} < y^{L_2} = \min X^{L_2}$, we obtain $\min X^{L_1} < \min X^{L_2}$. Then $\min X^{L_1 \cup L_2} = \max\{\min X^{L_1}, \min X^{L_2}\} = \min X^{L_2}$. The result follows from $y^{L_1 \cup L_2} \geq \min X^{L_1 \cup L_2} = \min X^{L_2}$ and assumption $y^{L_2} = \min X^{L_2}$.

(iii) $\min X^{L_1} < y^{L_2}$ and $y^{L_2} > \min X^{L_2}$. From $y^{L_1} \geq y^{L_2}$, we have that V^{L_1} is non decreasing at $x = y^{L_2}$. From $y^{L_2} > \min X^{L_2}$, so is V^{L_2} , and thus so is $V^{L_1 \cup L_2} = V_1^L + V^{L_2}$. This implies $y^{L_1 \cup L_2} \geq y^{L_2}$.

Next we show $y^{L_1 \cup L_2} \leq y^{L_1}$ by distinguishing the cases (i) $\max X^{L_2} \leq y^{L_1}$; (ii) $\max X^{L_2} > y^{L_1}$ and $y^{L_1} = \max X^{L_1}$; and (iii) $\max X^{L_2} > y^{L_1}$ and $y^{L_1} < \max X^{L_1}$.

(i) $\max X^{L_2} \leq y^{L_1}$. From $\max X^{L_2} \leq y^{L_1}$ and $y^{L_1} \leq \max X^{L_1}$, we obtain $\max X^{L_2} \leq \max X^{L_1}$. Therefore $\max X^{L_1 \cup L_2} = \min\{\max X^{L_1}, \max X^{L_2}\} = \max X^{L_2}$. Then the result follows from $y^{L_1 \cup L_2} \leq \max X^{L_1 \cup L_2} = \max X^{L_2}$ and assumption $\max X^{L_2} \leq y^{L_1}$.

(ii) $\max X^{L_2} > y^{L_1}$ and $y^{L_1} = \max X^{L_1}$. From $\max X^{L_2} > y^{L_1} = \max X^{L_1}$, we obtain $\max X^{L_1 \cup L_2} = \min\{\max X^{L_1}, \max X^{L_2}\} = \max X^{L_1}$. Then the result follows from $y^{L_1 \cup L_2} \leq \max X^{L_1 \cup L_2} = \max X^{L_1}$ and assumption $\max X^{L_1} = y^{L_1}$.

(iii) $\max X^{L_2} > y^L$ and $y^{L_1} < \max X^{L_1}$. From $y^{L_2} \leq y^{L_1}$ and $\max X^{L_2} > y^{L_1}$, we have $y^{L_1} \in [y^{L_2}; \max X^{L_2})$. Since y^{L_2} is the maximal argument of V^{L_2} on X^{L_2} , it is also the maximal argument of V^{L_2} on $[y^{L_2}; \max X^{L_2}) \subset X^{L_2}$. Thus from $y^{L_1} \in [y^{L_2}; \max X^{L_2})$, we have

$V^{L_2}(y^{L_2}) \geq V^{L_2}(y^{L_1})$. From that, we get that V^{L_2} is non increasing on $[y^{L_2}, +\infty)$. In particular, it is non increasing at $x = y^{L_1} \geq y^{L_2}$. Therefore $V^{L_1 \cup L_2} = V^{L_1} + V^{L_2}$ is non increasing at $x = y^{L_1}$. This implies $y^{L_1 \cup L_2} \leq y^{L_1}$. \square

Lemma 7. *Let $L, L' \subset \Theta$ with $X^L \neq \emptyset$, $X^{L'} \neq \emptyset$, and suppose that there exists $\theta \in L'$ with $\theta \notin L$. If $U^\theta(y^L) > U^\theta(y^{L'})$, then $X^{L \cup \{\theta\}} \neq \emptyset$.*

Proof. By definition, $y^{L'} \in X^{L'}$, and since $X^{L'} = X^{L' \setminus \{\theta\}} \cap X^\theta \subseteq X^\theta$, we have $y^{L'} \in X^\theta$. Therefore $U^\theta(y^{L'}) \geq 0$, so that $U^\theta(y^L) > U^\theta(y^{L'})$ implies $U^\theta(y^L) > 0$. Hence $y^L \in X^\theta$. Since $y^L \in X^L$, we obtain $y^L \in X^L \cap X^\theta = X^{L \cup \{\theta\}}$, and in particular $X^{L \cup \{\theta\}} \neq \emptyset$. \square

Lemma 8. *Either $\ell = |\pi^n|$ or, if $\ell < |\pi^n|$, then for every $j \in \{\ell + 1, \dots, |\pi^n|\}$, $y^{\theta_{i_j}} = \min X^{\theta_{i_j}}$. In that latter case, $y^{\pi^n} = y^{\theta_{i_{|\pi^n|}}} = \min X^{\theta_{i_{|\pi^n|}}}$.*

Proof. If $\ell \neq |\pi^n|$, let $j \in \{\ell + 1, \dots, |\pi^n|\} \neq \emptyset$. First, we show

$$y^{\pi^{n+1}} > x_+^{\theta_{i_j}}(y^{\pi^n}). \quad (17)$$

Since $j > \ell$, $y^{\pi^{n+1}} \notin X^{\theta_{i_j}}(y^{\pi^n})$. This means that either (17) holds, or $y^{\pi^{n+1}} < x_-^{\theta_{i_j}}(y^{\pi^n})$. However this latter inequality cannot occur. Indeed, from (P1), we have $y^{\pi^{n+1}} > y^{\pi^n}$. And from $y^{\pi^n} \in X^{\theta_{i_j}}(y^{\pi^n})$, we have $y^{\pi^n} \geq x_-^{\theta_{i_j}}(y^{\pi^n})$. We obtain $y^{\pi^{n+1}} \geq x_-^{\theta_{i_j}}(y^{\pi^n})$.

Second, we show

$$y^{\pi^n} = x_-^{\theta_{i_\ell}}(y^{\pi^n}). \quad (18)$$

If (18) does not hold, then $y^{\pi^n} = x_+^{\theta_{i_\ell}}(y^{\pi^n})$. If this holds, then from $y^{\pi^{n+1}} \in X^{\theta_{i_\ell}}(y^{\pi^n})$, we obtain $y^{\pi^{n+1}} \leq x_+^{\theta_{i_\ell}}(y^{\pi^n}) = y^{\pi^n}$. But from (P1), we have $y^{\pi^n} < y^{\pi^{n+1}}$.

Third, we show

$$\theta_{i_j} < \theta_{i_\ell}. \quad (19)$$

From (17), $x_+^{\theta_{i_j}}(y^{\pi^n}) < y^{\pi^{n+1}}$ and from $y^{\pi^{n+1}} \in X^{\theta_{i_\ell}}(y^{\pi^n})$, we have $y^{\pi^{n+1}} \leq x_+^{\theta_{i_\ell}}(y^{\pi^n})$. Then we obtain $x_+^{\theta_{i_\ell}}(y^{\pi^n}) > x_+^{\theta_{i_j}}(y^{\pi^n})$. This implies $x_+^{\theta_{i_\ell}}(y^{\pi^n}) \notin X^{\theta_{i_j}}(y^{\pi^n})$. Since $y^{\pi^n} \in X^{\theta_{i_j}}(y^{\pi^n})$, we get

$$U^{\theta_{i_j}}(x_+^{\theta_{i_\ell}}(y^{\pi^n})) - U^{\theta_{i_j}}(y^{\pi^n}) < 0. \quad (20)$$

From (18), $U^{\theta_{i_\ell}}(x_-^{\theta_{i_\ell}}(y^{\pi^n})) - U^{\theta_{i_\ell}}(y^{\pi^n}) = 0$, and since $U^{\theta_{i_\ell}}(x_+^{\theta_{i_\ell}}(y^{\pi^n})) = U^{\theta_{i_\ell}}(x_-^{\theta_{i_\ell}}(y^{\pi^n}))$, we get

$$U^{\theta_{i_\ell}}(x_+^{\theta_{i_\ell}}(y^{\pi^n})) - U^{\theta_{i_\ell}}(y^{\pi^n}) = 0.$$

Therefore, according to (20),

$$U^{\theta_{i_j}}(x_+^{\theta_{i_\ell}}(y^{\pi^n})) - U^{\theta_{i_j}}(y^{\pi^n}) < U^{\theta_{i_\ell}}(x_+^{\theta_{i_\ell}}(y^{\pi^n})) - U^{\theta_{i_\ell}}(y^{\pi^n}). \quad (21)$$

Now note that $x_+^{\theta_{i_\ell}}(y^{\pi^n}) > y^{\pi^n}$ so that from the single-crossing condition (A1), (21) holds only if (19) holds.

Fourth, we show

$$y^{\theta_{i_j}} = \min X^{\theta_{i_j}}. \quad (22)$$

Inequality (19) implies that either $y^{\theta_{i_j}} = \min X^{\theta_{i_j}}$ or $y^{\theta_{i_\ell}} = \min X^{\theta_{i_\ell}}$, since otherwise, $y^{\theta_{i_j}} > \min X^{\theta_{i_j}}$ and $y^{\theta_{i_\ell}} > \min X^{\theta_{i_\ell}}$, which implies, $y^*(\theta_{i_j}) = y^{\theta_{i_j}}$ and $y^*(\theta_{i_\ell}) = y^{\theta_{i_\ell}}$. But in that case, from the single-crossing assumption on $V^\theta(x)$, $y^*(\theta_{i_j}) < y^*(\theta_{i_\ell})$ which gives $y^{\theta_{i_j}} < y^{\theta_{i_\ell}}$, *i.e.* $j < \ell$, a contradiction. Hence if $y^{\theta_{i_\ell}} > \min X^{\theta_{i_\ell}}$ then necessarily (22) holds. It remains to show that $y^{\theta_{i_\ell}} = \min X^{\theta_{i_\ell}}$ also implies (22). If $y^{\theta_{i_\ell}} = \min X^{\theta_{i_\ell}}$ then $y^*(\theta_{i_\ell}) \leq \min X^{\theta_{i_\ell}}$. From (19) and the single-crossing condition, $y^*(\theta_{i_j}) < y^*(\theta_{i_\ell})$ and thus $y^*(\theta_{i_j}) < \min X^{\theta_{i_\ell}} = y^{\theta_{i_\ell}}$. Then from $y^{\theta_{i_\ell}} \leq y^{\theta_{i_j}}$ we obtain $y^*(\theta_{i_j}) < y^{\theta_{i_j}}$, which precisely implies (22).

Finally, consider $j = |\pi^n|$. Since for every $\theta \in \pi^n$, $y^\theta \leq y^{\theta_{i_j}}$, according to Lemma 5, we have, on the one hand,

$$y^{\pi^n} \leq y^{\theta_{i_j}}.$$

On the other hand, from $y^{\pi^n} \in \bigcap_{\theta \in \pi^n} X^\theta$, we also have $y^{\pi^n} \geq \min X^{\theta_{i_j}}$. That is, according to (22), $y^{\pi^n} \geq y^{\theta_{i_j}}$. Hence we obtain

$$y^{\pi^n} = y^{\theta_{i_{|\pi^n|}}} = \min X^{\theta_{i_{|\pi^n|}}}.$$

□

Lemma 9. • If $\pi^n \setminus \{\tilde{\theta}\} \neq \emptyset$, then

$$y^{\pi^n \setminus \{\tilde{\theta}\}} \leq y^{\pi^n}, \quad (1)$$

and, if moreover $\ell < |\pi^n|$,

$$y^{\pi^n \setminus \{\tilde{\theta}\}} = y^{\pi^n}. \quad (1a)$$

•

$$y^{\tilde{\theta}} \leq y^{\pi^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi^{n+1}} \quad (2)$$

• If $\ell = |\pi^n|$, then

$$y^{\pi^n} \leq y^{\pi^{n+1} \cup \{\tilde{\theta}\}}. \quad (3)$$

Proof of (1) and (1a). According to Lemma 8, either $\ell = |\pi^n|$, or $[\ell < |\pi^n|$ and $y^{\pi^n} = \min X^{\theta_{i|\pi^n}} = y^{\theta_{i|\pi^n}}$]. If $\ell = |\pi^n|$, then for every $\theta \in \pi^n$, $y^\theta \leq y^{\theta_{i\ell}}$. According to Lemma 6, this implies

$$y^{\pi^n \setminus \{\theta_{i\ell}\}} \leq y^{(\pi^n \setminus \{\theta_{i\ell}\}) \cup \{\theta_{i\ell}\}} = y^{\pi^n}.$$

If instead $\ell < |\pi^n|$ and $y^{\pi^n} = \min X^{\theta_{i|\pi^n}} = y^{\theta_{i|\pi^n}}$, then from $\theta_{i|\pi^n} \in \pi^n \setminus \{\theta_{i\ell}\}$, we have $y^{\pi^n \setminus \{\theta_{i\ell}\}} \in X^{\theta_{i|\pi^n}} \geq \min X^{\theta_{i|\pi^n}} = y^{\pi^n}$. Moreover, since for every $\theta \in \pi^n$, $y^\theta \leq y^{\theta_{i|\pi^n}}$, we also have for every $\theta \in \pi^n \setminus \{\theta_{i\ell}\}$, $y^\theta \leq y^{\theta_{i|\pi^n}}$. Therefore, according to Lemma 5, $y^{\pi^n \setminus \{\theta_{i\ell}\}} \leq y^{\theta_{i|\pi^n}} = \min X^{\theta_{i|\pi^n}}$. Thus, we obtain (1a). \square

Proof of (2). According to Lemma 6, inequalities (2) hold whenever

$$y^{\theta_{i\ell}} \leq y^{\pi^{n+1}} \tag{2a}$$

holds. To show (2a), we use (P2).

Suppose that $\theta_{i\ell}$ is such that $y^{\theta_{i\ell}} > \min X^{\theta_{i\ell}}$. Then from (P2), for every $\theta' \in \pi^{n+1}$, $y^{\theta_{i\ell}} \leq y^{\theta'}$. Then (2a) derives from Lemma 4.

If instead $\theta_{i\ell}$ is such that $y^{\theta_{i\ell}} = \min X^{\theta_{i\ell}}$, then from $y^{\pi^n} \in X^{\theta_{i\ell}}$, we have $y^{\pi^n} \geq \min X^{\theta_{i\ell}} = y^{\theta_{i\ell}}$. Then from $y^{\pi^{n+1}} > y^{\pi^n}$, we obtain $y^{\pi^{n+1}} > y^{\theta_{i\ell}}$, which implies (2a). \square

Proof of (3). If $\ell = |\pi^n|$, then for every $\theta \in \pi^n \setminus \{\theta_{i\ell}\}$, $y^\theta \leq y^{\theta_{i\ell}}$. Then From Lemma 5, $y^{\pi^n \setminus \{\theta_{i\ell}\}} \leq y^{\theta_{i\ell}}$, and then from Lemma 6,

$$y^{\pi^n} = y^{(\pi^n \setminus \{\theta_{i\ell}\}) \cup \{\theta_{i\ell}\}} \leq y^{\theta_{i\ell}}. \tag{23}$$

Now suppose that $\theta_{i\ell}$ is such that $y^{\theta_{i\ell}} > \min X^{\theta_{i\ell}}$. Then we have: for every $\theta' \in \pi^{n+1}$, $y^{\theta_{i\ell}} \leq y^{\theta'}$. Then from Lemma 4, we have $y^{\theta_{i\ell}} \leq y^{\pi^{n+1}}$, and from Lemma 10, we deduce $y^{\theta_{i\ell}} \leq y^{\pi^{n+1} \cup \{\theta_{i\ell}\}}$. Then from (23), we obtain $y^{\pi^n} \leq y^{\theta_{i\ell}} \leq y^{\pi^{n+1} \cup \{\theta_{i\ell}\}}$, so that (3) holds. If instead $y^{\theta_{i\ell}} = \min X^{\theta_{i\ell}}$, then from $y^{\pi^{n+1} \cup \{\theta_{i\ell}\}} \in X^{\theta_{i\ell}}$, we necessarily have $y^{\pi^{n+1} \cup \{\theta_{i\ell}\}} \geq \min X^{\theta_{i\ell}} = y^{\theta_{i\ell}}$, which implies (3) using (23). \square

Lemma 10. *Let $\Pi = \pi^1 \cup \dots \cup \pi^{|\Pi|}$ be a partition of Θ that satisfy (IR), (P1) and (L-IC). Let $n_2 \in \{2, \dots, |\Pi|\}$, $n_1 \in \{1, \dots, n_2 - 1\}$ and let $\theta \in \pi^{n_2}$. Then for every $y_1, y_2 \in \mathbb{R}$, if $y_1 < y_2$, $y_2 \leq y^{\pi^{n_2}}$ and $y_1 \leq y^{\pi^{n_1}}$, then $y_1 \notin X^\theta(y_2)$.*

Proof. Since Π satisfies (L-IC), given $n_2 \in \{2, \dots, |\Pi|\}$, $n_1 \in \{1, \dots, n_2 - 1\}$ and $\theta \in \pi^{n_2}$, we have $y^{\pi^{n_1}} \notin X^\theta(y^{\pi^{n_2}})$. Therefore either $y^{\pi^{n_1}} < x_-^\theta(y^{\pi^{n_2}})$ or $y^{\pi^{n_1}} > x_+^\theta(y^{\pi^{n_2}})$. Since Π satisfies

(P1), we have $y^{\pi^{n_1}} < y^{\pi^{n_2}} \leq x_+^\theta(y^{\pi^{n_2}})$, so that necessarily

$$y^{\pi^{n_1}} < x_-^\theta(y^{\pi^{n_2}}). \quad (24)$$

Let us first suppose $y_2 \in X^\theta(y^{\pi^{n_2}})$. This means $U^\theta(y_2) \geq U^\theta(y^{\pi^{n_2}})$, which implies $X^\theta(y_2) \subseteq X^\theta(y^{\pi^{n_2}})$. In particular, $x_-^\theta(y^{\pi^{n_2}}) \leq x_-^\theta(y_2)$. This gives, with $y_1 \leq y^{\pi^{n_1}}$ and (24),

$$y_1 \leq y^{\pi^{n_1}} < x_-^\theta(y^{\pi^{n_2}}) \leq x_-^\theta(y_2).$$

In particular, $y_1 \notin X^\theta(y_2)$.

Now suppose $y_2 \notin X^\theta(y^{\pi^{n_2}})$. This means $U^\theta(y_2) < U^\theta(y^{\pi^{n_2}})$, so that $y^{\pi^{n_2}} \in X^\theta(y_2)$. In particular,

$$y^{\pi^{n_2}} \leq x_+^\theta(y_2).$$

Let us consider the cases $y^{\pi^{n_2}} < x_+^\theta(y_2)$ and $y^{\pi^{n_2}} = x_+^\theta(y_2)$ respectively. If $y^{\pi^{n_2}} < x_+^\theta(y_2)$, then from $y_2 \leq y^{\pi^{n_2}}$, we obtain $y_2 < x_+^\theta(y_2)$. Then necessarily $y_2 = x_-^\theta(y_2)$. Then from $y_1 < y_2$ we have $y_1 < x_-^\theta(y_2)$, which implies $y_1 \notin X^\theta(y_2)$. If $y^{\pi^{n_2}} = x_+^\theta(y_2)$ then $X^\theta(y_2) = X^\theta(y^{\pi^{n_2}})$, and in particular $x_-^\theta(y^{\pi^{n_2}}) = x_-^\theta(y_2)$. This gives, with $y_1 \leq y^{\pi^{n_1}}$ and (24),

$$y_1 \leq y_1^{\pi^{n_1}} < x_-^\theta(y^{\pi^{n_2}}) = x_-^\theta(y_2),$$

and thus $y_1 \notin X^\theta(y_2)$. □